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On the Divisor Problem in an Arithmetic Progression

by

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1. Introduction

Denote by $d(n)$ the number of divisors of the positive integer n. Then the classical Dirichlet's divisor problem is the question for the infimum θ of all real numbers λ for which (as $x\to\infty$)

$$\sum_{n\leqq x} d(n)=x\log x+(2\gamma-1)x+O(x^{\lambda}) \tag{1}$$

where γ is the Euler-Mascheroni constant (cf. e.g. the monograph of Fricker [1], p. 67). It is known that $1/4\leqq\theta\leqq 35/108$, the upper bound having been established rather recently by Kolesnik [3] (cf. also Hafner [2] for the best Ω-result to date).

Ramanujan [8] was the first to consider the analogue of this problem which arises when n is restricted to a given residue class n_0 modulo a fixed integer k. He gave the estimate

$$D(n_0,k;x):=\sum_{\substack{n\leqq x\\ n\equiv n_0(\mathrm{mod}\,k)}} d(n)=\alpha(n_0,k)x\log x+\beta(n_0,\,k)x+O(x^{1/3}\log x) \tag{2}$$

(cf. also Theorem 12 on page 86 and the notes on page 93 of Fricker [1]). This was improved by Walfisz [9] who obtained the error term $O(x^{27/82}(\log x)^{11/41})$ via a representation for $\int_0^X D(n_0,\,k;\,x)\mathrm{d}x$, apparently the best result on this problem to date. Evaluations of the constants $\alpha(n_0,k)$ and $\beta(n_0,k)$ were given recently by Kopetzky [4].

2. Statement of our result

It is the objective of the present paper to improve the O-term in (2) as far as Kolesnik succeeded in improving the estimate for the original Dirichlet's divisor problem; i.e. we prove the following

THEOREM. *For arbitrary fixed integers $k\geqq 1$ and $1\leqq n_0\leqq k$, arbitrary $\varepsilon>0$ and $x\to\infty$ we have*

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$$D(n_0, k; x) = \sum_{\substack{n \leqq x \\ n \equiv n_0 (\mathrm{mod}\, k)}} d(n) = \alpha(n_0, k) x \log x + \beta(n_0, k) x + O(x^{35/108+\varepsilon}) . \tag{3}$$

Here the constant implied in the O-term may depend on n_0, k and ε—this will be the case for all O-constants throughout the whole paper. We remark parenthetically that the problem becomes much harder and leads to considerably weaker results if one allows k to grow with x (cf. e.g. Petečuk [6]).

Our proof combines classical methods and results due to Walfisz [9] and Landau [5] with Kolesnik's recent estimate for double exponential sums [3].

3. Proof of our theorem

3.1. Preliminaries. Throughout the whole paper, we denote by C any absolute numerical constant, not necessarily the same at each occurrence; ε and δ are suitable small positive real numbers, a and b equal 0 or 1. $P(x)$ denotes any expression of the form $\alpha x \log x + \beta x$ where the coefficients α and β depend on n_0, k, a and b and are not necessarily the same at each occurrence.

We note first that it is sufficient to prove that

$$D^*(u_0, v_0, k; x) := \sum_{\substack{uv \leqq x \\ u \equiv u_0,\, v \equiv v_0 (\mathrm{mod}\, k)}} 1 = P(x) + O(x^{35/108+\varepsilon}) \tag{4}$$

holds for any $u_0, v_0 \in \{1, \cdots, k\}$ because, summing over all pairs $(u_0, v_0) \in \{1, \cdots, k\}^2$ with $u_0 v_0 \equiv n_0 (\mathrm{mod}\, k)$, we immediately infer (3) from (4).

3.2. For complex $s = \sigma + it$ and real w, $0 < w \leqq 1$ we define the functions $\xi_0(s, w)$ and $\xi_1(s, w)$ for $\sigma > 1$ by the absolutely convergent series

$$\xi_a(s, w) = \frac{1}{2} \sum_{n=0}^{\infty} (n+w)^{-s} + \frac{1}{2} (-1)^a \sum_{\substack{n=1 \\ n \neq w}}^{\infty} (n-w)^{-s} \quad (a \in \{0, 1\}) \tag{5}$$

and for $\sigma \leqq 1$ by the corresponding analytic continuations. It is known (cf. Landau [5], p. 58) that $\xi_1(s)$ is an entire function, while $\xi_0(s)$ is meromorphic with exactly one pole, namely at $s = 1$ of order 1 with residue 1. Moreover, in any strip $\sigma_1 \leqq \sigma \leqq \sigma_2$ one has $\xi_a(\sigma + it, w) = O(e^{|t|})$ uniformly for $|t| \to \infty$ $(a \in \{0, 1\})$.

For given u_0, v_0 and k we further put

$$Z_{a,b}(s) = Z_{a,b}(s;\, u_0,\, v_0,\, k) := k^{-2s} \xi_a(s,\, u_0/k) \xi_b(s,\, v_0/k) . \tag{6}$$

Let $d_n(a, b)$ denote the coefficients in the Dirichlet series

$$Z_{a,b}(s) = \sum_{n=1}^{\infty} d_n(a, b) n^{-s} \quad (\sigma > 1) \tag{7}$$

then it is seen by an easy combinatorial argument that

$$D^*(u_0, v_0, k; x) = \sum_{a,b=0}^{1} \sum_{n \leqq x} d_n(a, b) =: \sum_{a,b=0}^{1} S(a, b) \quad \text{(say)} \tag{8}$$

where $d_n(a, b) \ll d(n) \ll n^\delta$.

3.3. We now suppose (as we obviously may w.l.o.g.) that $x \equiv 1/2 (\mathrm{mod}\ 1)$ and obtain by a well-known theorem (cf. e.g. Prachar [7], p. 376, Theorem 3.1.)

$$S(a, b) = \sum_{n \leqq x} d_n(a, b) = (2\pi i)^{-1} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} Z_{a,b}(s) x^s s^{-1} ds + O(x^{1+\varepsilon} T^{-1}), \tag{9}$$

where $T > 0$ rests at our disposition. In order to replace the line of integration of this integral by the line segment from $-\varepsilon - iT$ to $-\varepsilon + iT$, we first need some information about the order of the integrand for large t: From the relation

$$\xi_a(s, w) = \Gamma((1+a-s)/2)(\Gamma((a+s)/2))^{-1} \pi^{s-1/2} \sum_{u=1}^{\infty} u^{-1+s} \cos(2\pi u w - a\pi/2) \tag{10}$$

(which is valid for $\sigma < 0$ and $a \in \{0, 1\}$; cf. [5], p. 58, formulae (67) and (69)) we obtain (in view of (6))

$$Z_{a,b}(s) = k^{-2s} G(s) \pi^{2s-1} \sum_{n=1}^{\infty} c_n n^{-1+s} \quad (\sigma < 0), \tag{11}$$

where

$$G(s) = G_{a,b}(s) = \Gamma((1+a-s)/2)\Gamma((1+b-s)/2)/\Gamma((a+s)/2)\Gamma((b+s)/2), \tag{12}$$

$$c_n = \sum_{uv=n} \cos(2\pi u u_0 k^{-1} - a\pi/2) \cos(2\pi v v_0 k^{-1} - b\pi/2). \tag{13}$$

Hence, by the asymptotic formula for the Γ-function

$$\Gamma(\sigma \pm it) = e^{\pm i(\pi(\sigma-1/2)/2 + t\log t - t)} e^{c - \pi t/2} t^{\sigma-1/2} (1 + O(t^{-1})) \tag{14}$$

(for $t > 0$; see [5], p. 227; $c = \log(2\pi)/2$), we have

$$Z(-\varepsilon + it) \ll |G(-\varepsilon + it)| \ll t^{1+2\varepsilon} \tag{15}$$

and, since $Z(1+\varepsilon+it) \ll 1$, by the Phragmén-Lindelöf principle ([5], p. 229)

$$Z(\sigma + it) \ll t^{1+\varepsilon-\sigma} \tag{16}$$

uniformly in the strip $-\varepsilon \leqq \sigma \leqq 1+\varepsilon$. Therefore

$$\int_{-\varepsilon+iT}^{1+\varepsilon+iT} Z(s) x^s s^{-1} ds \ll \int_{-\varepsilon}^{1+\varepsilon} x^\sigma T^{\varepsilon-\sigma} d\sigma \ll x^{1+\varepsilon} T^{-1}, \tag{17}$$

if we suppose that $T < x$ for large x (in fact, T will be chosen to be of the order $x^{73/108}$); the integral from $-\varepsilon - iT$ to $1+\varepsilon-iT$ can be estimated in a completely analogous way. We observe finally that the integrand in (9) is everywhere regular with the possible exception of the point $s = 1$ where it may have a pole of order at most 2 with residue $P(x)$, and of the point $s = 0$ where it may have a simple pole with residue

$Z(0)=O(1)$. So the residue theorem yields (because of (9) and (17))

$$S(a,b)=P(x)+(2\pi i)^{-1}\int_{-\varepsilon-iT}^{-\varepsilon+iT} Z(s)x^s s^{-1}ds+O(x^{1+\varepsilon}T^{-1}). \tag{18}$$

3.4. Now we can use the representation (11) for $Z(s)$ and obtain

$$S(a,b)=P(x)+(2\pi^2 i)^{-1}\sum_{n=1}^{\infty} c_n n^{-1}\int_{-\varepsilon-iT}^{-\varepsilon+iT} G(s)s^{-1}z_n^s 2^{-2s}ds+O(x^{1+\varepsilon}T^{-1}) \tag{19}$$

where $z=z_n:=4\pi^2 nx/k^2$ for short. Writing $A=(a+b+1)/2$ and $\Psi(s)=\Gamma(A-s)/\Gamma(1+A+s)$, we infer from (12) and (14) by a short computation that

$$G(s)=2^{2s-1}s(s+1)\Psi(s)+O(|t|^{2\varepsilon}) \tag{20}$$

for $s=-\varepsilon+it$ and large $|t|$. Denoting the union of the two segments from $-\varepsilon-iT$ to $-\varepsilon-i$ and from $-\varepsilon+i$ to $-\varepsilon+iT$ by $\gamma(-\varepsilon)$, we observe that

$$\int_{\gamma(-\varepsilon)} |t|^{2\varepsilon}s^{-1}z_n^s ds \ll z_n^{-\varepsilon}\int_1^T t^{2\varepsilon-1}dt \ll T^{2\varepsilon}z_n^{-\varepsilon} \ll x^\varepsilon n^{-\varepsilon}, \tag{21}$$

so the contribution of the O-term in (20) to the sum in (19) is $O(x^\varepsilon)$ (since, by (13), $c_n \ll d(n) \ll n^\delta$) and therefore contained in the O-term of (19). Moreover,

$$\int_{-\varepsilon-i}^{-\varepsilon+i} G(s)s^{-1}z_n^s 2^{-2s}ds \ll z_n^{-\varepsilon}\max_{-1\leqq t\leqq 1}|G(-\varepsilon+it)/(-\varepsilon+it)| \ll x^{-\varepsilon}n^{-\varepsilon}, \tag{22}$$

so that we may simplify (19) to

$$S(a,b)=P(x)+(4\pi^2 i)^{-1}\sum_{n=1}^{\infty} c_n n^{-1}I_n(T)+O(x^{1+\varepsilon}T^{-1}), \tag{23}$$

$$I_n(T):=\int_{\gamma(-\varepsilon)}(s+1)\Psi(s)z_n^s ds. \tag{24}$$

3.5. We now define N by the relation $k^2T^2(4\pi^2x)^{-1}=N+1/2$ and assume w.l.o.g. that T is chosen in such a way that N is an integer. Denoting the integrand of this last integral by $H_n(s)$ and approximating $\Psi(s)$ by (14) we get

$$H_n(-\varepsilon+it)=iz_n^{-\varepsilon}e^{iF_n(t)}t^{2\varepsilon}(1+O(t^{-1})) \qquad (t\geqq 1) \tag{25}$$

where

$$F_n(t)=-A\pi-2t\log t+2t+t\log(4\pi^2nx/k^2), \tag{26}$$

hence

$$F_n'(t)=-2\log t+\log(4\pi^2nx/k^2)=\log(4\pi^2nx/k^2t^2). \tag{27}$$

We now estimate the integral corresponding to the main term of (25) in the usual way using the second mean-value theorem: For $n>N$ we get

$$z_n^{-\varepsilon}\int_1^T t^{2\varepsilon}e^{iF_n(t)}dt \ll z_n^{-\varepsilon}\max_{1\leqq t\leqq T}|t^{2\varepsilon}F_n'(t)^{-1}|$$
$$\ll x^{-\varepsilon}n^{-\varepsilon}T^{2\varepsilon}(\log(4\pi^2nx/k^2T^2))^{-1}$$
$$\ll n^{-\varepsilon}x^{\varepsilon}\left(\log n-\log\left(N+\frac{1}{2}\right)\right)^{-1}. \tag{28}$$

For $n\geqq 2N$ this contributes to the sum in (23) obviously $O(x^{\varepsilon})$ and for $N<n<2N$ we have

$$x^{\varepsilon}\sum_{N<n<2N} c_n n^{-1-\varepsilon}\left(\log n-\log\left(N+\frac{1}{2}\right)\right)^{-1} \ll x^{\varepsilon}\sum_{N<n<2N} c_n n^{-\varepsilon}(n-N)^{-1}$$
$$\ll x^{\varepsilon}N^{-\varepsilon/2}\log N \ll x^{\varepsilon}. \tag{29}$$

Noting that the contribution of the O-term in (25) to $I_n(T)$ is $O(x^{\varepsilon}n^{-\varepsilon})$ (cf. (21)) we finally obtain

$$S(a,b)=P(x)+(4\pi^2 i)^{-1}\sum_{n=1}^{N} c_n n^{-1}I_n(T)+O(x^{1+\varepsilon}T^{-1}). \tag{30}$$

3.6. Next we replace the way of integration $\gamma(-\varepsilon)$ in (24) by $\gamma(\varepsilon)$ which we define as the union of the segments from $\varepsilon-iT$ to $\varepsilon-i$ and from $\varepsilon+i$ to $\varepsilon+iT$. Using (25) (with σ instead of $-\varepsilon$) we infer that

$$H_n(\sigma+it)\ll z_n^{\sigma}|t|^{-2\sigma} \qquad (|t|\geqq 1) \tag{31}$$

uniformly in any strip $\sigma_1\leqq\sigma\leqq\sigma_2$, hence (for $n\leqq N$)

$$\int_{-\varepsilon+i\tau}^{\varepsilon+i\tau} H_n(s)ds \ll z_n^{\varepsilon}\tau^{2\varepsilon} \ll x^{C\varepsilon} \quad (\tau=1 \text{ or } T). \tag{32}$$

Since the argument also applies to the lower half plane and since $H_n(s)$ is regular in each of the rectangles involved we get by Cauchy's theorem

$$I_n(T)=\int_{\gamma(\varepsilon)} H_n(s)ds+O(x^{C\varepsilon}). \tag{33}$$

Now we want to replace $\gamma(\varepsilon)$ by the line from $\varepsilon-i\infty$ to $\varepsilon+i\infty$. To this end we make again use of (25) (with ε instead of $-\varepsilon$) and obtain for the contribution of the main term

$$z_n^{\varepsilon}\int_T^{\infty} e^{iF_n(t)}t^{-2\varepsilon}dt \ll z_n^{\varepsilon}\max_{t\geqq T}(t^{-2}t^{-2\varepsilon}|\log(4\pi^2nx/k^2t^2)|^{-1})$$
$$\ll z_n^{\varepsilon}T^{-2\varepsilon}\left|\log\left(n\Big/\left(N+\frac{1}{2}\right)\right)\right|^{-1} \ll x^{C\varepsilon}n^{\varepsilon}\left(\log\left(N+\frac{1}{2}\right)-\log n\right)^{-1}$$
$$\tag{34}$$

(provided that $n \leqq N$). The contribution of the O-term in (25) is easily seen to be also $\ll x^{C\varepsilon}$, and the same error term arises by adding the segment from $\varepsilon - i$ to $\varepsilon + i$ to our path of integration (cf. (22)). Altogether we arrive at

$$I_n(T) = I_n + O(x^{C\varepsilon} N(N-n+1)^{-1}), \qquad I_n := \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} H_n(s) ds, \tag{35}$$

where the convergence of I_n follows from the above arguments.

3.7. We are now going to evaluate I_n by means of the residue theorem. To this end we notice first that the singularities of

$$H_n(s) = (s+1) z_n^s \Gamma(A-s)/\Gamma(1+A+s) \tag{36}$$

are exactly the (simple) poles of $\Gamma(A-s)$ at $s = A+p$, $p \in \mathbf{N}_0$, with residue

$$(1+A+p) z_n^{A+p} (-1)^p (p!\,\Gamma(1+2A+p))^{-1}.$$

We put $U_m = m + \varepsilon$ for $m \in \mathbf{N}$ and consider the rectangle with vertices $\varepsilon \pm i\tau$, $U_m \pm i\tau$ for some fixed m. By virtue of (31), we have

$$\int_{\varepsilon + i\tau}^{U_m + i\tau} H_n(s) ds \ll z_n^{U_m} \tau^{-2\varepsilon} \tag{37}$$

and this tends to 0 as $\tau \to \infty$. Treating the integral from $\varepsilon - i\tau$ to $U_m - i\tau$ in the same way, we get (for arbitrary $m \in \mathbf{N}$)

$$I_n = -2\pi i \sum_{0 \leqq p < U_m - A} (1+A+p) z_n^{A+p} (-1)^p (p!(2A+p)!)^{-1} + \int_{U_m - i\infty}^{U_m + i\infty} H_n(s) ds. \tag{38}$$

In order to show that this last integral tends to 0 for $m \to \infty$, we deduce from the functional equation of the Γ-function that

$$\begin{aligned} H_n(U_m + it) &= (1 + U_m + it) z_n^{U_m + it} \Gamma(A - \varepsilon - m - it)/\Gamma(1 + A + \varepsilon + m + it) \\ &= (1 + \varepsilon + m + it) z_n^{\varepsilon + m + it} \Psi(1 + \varepsilon + it) \\ &\quad \times \prod_{j=2}^{m} (A - \varepsilon - j - it)^{-1} \prod_{j=1}^{m-1} (1 + A + \varepsilon + j + it)^{-1} \end{aligned}$$

and note that the moduli of the factors in the first product are all $\leqq \varepsilon^{-1}$ and that the modulus of the j-th factor in the second product is $\leqq (j+1)^{-1}$. Hence

$$H_n(U_m + it) \ll (m + |t|)\, |\Psi(1 + \varepsilon + it)|\, z_n^{\varepsilon + m} \varepsilon^{-m} (m!)^{-1}. \tag{39}$$

Denoting the maximum of $|\Psi(1+\varepsilon+it)|$ for $-1 \leqq t \leqq 1$ by μ and estimating $|\Psi(1+\varepsilon+it)|$ for $|t| \geqq 1$ by (14), we obtain

$$\int_{-\infty}^{\infty} H_n(U_m + it) dt \ll m \varepsilon^{-m} z_n^{\varepsilon + m} (m!)^{-1} \left(\int_1^{\infty} t^{-2-2\varepsilon} dt + \mu \right), \tag{40}$$

where the right-hand side tends to 0 (for $m \to \infty$) as asserted. Thus (38) yields

$$I_n = -2\pi i\left(\sum_{p=0}^{\infty}(-1)^p z_n^{A+p}(p!(2A+p)!)^{-1}(1+A)\right.$$
$$\left.-\sum_{p=0}^{\infty}(-1)^p z_n^{1+A+p}(p!(1+2A+p)!)^{-1}\right)$$
$$= -2\pi i((1+A)J_{2A}(2\sqrt{z_n}) - \sqrt{z_n}J_{2A+1}(2\sqrt{z_n})) \tag{41}$$

where J_K are the usual Bessel functions (of the first kind) of order K (cf. e.g. Fricker [1], p. 195). Using their asymptotic expansions ([5], p. 244) we get

$$I_n = 2\sqrt{\pi}\, iz_n^{1/4}\cos(2\sqrt{z_n} - A\pi - 3\pi/4) + O(z_n^{-1/4})$$
$$= 2^{3/2}\pi i(x/k^2)^{1/4}n^{1/4}\cos(4\pi(nx/k^2)^{1/2} - A\pi - 3\pi/4) + O(x^{-1/4}n^{-1/4}). \tag{42}$$

Entering this (together with (35)) into (30) we finally arrive at

$$S(a,b) = P(x) + 2^{-1/2}\pi^{-1}k^{-1/2}x^{1/4}\sum_{n=1}^{N} c_n n^{-3/4}\cos(4\pi(nx/k^2)^{1/2}$$
$$-A\pi - 3\pi/4) + O(x^{1+\varepsilon}T^{-1}) + O(x^{C\varepsilon}) \tag{43}$$

which is a generalization of the corresponding well-known formula for $D(x)$ in the case of the classical Dirichlet's divisor problem.

3.8. The last step of our proof is to apply a deep theorem recently established by Kolesnik [3] to the trigonometric sum in (43). Recalling the definition (13) of c_n we have to estimate

$$\sum_{uv \leqq N}(uv)^{-3/4}\cos(2\pi uu_0/k - \pi a/2)\cos(2\pi vv_0/k - \pi b/2)$$
$$\times\cos(4\pi(x/k^2)^{1/2}(uv)^{1/2} - A\pi - 3\pi/4)$$

which is

$$\ll \sum\left|\sum_{\substack{uv \leqq N\\ u \geqq v}}(uv)^{-3/4}e(f(u,v))\right| + O(1), \tag{44}$$

where $e(f) = e^{2\pi i f}$ as usual, $R := x/k^2$ for short,

$$f(u, v) := 2R^{1/2}(uv)^{1/2} \pm uu_0/k \pm vv_0/k \tag{45}$$

and the first sum in (44) is to be taken over all possible choices of the $\pm$-sign in $f(u, v)$. We put $U_i = 2^i/3$, $V_j = 2^j/3$, $N_{ij} = U_iV_j$ and

$$D_{ij} := \{(u, v) \in \mathbf{Z}^2 :\ U_i \leqq u \leqq 2U_i,\ V_j \leqq v \leqq 2V_j,\ u \geqq v,\ uv \leqq N\} \tag{46}$$

and infer from Kolesnik's Lemma 3 ([3], p. 108) that

$$\left|\sum_{D_{ij}}(uv)^{-3/4}e(f(u,v))\right| \ll N_{ij}^{-3/4}\left|\sum_{D_{ij}^*}e(f(u,v))\right| \tag{47}$$

where D_{ij}^* is a certain subset of D_{ij} determined by additional restrictions of the form $U_i' \leqq u \leqq U_i''$, $V_j' \leqq v \leqq V_j''$. Then crucial point in the application of Kolesnik's theorem ([3], p. 112) is to verify that, for $(u, v) \in D_{ij}$,

$$\frac{\partial^{p+q} f(u, v)}{\partial u^p \partial v^q} = C_{p,q} f(u, v) u^{-p} v^{-q} + O(N_{ij}^{-1/3} F_{ij} U_i^{-p} V_j^{-q}) \tag{48}$$

where the constants $C_{p,q}$ have to satisfy a rather long list of very specific conditions and F_{ij} is the maximum of $|f(u, v)|$ on D_{ij}, so $(xN_{ij})^{1/2} \ll F_{ij} \ll (xN_{ij})^{1/2}$. We now choose T (occurring in (43)) exactly of the order $x^{73/108}$, thus N (defined in 3.5.) is of the order $x^{38/108}$. By (45) we have for $p^2 + q^2 > 1$

$$\frac{\partial^{p+q} f(u, v)}{\partial u^p \partial v^q} = 2C_{p,q} R^{1/2} u^{1/2-p} v^{1/2-q} \tag{49}$$

(which defines the constants $C_{p,q}$). Therefore, by (45),

$$\frac{\partial^{p+q} f(u, v)}{\partial u^p \partial v^q} - C_{p,q} f(u, v) u^{-p} v^{-q} \ll (U_i + V_j) U_i^{-p} V_j^{-q} \tag{50}$$

for $(u, v) \in D_{ij}$, and this is less than the O-term in (48), since

$$U_i + V_j \ll U_i V_j = N_{ij} \ll x^{1/2} N_{ij}^{1/6} \ll N_{ij}^{-1/3} F_{ij}$$

because of $N_{ij} \ll N \ll x^{38/108}$. The cases $p=1$, $q=0$ and $p=0$, $q=1$ can be dealt with in a completely analogous way. We now further note that our constants $C_{p,q}$ obviously are the same as those occurring in connection with Dirichlet's divisor problem (where $f(u, v)$ is defined without the terms $\pm uu_0/k \pm vv_0/k$) — and for this case Kolesnik has already verified his numerous conditions on the $C_{p,q}$. Therefore, we may apply his estimate ([3], p. 113) to the exponential sum over D_{ij}^* and infer from (47) that

$$\begin{aligned} \sum_{D_{ij}} (uv)^{-3/4} e(f(u, v)) &\ll N_{ij}^{1/4+\delta} (N_{ij}^{61/38} F_{ij}^{-1} + F_{ij} N_{ij}^{-85/38})^{1/8} \\ &\ll N_{ij}^{1/4+\delta} (x^{-1/2} N_{ij}^{21/19} + x^{1/2} N_{ij}^{-33/19})^{1/8} \\ &\ll x^{-1/16} N_{ij}^{59/152+\delta} + x^{1/16} N_{ij}^{5/152+\delta} \ll x^{2/27+\delta}. \end{aligned}$$

Summing over all D_{ij} which are non-empty (their number being $O(x^\delta)$) and going back to (44) and finally to (43) we complete the proof of the theorem.

4. A corollary concerning the circle problem

As an easy consequence of our formula (4) we obtain the following result for the classical circle problem:

COROLLARY. *Let $r(n)$ denote the number of integer pairs (u, v) for which $u^2 + v^2 = n$, then (for $x \to \infty$)*

$$\sum_{n \leqq x} r(n) = \pi x + O(x^{35/108+\varepsilon}) \qquad (\varepsilon > 0). \tag{51}$$

Proof. Denote by $d_j(n)$ the number of (positive) divisors of n which are congruent to j modulo 4. Then, by a classical elementary relation (see Fricker [1], p. 16) and our formula (4), we get

$$\begin{aligned}\sum_{n \leqq x} r(n) &= 4 \sum_{\substack{uv \leqq x \\ u \equiv 1 \,(\mathrm{mod}\, 4)}} 1 - 4 \sum_{\substack{uv \leqq x \\ u \equiv 3 \,(\mathrm{mod}\, 4)}} 1 \\ &= 4 \sum_{r=1}^{4} (D^*(1, r, 4; x) - D^*(3, r, 4; x)) = P(x) + O(x^{35/108+\varepsilon}).\end{aligned}$$

Comparing this with an elementary version of (51) (with $O(x^{1/2})$, say) we conclude that necessarily $P(x) = \pi x$ in this case.

Remark. This result apparently improves upon all previously known O-estimates for the circle problem, the former "record" being due to J. R. Chen from the year 1963 (see the historical and bibliographic remarks in Fricker [1], p. 87).

A self-contained proof of the above corollary (somewhat simpler in technical details than the general case considered here) is also given in another paper by the author which is in course of publication elsewhere.

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