Representations of the Fundamental Groups of 3-Manifolds, I

by

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In [4], we have considered the representations of the fundamental groups of 3-manifolds obtained by Dehn surgeries along 2-bridge knots. In this paper, we shall show that this method can be applied also to manifolds which are not obtained by Dehn surgery along a knot.

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§ 1. Lens space conjecture

Unless otherwise stated, we denote by $M$ a closed orientable connected 3-manifold and by $\pi_1(M)$ its fundamental group. By a lens space we mean a closed 3-manifold obtained by glueing the boundaries of two solid tori. Thus we include $S^3$ and $S^2 \times S^1$ in lens spaces. Let $Z_n$ be the finite cyclic group of order $n$.

First we consider the following conjecture:

Conjecture 1 (Haken). If $\pi_1(M) \cong Z_n$, then $M$ is a lens space.

We call this conjecture the lens space conjecture. Obviously this conjecture for $n = 1$ is just the Poincaré conjecture. We first derive some consequences from this conjecture.

Theorem 1. Suppose that the lens space conjecture is true. Then, if $\pi_1(M)$ is abelian, then either $M$ is a lens space or $M$ is homeomorphic to $S^3 \times S^1 \times S^1$.

Proof. Suppose that the lens space conjecture is true and that $\pi_1(M)$ is abelian. Then by Epstein [1], $\pi_1(M)$ is isomorphic to one of the following groups:

$Z_n$, $Z$, $Z_n \times Z$, $Z \times Z$, $Z \times Z \times Z$.

If $\pi_1(M)$ is finite, then it must be isomorphic to $Z_n$, and hence by the lens space conjecture $M$ is a lens space. If $\pi_1(M)$ is infinite, then $H_1(M)$ ($\cong \pi_1(M)$) is infinite and hence by Waldhausen [6], $M$ is sufficiently large. So $M$ contains an incompressible surface $F$. Since $\pi_1(M)$ is abelian, the genus of $F$ must be 0 or 1. First suppose that the genus of $F$ is 0, that is, $F$ is a 2-sphere. By a standard argument we can assume that $F$ is separating in $M$, unless $M$ is homeomorphic to $S^2 \times S^1$. Then $M$ is the connected
sum of two closed 3-manifolds $M_1$ and $M_2$ which are not homeomorphic to $S^3$, and 
$\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$, (a free product). Since the lens space conjecture implies
Poincaré conjecture, $\pi_1(M_1)$ and $\pi_1(M_2)$ are non-trivial. But then $\pi_1(M)$ cannot be
abelian. Next suppose that the genus of $F$ is 1, that is, $F$ is a torus. If $F$ is separating, then
$\pi_1(M)$ is an amalgamated free product

$$\pi_1(M) = \pi_1(M_1) *_{\pi_1(F)} \pi_1(M_2)$$

where $\partial M_1 = \partial M_2 = F$. Since $\pi_1(M)$ is abelian, this amalgamated free product must be
trivial:

$$\pi_1(M_1) \ (\text{or} \ \pi_1(M_2)) \cong \pi_1(F) \cong \mathbb{Z} \times \mathbb{Z}.$$

But there does not exist a 3-manifold $M_1$ such that $\partial M_1$ is a torus and $\pi_1(M_1) = \mathbb{Z} \times \mathbb{Z}$. For, by Waldhausen [7], there is an incompressible surface $G$ in $M_1$ such that $0 \not\in [\partial G] \in H_1(\partial M_1)$. $G$ must be a 2-disk or an annulus. In either case, a contradiction arises. Finally suppose that $F$ is a torus and non-separating in $M$. We choose a base
point $P$ on $F$. Let $a$ and $b$ be loops on $F$ which represent independent generators of
$\pi_1(F) = \mathbb{Z} \times \mathbb{Z}$. Let $c$ be a loop in $M$ such that $c$ intersects $F$ transversely only at one
point $P$. Since $\pi_1(M)$ is abelian, we have $ab = ba$, $ac = ca$, $bc = cb$. Moreover
$d^2b^m c^n = 1$ implies $n = 0$ (since the intersection number of $d^2b^m c^n$ with $F$ must be 0) and
hence $l = 0$ and $m = 0$ (since $\pi_1(F) \rightarrow \pi_1(M)$ is injective). Thus $\pi_1(M)$ contains a
subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Hence by Epstein's result mentioned above,
$\pi_1(M)$ must be isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Since we are assuming the lens space
conjecture and hence Poincaré conjecture, $M$ must be irreducible. And $M$ is also
sufficiently large. So it is determined by $\pi_1(M)$ (Waldehausen [7]). Hence $M$ must be
homeomorphic to $S^1 \times S^1 \times S^1$. This completes the proof of the theorem.

**Corollary 2.** Suppose that the lens space conjecture is true. Then, if $M$ has a
Heegaard splitting of genus 2 and $\pi_1(M)$ is abelian, then $M$ is a lens space. In other
words, if $M$ is of Heegaard genus 2, then $\pi_1(M)$ is non-abelian.

**Proof.** This follows immediately from the Theorem 1, since $S^1 \times S^1 \times S^1$
does not have Heegaard splittings of genus 2.

**Corollary 3.** Suppose that the lens space conjecture is true. Then, (i) if
$\pi_1(M) \cong \mathbb{Z}$, then $M$ is homeomorphic to $S^2 \times S^1$, and (ii) if $\pi_1(M) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, then $M$
is homeomorphic to $S^1 \times S^1 \times S^1$.

The following conjecture is well-known:

**Conjecture 2.** If $Z_n$ acts freely on $S^3$, then the quotient space is a lens space.

**Theorem 4 (Haken).** The lens space conjecture is equivalent to the conjunction
of Poincaré conjecture and Conjecture 2.

**Proof.** Clearly the lens space conjecture implies Poincaré conjecture and
Conjecture 2. Conversely suppose that Poincaré conjecture and Conjecture 2 are true
but the lens space conjecture is false. Then there exists a 3-manifold \( M \) with \( \pi_1(M) \cong \mathbb{Z}_n \) which is not a lens space. Consider the universal cover \( \tilde{M} \) of \( M \). By Poincaré conjecture \( \tilde{M} \) is homeomorphic to \( S^3 \) since \( \tilde{M} \) is compact. The covering translations constitute a group isomorphic to \( \mathbb{Z}_n \) and this group acts freely on \( \tilde{M} \) and the quotient space is \( M \). This contradicts Conjecture 2.

\section{Representations of \( \pi_1(M) \)}

First we define the following four groups:

\[
\begin{align*}
PGL(2, \mathbb{C}) &= GL(2, \mathbb{C})/\{ \lambda E \}, \\
PSL(2, \mathbb{C}) &= SL(2, \mathbb{C})/\{ \pm E \}, \\
\mathfrak{M} &= \text{the group of all Möbius transformations} \\
&= \frac{az+b}{cz+d},
\end{align*}
\]

where \( a, b, c, d \in \mathbb{C} \) and \( ad - bc \neq 0 \).

\[
I^+(H^3) = \text{the group of all orientation-preserving isometries of the hyperbolic 3-space} \, H^3.
\]

Then it is known that these four groups are all isomorphic:

\[
PGL(2, \mathbb{C}) \cong PSL(2, \mathbb{C}) \cong \mathfrak{M} \cong I^+(H^3).
\]

Hereafter, by a representation of \( \pi_1(M) \) we shall mean a representation of \( \pi_1(M) \) into \( PGL(2, \mathbb{C}) \). Let \( h \) and \( h' \) be two representations of \( \pi_1(M) \). \( h \) and \( h' \) are said to be equivalent if there exists an \( A \in PGL(2, \mathbb{C}) \) such that for all \( x \in \pi_1(M) \),

\[
h'(x) = Ah(x)A^{-1}.
\]

In many cases, the number of the equivalence classes of representations of \( \pi_1(M) \) is finite with the exception of the connected sums of lens spaces, some of sufficiently large manifolds, etc. Let \( \delta(M) \) be the number of the equivalence classes of representations of \( \pi_1(M) \). Then \( \delta(M) \) is a (computable) invariant of \( M \). We conjecture the following:

\textbf{Conjecture 3.} If \( M \) is irreducible but not sufficiently large, then \( \delta(M) \) is finite.

A representation is said to be abelian, cyclic, trivial, etc., if so is its image.

\textbf{Conjecture 4.} If \( M \) is not homeomorphic to \( S^3 \), then there exists a non-trivial representation of \( \pi_1(M) \).

Obviously this conjecture implies Poincaré conjecture.

\textbf{Conjecture 5.} If \( M \) is irreducible but not sufficiently large, and not a lens space, then there exists a non-abelian representation of \( \pi_1(M) \).

\textsuperscript{*1} \( E \) is the identity matrix.
This conjecture implies the lens space conjecture.

Example. There exists an irreducible, sufficiently large, closed 3-manifold $M$ such that
\[
\pi_1(M) \cong \langle a, b \mid a^3b^{-2}a^3b^{-1} = b^3a^2b^3a^{-1} = 1 \rangle.
\]
This $\pi_1(M)$ is non-abelian (in fact, a non-trivial amalgamated free product, see [2]), but it can be shown that all the representations of $\pi_1(M)$ are abelian.

§ 3. A class of 3-manifolds

For 3-manifolds obtained by Dehn surgeries along 2-bridge knots, the computation of the representations of $\pi_1(M)$ is carried out in [4]. The remainder of this paper is devoted to computing all the representations of $\pi_1(M)$ for a certain class of 3-manifolds. The class of 3-manifolds we will consider appears in [3], and each manifold in this class has a Heegaard splitting of genus 2 and has the corresponding presentation of the fundamental group in which one of the relators is of length 10.

In order to describe the class of 3-manifolds, first we consider a solid torus $V$ of genus 2. $V$ can be viewed as obtained from a 3-disk $D^3$ by gluing $\alpha^+$ to $\alpha^-$ and $\beta^+$ to $\beta^-$, where $\alpha^+, \alpha^-, \beta^+, \beta^-$ are disjoint 2-disks on $\partial D^3$. Then, $\alpha = \alpha^+ = \alpha^-$ and $\beta = \beta^+ = \beta^-$ (in $V$) constitute a system of meridian disks of $V$. Let $c$ be the loop on $V$ as shown in Fig. 1. (We glue $\alpha^+$ to $\alpha^-$ and $\beta^+$ to $\beta^-$ so that the points with the same number coincide.) We attach a 2-handle $D^2 \times D^1$ to $V$ along $c$, that is, we glue $\partial D^2 \times D^1$ to
$\mathcal{N}(c)$, where $D^2$ is a 2-disk, $D^1$ is $[0, 1]$ and $\mathcal{N}(c)$ is the closure of a regular neighborhood of $c$ in $\partial V$. Then we obtain a 3-manifold $N$ with a torus as its boundary. $\pi_1(M)$ has the following presentation:

$$\pi_1(M) \cong \langle a, b \mid a^3 b^{-1} a b^3 a b^{-1} = 1 \rangle,$$

where $a$ and $b$ are generators corresponding to the meridian disk $\alpha$ and $\beta$ respectively, and the relator corresponds to the loop $c$ and is read from Figure 1. Let $\gamma^+ = D^2 \times \{0\}$ and $\gamma^- = D^2 \times \{1\}$. $\gamma^+ \cup \gamma^- \cup \partial \beta \subseteq \partial N$ is called the reverse graph of $c$. Since $\partial N$ is a torus, its universal covering space $P$ is a plane. The reverse graph of $c$ induces an infinite graph on $P$, as shown in Fig. 2. This is called the reverse development of $c$. (Cf. [3].)

![Fig. 2](image)

Now let $T$ be a solid torus. If we glue $\partial T$ to $\partial N$ in any way, then we obtain a closed orientable 3-manifold. It is determined by the homotopy type of a loop $d$ on $\partial N$ which is identified with a meridian of $T$ by the gluing. This homotopy type is, in
turn, determined by a pair \((m, n)\) of relatively prime integers, where \(d\) is homotopic to \(mx+ny\) and, \(x\) and \(y\) are loops on \(\partial N\) as shown in Figure 2. The closed manifold obtained is denoted by \(M_{m,n}\). Obviously, \(M_{-m,-n}=M_{m,n}\). Hence we can assume \(m \geq 0\). Since \(x\) and \(y\) correspond to the words \(ab^{-1}a^2b^{-1}\) and \(ab^2\) respectively and they commute in \(\pi_1(N)\), we have presentations
\[
\pi_1(M_{m,n}) \cong \langle a, b \mid a^2b^{-1}ab^3ab^{-1} = (ab^{-1}a^2b^{-1})^m(ab^2)^n = 1 \rangle
\]
\[
\cong \langle a, b \mid a^2b^{-1}ab^3ab^{-1} = (b^{-3}a^2b^{-1})^m(ab^2)^j = 1 \rangle,
\]
where \(j = m + n\).

§ 4. Computation of representations of \(\pi_1(N)\)

We shall find all the representations of \(\pi_1(N)\) and of \(\pi_1(M_{m,n})\). First we determine all the representations of \(\pi_1(N)\).

**Lemma 4.** For any non-negative integer \(n\), let
\[
\begin{pmatrix}
p_n \\ q_n \\ r_n \\ s_n
\end{pmatrix} = \begin{pmatrix}
p \\ q \\ r \\ s
\end{pmatrix}^n.
\]
Moreover we define polynomials \(\rho_n = \rho_n(x, y)\), inductively as follows:
\[
\rho_0 = 0, \quad \rho_1 = 1, \quad \rho_{n+2} = x\rho_{n+1} + y\rho_n.
\]
Let \(x = p + s\) and \(y = qr - ps\). Then, we have
\[
p_n = p\rho_n + y\rho_{n-1}, \quad q_n = q\rho_n,
\]
\[
r_n = r\rho_n, \quad s_n = s\rho_n + y\rho_{n-1}.
\]

**Proof.** By the induction on \(n\).

Note that
\[
\rho_2 = x, \quad \rho_3 = x^2 + y, \quad \rho_4 = x^3 + 2xy, \quad \rho_5 = x^4 + 3x^2y + y^2, \ldots.
\]

**Corollary.** \((p, q)^n\) is a scalar matrix \(\lambda E\), if and only if \(\rho_n(x, y) = 0\), where \(x = p + s\) and \(y = qr - ps\).

**Proof.** \((p, q)^n\) is a scalar matrix if and only if \(p_n - s_n = q_n - r_n = 0\). By Lemma 1, this condition is equivalent to \(\rho_n = 0\).

q.e.d.

Here we note that any matrix \(A\) in \(GL(2, C)\) has its Jordan normal form
\[
\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
\lambda & 1 \\
0 & \lambda
\end{pmatrix}.
\]
If the latter is the case, \(A\) is called parabolic.

Now let \(A, B \in GL(2, C)\). We define
\[
A \approx B \iff \exists \lambda \neq 0, \quad \lambda A = B.
\]
That is, \( A \approx B \) if and only if \( \bar{A} = \bar{B} \), where \( \bar{A} \) and \( \bar{B} \) are elements of \( PGL(2, C) \) corresponding to \( A \) and \( B \), respectively.

Let \( h \) be a representation of \( \pi_1(N) \), and let \( h(a) = \bar{A} \) and \( h(b) = \bar{B} \). Then, by (1),
\[
\bar{A}^3 \bar{B}^{-1} \bar{A} \bar{B}^3 \bar{A} \bar{B}^{-1} = 1,
\]
that is, \( A^3 B^{-1} AB^3 AB^{-1} \approx \bar{E} \). Conversely, if \( \bar{A}, \bar{B} \in PGL(2, C) \) are such that (3) holds, then the equations \( h(a) = \bar{A} \) and \( h(b) = \bar{B} \) define a representation of \( \pi_1(N) \). \(^*\)

We denote these equations by
\[
a \rightarrow A, \quad b \rightarrow B.
\]

**Theorem 6.**

(i) Let \( A, B \in GL(2, C) \) and \( \bar{B} = \bar{A}^{-5} \), then (4) defines an abelian representation of \( \pi_1(N) \). Every abelian representation is obtained in this way. Two such representations are equivalent if and only if the corresponding \( \bar{A} \)'s are conjugate in \( PGL(2, C) \).

(ii) Let \( \lambda, \mu \in C \) be such that \( \lambda \mu \neq 0 \) and \( \lambda^3 \neq \mu^3 \). Let
\[
p = \lambda^5 + 2\lambda^5 \mu + 3\lambda^4 \mu^2 + 2\lambda^3 \mu^3 + 2\lambda^2 \mu^4 + \lambda \mu^5,
\]
\[
s = -\lambda^5 \mu - 2\lambda^4 \mu^2 - 2\lambda^3 \mu^3 - 3\lambda^2 \mu^4 - 2\lambda \mu^5 - \mu^6,
\]
and let \( q, r \in C \) be such that both are not zero and \( qr = ps - \lambda^3 \mu^2(\lambda^3 - \mu^3)^2 \). Then the correspondence
\[
a \rightarrow A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad b \rightarrow B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},
\]
defines a non-abelian representation of \( \pi_1(N) \).

(iii) Also the correspondence
\[
a \rightarrow A = \begin{pmatrix} 10 & 1 \\ 11 & 2 \end{pmatrix}, \quad b \rightarrow B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
defines a non-abelian representation of \( \pi_1(N) \).

(iv) Every non-abelian representation of \( \pi_1(N) \) is equivalent to one of the representations defined in (ii) and (iii).

(v) A representation defined in (ii) and the one defined in (iii) are not equivalent.

(iv) Two representations

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\(^*\) \( h(a) = \bar{A} \) and \( h(b) = \bar{B} \) clearly define a representation of the free group \( G \) generated by \( a \) and \( b \). Let \( \mathcal{N} \) be the least normal subgroup generated by \( a^3 b^{-1} ab^3 ab^{-1} \). By (3), \( h(\mathcal{N}) = \{ \bar{E} \} \). So \( h \) is uniquely defined by the following commutative diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{h} & PGL(2, C) \\
j & \Bigg\downarrow \Bigg\uparrow \hspace{1cm} h \\
G/\mathcal{N} = \langle a, b | a^3 b^{-1} ab^3 ab^{-1} \rangle
\end{array}
\]

where \( j \) is the natural homomorphism.
\[ a \to \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad b \to \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \]

and

\[ a \to \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}, \quad b \to \begin{pmatrix} \lambda' & 0 \\ 0 & \mu' \end{pmatrix}, \]

which are defined as in (ii) are equivalent, if and only if one of the following is satisfied:

1. \( \lambda : \mu = \lambda' : \mu' \) or \( \lambda : \mu = \mu : \lambda' \)
2. \( \lambda : \mu = \lambda' : \mu' \) and \( q = q' = 0 \) or \( r = r' = 0 \)
3. \( \lambda : \mu = \mu' : \lambda' \) and \( q = r' = 0 \) or \( r = q' = 0 \)

**Proof.** The abelian case (i) is obvious. Suppose that \( h \) is a non-abelian representation of \( \pi_1(N) \) defined by

\[ a \to A, \quad b \to B. \]

First suppose that \( B \) is not parabolic. Then we can assume that \( B \) is in Jordan normal form \((\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix})\) with \( \lambda \neq \mu, \lambda \mu \neq 0 \). Let \( A = (\begin{smallmatrix} q & 0 \\ 0 & p \end{smallmatrix}) \). Then we must have \( A^3 B^{-1} A B^{-1} A^3 B^{-1} \approx E \). Let \( x = p + s \) and \( y = q r - p s \). Then by Lemma 1 we have

\[ p_3 = p \rho_3 + y \rho_2, \quad q_3 = q \rho_3, \quad r_3 = r \rho_3, \quad s_3 = s \rho_3 + y \rho_2. \]

By computation we have

\[
A^3 B^{-1} A B^3 A B^{-1} \approx \begin{pmatrix} p_3 & q_3 \\ r_3 & s_3 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \lambda^3 & 0 \\ 0 & \mu^3 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} q_3 p r \lambda^4 \mu + p_3 p r^2 \lambda^3 \mu^2 + q_3 r s \lambda \mu^4 + p_3 s q r \mu^5 \\ s_3 r p \lambda^4 \mu + r_3 p r^2 \lambda^3 \mu^2 + s_3 r s \lambda \mu^4 + r_3 s q r \mu^5 \end{pmatrix} \]

Hence we must have

\[
q_3 p r \lambda^4 \mu + p_3 p r^2 \lambda^3 \mu^2 + q_3 r s \lambda \mu^4 + p_3 s q r \mu^5 = s_3 q r \lambda^5 + r_3 p q \lambda^4 \mu + s_3 s^2 \lambda^2 \mu^3 + r_3 q s \lambda \mu^4,
\]

\[
q_3 r \lambda^5 + p_3 p q \lambda^4 \mu + q_3 s^2 \lambda^2 \mu^3 + r_3 q s \lambda \mu^4 = 0,
\]

\[
s_3 q r \lambda^5 + r_3 q p \lambda^4 \mu + s_3 s^2 \lambda^2 \mu^3 + r_3 q s \lambda \mu^4 = 0;
\]

or,

\[
X \equiv s_3 q r \lambda^5 - p_3 p \lambda^3 \mu^2 + s_3 s^2 \lambda^2 \mu^3 - p_3 q r \mu^5 = 0, \tag{5}
\]

\[
q(p_3 q r \lambda^4 + p_3 p \lambda^3 \mu + r_3 s^2 \lambda \mu^3 + p_3 q s \mu^4) = 0, \tag{6}
\]

\[
r(s_3 q \lambda^4 + p_3 p \lambda^3 \mu + s_3 s \lambda \mu^3 + p_3 q s \mu^4) = 0. \tag{7}
\]

Suppose that \( q \neq 0 \) and \( r \neq 0 \). From (6) and (7), it follows that

\[
Y \equiv p_3 q r \lambda^4 + p_3 p \lambda^3 \mu + r_3 s^2 \lambda \mu^3 + p_3 q s \mu^4 = 0, \tag{8}
\]

\[
Z \equiv s_3 q \lambda^4 + p_3 p \lambda^3 \mu + s_3 s \lambda \mu^3 + p_3 q s \mu^4 = 0. \tag{9}
\]
Note that
\[ \rho_3 X = s_3 \lambda Y + p_3 \mu Z. \] (10)

Now
\[ q = y + ps, \quad \rho_3 = x^2 + y, \quad p_3 = px^2 + (p + x)y. \]

Substituting these into (8), we obtain
\[ y^2 \mu^4 + y\{p(s + x)\lambda^2 + p^2 \lambda^3 \mu + s(s + x)\lambda \mu^3 + (x^2 + ps)\mu^4\} \]
\[ + \{psx^2 \lambda^4 + p^2 x^2 \lambda^3 \mu + s^2 x^2 \lambda \mu^3 + psx^2 \mu^4\} = 0. \] (11)

Similarly we obtain from (9)
\[ y^2 \mu^4 + y\{p(s + x)\lambda^2 + p^2 \lambda^3 \mu + s(s + x)\lambda \mu^3 + (x^2 + ps)\mu^4\} \]
\[ + \{psx^2 \lambda^4 + p^2 x^2 \lambda^3 \mu + s^2 x^2 \lambda \mu^3 + psx^2 \mu^4\} = 0. \] (12)

Subtracting (12) from (11), we have
\[ y^2(\lambda^4 - \mu^4) + y(sx \lambda^4 + px \lambda^3 \mu - sx \lambda \mu^3 - px \mu^4) = 0. \]

Since \( y \neq 0 \), we have
\[ y(\lambda^4 - \mu^4) + (sx \lambda^4 + px \lambda^3 \mu - sx \lambda \mu^3 - px \mu^4) = 0. \]

Hence, if \( \lambda^4 \neq \mu^4 \), we have
\[ y = -\frac{(\lambda^2 + \lambda \mu + \mu^2)(s \lambda + p \mu)}{(\lambda + \mu)(\lambda^2 + \mu^2)} x. \] (13)

On the other hand we must also have
\[ B^3AB^{-1}A^3B^{-1}A \cong E. \]

And
\[ B^3AB^{-1}A^3B^{-1}A \cong \begin{pmatrix} \lambda^3 & 0 \\ 0 & \mu^3 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} p_3 & q_3 \\ r_3 & s_3 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \]
\[ = \begin{pmatrix} \lambda^3 (s_3 s \lambda + 2 p q r \rho_3 \lambda \mu + p_3 p^2 \mu^2) \\ \mu^3 (s_3 s \lambda + 2 p q r \rho_3 \lambda \mu + p_3 p^2 \mu^2) \end{pmatrix}. \]

So we must have
\[ s_3 s \lambda^2 + \rho_3 (ps + qr) \lambda \mu + p_3 p \mu^2 = 0, \] (14)
or,
\[ y^2 \lambda \mu + y(s \lambda + p \mu)((s \lambda + p \mu) + (p + s)(\lambda + \mu)) + x^2(s \lambda + p \mu)^2 = 0. \] (15)

Substituting (13) into (15), we easily obtain
\[ \lambda^3 \mu^3 \mu = (\lambda + \mu)(\lambda^2 + \mu^2)(\lambda^2 + \lambda \mu + \mu^2)(s \lambda + p \mu). \] (16)
From this it follows that
\[ p(\lambda^5 \mu + 2\lambda^4 \mu^2 + 2\lambda^3 \mu^3 + 3\lambda^2 \mu^4 + 2\lambda \mu^5 + \mu^6) \]
\[ + s(\lambda^6 + 2\lambda^5 \mu + 3\lambda^4 \mu^2 + 2\lambda^3 \mu^3 + 2\lambda^2 \mu^4 + \lambda \mu^5 + \mu^6) = 0. \]  
(17)

Since this holds also when some scalar is multiplied to the matrix \((p \quad q)\), we may assume from (17) that
\[ p = \lambda^6 + 2\lambda^5 \mu + 3\lambda^4 \mu^2 + 2\lambda^3 \mu^3 + 2\lambda^2 \mu^4 + \lambda \mu^5 + \mu^6, \]  
(18)
\[ s = -(\lambda^5 \mu + 2\lambda^4 \mu^2 + 2\lambda^3 \mu^3 + 3\lambda^2 \mu^4 + 2\lambda \mu^5 + \mu^6). \]  
(19)

Then, we have
\[ x = p + s = (\lambda + \mu)(\lambda^2 + \mu^2)(\lambda^3 - \mu^3), \]  
(20)
\[ s\lambda + p\mu = \lambda^3 \mu^2(\lambda - \mu). \]  
(21)

Hence by (13),
\[ y = qr - ps = -\lambda^3 \mu^3(\lambda^3 - \mu^3)^2. \]  
(22)

It follows that \(\lambda^3 \neq \mu^3\). In the above, the case \(\lambda^4 = \mu^4\) was excluded. In this case, we must have \(s\lambda + p\mu = 0\) or \(x = 0\). But, if \(s\lambda + p\mu = 0\), then by (15) we must have \(y^2 \lambda \mu = 0\). This is impossible. If \(x = 0\), then (16) holds (since \(\lambda \neq \mu\)) and hence we obtain (18), (19) and (22) also in this case.

It remains the case \(q = 0\) or \(r = 0\). We shall show that (18), (19) and (22) hold also in this case. Since the case \(q = 0\) is treated similarly, we only treat the case \(r = 0\). Then we have \(q \neq 0\) (otherwise we would have \(AB = BA\)) and
\[ A = \begin{pmatrix} p & q \\ 0 & s \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}. \]

(5), (8) and (14) are available in this case. So, if we show that (9) is also available, then (18), (19) and (22) will follow. We show it by using (10). Since \(y = -ps \neq 0\), we have
\[ p \lambda = p(x^2 + y) + xy = p^3. \]

So, we have \(p \lambda \mu \neq 0\), \(X = 0\), \(Y = 0\). Hence by (10), we have \(Z = 0\). Thus (9) is available, as desired.

Thus, we have proved that if \(h\) is a non-abelian representation of \(\pi_1(N)\) in which \(h(b)\) is not parabolic, then \(h\) is equivalent to a representation defined in (ii) of Theorem 5.

Conversely, suppose that \(\lambda \mu = 0\), \(\lambda^3 \neq \mu^3\), \((q \neq 0\) or \(r \neq 0)\) and that (18), (19) and (22) hold. Direct computation shows that the correspondence
\[ a \to A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad b \to B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \]
defines a non-abelian representation of \(\pi_1(N)\). Thus (ii) of Theorem 5 is proved.

Next suppose that \(h\) is a non-abelian representation of \(\pi_1(N)\) defined by \(a \to A\),
$b \rightarrow B$, and that $B$ is parabolic. Then we may assume that $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

Since $A^{3}B^{-1}AB^{3}A^{-1} \approx E$, we have $BA^{-3}B \approx AB^{3}A$, that is,

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
s_{3} & -q_{3} \\
r_{3} & p_{3}
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\approx
\begin{pmatrix}
p & q \\
r & s
\end{pmatrix}
\begin{pmatrix}
1 & 3 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
p & q \\
r & s
\end{pmatrix},
\]

or,

\[
\begin{pmatrix}
s_{3} - r_{3} & s_{3} + p_{3} - r_{3} - q_{3} & p^{2} + 3pr + qr & pq + 3ps + qs \\
r_{3} & p_{3} - r_{3} & pr + 3r^{2} + rs & qr + 3rs + s^{2}
\end{pmatrix}.
\]  

(23)

We show that $r \neq 0$. Suppose that $r = 0$. Then,

\[
\begin{align*}
s_{3} &= s \rho_{3} + y \rho_{2} \\
&= s(x^{2} + y) + yx \\
&= s(x^{2} - ps) - psx \\
&= s^{3},
\end{align*}
\]

and similarly we have $p_{3} = p^{3}$. Hence (23) becomes

\[
\begin{pmatrix}
s^{3} & s^{3} + p^{3} - q_{3} \\
0 & p^{3}
\end{pmatrix}
\approx
\begin{pmatrix}
p^{2} & pq + 3ps + qs \\
0 & s^{2}
\end{pmatrix}.
\]

So we must have

\[
p^{5} = s^{5}
\]  

(24)

and

\[
s^{2}(s^{3} + p^{3} - q_{3}) = p^{3}(pq + 3ps + qs).
\]  

(25)

Since $AB \approx BA$, we must have $p \neq s$. Hence it follows that

\[
p^{4} + p^{3}s + p^{2}s^{2} + ps^{3} + s^{4} = 0.
\]  

(26)

Moreover from (25) we have

\[
-3p^{4}s + p^{3}s^{2} + s^{5} = s^{2}q_{3} + p^{4}q + p^{3}sq
\]

\[
= s^{2}q(x^{2} + y) + p^{4}q + p^{3}sq
\]

\[
= q(p^{4} + p^{3}s + p^{2}s^{2} + ps^{3} + s^{4})
\]

\[
= 0.
\]

From this and (26), we must have $p = s = 0$, a contradiction. Thus $r \neq 0$.

Now from (23) we have

\[
(s_{3} - r_{3}) : (p^{2} + 3pr + qr) = (s_{3} + p_{3} - r_{3} - q_{3}) : (pq + 3ps + qs)
\]

\[
= (- \rho_{3}) : (p + 3r + s)
\]

\[
= (p_{3} - r_{3}) : (qr + 3rs + s^{2}),
\]
or,

\[(s_3 - r_3)(p + 3r + s) + (p^2 + 3pr + qr)r_3 = 0 \quad (27)\]
\[(p_3 - r_3)(p + 3r + s) + (s^2 + 3sr + qr)r_3 = 0 \quad (28)\]
\[(s_3 + p_3 - r_3 - q_3)(p + 3r + s) + (pq + 3ps + qs)r_3 = 0 \quad (29)\]

But,

\[r_3 = x^2 + y,\]
\[p + 3r + s = x + 3r,\]
\[s_3 - r_3 = (s - r)(x^2 + y) + xy,\]
\[p_3 - r_3 = (p - r)(x^2 + y) + xy,\]
\[s_3 + p_3 - r_3 - q_3 = (x - r - q)(x^2 + y) + 2xy,\]
\[p^2 + 3pr + qr = p(x + 3r) + y,\]
\[a^2 + 3sr + qr = s(x + 3r) + y,\]
\[pq + 3ps + qs = q(x + 3r) - 3y.\]

Hence, (27) and (28) become the same equation

\[(x + 3r)(x - r)(x^2 + y) + xy(x + 3r) + y(x^2 + y) = 0, \quad (30)\]

while (29) becomes

\[(x + 3r)(x - r)(x^2 + y) + 2xy(x + 3r) - 3y(x^2 + y) = 0. \quad (31)\]

From (30) and (31), we have

\[x(x + 3r) = 4(x^2 + y), \quad (32)\]

since \(y \neq 0\). Hence \(x \neq 0\) and

\[r = x + \frac{4y}{3x}. \quad (33)\]

Substituting it in (30) we have

\[(x^2 + y)(x^2 + 16y) = 0.\]

But if \(x^2 + y = 0\), then

\[
\begin{pmatrix}
  s_3 - r_3 & s_3 + p_3 - r_3 - q_3 \\
  -r_3 & p_3 - r_3
\end{pmatrix}
\begin{pmatrix}
  x y \\
  0
\end{pmatrix}
= \begin{pmatrix}
  2xy \\
  xy
\end{pmatrix},
\]
\[
\begin{pmatrix}
  p^2 + 3pr + qr & pq + 3ps + qs \\
  pr + 3r^2 + rs & qr + 3rs + s^2
\end{pmatrix}
\begin{pmatrix}
  y \\
  -3y
\end{pmatrix}
= \begin{pmatrix}
  y \\
  0
\end{pmatrix},
\]

and
\[
\begin{pmatrix}
xy & 2xy \\
0 & xy
\end{pmatrix} \cong \begin{pmatrix}
y & -3y \\
0 & y
\end{pmatrix}.
\]

This contradicts (23). Hence
\[x^2 + 16y = 0.\]  \hfill (34)

From this and (33) we have
\[r = \frac{11}{12} x.\]  \hfill (35)

Conversely, (34) and (35) together with \(y \neq 0\), \(p \neq s\) are sufficient for a non-abelian representation of \(\pi_1(N)\). Now from (34) and (35), the values of \(p\) and \(s\) determine the values of \(q\) and \(r\). We shall show the resulting representations are all equivalent, irrespective of the values of \(p\) and \(s\). Now the correspondence
\[
a \rightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p + tr & q - tp + ts - t^2r \\ r & s - tr \end{pmatrix},
\]
\[
b \rightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
gives an equivalent representation to the original one. That is, \((p, s)\) and \((p + tr, s - tr)\) give equivalent representations. But, \((p + tr)/(s - tr)\) takes arbitrary values at \(t\) varies. Thus, all the representations considered in this case are equivalent. So, as a representative of these we can choose the one defined by
\[
a \rightarrow \begin{pmatrix} 10 & 1 \\ 11 & 2 \end{pmatrix}, \quad b \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Thus we have proved (iii) and (iv) of the theorem. Moreover (v) is obvious since \(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\) are not conjugate.

Finally we shall show (vi). Let \(h\) and \(h'\) be two non-abelian representations defined respectively by
\[
a \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad b \rightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix};
\]
and
\[
a \rightarrow \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}, \quad b \rightarrow \begin{pmatrix} \lambda' & 0 \\ 0 & \mu' \end{pmatrix}.
\]

Since they are non-abelian it follows that \(\lambda \neq \mu\) and \(\lambda' \neq \mu'\). And \(\{\lambda, \mu\}\) and \(\{\lambda', \mu'\}\) are eigenvalues of \(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\), respectively. The last two matrices are conjugate in \(GL(2, \mathbb{C})\) if and only if \(\{\lambda, \mu\} = \{\lambda', \mu'\}\). Hence these are conjugate in \(PGL(2, \mathbb{C})\) if and only if \(\lambda: \mu = \lambda': \mu'\) or \(\lambda: \mu = \mu': \lambda'\). So, this condition is necessary for the equivalence of \(h\) and \(h'\).
Suppose that $h$ and $h'$ are equivalent. Then, for some $A \in \text{GL}(2, C)$,
\[ A \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} A^{-1} \approx \begin{pmatrix} \lambda' & 0 \\ 0 & \mu' \end{pmatrix}, \]
\[ A \begin{pmatrix} p & q \\ r & s \end{pmatrix} A^{-1} \approx \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}. \]
(36)
(37)

If $\lambda: \mu = \lambda': \mu' \neq -1$, then by (36), $A$ must be of the form $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So, $q = 0$ implies $q' = 0$, and $r = 0$ implies $r' = 0$. If $\lambda: \mu = \mu': \lambda' \neq -1$, then by (36), $A$ must be of the form $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So, $q = 0$ implies $r' = 0$, and $r = 0$ implies $q' = 0$. If $\lambda: \mu = -1$, then we may assume that $\lambda = 1$, $\mu = 1$. Then by (18), (19) and (22), we have $p = 1$, $s = -1$, $qr - ps = 4$. So, $qr \neq 0$.

These considerations show that if $h$ and $h'$ are equivalent, then one of the conditions (I), (II), (III) of Theorem 5, (vi), holds. Next suppose that (I) holds. Since
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \]
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} s & r \\ q & p \end{pmatrix}, \]
(38)
we may assume that $\lambda: \mu = \lambda': \mu'$. But then we may also assume that $\lambda = \lambda'$ and $\mu = \mu'$. Then we have by (ii) that $p = p'$, $s = s'$ and $qr = q'r' \neq 0$. So there exists an $\alpha \neq 0$ such that $\alpha^2 q = q'$. Now
\[ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \]
\[ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} p & \alpha^2 q \\ r'^{-1} & s \end{pmatrix} = \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}. \]

Hence the two representations are equivalent.

Next we assume that (II) holds. We may assume that $\lambda = \lambda'$, $\mu = \mu'$. If $r = r' = 0$, then $q = 0$, $q' \neq 0$, for otherwise the representations become abelian. So by the same reason as above, the two representations are equivalent. Similarly for the case $q = q' = 0$. The case (III) is reduced to the case (II) by (38). This completes the proof of Theorem 5.

§ 5. Computation of representations of $\pi_1(M_{m, n})$

Theorem 7. If $(m, n) \neq (0, 1), \pm (1, 0)$, then $\pi_1(M_{m, n})$ has a non-abelian representation and hence it is non-abelian. Moreover $M_{0, 1}$ is the lens space of type $(9, 2)$ and $M_{1, 0}$ is the lens space of type $(13, 3)$. Therefore the lens space conjecture holds for the class of 3-manifolds $\{M_{m, n}\}$.

The rest of this section is devoted to proving this theorem.

We consider the representations of
\[ \pi_1(M_{m,n}) = \langle a, b \mid a^2b^{-1}ab^3ab^{-1} = (ab^{-1}a^{-1}a^{-1})^m(ab^2)^n = 1 \rangle = \langle a, b \mid a^3b^{-1}ab^3ab^{-1} = (b^{-3}a^2b^{-1})^m(ab^2)^n = 1 \rangle, \]

where \( j = m + n \). Let

\[
A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}
\]

be a representation of \( \pi_1(N) \) defined in Theorem 6(ii). Then it becomes a representation of \( \pi_1(M_{m,n}) \) if and only if

\[
(B^{-3}A^2B^{-1})^m(AB^2)^n \approx E.
\]

Let

\[
ab^2 \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \lambda^2 & 0 \\ 0 & \mu^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]

\[
b^{-3}a^2b^{-1} \rightarrow \begin{pmatrix} \mu^3 & 0 \\ 0 & \lambda^3 \end{pmatrix} \begin{pmatrix} p_2 & d_2 \\ r_2 & s_2 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}.
\]

Then,

\[
\alpha' = \lambda^2p = \lambda^3(\lambda^3 + 2\lambda^4\mu + 3\lambda^2\mu^2 + 2\lambda^2\mu^3 + 2\lambda^4\mu^4 + \mu^5)
\]

(abbreviated by \( \lambda^3 (1, 2, 3, 2, 2, 1) \)),

\[
\beta' = \mu^2q, \quad \gamma' = \lambda^2r, \quad \delta' = \mu^2s = -\mu^3 (1, 2, 2, 3, 2, 1),
\]

\[
\alpha'' = \mu^4p_2 = \lambda^3\mu^4(\lambda^3 - \mu^3) (1, 3, 6, 7, 9, 8, 6, 3, 1),
\]

\[
\beta'' = \mu^3xq = \lambda^3(\lambda + \mu)(\lambda^2 + \mu^2)(\lambda^3 - \mu^3)q,
\]

\[
\gamma'' = \lambda^3\mu x r = \lambda^3(\lambda + \mu)(\lambda^2 + \mu^2)(\lambda^3 - \mu^3)r,
\]

\[
\delta'' = \lambda^4s_2 = -\lambda^4(\lambda^3 - \mu^3) (1, 3, 6, 8, 9, 7, 6, 3, 1).
\]

Let

\[
\alpha' = \lambda\mu^4 (1, 3, 6, 7, 9, 8, 6, 3, 1),
\]

\[
\beta' = \lambda\mu^3(\lambda + \mu)(\lambda^2 + \mu^2)q,
\]

\[
\gamma' = \lambda^3(\lambda + \mu)(\lambda^2 + \mu^2)r,
\]

\[
\delta' = -\lambda^4\mu (1, 3, 6, 8, 9, 7, 6, 3, 1).
\]

Then,
\[
\left(\begin{array}{c}
\alpha'' \\
\beta'' \\
\gamma'' \\
\delta''
\end{array}\right) \approx \left(\begin{array}{c}
\alpha' \\
\beta' \\
\gamma' \\
\delta'
\end{array}\right)
\]

Note that \(ab^2\) and \(b^{-3}a^2b^{-1}\) commute in \(\pi_1(N)\). Hence
\[
\left(\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta
\end{array}\right) \text{ and } \left(\begin{array}{c}
\alpha' \\
\beta' \\
\gamma' \\
\delta'
\end{array}\right)
\]

must commute in \(PGL(2, C)\).

Let \(\kappa\xi, \kappa\eta\) be the eigen-values of \((\xi \ y)\), and \(\theta\zeta', \theta\eta'\) be those of \((\zeta' \ y')\). Then,
\[
\kappa(\xi + \eta) = \alpha + \beta, \quad \kappa^2 \xi \eta = \alpha \delta - \beta \gamma, \quad \text{(39)}
\]
\[
\theta(\zeta' + \eta') = \alpha' + \delta', \quad \theta^2 \xi' \eta' = \alpha' \delta' - \beta' \gamma'.
\]

Hence we have
\[
(\xi + \eta)^2(\alpha \delta - \beta \gamma) = \xi \eta (\alpha + \delta)^2, \quad (\xi' + \eta')^2(\alpha' \delta' - \beta' \gamma') = \xi' \eta'(\alpha' + \delta')^2. \quad \text{(40)}
\]

We first assume that \(\xi \neq \eta, \xi \neq -\eta\). Then, for some \(P \in GL(2, C)\),
\[
P \left(\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta
\end{array}\right) P^{-1} = \left(\begin{array}{c}
\kappa \xi \\
0 \\
0 \\
\kappa \eta
\end{array}\right).
\]

Since \((x \ y)\) and \((z' \ y')\) commute in \(PGL(2, C)\), \(P(x' \ y')P^{-1}\) is also diagonal and equal to either \((\theta \xi' \ 0 \ 0 \ \eta')\) or \((\theta' \ 0 \ 0 \ \eta)\). We may assume that \(P(x' \ y')P^{-1} = (\theta \xi' \ 0 \ 0 \ \eta)\). Then
\[
P \left(\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta
\end{array}\right) \left(\begin{array}{c}
\alpha' \\
\beta' \\
\gamma' \\
\delta'
\end{array}\right) P^{-1} = \left(\begin{array}{c}
\kappa \theta \xi' \\
0 \\
0 \\
\kappa \theta \eta'
\end{array}\right).
\]

So the eigen-values of \((x \ y)\) are \(\kappa \theta \xi'\) and \(\kappa \theta \eta'\), and we have
\[
\alpha \alpha' + \beta \gamma' + \gamma \beta' + \delta \delta' = \kappa \theta (\xi' + \eta'). \quad \text{(41)}
\]

Also by (39) we have
\[
\kappa \theta (\xi + \eta)(\xi' + \eta') = (\alpha + \delta)(\alpha' + \delta'). \quad \text{(42)}
\]

From (41) and (42) we have
\[
(\alpha \alpha' + \beta \gamma' + \gamma \beta' + \delta \delta')(\xi + \eta)(\xi' + \eta') = (\alpha + \delta)(\alpha' + \delta')(\xi' + \eta'),
\]
or
\[
P_0(\xi \eta' + \xi \eta') = Q_0(\xi \xi' + \eta \eta'),
\]
where
\[
P_0 = \alpha \alpha' + \beta \gamma' + \gamma \beta' + \delta \delta' \quad \text{and} \quad Q_0 = \alpha \delta' - \beta \gamma' - \gamma \beta' + \delta \alpha'.
\]

From (43) we have
\[ \xi' : \eta' = (P_0 \xi - Q_0 \eta) : (Q_0 \xi - P_0 \eta). \]

So if \( \theta \) is suitably chosen, we may assume
\[ \xi' = P_0 \xi - Q_0 \eta, \quad \eta' = Q_0 \xi - P_0 \eta, \]
unless they are both zero.

Now the second relator becomes
\[ (b^{-3}a^2b^{-1})^m(ab^2)^l \rightarrow \left( \begin{array}{c} \alpha' \\ \beta' \\ \gamma \\ \delta' \end{array} \right)^m \left( \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} \right)^l \approx E. \]

From this, it is necessary that two eigen-values of
\[ \left( \begin{array}{c} \alpha' \\ \beta' \\ \gamma \\ \delta' \end{array} \right)^m \left( \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} \right)^l \]
coincide, that is, \( \xi'^m \xi^l = \eta'^m \eta^l \), or
\[ (P_0 \xi - Q_0 \eta)^m \xi^l = (P_0 \xi - Q_0 \eta)^m \eta^l. \quad (43) \]

But this condition is also sufficient in this case. For, if \( (\xi', \eta')^m(\xi, \eta)^l \approx E \) does not hold but (44) holds, then \( (\xi', \eta')^m(\xi, \eta)^l \) must be parabolic and hence \( (\xi', \eta') \) is also parabolic, since \( (\xi', \eta') \) and \( (\xi', \eta') \) commute. But it is impossible since we assume \( \xi \neq \eta \).

In order to obtain representations of \( \pi_1(M_{m,n}) \), we solve the simultaneous homogeneous equations
\[ (\xi + \eta)^2(\alpha \delta - \beta \gamma) = \xi \eta(\alpha + \delta)^2, \quad (40) \]
\[ (P_0 \xi - Q_0 \eta)^m \xi^l = (P_0 \xi - Q_0 \eta)^m \eta^l. \quad (43) \]

As in [4], we consider the solutions of these equations as the intersection of two algebraic curves in \( CP^1 \times CP^1 \) with the coordinate system \( \{ (\lambda, \mu; \xi, \eta) \} \), where \( \lambda, \mu \) are not both zero, and \( \xi, \eta \) are not both zero, and \( (\lambda, \mu; \xi, \eta) \) and \( (\lambda', \mu'; \xi', \eta') \) denote the same point iff \( \lambda' = \sigma \lambda, \mu' = \sigma \mu, \xi' = \tau \xi, \eta' = \tau \eta \), for some \( \sigma \neq 0 \) and \( \tau \neq 0 \). These solutions give desired representations iff \( \lambda \mu \neq 0, \lambda^3 \neq \mu^3, \xi \eta \neq 0, \xi \neq \eta, \xi \neq -\eta. \)

Now,
\[ \alpha \delta - \beta \gamma = - \lambda^2 \mu^2 \gamma = \lambda^5 \mu^5 (\lambda^3 - \mu^3)^2, \]
\[ \alpha + \delta = (\lambda^2 - \mu^2) (1, 2, 4, 3, 4, 2, 1). \]

So, (40) becomes
\[ (\xi + \eta)^2 \lambda^5 \mu^5 (\lambda^2 + \lambda \mu + \mu^2)^2 = \xi \eta (\lambda + \mu)^2 (1, 2, 4, 3, 4, 2, 1)^2. \quad (44) \]

Moreover,
\[ P_0 = \alpha \lambda' + \beta \gamma' + \gamma \beta' + \delta \delta' \]
\[ = - \lambda^5 \mu^5 (\lambda - \mu)^2 (\lambda + \mu) (1, 2, 4, 5, 4, 2, 1). \]
\[ Q_0 = -\lambda \mu (\lambda - \mu)^2 (\lambda + \mu) (1, 6, 113, 185, 261, 316, 339, 316, 261, 185, 113, 56, 22, 6, 1) \]
\[ = -\lambda \mu (\lambda - \mu)^2 (\lambda + \mu)(\lambda^2 + \lambda \mu + \mu^2)^2 \]
\[ (1, 4, 11, 20, 31, 37, 43, 37, 31, 20, 11, 4, 1) . \]

In (44), if \( \lambda + \mu = 0 \), then \( \xi + \eta = 0 \). So it does not give representations. So we can divide (43) by \(-\lambda \mu (\lambda - \mu)^2 (\lambda + \mu)\). Then we have

\[ (P \xi - Q \eta)^m \xi^j = (Q \xi - P \eta)^m \eta^j , \] (45)

where \( P = \lambda^2 \mu^5 \) (1, 2, 4, 5, 4, 2, 1) and

\[ Q = (\lambda^2 + \lambda \mu + \mu^2)^2 (1, 4, 11, 20, 31, 37, 43, 37, 31, 20, 11, 4, 1) . \]

So we solve the simultaneous equations (44) and (45).

In (44),

(i) if \( \lambda \mu = 0 \), then \( \xi \eta = 0 \);
(ii) if \( \lambda^2 + \lambda \mu + \mu^2 = 0 \), then \( \xi \eta = 0 \);
(iii) if \( \lambda = \mu \), then \( 9(\xi + \eta)^2 = 1156 \xi \eta \);
(iv) if \( \xi \eta = 0 \), then \( \lambda \mu = 0 \) or \( \lambda^2 + \lambda \mu + \mu^2 = 0 \);
(v) if \( \xi = \eta \), then \( 4\lambda^2 \mu^5 (\lambda^2 + \lambda \mu + \mu^2)^2 = (\lambda + \mu)^2 (1, 2, 4, 3, 4, 2, 1) \);
(vi) if \( \xi = -\eta \), then \( \mu = -\lambda \) or (1, 2, 4, 3, 4, 2, 1) = 0.

By the way there are only finitely many such exceptional points.

First we remark that \( \lambda = \mu \) does not occur in any solution of (44) and (45). For, if \( \lambda = \mu \), then from (44) we have \( 9(\xi + \eta)^2 = 1156 \xi \eta \), that is,

\[ 9\xi^2 - 1138 \xi \eta + 9\eta^2 = 0 . \]

So if we put \( x = \xi / \eta \), then \( x \) satisfies the equation

\[ 9x^2 - 1138x + 9 = 0 . \] (46)

Moreover, by (45) we have

\[ (Px - Q)^m x^j = (Qx - P)^m , \]

\[ x^j = \left( \frac{Qx - P}{Px - Q} \right)^m = \left( \frac{2259x - 19}{19x - 2259} \right)^m . \]

So, if we put

\[ y = \frac{2259x - 19}{19x - 2259} , \]

we have

\[ 9y^2 - 17938y + 9 = 0 , \] (47)

and
\[ x^j = y^n. \] (48)

And each of \( x^n \) and \( y^n \) satisfies the equation of form
\[ 9^n r^2 - k t + 9^n = 0, \]
where \( k \) is an integer relatively prime to 9. So we must have \(|j| = |m| = 1\) and \( x = y \) or \( y^{-1} \), by (48). But this is impossible by (46) and (47). So \( \lambda \neq \mu \) in any solution of (44) and (45).

Now, let
\[ f(\lambda, \mu; \zeta, \eta) = (\xi + \eta)^2 \xi^2 \lambda^2 + \xi^2 \lambda \mu + \mu^2)^2 - \xi \eta (\lambda + \mu)^2 (1, 2, 4, 3, 4, 2, 1)^2 \] (49)

and
\[ g_{m, j}(\lambda, \mu; \zeta, \eta) = (P \xi - Q \eta)^m \xi^j - (Q \xi - P \eta)^m \eta^j \] (50)

and we consider the simultaneous equations
\[ f(\lambda, \mu; \zeta, \eta) = 0, \quad g_{m, j}(\lambda, \mu; \zeta, \eta) = 0, \] (51)
in \( CP^1 \times CP^1 \). \( f \) is of degree \((2, 14)\) and \( g_{m, j} \) is of degree \((m + |j|, 16m)\). By Bezout's theorem for \( CP^1 \times CP^1 \) (cf. [4]), the total sum of the number of intersection is
\[ 2 \cdot 16m + 14(m + |j|) = 46m + 14 |j|. \]

First we compute the numbers of intersections at
\[ (\lambda, \mu; \zeta, \eta) = (0, 1; 0, 1), \quad (0, 1; 1, 0), \quad (1, 0; 0, 1), \quad (1, 0; 1, 0). \]

We easily see that the numbers of intersections at these points are the same. So we only compute that of the point \( F = (0, 1; 0, 1) \).

Now the only parametrization of (49) with center at \( F \) is given by
\[ \lambda = t, \quad \mu = 1; \quad \zeta = t^5 - 4t^6 + 6t^7 - t^9 + 34t^{10} + \cdots, \quad \eta = 1. \]

Then
\[ (P \xi - Q \eta)^m \xi^j = \pm t^j + \cdots, \]
\[ (Q \xi - P \eta)^m \eta^j = \pm t^m + \cdots. \]

So, if \( 5j \neq 8m, j \geq 0 \), then
\[ \text{ord } (g_{m, j}) = \text{min } (5j, 8m). \]

Moreover we can show that if \( 5j = 8m \), i.e. \((m, j) = (5, 8)\), then
\[ \text{ord } (g_{m, j}) = 41. \]

If \( j < 0 \), we easily obtain
\[ \text{ord } (g_{m, j}) = 0. \]
So, if we denote the number intersection at \( F \) by \( \tau(F) \), then
\[
\tau(0, 1; 0, 1) = \tau(0, 1; 0, 1) = \tau(0, 1; 0, 1) = \tau(1, 0; 1, 0)
\]
\[
= \begin{cases} 
\min (5j, 8m), & \text{if } j \geq 0, 5j \neq 8m, \\
41, & \text{if } (m,j) = (5, 8), \\
0, & \text{if } j < 0.
\end{cases}
\]

Next we consider the exceptional points with \( \lambda^2 + \mu + \mu^2 = 0, \xi \eta = 0 \). Let \( \omega \) denote a root of \( \omega^2 + \omega + 1 = 0 \). Then there are four points under consideration:
\[
(\lambda, \mu; \xi, \eta) = (\omega, 1; 0, 1), \quad (\omega^2, 1; 0, 1), \quad (\omega, 1; 1, 0), \quad (\omega^2, 1; 1, 0).
\]

Of course the numbers of intersection at these points are equal. So we only treat the point \((\omega, 1; 0, 1)\). The place with center at this point is given by
\[
\lambda = \omega + t, \quad \mu = 1; \quad \xi = -3\omega t^2 + \cdots, \quad \eta = 1.
\]
If \( j \geq 0 \), then \( \operatorname{ord}(g_{m,j}) = 0 \). So the number of intersection \( \tau(\omega, 1; 0, 1) = 0 \). If \( j < 0 \), then
\[
(P_x - Q_y)^{m} \eta^{j} = 3^{m} \eta^{2m} + \cdots,
\]
\[
(Q_x - P_y)^{m} \xi^{j} = (-3\omega)^{-j} t^{2j} + \cdots.
\]

So if \( 2m \neq -2j \), then \( \tau(\omega, 1; 0, 1) = \min (2m, -2j) \). If \( 2m = -2j \), i.e. \( (m,j) = (1, -1) \), then \( \tau(\omega, 1; 0, 1) = 3 \). Thus
\[
\tau(\omega, 1; 0, 1) = \tau(\omega^2, 1; 0, 1) = \tau(\omega, 1; 1, 0) = \tau(\omega^2, 1; 1, 0)
\]
\[
= \begin{cases} 
0, & \text{if } j \geq 0, \\
\min (2m, -2j), & \text{if } j < 0, (m,j) \neq (1, -1), \\
3, & \text{if } (m,j) = (1, -1).
\end{cases}
\]

Next we compute the number of intersection at \((-1, 1; -1, 1)\). Let \( \lambda = -1 + t, \mu = 1, \eta = 1 \). Then, by (44), \( \xi = -1 + 3t - 3t^2 + \cdots \), or \( \xi = -1 - 3t - 6t^2 + \cdots \). If \( \xi = -1 + 3t - 3t^2 + \cdots \), then
\[
(P_x - Q_y)^{m} \xi^{j} - (Q_x - P_y)^{m} \eta^{j}
\]
\[
= \{(1 - 1 + t)^2 (3 - 9t)(-1 + 3t - (1 - 2t)(7 - 42t))\}^{m} (1 + 3t)^{j}
\]
\[
- \{(1 - 2t)(7 - 42t)(-1 + 3t) - (1 + t)^2 (3 - 9t)\}^{m} + \cdots
\]
\[
= (-4 + 23t) (1 + 3t)^{j} - (-4 + 101t)^{m} + \cdots
\]
\[
= \{-4)^{m} (1)^{j} - (-4)^{m} + 2(-4)^{m-1} (39m - 2(-3)^{j}) t + \cdots.
\]

So
ord \( (g_{m,j}) = \begin{cases} 0, & \text{if } j \text{ is odd}, \\ 1, & \text{if } j \text{ is even}. \end{cases} \)

The same holds when \( \xi = -1 - 3t - 6t^2 + \cdots \). Thus

\[ i(-1, 1; -1, 1) = \begin{cases} 0, & \text{if } j \text{ is odd}, \\ 1 \times 2 = 2, & \text{if } j \text{ is even}. \end{cases} \]

But if \( j \) is even, there exists a homomorphism

\[ \pi_1(M_{m,n}) \cong \langle a, b \mid a^2 b^{-1} ab b^{-1} = (b^{-3} a^2 b^{-1})^m (ab)^2 \rangle = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle. \]

The latter group has the representations

\[ a \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}, \]

where

\[ ab \rightarrow \begin{pmatrix} ps + qr & -2pq \\ -2rs & ps + qr \end{pmatrix} \]

and \( 3(ps)^2 + 10(ps)(qr) + 3(qr)^2 = 0 \), so that \((ab)^3 = 1\). In some sense, these representations may be counted twice. So we do not subtract \( i(-1, 1; -1, 1) \) from the total number of intersection.

Next we compute the number of intersection at \((\lambda, \mu; \xi, \eta) (\alpha, 1; -1, 1)\), where \( \alpha \) is a root of

\[ \phi(t) = t^6 + 2t^5 + 4t^4 + 3t^3 + 4t^2 + 2t + 1 = 0. \]

First we remark that \( \phi(t) \) is an irreducible polynomial in \( \mathbb{Z}[t] \) and hence does not have double roots. So there exist six roots of it and they determine six points \((\alpha, 1; -1, 1)\)'s in \( CP^1 \times CP^1 \).

We shall show that

\[ i(\alpha, 1; -1, 1) = 2m, \]

for each \( \alpha \). Now there are two places with center at this point. These are parametrized by

\[ \xi = -1 + t, \quad \lambda = \alpha + at + bt^2 + \cdots, \quad (52) \]

\[ \eta = 1, \quad \mu = 1. \]

In order to find the values of \( a \), we substitute \( (52) \) into \( f(\lambda, \mu; \xi, \eta) \). Then

\[ 0 = f(-1 + t, 1; \alpha + at + bt^2 + \cdots, 1) \]

\[ \equiv t^2(\alpha + at)^5 \{(\alpha + at)^2 + (\alpha + at) + 1\}^2 \]

\[ -(-1 + t)(1 + \alpha + at)^2 \phi(\alpha + at)^2 \]
Thus, we have
\[ a = \pm \frac{\frac{-\alpha^2(\alpha^2 + \alpha + 1)^2}{(\alpha + 1)^2} \frac{1}{\phi'(\alpha)}}{(\alpha + 1)\phi'(\alpha)} \sqrt{-\alpha} \sqrt{\alpha^2 + \alpha + 1} \].

Next we compute the order of \( g_{m,j}(\lambda, \mu; \xi, \eta) \). We assume \( j > 0 \). The case \( j < 0 \) can be treated similarly.

\[ g_{m,j}(-1 + t, 1, \alpha + at + b t^2 + \cdots, 1) \]
\[ = (P \cdot (-1 + t) - Q)^m (-1 + t)^j - (Q \cdot (-1 + t) - P)^m \]
\[ = (P t - (P + Q))^m (-1 + t)^j - (Q t - (P + Q))^m. \]

Here we notice that
\[ P(\lambda, 1) + Q(\lambda, 1) = \phi(\lambda) \chi(\lambda), \]

where
\[ \chi(\lambda) = \lambda^{10} + 4\lambda^9 + 10\lambda^8 + 17\lambda^7 + 23\lambda^6 + 24\lambda^5 + 23\lambda^4 + 17\lambda^3 + 10\lambda^2 + 4\lambda + 1. \]

Thus,
\[ (P t - (P + Q))^m (-1 + t)^j - (Q t - (P + Q))^m \]
\[ \equiv (P(\lambda, 1)t - a\phi'(\lambda)\chi(\lambda)t)^m (-1)^j - (Q(\lambda, 1)t - a\phi'(\lambda)\chi(\lambda)t)^m \]
\[ \equiv \{(P(\lambda, 1) - a\phi'(\lambda)\chi(\lambda))^m (-1)^j - (Q(\lambda, 1) - a\phi'(\lambda)\chi(\lambda))^m\} t^m \text{ (mod. } t^{m+1}). \]

We show \( \text{ord } (g_{m,j}) = m \) by checking that the coefficient
\[ c = (P(\lambda, 1) - a\phi'(\lambda)\chi(\lambda))^m (-1)^j - (Q(\lambda, 1) - a\phi'(\lambda)\chi(\lambda))^m \neq 0. \]

Suppose that \( c = 0 \). Then
\[ d = \frac{P(\lambda, 1) - a\phi'(\lambda)\chi(\lambda)}{Q(\lambda, 1) - a\phi'(\lambda)\chi(\lambda)} \]
is a \( 2m \)-th root of unity and hence \( d \) must satisfy a cyclotomic equation. Thus if we show that \( d \) satisfies an irreducible non-cyclotomic equation in \( Q \), then we have a contradiction and it shows that \( c \neq 0 \). Now
\[ P(\lambda, 1) = 2\lambda^8, \quad \text{(since } P(\lambda, 1) \equiv 2\lambda^8 \text{ (mod. } \phi(\lambda)) \]
\[ Q(\lambda, 1) = -2\lambda^8, \quad \text{(by the same reason as above)} \]
\[ \chi(\lambda) = -2\lambda^5, \quad \text{(by the same reason as above)} \]
\[ a\phi'(\lambda) = \frac{\pm \alpha^2(\alpha^2 + \alpha + 1)\sqrt{-\alpha}}{\alpha + 1}. \]
\[
\pm \frac{\alpha^2(\alpha^2 + \alpha + 1) \sqrt{-\alpha}}{\alpha + 1} (-2\alpha^5) - 2\alpha^8 \\
\pm \frac{\alpha^2(\alpha^2 + \alpha + 1) \sqrt{-\alpha}}{\alpha + 1} (-2\alpha^5) + 2\alpha^8 \\
= \frac{(\alpha^2 + \alpha + 1) \sqrt{-\alpha + \alpha(\alpha + 1)}}{(\alpha^2 + \alpha + 1) \sqrt{-\alpha + \alpha(\alpha + 1)}} \\
\]

A further (rather long) calculation shows that \(d\) satisfies the irreducible equation
\[
d^6 + 14d^5 + 63d^4 + 36d^3 + 63d^2 + 14d + 1 = 0 ,
\]
which is not cyclotomic.

Thus we have shown that \(\text{ord} (g_m, \iota) = m\), for each place with center at \((\alpha, 1; -1, 1)\). Since there are exactly two places with center at this point, we have that
\[
i(\alpha, 1; -1, 1) = 2 | m |.
\]

Since there are 6 different roots of \(\phi(\alpha) = 0\), we have
\[
\sum_{\phi(\alpha) = 0} (-1, 1; \alpha, 1) = 12 | m |.
\]

Next note that
\[
f(\lambda, \mu; \xi, \eta) = (\phi - \eta)^2 \lambda^5 \mu^5 (\lambda^2 + \lambda \mu + \mu^2)^2 - \xi \eta \cdot \psi(\lambda, \mu),
\]
where
\[
\psi(\lambda, \mu) = (1, 6, 21, 50, 92, 134, 167, 178, 167, 134, 92, 50, 21, 6, 1)
\]
\[
= (1, 4, 8, 9, 8, 4, 1) (1, 2, 5, 6, 5, 5, 2, 1).
\]

Since these factors are reciprocal we can easily compute all the roots of \(\psi(\lambda, 1)\), (e.g. by the aid of programable electronic calculator) and see that \(\psi(\lambda, 1)\) does not have multiple roots.

Let \(\beta\) be any solution of \(\psi(\lambda)\). Then by Walker [8], there exists exactly one place with center at \((\beta, 1; 1, 1)\), which is parametrized by
\[
\xi = 1 + t, \quad \lambda = \beta + at^2 + bt^3 + \cdots ,
\]
\[
\eta = 1, \quad \mu = 1.
\]
Substituting these in \(f(\lambda, \mu; \xi, \eta)\), we obtain
\[
0 = f(\beta + at^2 + bt^3 + \cdots, 1; 1 + t, 1)
\]
\[
= t^2 (\beta + at^2 + bt^3 + \cdots)^5 ((\beta + at^2 + bt^3 + \cdots)^2 + (\beta + at^2 + bt^3 + \cdots) + 1)^2
\]
\[
- (1 + t) \psi(\beta + at^2 + bt^3 + \cdots, 1)
\]
\[
\equiv \{\beta^2 (\beta + 1)^2 - \psi(\beta) a\} t^2 \quad \text{(mod.} t^3) .
\]
Thus

\[ a = \frac{\beta^5(\beta^2 + \beta + 1)}{\psi(\beta)}. \]

Note that

\[ Q - P = (\lambda^2 + \lambda \mu + \mu^2)\psi(\lambda, \mu) \]
\[ = ((\beta + at^2 + bt^3 + \cdots)^2 + (\beta + at^2 + bt^3 + \cdots) + 1)(\psi'(\beta)at^2 + \cdots) \]
\[ \equiv (\beta^2 + \beta + 1)\psi'(\beta)at^2 \]
\[ = \beta^5(\beta^2 + \beta + 1)^3 t^2 \quad (\text{mod. } t^3), \]
\[ P \equiv \beta^5(\beta^6 + 2\beta^5 + 4\beta^4 + 5\beta^3 + 4\beta^2 + 2\beta + 1) \quad (\text{mod. } t^2), \]
\[ Q \equiv \lambda^5 \mu^5(\lambda^6 + 2\lambda^5 \mu + 4\lambda^4 \mu^2 + 5\lambda^3 \mu^3 + 4\lambda^2 \mu^4 + 2\lambda \mu^5 + \mu^6) \quad (\text{mod. } \psi) \]
\[ \equiv \beta^5(\beta^6 + 2\beta^5 + 4\beta^4 + 5\beta^3 + 4\beta^2 + 2\beta + 1) \quad (\text{mod. } t^2). \]

Thus (assuming \( j > 0 \))

\[ g_{m, j}(\beta + at^2 + bt^3 + \cdots, 1; 1 + t, 1) \]
\[ = (P \cdot (1 + t) - Q)^m (1 + t)^j - (Q \cdot (1 + t) - P)^m \]
\[ = (Pt - (Q - P))^m (1 + t)^j - (Qt + (Q - P))^m \]
\[ \equiv (P^m t^m - mP^{m-1} \beta^5(\beta^2 + \beta + 1)^3 t^{m-1}) (1 + jt) \]
\[ - (P^m t^m + mP^{m-1} \beta^5(\beta^2 + \beta + 1)^3 t^{m+s}) \quad (\text{mod. } t^{m+2}) \]
\[ = \{-2mP^{m-1} \beta^5(\beta^2 + \beta + 1)^3 + jP^m \} t^{m+1}, \]

where

\[ \bar{P} = \beta^5(\beta^6 + 2\beta^5 + 4\beta^4 + 5\beta^3 + 4\beta^2 + 2\beta + 1). \]

But it is easily seen that

\[ \frac{\beta^5(\beta^2 + \beta + 1)^3}{\bar{P}} = \frac{(\beta^2 + \beta + 1)^3}{\beta^6 + 2\beta^5 + 4\beta^4 + 5\beta^3 + 4\beta^2 + 2\beta + 1} \]

is irrational. So the coefficient

\[ -2mP^{m-1} \beta^5(\beta^2 + \beta + 1)^3 + jP^m \neq 0. \]

Thus \( i(\beta, 1; 1, 1) = \text{ord} (g_{m, j}) = m + 1. \) There are 14 solution of the equation \( \psi(\lambda) = 0. \) So

\[ \sum_{\psi(\beta) = 0} i(\beta, 1; 1, 1) = 14m + 14. \]

Recall that \( j = m + n. \) Now
\[ d_{m,n} = \text{(the total sum of the numbers of intersections)} \]
\[ -\nu(0,1;0,1) - \nu(0,1;1,0) - \nu(1,0;0,1) - \nu(1,0;1,0) \]
\[ -\nu(\omega,1;0,1) - \nu(\omega^2,1;0,1) - \nu(\omega,1;1,0) - \nu(\omega^2,1;1,0) \]
\[ - \sum_{\phi(\alpha) = 0} \nu(\alpha,1;0,1) - \sum_{\phi(\beta) = 0} \nu(\beta,1;0,1) \]
\[ = 46m + 14 |j| - 4 \max(0, \min(5j, 8m)) - 4 \max(0, \min(-2j, 2m)) \]
\[ - 12m - 14m - 14 \]
\[ = \begin{cases} 14j - 12m - 14, & \text{if } 5j > 8m, \\ -6j + 20m - 14, & \text{if } 8m > 5j > 0, \\ -6j + 20m - 14, & \text{if } 0 > 2j > -2m, \\ -14j + 12m - 14, & \text{if } -2m > 2j. \end{cases} \]
\[ = 2|5j - 8m| + 4|j + m| - 14 = 2|5n - 3m| - 4|n + 2m| - 14, \]
if \((m, j) \neq (5, 8), (1, -1), (0, 1), (1, 0)\). Moreover,
if \((m, j) = (5, 8)\), then \[ d_{m,n} = 32 = 2|5n - 3m| + 4|n + 2m| - 18, \]
if \((m, j) = (1, -1)\), then \[ d_{m,n} = 8 = 2|5n - 3m| + 4|n + 2m| - 18, \]
if \((m, j) = (0, 1)\), then \[ d_{m,n} = 0 = 2|5n - 3m| + 4|n + 2m| - 14, \]
if \((m, j) = (1, 0)\), then \[ d_{m,n} = 6 = 2|5n - 3m| + 4|n + 2m| - 14, \]

Thus
\[ d_{m,n} = 2|5n - 3m| + 4|n + 2m| - 14 - \delta_{m,n}, \]
where
\[ \delta_{m,n} = \begin{cases} 0, & \text{if } (m,n) \neq (5,3), (1, -2), \\ 4, & \text{if } (m,n) = (5,3), (1, -2). \end{cases} \]

Let
\[ h(x, y) = 2|5x - 3y| + 4|x + 2y| - 14, \]
where \(x, y \in \mathbb{R}\). Then \(h(x, y)\) is a continuous, piecewise linear function of \(x, y\). In \((x, y)\)-plane, \(\{(x, y) | h(x, y) = 0\}\) is the parallelogram \(L\) illustrated in Figure 3. \(h(x, y) > 0\) outside \(L\), and \(h(x, y) < 0\) inside \(L\). The interior of \(L\) does not contain any lattice point other than the origin and on \(L\) there are four lattice points \(\pm (0, 1), \pm (1, 0)\). This means that \(d_{m,n} > 0\) except \((m,n) = (0,1), (1,0)\).

So if \((m,n) \neq (0,1), (1,0)\), then there exists a non-abelian representation of \(\pi_1(M_{m,n})\) and hence \(\pi_1(M_{m,n})\) is non-abelian and \(M_{m,n}\) is not a lens space. Moreover, by examining Heegaard diagrams we can show that \(M_{(1,0)}\) is the lens space of type
(13, 3) and $M_{(0,1)}$ is the lens space of type $(9, 2)$. Thus we conclude that for the class of manifolds $\{M_{m,n}\}$ the lens space conjecture holds.

§ 6. Remark

During the writing of this paper, we knew Thurston’s theory [5]. If we used his theory with some device, the arguments of this paper could be fairly simplified. Moreover we can show that the interior of $N$ admits a (complete) hyperbolic structure (with finite volume). This structure can be constructed by glueing together the faces of three ideal tetrahedra. We can also show that the critical cases $(m, n) = (5, 3), (1, -2)$, the manifold $M_{m,n}$ is sufficiently large. Indeed $M_{(1,-2)}$ contains an incompressible torus and $M_{(5,3)}$ contains an incompressible surface of genus 2. $M_{(9,13)}$ is also sufficiently large since $H_1(M_{(9,13)})$ is infinite. It seems likely that any other $M_{m,n}$ is not sufficiently large. There is no theoretical difficulty to check it but only a tedious effort would be necessary.

Also it can be shown that when $(m, j) = (1, -1), (1, 1), (2, 1)$, the manifold $M_{m,n}$ is a (special) Seifert fibered space and hence does not admit hyperbolic structure. $M_{(1,-2)}$ does not admit hyperbolic structure since it contains an incompressible torus. It seems likely that when $(m, n) \neq (0, 1), (1, 0), (1, -2), (1, -1), (1, 1), (2, 1), M_{m,n}$ does admit hyperbolic structure. Thurston’s hyperbolic Dehn surgery argument can
apply. However it causes some difficulty when both positively oriented simplexes and negatively oriented simplexes occur.

References


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