Note on purity of bi-ideals on semigroups

by

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1. A subsemigroup $A$ of a semigroup $S$ is called a bi-ideal of $S$ if $ASA \subseteq A$. The notion of bi-ideal was introduced by R. A. Good and D. Hughes [2]. It was also a special case of the $(m, n)$-ideal introduced by S. Lajos [5]. Let $B(S)$ be the set of all bi-ideals of a semigroup $S$. We define a binary operation on $B(S)$ as follows: For $X, Y \in B(S)$,

$$XY = \{xy : x \in X \text{ and } y \in Y\}.$$  

Then the product $XY$ is a bi-ideal of $S$ ([6] Theorem 8), and $B(S)$ is a semigroup.

A semigroup $S$ is called regular if, for any element $a$ of $S$, there exists an element $x$ in $S$ such that $a = axa$.

J. Luh has given in [8] the following:

**Proposition 1.** For a semigroup $S$ the following conditions are equivalent:

1. $S$ is regular;
2. $B(S)$ is regular.

The following is evident:

**Proposition 2.** If $B(S)$ is idempotent, then it is regular.

The present note is concerned with the problem: For what kind of semigroups does the converse of Proposition 2 hold?

A semigroup $S$ is called intra-regular if, for any element $a$ of $S$, there exist $x$ and $y$ in $S$ such that $a = xa^2y$. S. Lajos proved that, for an intra-regular semigroup $S$, $B(S)$ is idempotent if and only if $B(S)$ is regular ([7] Theorem 36), and gave an example of a semigroup $S$ such that $B(S)$ is regular but not idempotent ([7] Example 4). A semigroup $S$ is called normal if $aS = Sa$ for all elements $a$ of $S$ ([9]). The author proved that, for a normal semigroup $S$, $B(S)$ is idempotent if and only if $B(S)$ is regular ([4]).

In discussing the above problem we shall introduce the notion of purity of bi-ideals of a semigroup, which is called the $T$-purity. It is an analogous notion of $I$-pure ideal of a semigroup introduced by the author [3]. In this note we shall give some properties of $T$-pure bi-
ideals and prove that, for a semigroup $S$ such that every bi-ideal of it is $T$-pure, $B(S)$ is idempotent if and only if $B(S)$ is regular. For the terminology not defined here we refer to the book by A. H. Clifford and G. B. Preston [1].

2. A bi-ideal $A$ of a semigroup $S$ is called $T$-pure if

$$A \cap xSy = xAy$$

for all elements $x$ and $y$ of $S$. A semigroup $S$ is called $T^*$-pure if every bi-ideal of it is $T$-pure. The semigroup $S$ itself is a trivial example of a $T$-pure bi-ideal of $S$. It is clear that a group is a $T^*$-pure semigroup (see, [1] p. 84).

We denote by $[a]$ the principal bi-ideal of a semigroup $S$ generated by $a$ in $S$. Then, by S. Lajos [5],

$$[a] = a \cup a^2 \cup aSa.$$ 

**Lemma 3.** For any bi-ideal $A$ of a semigroup $S$, the following conditions are equivalent:

1. $A \cap XSY = XAY$ for all $X, Y \in B(S)$;
2. $A \cap [x]S[y] = [x]A[y]$ for all $x, y \in S$.

**Proof.** It is clear that (1) implies (2). Assume that (2) holds. Let $X$ and $Y$ be any bi-ideals of $S$ and $a = xsy(a \in A, x \in X, s \in S, y \in Y)$ any element of $A \cap XSY$. Then we have

$$a = xsy \in A \cap [x]S[y] = [x]A[y] \subseteq XAY.$$ 

Thus we have

$$A \cap XSY \subseteq XAY$$

for all $X, Y \in B(S)$. We note that (2) implies that

$$[x]A[y] \subseteq A$$

for all $x, y \in S$. Then in order to prove that

$$XAY \subseteq A$$

for all $X, Y \in B(S)$, let $xay(x \in X, a \in A, y \in Y)$ be any element of $XAY$. Then we have

$$xay \in [x]A[y] \subseteq A$$

and so we have

$$XAY \subseteq A.$$ 

Since the inclusion

$$XAY \subseteq XSY$$

always holds, we have
\[XAY \subseteq A \cap XSY\]
for all \(X, Y \in B(S)\). Therefore we have
\[A \cap XSY = XAY\]
for all \(X, Y \in B(S)\). Thus we obtain that (2) implies (1).

The proof is complete.

**Lemma 4.** Let \(A\) be any \(T^*\)-pure bi-ideal of a smigroup \(S\). Then any one of the conditions (1), (2) of Lemma 3 holds.

**Proof.** It suffices to prove that (1) of Lemma 3 holds. Let \(X\) and \(Y\) be any bi-ideals of \(S\), and \(a = xsy(a \in A, x \in X, s \in S, y \in Y)\) any element of \(A \cap XSY\). Then we have
\[a = xsy \in A \cap xSY = xAy \subseteq XAY\]
and so we have
\[A \subseteq XSY \subseteq XAY.\]
Let \(xay(x \in X, a \in A, y \in Y)\) be any element of \(XAY\). Then we have
\[xay \in xAy = A \cap xsy \subseteq A \cap XSY\]
and so we have
\[XAY \subseteq A \cap XSY.\]
Thus we obtain that
\[A \cap XSY = XAY\]
for all \(X, Y \in B(S)\), and that the \(T^*\)-purity of the bi-ideal \(A\) implies that (1) of Lemma 3 holds.

**Theorem 5.** For a smigroup \(S\) the following conditions are equivalent:

1. \(S\) is \(T^*\)-pure.
2. Every principal bi-ideal of \(S\) is \(T^*\)-pure.

**Proof.** It is clear that (1) implies (2). Assume that (2) holds. Let \(A\) be any bi-ideal of \(S\), and \(x\) and \(y\) any elements of \(S\). Let \(a = xsy\) \((a \in A, s \in S)\) be an element of \(A \cap xSy\). Then we have
\[a = xsy \in [a] \cap xSy = x[a]y \subseteq xAy\]
and so we have
\[A \cap xSy \subseteq xAy\]
for all \(x, y \in S\). We note that (2) implies
\[x[a]y \subseteq [a]\]
for all \( x, y \in S \). In order to prove that
\[
xAy \subseteq A
\]
for all \( x, y \in S \), let \( xay (a \in A) \) be any element of \( xAy \). Then we have
\[
xay \in x[a]y \subseteq [a] \subseteq A
\]
and so we have
\[
xAy \subseteq A
\]
for all \( x, y \in S \). Since the inclusion
\[
xAy \subseteq xSy
\]
always holds, we have
\[
xAy \subseteq A \cap xSy
\]
for all \( x, y \in S \). Thus we obtain that
\[
A \cap xSy = xAy
\]
for all \( x, y \in S \). Therefore we obtain that (2) implies (1).
This complete the proof of the theorem.

3. In this section we consider some properties of minimal \( T \)-pure bi-ideals of a semigroup.

**Theorem 6.** For any minimal bi-ideal \( A \) of a semigroup \( S \), the following conditions are equivalent:

1. \( A \) is \( T \)-pure:
2. \( A = xAy \) for all \( x, y \in S \).

*Proof.* Assume that \( A \) is \( T \)-pure. Then, for all \( x, y \in S \), we have
\[
xAy = A \cap xSy \subseteq A.
\]
Since \( xAy \) is a bi-ideal of \( S \) by Theorem 8 of [6], it follows from the minimality of \( A \) that
\[
xAy = A.
\]
Thus we obtain that (1) implies (2).

Conversely we assume that (2) holds. Then, for all \( x, y \in S \), we have
\[
A \cap xSy = xAy \cap xSy = xAy.
\]
This means that \( A \) is \( T \)-pure. Therefore we obtain that (2) implies (1).

**Corollary 7.** The minimal \( T \)-pure bi-ideal of a semigroup is regular.

*Proof.* Let \( A \) be the minimal \( T \)-pure bi-ideal of a semigroup \( S \), and \( a \) any element of \( A \). Then by Theorem 6 we have
This means that $A$ is regular.

**Lemma 8.** For any bi-ideal $A$ of a semigroup $S$, the following conditions are equivalent:

1. $A = XAY$ for all $X, Y \in B(S)$;

**Proof.** It is clear that (1) implies (2). Assume that (2) holds. Let $X$ and $Y$ be any bi-ideals of $S$, and $x$ and $y$ respectively elements of $X$ and $Y$. Then we have

$$A = [x]A[y] \subseteq XAY.$$ 

Let $xay$ $(x \in X, a \in A, y \in Y)$ be any element of $XAY$. Then we have

$$xay \in [x]A[y] = A$$

and so we have

$$XAY \subseteq A.$$ 

Thus we have

$$A = XAY$$

for all $X, Y \in B(S)$. Therefore we obtain that (2) implies (1).

**Theorem 9.** Let $A$ be any minimal $T$-pure bi-ideal of a semigroup $S$. Then any one of the conditions (1), (2) of Lemma 8 holds.

**Proof.** It suffices to prove that (1) of Lemma 8 holds. Let $X$ and $Y$ be any bi-ideals of $S$. Then, since $A$ is $T$-pure, it follows from Lemma 4 that

$$XAY = A \cap XSY \subseteq A.$$ 

Since $XAY$ is a bi-ideal of $S$, it follows from the minimality of $A$ that

$$XAY = A.$$ 

Therefore we obtain that (1) of Lemma 8 holds.

4. A semigroup $S$ is called $T$-pure-free if it does not properly contain any $T$-pure bi-ideal. In this section we give a class of a $T$-pure-free semigroup.

A semigroup $S$ is called archimedean if, for any elements $a$ and $b$ of $S$, there exists a positive integer $n$ for which

$$a^n \in SbS.$$ 

**Theorem 10.** A cancellative archimedean semigroup without idempotent is $T$-pure-free.

**Proof.** Let $A$ be any $T$-pure bi-ideal of a cancellative archimedean semigroup $S$ without idempotent, and $a$ and $s$ respectively any elements
of $A$ and $S$. Since $S$ is archimedean, there exist elements $x$ and $y$ in $S$ and a positive integer $n$ such that
\[ a^n = xsy. \]
Since $A$ is $T$-pure, we have
\[ a^n = xsy \in A \cap xSy = xAy. \]
This implies that there exists an element $b$ in $A$ such that
\[ xsy = xby. \]
Since $S$ is cancellative, we have
\[ s = b \in A \]
and so we have
\[ S \subseteq A. \]
This we obtain that
\[ A = S. \]
Since $A$ is any $T$-pure bi-ideal of $S$, this means that $S$ is $T$-pure-free. This completes the proof of the theorem.

5. In this section we give our main result.

**Theorem 11.** For a $T^*$-pure semigroup $S$ the following conditions are equivalent:

1. $S$ is regular;
2. $B(S)$ is regular;
3. $B(S)$ is idempotent.

*Proof.* By Propositions 1 and 2, it suffices to prove that (2) implies (3). We assume that $B(S)$ is regular. Let $A$ be and bi-ideal of $S$. Then for some $X \in B(S)$ we have
\[ A = AXA \subseteq ASA \subseteq A \]
and so we have
\[ A = ASA. \]
Since $A$ is $T$-pure, it follows from Lemma 4 that
\[ XAY = A \cap XSY \]
for all $X, Y \in B(S)$. This holds for $X = Y = A$. Then we have
\[ A^3 = A \cap ASA = A \cap A = A \]
and so we have
\[ A = A^3 \subseteq A^4 \subseteq A. \]
Thus we have

\[ A = A^2. \]

Therefore we obtain that \( B(S) \) is idempotent. This completes the proof of the theorem.

References


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