Definability in $L^p$-Spaces

by

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This is a sequel to the author’s previous paper [9], in which we developed the “definable” theory of Daniell integration. As an application of it, we here present the definable theory of $L^p$-spaces, including the Radon-Nikodym theorem, the Riesz representation theorem and a theorem on the derivatives of bounded linear functionals.

The acquaintance with [5]–[9] is assumed, and the definitions and the propositions in [9] will be quoted with the asterisk affixed. Thus, for example, “Proposition 4.1*” stands for “Proposition 4.1 in [9].”

Mathematically we have followed [1], [3] and [4]. [2] is quoted as a reference, since it deals with the Randon-Nikodym theorem within the framework of Bishop’s constructive mathematics.

§ 1. $L^p$-spaces

[Assumption] 1) Throughout this paper the definitions in Section 1* will be assumed. They include in particular the axioms on the integration space $\mathcal{X} = (X, L, J)$.

2) The letters $p$ and $q$ will stand for the parameters ranging over the “extended reals” which are no less than 1; $1 \leq p, q \leq \infty$.

DEFINITION 1.1.

$$\exp (|f|, p) : \{ x \} \exp (|f(x)|, p),$$

where $\exp (a, b) = a^b$.

$$L(p; f, W_1, W_2) : \text{mbl} (f, W_2) \wedge [(1 \leq p < \infty \wedge \text{itg} (\exp (|f|, p), W_1) )$$

$$\lor (p = \infty \wedge \exists r \ \text{ae} (x, |f(x)| \leq r, W_1))],$$

where $W_1$ and $W_2$ stand for finite sequences of parameters and $W_1'$ is a subset of $W_1$. See Definitions 6.2*, 3.1* and 2.2* for mbl, itg and ae.

$$\text{esssup} (f, W_1) : \inf \{r : \text{ae} (x, |f(x)| \leq r, W_1')\}$$

$$\text{norm} (p, f, W_1, t) : [1 \leq p < \infty \wedge t < \exp (J^1(\exp (|f|, p), W_1), 1/p)]$$
\[ \forall [p = \infty \wedge t < \text{esssup} \, (f, W_1)], \]

where \( J^1 \) is defined in Definition 3.1*. The 1 in \( J^1 \) may be omitted.

We write \( \text{norm} \, (p; f, W_1) \) or, for short, \( \text{norm} \, (p; f) \) for \( \{t\} \text{norm} \, (p; f, W_1, t) \). We may even omit \( p \) when it is fixed. Notice that \( \text{norm} \, (p; f, W_1) \) is definable.

\[ \mathcal{B}(f, W_1) : \exists r \, \forall x \, (|f(x)| \leq r, W_1) \]

\[ \text{Note.} \] Let \( \mathcal{J} \) be the theory defined in the Theorem in Section 1*. In the following the propositions are meant to be provable in \( \mathcal{J} \).

**Proposition 1.1.**
1. \( L(p; f, W_1, W_2) \to 0 \leq \text{norm} \, (p; f) < \infty \).
2. \( \text{itg} \, (f, W_1), \mathcal{B}(f, W_1) \to L(p; f, W_1, W_2^*) \)
   for a definable \( W_2^* \).
3. \( L(p): \{f, W_1, W_2\} \subseteq \text{L}(p; f, W_1, W_2) \) is a linear space. The precise meaning of this will become clear in the course of the proof.
4. \( L(p; f, W_1, W_2), L(p; f, U_1, U_2) \to \text{norm} \, (p; f, W_1) = \text{norm} \, (p; f, U_1) \).
5. \( L(p) \) is closed with respect to \( f^+ \) and \( f^- \).
6. \( L(p; f, W_1, W_2), a \in R \to \text{norm} \, (p; af, W^*) = |a| \text{norm} \, (p; f, W_1) \)
   for some definable \( W^* \).

**Proof of 3) where \( p < \infty \).** Suppose

\[ \forall i \leq k L(p; F(i), V_1(i), V_2(i)). \]

Then \( \text{mbl} \, (\Sigma[F(i); i \leq k], V^*) \) for a \( V^* \) by 4) of Proposition 6.2*.

\[ \exp \,(|\Sigma[F(i); i \leq k]|, p) \leq \exp \,(k, p) \Sigma[\exp \,(|F(i)|, p); i \leq k] \]

and \( \text{itg} \, (\Sigma[\exp \,(|F(i)|, p); i \leq k], W_2^*) \) by Proposition 4.1*, and hence Proposition 6.5* implies that \( \exp \,(|\Sigma[F(i); i \leq k]|, p) \) is integrable. From this follows that \( L(p) \) is closed under the finite sum. Other conditions can be proved in a similar manner.

**Definition 1.2.**

\[ \text{cnjg} \, (p, q): (p = 1 \wedge q = \infty) \vee (p = \infty \wedge q = 1) \vee (1 < p, q < \infty \wedge (1/p) + (1/q) = 1) \]

[Assumption] We shall henceforth assume \( \text{cnjg} \, (p, q) \).

**Proposition 1.2** (Hölder’s inequality).

\[ L(p; f, W_1, W_2), L(q; g, U_1, U_2) \]
\[ \to \text{itg} \, (fg, W^*) \wedge (|J(fg, W^*)| \leq \text{norm} \, (p; f) \text{norm} \, (q; g)). \]

The mathematical proof goes through. One has only to note that Proposition 4.1* applies (for the functions \(|f|, g \), \( \exp \,(|f|, p) \) and \( \exp \,(|g|, q) \).

**Proposition 1.3** (Minkowski’s inequality).

\[ \forall k \leq n \text{L}(p; F(k), V_1(k), V_2(k)) \]
Definability in $L^p$-Spaces

\[ \text{norm}(p; \Sigma[F(k); k \leq n], W^*) \leq \Sigma\{\text{norm}(p; F(k), V_1(k)); k \leq n\}. \]

**Proof.** $\forall m \leq n L(p; \Sigma[F(k); k \leq m], U_1(m), U_2(m))$ for some $U_1$ and $U_2$ (by Proposition 1.1).

\[ \forall m \leq n (\text{norm}(\Sigma[F(k); k \leq m], U_1(m)) \leq \Sigma\{\text{norm}(F(k), W_1(k)); k \leq m\} \]

is a definable formula, and is provable by induction on $m$. Put $W^* \equiv U_1(n)$.

**PROPOSITION 1.4.** $\forall k L(p; F(k), V_1(k), V_2(k))$,

\[ \Sigma\{\text{norm}(p; F(k)); k \} < \infty \rightarrow \exists (x, \Sigma|F(k, x)| < \infty, U^*) \]

\[ \land [\exists (x, f(x) = \Sigma F(k, x), U^*) \land L(p; f, W_1^*, W_2^*)] \]

\[ \land \text{norm}(p; f) \leq \Sigma \text{norm}(p; F(k))]. \]

**Proof.** The necessary parameters can be constructed as applications of Proposition 1.3 above and Propositions 4.5* and 4.8*.

**PROPOSITION 1.5 (Riesz-Fischer).** $\forall k L(p; F(k), V_1(k), V_2(k))$,

\[ \lim \{\text{norm}(p; F(k) - F(l)); k, l\} = 0 \]

\[ \rightarrow L(p; f^*, W_1^*, W_2^*) \land \lim \text{norm}(p; f^* - F(k)) = 0, \]

for a definable $f^*$.

**Proof.** As applications of DDI (See Definition 1.3* and Definition 1.6 of [8].), we define $\nu$ and $G$ as follows.

\[ \nu(1) = \min (l, \forall m \geq l (\text{norm}(F(m) - F(l)) \leq \exp(2, -1))), \]

\[ \nu(n + 1) = \min (l, l > \nu(n) \land \forall m \geq l (\text{norm}(F(m) - F(l)) \leq \exp(2, -(n + 1)))), \]

\[ G(1) = F(\nu(1)), \]

\[ G(n + 1) = F(\nu(n + 1)) - F(\nu(n)). \]

Then $\{\nu(n)\}_n$ is increasing,

\[ \text{norm}(F(m) - F(\nu(n))) \leq \exp(2, -n) \]

if $m \geq \nu(n), G(n) \in L(p)$ for each $n$ and

\[ \Sigma \text{norm}(G(n)) \leq \text{norm}(F(\nu(1))) + \Sigma \exp(2, -(n - 1)) \]

\[ = \text{norm}(F(\nu(1))) + 1. \]

Thus, by virtue of Proposition 1.4, $\Sigma G(n)$ is absolutely convergent "almost everywhere." Now define $f^*$ by
\[ f^*(x) = \begin{cases} \Sigma G(n, x) = \lim F(n, x) & \text{if } \Sigma |G(n, x)| < \infty, \\ 0 & \text{otherwise}. \end{cases} \]

The properties required of \( f^* \) follow from Proposition 1.4 if the \( f \) there is taken to be this \( f^* \).

**Proposition 1.6.** \( 1 \leq p < \infty, \epsilon > 0, \)

\[ L(p; f, W_1, W_2) \rightarrow \phi^* \in L \wedge L(p; \phi^*, W_1^*, W_2^*) \wedge \text{norm } (f - \phi^*) \leq \epsilon \]

for a definable \( \phi^* \), where \( \epsilon \) stands for a rational number.

**Proof.** It suffices to prove for the case \( f \geq 0 \). Define

\[ H(n) \equiv \min (n, n \exp (f, p), f), \]

and

\[ n_0 = \min (n, \text{norm } (f - H(n)) \leq \epsilon). \]

Then \( H(n) \in L(p) \wedge \text{itg} \), and \( n_0 \) is meaningful. By virtue of Proposition 4.2*, there is a \( \psi \in L \) (in fact \( \psi = \Psi(1) \)) such that

\[ J(|H(n_0) - \psi|) \leq \exp (\epsilon, p)/\exp (2n, p - 1). \]

Define \( \phi^* \equiv \min (n_0, \psi^+) \). This will do.

**Proposition 1.7.** The following are mutually definably interpretable for any \( p, 1 \leq p < \infty \).

(5) \[ L(p; f, W_1, W_2). \]

(6) \[ \forall n(F(n) \in L \wedge L(p; F(n), V_1(n), V_2(n)) \wedge \text{ae } (x, f(x) = \Sigma F(n, x), U) \wedge \Sigma \text{ norm } (F(n)) < \infty. \]

**Proof.** (1) follows from (2) by Proposition 1.4. (2) follows from (1) by Proposition 1.6. Notice that the \( \phi^* \) above is defined uniformly in \( \epsilon \).

§ 2. Linear functionals on \( L^p \)

**Definition 2.1.** \( \text{Inf}l (p; T); \forall f \forall W_1 \forall W_2 \)

\[ (L(p; f, W_1, W_2) \vdash T(f) \in R) \wedge \forall k \forall F \forall V_1 \forall V_2 \forall a(\forall i \leq k L(p; F(i), V_1(i), V_2(i)) \wedge a \in R \vdash [T(aF(1)) = aT(F(1)) \wedge T(\Sigma F(i); i \leq k)] = \Sigma \{T(F(i); i \leq k)\}) \]

\( \text{Bdf} (p; T, a); a > 0 \wedge \forall f \forall W_1 \forall W_2 \)

\[ (L(p; f, W_1, W_2) \vdash |T(f)| \leq a \text{ norm } (f)) \]
Definability in $L^p$-Spaces

$$
\text{blf}(p; T, a) : \text{lnfl}(p; T) \land \text{bdf}(p; T, a)
$$

$$
\text{cntf}(p; T, \delta) : \forall f \forall W_1 \forall W_2 \forall \varepsilon > 0
\quad \left[ \delta(\varepsilon) > 0 \land (L(p; f, W_1, W_2) \land \text{norm}(f) < \delta(\varepsilon) \implies |T(f)| < \varepsilon) \right]
$$

**Proposition 2.1.**
1) $\text{lnfl}(p; T) \implies T(0) = 0$.
2) $\text{blf}(p; T, a)$, $\forall n L(p; F(n), V_1(n), V_2(n))$,
   $$
   L(p; f, W_1, W_2), \lim \text{norm}(F(n) - f) = 0
   \implies \lim T(F(n)) = T(f).
   $$

**Proposition 2.2.** Under the assumption of $\text{lnfl}(p; T)$, $\text{bdf}(p; T, a)$ and
$\text{cntf}(p; T, \delta)$ are mutually definably interpretable.

**Proof.** Suppose $\text{bdf}(p; T, a)$. Then $\text{cntf}(p; T, \{\varepsilon/2a\})$. Suppose next $\text{cntf}(p; T, \delta)$. Then $\text{bdf}(p; T, 2/\delta(1))$.

As an application of Hölder's inequality (Proposition 1.2) we have

**Proposition 2.3.** $L(q; g, U_1, U_2)$

$$
\rightarrow \text{blf}(p; f) J(fg, W^*), \text{norm}(q; g))
$$

for a definable $W^*$. (Recall that $\text{norm}(q; g)$ is definable.)

**Definition 2.2.** $\text{nrm}(p; T, b) : \text{bdf}(p; T, b)$

$$
\land \forall a (\text{bdf}(p; T, a) \implies b \leq a)
$$

**Proposition 2.4.**
1) $\text{blf}(p; T, a) \land \text{nrm}(p; T, b), \text{nrm}(p; T, c)$

$$
\rightarrow b = c \geq 0.
$$

2) $\text{blf}(p; T, a) \land \text{nrm}(p; T, b)$

$$
\rightarrow H(b) \land \forall c (H(c) \implies b \leq c),
$$

where $H(b)$ abbreviates

$$
\forall f \forall W_1 \forall W_2 (L(p; f, W_1, W_2) \land \text{norm}(f) \neq 0 \implies |T(f)|/\text{norm}(f) \leq b).
$$

Notice that, although the notions such as $\text{blf}(p; T, a)$ and $\text{nrm}(p; T, b)$ are not definable, the basic properties concerning them can be proved in our theory.

In the next two propositions, we deal with the case where $p = q = 2$.

**Proposition 2.5.**
1) $L(2; g, U_1, U_2)$

$$
\rightarrow \text{nrm}(2; f) J(fg, \text{norm}(2; g)).
$$

2) Let $H(f, W_1, W_2)$ denote

$$
L(2; f, W_1, W_2) \land J(fg) = \exp(\text{norm}(2; g), 2).
$$
Then

\[ L(2; g, U_1, U_2) \rightarrow H(g, U_1, U_2) \]

\[ \land \forall f \forall W_1 \forall W_2 (H(f, W_1, W_2) \vdash [\text{norm} (2; g) \leq \text{norm} (2; f)] \]

\[ \land (\text{norm} (g) = \text{norm} (f) \vdash \text{ae} (x, f(x) = g(x), E^*, \chi^*)) \] for some \( E^* \) and \( \chi^* \).

**Proof.** First notice that \( L(2; f, W_1, W_2) \) and \( L(2; g, U_1, U_2) \) imply its \((fg, W^*)\) and \(\text{blf}(2; \{f\} J(fg, W^*), a^*)\) for some \( W^* \) and \( a^* \) (Proposition 1.2 and 2.3), and hence the statements above make sense. The required properties then follow as consequences of Proposition 1.2 where \( p = q = 2 \), or Schwarz inequality.

The representation theorem for the bounded linear functional on \( L^2 \) assumes the following form.

**PROPOSITION 2.6.** \( \text{blf}(2; T, a), L(2; h, Z_1, Z_2), T(h) \neq 0, \)

\[ \text{norm} (2; T, b), \quad \forall n (L(2; G(n), V_1(n), V_2(n)) \]

\[ \land T(G(n)) = \exp (b, 2) \]

\[ \land \text{norm} (G(n + 1)) \leq \text{norm} (G(n)) \]

\[ \forall f \forall W_1 \forall W_2 (L(2; f, W_1, W_2) \land T(f) = \exp (b, 2) \]

\[ \vdash \exists n (\text{norm} (G(n)) \leq \text{norm} (f)) \]

\[ \rightarrow L(2; g^*, U_1^*, U_2^*) \land T(g^*) = \exp (b, 2) \]

\[ \land \forall f \forall W_1 \forall W_2 (L(2; f, W_1, W_2) \land T(f) = \exp (b, 2) \]

\[ \vdash [\text{norm} (g^*) \leq \text{norm} (f)] \]

\[ \land (\text{norm} (g^*) = \text{norm} (f) \]

\[ \vdash \text{ae} (x, f(x) = g^*(x), E^*, \chi^*)) \]

\[ \land \forall f \forall W_1 \forall W_2 (L(2; f, W_1, W_2) \vdash T(f) = J(fg^*, W^*) \] for some definable \( g^*, U_1^*, U_2^*, E^*, \chi^* \) and \( W^* \).

**Proof.** Put \( c = \lim \text{norm} G(n) \). Then \( c \geq 0 \) and \( c \) satisfies that

\[ c = \inf \{ \text{norm} (f) ; L(2; f, W_1, W_2) \} . \]

Also, \( 0 < b \leq c \), and hence \( c > 0. \)

\[ \lim \{ \text{norm} (G(k) - G(l)); k, l \} = 0 , \]

and hence, by virtue of the Riesz-Fischer theorem (Proposition 1.5), there are \( g^*, U_1^* \) and \( U_2^* \) such that \( L(2; g^*, U_1^*, U_2^*) \) and
Definability in $L^p$-Spaces

(*) \[ \lim \text{norm } (g^* - G(k)) = 0. \]

As a corollary of (*)& and Proposition 2.2, we have

\[ T(g^*) = \lim T(G(k)) = \exp(b, 2). \]

Other properties which are required of $g^*$ can be proved in a similar manner, by

following the mathematical proof and by using the facts claimed above and

Proposition 2.5.

§ 3. Integration of complex-valued functions

**Definition 3.1.** $C(a)$: "$a$ is a complex number."

- $\text{Re}(a)$: the real part of $a$
- $\text{Im}(a)$: the imaginary part of $a$
- $\text{mp}(f, X, C)$: "$f$ is a map from $X$ to the complex numbers."
- $\text{Re}(f): \{x\} \text{re}(f(x))$
- $\text{Im}(f): \{x\} \text{Im}(f(x))$

*Note.* The theory of complex numbers can be developed in a conservative


**Proposition 3.1.**

1) $\text{Re}(f)$ and $\text{Im}(f)$ are definable.

2) $\text{mp}(f, X, C) \rightarrow f = \text{Re}(f) + i \text{Im}(f)$, where $i$ denotes the pure imaginary

number.

**Definition 3.2.** $\text{citg}(f, W)$: $\text{mp}(f, X, C)$

\[ \land \text{itg}(\text{Re}(f), W) \land \text{itg}(\text{Im}(f), W) \]

$\text{cJ}(f, W)$: $J(\text{Re}(f), W) + iJ(\text{Im}(f), W)$

$\text{cmbl}(f, U)$: $\text{mp}(f, X, C)$

\[ \land \text{mbl}(\text{Re}(f), U) \land \text{mbl}(\text{Im}(f), U) \]

*Note.* Due to the definitions, most of the properties concerning integration

and measurability of complex-valued functions are definable consequences of the

counterparts of real-valued functions.

**Proposition 3.2.**

1) $\text{cJ}(f)$ is independent of the parameters.

2) $\text{citg}$ is a linear space over the complex numbers, and $\text{cJ}$ is a linear functional

on $\text{citg}$.

3) $\text{cmbl}$ forms an algebra.

These are immediate consequences of Propositions 3.1*, 4.1* and 6.6*.

**Proposition 3.3.** Under the assumption of $\text{cmbl}(f)$, $\text{citg}(f)$ and $\text{itg}(|f|)$ are

mutually definably interpretable, and, if either holds, then
\begin{align*}
|cJ(f)| & \leq J(|f|) . \\
\text{Proof.} & \quad \text{Mutual interpretability is a consequence of Proposition 6.5*}, \text{ since} \\
|\text{Re}(f)|, |\text{Im}(f)| & \leq |f| \leq |\text{Re}(f)| + |\text{Im}(f)| .
\end{align*}

To prove (1), first suppose that
\begin{equation}
\exists m = (r_1, \cdots, r_n) \forall x \exists k \leq n (f(x) = r_k) ,
\end{equation}
where each $r_k$ denotes a rational complex. Put
\begin{align*}
D(k) &= \{ x; \ f(x) = r_k \} \\
&= \{ x; \ \text{Re}(f(x)) = \text{Re}(r_k) \} \cap \{ x; \ \text{Im}(f(x)) = \text{Im}(r_k) \} .
\end{align*}
Then $|f| = \Sigma \{ |r_k| \chi_{D(k)} ; \ k \leq n \}$. (See Definition 7.1* for $\chi_D$, the characteristic function of $D$.) From this follows (1) when (2) is assumed.

Next consider $f$ any complex-valued integrable function. Then there are definable $x^*$ and $\Phi^*$ such that
\begin{equation}
\forall n (\text{citg} (\Phi^*(n)) \land x^*(n) = (r(n, 1), \cdots, r(n, l(n))) \\
\land \forall x \exists i \leq l(n) (\Phi^*(n, x) = r(n, i)) \\
\land \text{ae} (x, f(x) = \lim \Phi^*(n, x)) \\
\land \lim J(|\Phi^*(n) - \Phi^*(m)|) = 0 \\
\land cJ(f) = \lim J(\Phi^*(n)) ,
\end{equation}
where $l(n)$ represents a natural number depending on $n$ and $r(n, i)$ denotes a rational complex for each $i \leq l(n)$. (3) is a consequence of Section 9*, and (1) follows from (3).

\textbf{Note.} The $L^p$-theory for the complex-valued functions can be developed similarly to Sections 1 and 2.

\section*{§ 4. Radon-Nikodym theorem}

\textbf{Definition 4.1.} Let $\mathcal{X} = (X, L, J)$ denote a (real) integration space satisfying $1^\circ$ to $5^\circ$ (Definitions 1.3* and 6.1*). We place further conditions on $(X, L)$ as listed below.

$6^\circ$. \quad \eta \subset L(+) \land X = \bigcup X_n ,

where $X_n = \{ x; \eta(n, x) > 0 \}$, and $\eta$ is a parameter.

$7^\circ$. \quad \forall \phi , \ \psi \in L (\phi \psi \in L \land (\psi \neq 0 \lor \phi / \psi \in L)) ,

where $\psi \neq 0$ means $\forall x (\psi(x) \neq 0)$.

\textbf{Note.} 1) The assumptions $6^\circ$ and $7^\circ$ are not essential restrictions, for they are
satisfied by the class of integrable, simple functions in most of the interesting spaces.

2) For any notion \( \mathcal{N} \) which depends on the integrals, we distinguish it for
different integrals by affixing their names to \( \mathcal{N} \). Thus, for example, if a function \( f \) is
integrable with respect to \( J \), then we write \( \text{itg} (J; f, W) \) (for some \( W \)).

**Proposition 4.1.** 1) For each pair of \( m \) and \( n \), \( \{ x; \eta(n, x) \geq 1/m \} \) is \( J \)-
integrable (for any \( J \) satisfying \( 1^\circ - 5^\circ \)), hence \( X \) is the union of a sequence of \( J \)-integrable
sets.

2) If \( h \) is integrable and \( \phi \in L \), then \( \phi h \) is also integrable.

**Proof.** 1) is an immediate consequence of Proposition 6.5*. 2) is a consequence
of \( 7^\circ \) in Definition 4.1.

**Definition 4.2.** absent \( (J, I; \Omega) \): \( \forall E \forall \chi (\text{nls} (I; E, \chi) \vdash \text{nls} (J; E, \Omega(\chi))) \)

\[ \text{litg} (I; h, \Xi) \equiv \text{mp} (h, X, R) \wedge \forall \phi \in L \text{itg} (I; h \phi, \Xi (\phi)) \]

(See Definition 2.1* for nls. litg abbreviates "locally integrable.")

\[ \Gamma (J, I, h, \Xi) : h \geq 0 \wedge \text{litg} (I; h, \Xi) \wedge \forall \phi \in L (J(\phi) = I(\phi h)) \]

[Assumption] We shall henceforth assume that \( I \) and \( J \) are integrals on \( (X, L) \),
so that \( \Xi_I = (X, L, I) \) and \( \Xi_J = (X, L, J) \) satisfy \( 1^\circ \) to \( 7^\circ \).

**Lemma.** mbl is closed with respect to the quotient, where mbl is understood to be
relative to \( I \).

**Proof.** It suffices to show that for any measurable function \( f \) for which
\( \forall x (f(x) \neq 0) \), \( 1/f \) is also measurable. This is a consequence of the definable theory of
reals and various properties of measurable functions which were obtained in Section
6*. We list some of these.

1) The constant functions are measurable.

2) mbl is closed with respect to \( \wedge, ( \ )^* \), the linear combination and the limit.

3) If \( \pi_n (a) = n/[na] \), where \( [c] \) represents the Gaussian of \( c \), then \( \pi_n (f) \) is a linear
combination of functions of the form \( K \circ f \), where \( K \) is the characteristic function of a
closed interval, and hence is measurable.

4) \( 1/f = \lim \pi_n (f) \); so \( 1/f \) is measurable.

**Proposition 4.2.** 1) \( \text{litg} (I; h, \Xi) \rightarrow \text{mbl} (I; h, W_1^*, W_2^*) \) for some \( W_1^* \) and
\( W_2^* \).

2) \( \text{itg} (I; h, A) \rightarrow \text{litg} (I; h, \{ \phi \} A) \).

**Proof.** 1) If \( \text{litg} (I; h, \Xi) \), then \( 6^\circ \) and \( 5^\circ \) imply that

\[ \text{litg} (I; h(\eta(n) \wedge 1), \Xi (\eta(n) \wedge 1)) \].

Define \( g \) to be \( \Sigma \exp (2, -n)h(\eta(n) \wedge 1) \). \( g \) is well-defined and positive. By virtue of 1)
and 5) of Proposition 6.2*, \( g \) is "\( I \)-measurable." Since

\[ h = (\Sigma \exp (2, -n)h(\eta(n) \wedge 1))/g \],
the lemma above and Proposition 6.2 ensure that $h$ is "I-integrable."

2) This is an immediate consequence of 2) of Proposition 4.1.

PROPOSITION 4.3. $\text{nls}(I; E, \chi)$ and the following are mutually definably interpretable:

$$\Psi \in L(+) \land \forall n(0 \leq \Psi(n) \leq 1 \text{ on } E) \land \Sigma I(\Psi(n)) < \infty.$$ 

Proof. Assuming $\text{nls}(I; E, \chi)$, define $\Psi(n)$ to be $|\chi(n) \land 1|$. The converse is trivial.

PROPOSITION 4.4. $\Gamma(J, I; h, \Xi) \rightarrow \text{absent } (J, I; \Omega^*)$ for a definable $\Omega^*$.

Proof. Assume $\Gamma(J, I; h, \Xi)$ and $\text{nls}(I; E, \chi)$, and show $\text{nls}(J; E, \chi^*)$ for some $\chi^*$. By virtue of Proposition 4.3, we may assume that every function on $L$ is bounded. By 6, it suffices to consider the case where $E$ is contained in an $X_m$, which we shall call $S$. (See Propositions 4.1 and 2.2.) Let $\xi$ denote $\eta(n) \land 1$. Then $\xi \in L$, and hence $\text{itg}(I; \xi h, \Psi, W)$ for some $\Psi$ and $W$.

Define

$$g(x) = \begin{cases} \xi(x) + \Sigma |\Psi(m, x)| & \text{if } \Sigma |\Psi(m, x)| < \infty, \\ \xi(x) & \text{otherwise}. \end{cases}$$

$\xi h \leq g$ "almost everywhere" with respect to $I$, and $g$ is "$I$-integrable," and hence is "locally integrable" due to 2) of Proposition 4.2. $K \equiv \{\phi\} I(\phi g)$ therefore defines an integral on $(X, L)$. (The mathematical proof goes through for these claims.)

Suppose $\text{nls}(K; E, \Theta)$ for some $\Theta$. Then $\Sigma \Theta(m)$ is divergent on $E$ and $\Sigma I(|\Theta(m)| g < \infty$, and hence $\Sigma I(\xi h |\Theta(m)|) < \infty$. Define $\chi^*(m) = \xi \Theta(m)$. Then $\text{nls}(J; E, \chi^*)$. We shall therefore establish $\text{nls}(K; E, \Theta)$.

Suppose $\Phi \subset L(+)$. $\Sigma \Phi(i)$ is divergent on $E$ and $\Sigma I(\Phi(i)) < \infty$. Define

$$\omega(m) = \xi + \Sigma [|\Psi(k)|; k \leq m].$$

$\omega(m) \in L$ and $\omega(m) > 0$ on $S$.

$$v(i) = \min (m, g \Phi(i)/\omega(m) < I(\Phi(i)) + \exp (2, -m))$$

is meaningful, and $\Sigma I(g \Phi(i)/\omega(v(i)) < \infty$. If $D = \{x; \omega(m, x) \text{ diverges}\}$, then $\Sigma \Phi(i)/\omega(v(i))$ is divergent on $E-D$. $\Phi(i)/\omega(v(i)) \in L$ by the condition 7 in Definition 4.1, and

$$\text{nls}(K; E-D, \{i\} (\Phi(i)/\omega(v(i)))) .$$

If we can show that $D$ is "$K$-null," then we are done. Let $\zeta(m)$ be the characteristic function of $\{x; g(x) < m\}$. Then $\zeta(m)g$ is $I$-integrable and $I(\zeta(m)g)$ increases to $I(g)$, and so one can define an increasing sequence of natural numbers $\sigma$ so that, if $\theta(m)$ is the characteristic function of $\{x; g(x) \geq \sigma(m)\}$, then $I(\theta(m)G) \leq \exp (2, -m)$.

Define a function $u$ on the reals by
Definability in $L^p$-Spaces

$$u(b) = \begin{cases} m & \text{if } \sigma(m) \leq b < \sigma(m+1), \\ 0 & \text{if } b < \sigma(1), \end{cases}$$

and let $\gamma(m)$ be the characteristic function of $\{x; \sigma(m, x) \leq g(x) < \sigma(m+1, x)\}$. As an application of Fatou's lemma, the "$I$-integrability" of $(u \circ g)g$ can be derived, and

$$I((u \circ g)g) = \Sigma I(\gamma(m)(u \circ g)g) \leq 2.$$

$u \circ \omega(m)$ is a "simple" function in the sense of Definition 9.1*, and hence the conclusion in Section 9* ensures that $(u \circ \omega(m))g$ is $I$-integrable, and

$$I((u \circ \omega(m))g) \leq I((u \circ g)g) \leq 2$$

for every $m$. On the other hand $u \circ \omega(m)$ diverges on $D$. Thus we have obtained that $\text{nls}(K; D, u \circ \omega)$.

As a consequence of Proposition 4.4 and Lebesgue's dominated convergence theorem we have

**Proposition 4.5.** Every $I$-measurable function is $J$-measurable, provided that $\Gamma(J, I; h, \Xi)$ holds for some $h$ and $\Xi$.

**Proposition 4.6.** Under the assumption of $\Gamma(J, I; h, \Xi)$ and mp $(f, X, R)$, "$f$ is $J$-integrable" and "$fh$ is $I$-integrable" are mutually definably interpretable, and, if either condition holds, then $J(f) = I(fh)$.

This is a consequence of Proposition 4.4 as well as other preceding results. The mathematical proof goes through.

**Proposition 4.7.** If we define $K$ to be $J + I$, then $K$ is an integral on $(X, L)$. If $f$ is $K$-integrable, then $f$ is integrable for $J$ and $I$ also, and in that case $K(f) = J(f) + I(f)$.

Let $L(2; K)$ denote the $L^2$-space with respect to $K$. Then $1 \in L(2; K)$ and, if $f \in L(2; K)$, then $f$ is $K$-integrable, and hence is $J$-integrable.

**Definition 4.3.**

$$\text{nrm}(I; J, a): \text{nrm}(2; J, a),$$

where $J$ is regarded as a linear functional on $L(2; K)$ and nrm (2) is taken with respect to $K$, $K$ being $J + I$. See Definition 2.2 for nrm (2).

**Proposition 4.8 (Radon-Nikodym).** Let $J$ and $I$ be as in our [Assumption]. Then (a) and (b) below are mutually definably interpretable, provided that $\text{nrm}(I; J, a_0)$ is assumed.

(a) $\Gamma(J, I; h, \Xi)$.

(b) absct (J, I; $\Omega$).

The $h$ in (a) is unique up to the addition of an $I$-null function.

**Proof.** (a) implies (b) by virtue of Proposition 4.4 above. nrm $(I; J, a_0)$ is not necessary for this direction.
Assume (b).

Case 1. 1 is integrable both for $J$ and $I$. Define $K$ to be $J + I$. Due to the assumption nrm $(I; J, a_0)$ and Proposition 2.6, there is a definable $g^*$ such that $J(f) = K(fg^*)$ for $f \in L(2; K)$. Define $D = \{ x; g^*(x) \geq 1 \}$, and

$$h(x) = \begin{cases} \Sigma \exp (g^*(x), k) & \text{if } x \notin D, \\ 0 & \text{if } x \in D. \end{cases}$$

For an $f \geq 0$, $K$-integrable,

$$J(f) = I(fh) + J(\chi_D f)$$

by the monotone convergence theorem, where $\chi_D$ represents the characteristic function of $D$. The result can be extended to all $K$-integrable functions. (b) then implies $J(\chi_D f) = 0$, since $D$ is $J$-null and $\chi_D f$ is $J$-integrable. Thus, $J(f) = I(fh)$, and hence in particular $J(\phi) = I(\phi h)$ for $\phi \in L$.

Case 2. When we do not have the condition in Case 1, consider

$$\pi = \Sigma \exp (2, -n) \exp (1 + K(\eta(n)), -1)(\eta(n) \land 1).$$

$0 < \pi \leq 1$, and $\pi$ is $K$-integrable, hence is integrable both for $J$ and $I$; also, $\pi$ is locally $J$-integrable. Define

$$J'(\phi) = I(\phi \pi), \quad I'(\phi) = I(\phi \pi), \quad K'(\phi) = K(\phi \pi).$$

$1$ is $J'$, $I'$, $K'$-integrable, and $J'$ is "absolutely continuous" with respect to $J$ (by Proposition 4.4 applied to $J$ and $J'$). $I$ is "absolutely continuous" with respect to $I'$.

(b) then implies that $J'$ is "absolutely continuous" with respect to $I'$, and nrm $(I'; J', a^*)$ for a definable real $a^*$. Thus, Case 1 holds for $I'$ and $J'$, and so there is an $h^* \geq 0$, which is $I'$-measurable and which satisfies

$$J(f \pi) = J'(f) = I'(fh^*) = I(f \pi h^*)$$

for every $f K'$-integrable. In particular, $\phi \in L$ implies that $\phi / \pi$ is $K'$-integrable. So $J(\phi) = I(\phi h^*)$. This means that $h^*$ is "locally $I$-integrable."

The uniqueness proof is straightforward.

§ 5. Applications of Radon-Nikodym theorem

In this section we work under the same assumptions as in Section 4 for $\mathcal{X}_I$ and $\mathcal{X}_J$.

DEFINITION 5.1. $\text{sglr}(I; T, E, \theta)$:

$$\text{nls}(I; E, \theta) \land \forall \phi \in L(T(\phi) = T(\chi_E \phi)), $$

where $\chi_E$ represents the characteristic function of $E$.

$$\text{sglr}(I; T, \chi_0): I(\chi_0) = 0 \land \forall \phi \in L(T(\phi) = T(\chi_0 \phi))$$
Proposition 5.1 (Lebesgue's decomposition theorem). There are definable $h$, $\mathcal{E}$, $\chi_0$ and $J_s$ such that

$$
\begin{align*}
h \geq 0 \land \text{litg} \left( I; h, \mathcal{E} \right) \\
\land \text{"$J_s$ is an integral on $(X, L)$"} \\
\land \text{sglr} \left( J_s; \{\phi \} I(\phi h), \chi_0 \right) \\
\land \forall \phi \in L, J(\phi) = I(\phi h) + J_s(\phi) \).
\end{align*}
$$

Proof. By reviewing the proof of Proposition 4.8, we can define a $g^*$ and $D = \{x; g^*(x) \geq 1\}$ as we did there. Let $\chi_0$ be $\chi_D$ and define $J_s$ to be $\{\phi \} J(\chi_0 \phi)$.

Definition 5.2. $\Delta(p; T, \alpha, H, \Phi, \Psi)$:

$$
\forall f \geq 0 \forall W_1 \forall W_2 (L(p; f, W_1, W_2) \\
\vdash \alpha(f) \in R \land \forall h \forall U_1 \forall U_2 (L(p; h, U_1, U_2) \\
\land 0 \leq h \leq f + T(h) \leq \alpha(f)) \\
\land \forall \varepsilon > 0 (L(p; H(\varepsilon, f), \Phi(\varepsilon, f), \Psi(\varepsilon, f)) \\
\land 0 \leq H(\varepsilon, f) \leq f \land \alpha(f) - \varepsilon < T(H(\varepsilon, f))))]
$$

Proposition 5.2. 1) $\text{blf} (p; T, a), \Delta(p; T, \alpha, H, \Phi, \Psi)$,

$$
\begin{align*}
b \geq 0, f, g \geq 0, L(p; f, W_1, W_2), L(p; g, V_1, V_2) \\
\rightarrow \alpha(bf) = b\alpha(f) \land \alpha(f + g) = \alpha(f) + \alpha(g).
\end{align*}
$$

2) $\text{blf} (p; T, a), \Delta(p; T, \alpha, H, \Phi, \Psi)$

$$
\begin{align*}
\rightarrow \text{blf} (p; \alpha^*, \alpha^*) \land \alpha^* \upharpoonright L(p; +) \equiv \alpha \\
\land \forall \beta \forall b (\text{blf} (p; \beta, b) \land \beta \upharpoonright L(p; +) \equiv \alpha + \beta \equiv \alpha^*)
\end{align*}
$$

For definable $\alpha^*$ and $\alpha^*$, where $\alpha^* \upharpoonright L(p; +)$ represents the restriction of $\alpha^*$ to $L(p; +)$ and $L(p; +)$ abbreviates $\{f, W_1, W_2\}(f \geq 0 \land L(p; f, W_1, W_2))$.

Proof. 1) We work $f + g$ as an example. Suppose $0 \leq h \leq f + g$. Put $d \equiv h \land f$ and $e \equiv (h - f)$. $h \equiv d + e$, $0 \leq d \leq f$, $0 \leq e \leq g$ and $d, e \in L(p)$ (with appropriate parameters). So $T(h) = T(d) + T(e) \leq \alpha(f) + \alpha(g)$ and $T(h) \leq \alpha(f + g)$. $\alpha(f + g) - \varepsilon < T(H(\varepsilon, f + g))$ implies

$$
\alpha(f + g) - \varepsilon \leq \alpha(f) + \alpha(g),
$$

and hence $\alpha(f + g) \leq \alpha(f) + \alpha(g)$. On the other hand,

$$
0 \leq H(\varepsilon, f) + H(\varepsilon, g) \leq f + g
$$

and

$$
H(\varepsilon, f) + H(\varepsilon, g) \in L(p)
$$
The condition on $H$ implies that
\[ \alpha(f) + \alpha(g) - 2\varepsilon < T(H(e, f)) + T(H(e, g)) = T(H(e, f) + H(e, g)) \leq \alpha(f + g), \]
and hence $\alpha(f) + \alpha(g) \leq \alpha(f + g)$.

2) Define $\alpha^* \equiv (\alpha(f^+) - \alpha(f^-))$ and $a^* = 2a$.

**Proposition 5.3.** $\text{blf} (p; T, a), \Delta(p; T, \alpha, H, \Phi, \Psi)$
\[ \rightarrow \text{blf} (p; \alpha^*, a^*) \land \text{blf} (p; \beta^*, b^*) \]
\[ \land \forall f \in L(p)(T(f) = \alpha^*(f) - \beta^*(f)) \]
\[ \land \forall f \in L(p; +) (\alpha^*(f) \geq 0 \land \beta^*(f) \geq 0) \]

for some definable $\alpha^*$ and $\beta^*$.

**Proof.** Let $\alpha^*$ and $a^*$ be as in Proposition 5.2 and let $\beta^*$ be defined by
\[ \beta^*(f) \equiv \alpha^*(f) - T(f), \]
and let $b^*$ be 3a.

**Proposition 5.4** (Riesz representation theorem; general cases). Under the assumptions for $\mathcal{X}_f$, $p$ and $q$,
\[ 1 \leq p < \infty, \ \text{blf} (p; T, a), \ \Delta(p; T, \alpha, H, \Phi, \Psi) \]
\[ \rightarrow g^* \in L(q) \land \forall f \in L(p)(T(f) = J(fg^*)) \]
\[ \land "g^* is unique up to the addition of a null function" \]
\[ \land \text{nrm} (p; T, \text{norm} (q; g^*)) \]
for a definable $g^*$.

**Note.** As is remarked in the mathematical proof, the condition $6^c$ is necessary only for $p=1$; see [1].

**Proof.** According to Proposition 5.3, it suffices to deal with the case where $T$ is positive: $\forall f \in L(p; +) (T(f) \geq 0)$.

Define $L(0; f, W_1, W_2)$ to be
\[ L(1; f, W_1, W_2) \land \exists r \forall x (|f(x)| < r). \]
$L(0) \subset L(1)$ and $L(0)$ satisfies $1^c$ to $5^c$, and hence we may assume $L(0)$ as the class of elementary functions. (See Corollary of Theorem 5.1.*) $6^c$ is also satisfied by $L(0)$ by $\{\eta(n) \land 1\}$. By Proposition 1.1 $L(0) \subset L(p)$, and hence $T$ is a positive linear functional on $L(0)$, which can be regarded as an integral on $L(0)$. Rewriting $T$ as $I$, we can easily see that $I$ is "absolutely continuous" with respect to $J$. By the Radon-Nikodym theorem, there is a $g^* \geq 0$, locally $J$-integrable, such that
\[ I(f) = T(f) = J(fg^*) \]
for any $f \in L(0)$. Claim $g^* \in L(q)$. If $p > 1$, define
Definability in $L^p$-Spaces

$E(n) = \{x; \max \{\eta(i, x); i \leq n\} > 1/n\}$,

and put $H(n) = \chi_{E(n)} (g^* \land n)$. Then $X = \bigcup E(n)$, $H(n) \in L(0)$, $H(n)$ increases to $h$, $\exp (H(n), q)$ is $J$-integrable (by the monotone convergence theorem). If $p = 1$, then notice that

$$\forall f \in L(0)(|J(fg^*)| = |T(f)| aJ(|f|)).$$

Define $E' = \{x; g^*(x) > a + 1\}$. Then

$$J(\chi_{E'} E(n)) \leq (a/(a + 1)) J(\chi_{E} E(n)),$$

so $J(\chi_{E} E(n)) = 0$ for every $n$. Thus, $E$ is a $J$-null set, and $g^*(x) \leq a + 1$ "almost everywhere" with respect to $J$; this implies $g^* \in L(\infty) = L(q)$. Hölder's inequality, that is,

$$\forall f \in L(p)(|J(fg^*)| \leq \text{norm} (p; f) \text{ norm} (q; g^*)),$$

implies $\text{blf} (p; \{f\} J(fg^*))$, $\text{norm} (q; g^*)$). $T(f) = J(fg^*)$ on $L(0)$. Suppose $f \in L(p)$. According to Proposition 1.6, we can construct a sequence $\psi^* \in L(0)$ such that $\lim \text{norm} (f - \psi^*(n)) = 0$, and then Proposition 2.1 ensures

$$T(f) = \lim T(\psi^*(n)) = \lim J(\psi^*(n)g^*) = J(fg^*).$$

The uniqueness easily follows.

§ 6. Derivatives of linear functionals

In this section, we shall present a sufficient condition for the differentiability of linear functionals of a certain kind. It is a modified version of the implication "$B \rightarrow C$" in [3]. As in Definition 4.1, we assume that the integration space $\mathcal{X} = (X, L, J)$ satisfies $1^\circ \sim 7^\circ$. Recall that $6^\circ$ claims the $\sigma$-finiteness of $\mathcal{X}$.

**DEFINITION 6.1.** $\mathcal{R}(h, U, V)$: $\text{itg} (h, U) \land \text{ae} (x, 0 \leq h(x) \leq 1, V)$

$$\text{dsj} (F): \forall x \forall i \forall j(i \neq j \vdash F(i, x)F(i, y) = 0)$$

$$\text{cad} (T): \forall F \forall U \forall W(\forall i (\text{itg} (F(i), W(i))))$$

$$\land \text{dsj} (F) \land \text{itg} (\{x\} \Sigma \{F(i, x); i = 1, 2, \ldots\}, U)$$

$$\vdash T(\{x\} \Sigma \{F(i, x); i = 1, 2, \ldots\})$$

$$= \Sigma \{T(F(i)); i = 1, 2, \ldots\}$$

[Assumption] In the following, we assume

$$\text{blf} (1; T, K) \land \text{cad} (T).$$

Recall that $\text{norm} (1; f) = J(|f|)$. See Definitions 1.1 and 2.1.
**Definition 6.2.** \( \Lambda(\rho, \sigma): \forall n, \forall (\sigma_1(n), \sigma_2(n), \sigma_3(n)) \)
\( \land \forall n(\rho(\sigma_1(n)) > 0) \)
\( \land \forall x \forall \varepsilon > 0 \exists m(\sigma_1(n, x) \neq 0 \land \rho(\sigma_1(n)) < \varepsilon) \)

\( SSTM(h, \rho, \sigma, v): "\nu(h) is a sequence of natural numbers" \)
\( \land \nu \forall \varepsilon > 0 (h(x) \neq 0 \lor \exists m(\sigma_1(v(h, m), x) \neq 0) \land \rho(\sigma_1(v(h, m))) < \varepsilon) \)

\( B(\rho, \sigma, \tau): \forall h \forall U \forall V \forall r > 0 (\forall x, U, V) \)
\( \land J(h) > 0 \land SSTM(h, \rho, \sigma, v) \)
\( \lor \forall \exists m(\tau(h, r, i) = v(h, m)) \land dsj([i] \sigma_1(\tau(h, r, i))) \land J(h) \)
\( = J(h \{x \sum_i \{\sigma_1(\tau(h, r, i), x); i = 1, 2, \ldots\}) \land \sum_i \{J(\sigma_1(\tau(h, r, i))); i = 1, 2, \ldots\} < J(h) + r \}

As a consequence of the definition and Proposition 4.3* we have

**Proposition 6.1.**
1) If \( h \in \mathcal{X} \), then norm \((1; k) = J(h) \geq 0\), and hence the boundedness of \( T \) is reduced to \(|T(h)| \leq KJ(h)\).
2) Let us abbreviate \( \{x \sum_i \{\sigma_1(\tau(h, r, i), x); i = 1, 2, \ldots\} \to \Sigma \omega(i) \text{. Then, in } B, f_0 = \Sigma \omega(i) \in \mathcal{X}, J(f_0) = \Sigma J(\omega(i)) \text{ and } J(h) \leq J(f_0), \text{ presuming that } \Lambda(\rho, \sigma) \text{ holds.} \)

**Definition 6.3.**
\( uA(T, \rho, \sigma; x, l): \)
\( \limzup \{T(\sigma_1(n))/J(\sigma_1(n)); n = 1, 2, \ldots, \sigma_1(n, x) \neq 0, \rho(\sigma_1(n)) < 1/l\} \)
\( lA(T, \rho, \sigma; x, l): \)
\( \limzinf \{T(\sigma_1(n))/J(\sigma_1(n)); n = 1, 2, \ldots, \sigma_1(n, x) \neq 0, \rho(\sigma_1(n)) < 1/l\} \)
\( Ud(T, \rho, \sigma; x): \lim \{uA(T, \rho, \sigma; x, l); l = 1, 2, \ldots\} \)
\( Id(T, \rho, \sigma; x): \lim \{lA(T, \rho, \sigma; x, l); l = 1, 2, \ldots\} \)

We shall abbreviate \( \{x \} Ud(T, \rho, \sigma; x) \) to \( Ud \). Similarly for \( Id \).

\( C(T, \rho, \sigma, W_1, W_2, W, \theta, g): \mbi (uD, W_1) \land \mbi (Id, W_2) \)
\( \land L(\infty; g, W) \land ae (x, uD(x) = Id(x) = g(x), \theta) \)
\( \land \forall f \forall U(itg(f, U) + T(f) = J(f(uD))) \)
\( = J(f(Id)) = J(fg)) \)

**Proposition 6.2.** If \( \Lambda(\rho, \sigma) \) is assumed, then \( J(\sigma_1(n)) \geq 0 \) in the definitions of \( uA \) and \( lA \).
Definability in $L^p$-Spaces

[Assumption] In the subsequent discussion, we assume $A(r, o)$ and $B(r, o, w, W)$, and thus the propositions which follow are understood to "definably interpretable from $A$ and $B$.''

**Proposition 6.3.** $b \in R, h \in X$,

$$\forall x(h(x) \neq 0 \iff \text{uD}(x) \geq b) \rightarrow T(h) \geq bJ(h).$$

The mathematical proofs of this proposition and the next one are more or less due to [4]. The author also owes to Mamoru Kanda for his comments in this regard.

**Proof.** It suffices to consider the case where $J(h) > 0$. (See 1) of Proposition 6.1.) Since $\text{uD}(x, l)$ is decreasing with respect to $l$, $\text{uD}(x) \geq b$ (where $h(x) \neq 0$) implies

$$h(x) > 0 \rightarrow \forall \varepsilon \exists n(\sigma_1(n, x) > 0 \land \rho(\sigma_1(n)) < 1/l \land T(\sigma_1(n)) \geq bJ(\sigma_1(n))).$$

Define $v^*$ by:

$$v^*(1) = \min(n, G(n)),$$

$$v^*(m + 1) = \min(n, n > v^*(m) \land G(n)),$$

where $G(n)$ stands for

$$\exists x \exists y(h(x) \neq 0 \land \sigma_1(n, x) > 0 \land \rho(\sigma_1(n)) < 1/l \land T(\sigma_1(n)) \leq bJ(\sigma_1(n))).$$

Then $s \in (h, r, o, w^*)$ follows. The condition $B$ applied to this $h$ and $v$: $v^*$ yields

$$\forall r > 0(\forall \varepsilon \exists n(\tau(h, r, v^* \land \text{dsj}(\langle i \rangle \sigma_1(\tau(h, r, i)))) \land J(h)) = J(h \Sigma \sigma(i) \land \Sigma J(\sigma(i)) < J(h) + r).$$

Proposition 6.1 assures us that $f_0 = \Sigma \sigma(i) \in X$, $J(f_0) = \Sigma J(\sigma(i))$ and $J(h) \leq J(f_0)$. "$h, f_0 \in X"$ implies that $h f_0, h - f_0, f_0 - h f_0 \in X$. From these facts and the complete additivity of $T$, we can easily obtain, successively,

$$T(h) = T(h f_0) > T(f_0) - Kr$$

$$= \Sigma T(\sigma(i)) - Kr \geq b \Sigma J(\sigma(i)) - Kr$$

$$= bJ(f_0) \geq bJ(h) - Kr;$$

that is, $\forall r > 0(T(h) \geq bJ(h) - Kr)$, from which follows $T(h) \geq bJ(h)$.

**Proposition 6.4.** $C(T, r, o, w_1, w_2, w, \theta, g)$.

**Proof.** $\{x\} \cup A(x, l)$ is measurable if and only if $A(s) = \{x; u A(x, l) > s\}$ for every rational $s$. But

$$A(s) = \bigcup \{\{x; \sigma_1(n, x) \neq 0\}; \rho(\sigma_1(n)) < 1/l \land T(\sigma_1(n))/J(\sigma_1(n)) > s\}$$

and $\sigma_1(n)$ is measurable.

By Proposition 5.4 (the Riesz-representation theorem) for $L(1)$, there is a definable $g^* \in L(\infty)$ such that
\[ T(f) = J(fg^*) \text{ for every } f \in L(1). \]
So, it suffices to show \( g^* = uD = 1D \) almost everywhere. For this it suffices to establish \( uD \leq g^* \) almost everywhere. Define, for \( n \geq 1 \) and \( k \geq 0 \),
\[ D(n, k) = \{ x ; uD(x) \geq (k+1)/n > k/n \geq g^*(x) \}, \]
and let \( \chi(n, k) \) denote the characteristic function of \( D(n, k) \). \( \chi(n, k) \) is measurable. By \( \sigma \)-finiteness (6°), \( \chi(n, k) = \Sigma \pi(j) \) for some \( \pi \), where \( 0 \leq \pi(j) \leq 1 \) and \( \pi(j) \) is integrable, and hence \( \pi(j) \in K \). \( \forall j(J(\pi(j)) = 0) \) will imply \( J(\chi(n, k)) = 0 \) for every \( (n, k) \), from which follows \( uD \leq g^* \) almost everywhere. \( J(\pi(i)) = 0 \) is a consequence of Proposition 6.3 where \( b = (k+1)/n \) and \( h = \pi(j) \).

References


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