Approximation Theorems in Categories

by

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1. Introduction

In a commutative algebra and in a theory of partially ordered groups there are several approximation theorems which have almost the same structure and serve as a very useful tool in these theories. Recall for example the Ribenboim approximation theorem for valuations of a field [5], the Krull approximation theorem for lattice-ordered groups [3] and the approximation theorem for d -groups [3]. Comparing these approximation theorems one may observe a common logical background of them. Roughly speaking, all these theorems are models of a theory in a suitable many sorted language and this enables us to unify the theory of approximation theorems in various branches of algebra. Moreover, instead of a classical model theory we propose here a categoric substitute as a natural generalization of approximation theorems in fields, lattice-ordered groups etc. Since the axioms of a theory of approximation theorems are rather simple, the interpretation of this theory requires only several additional properties of a category which then may serve as a natural generalization of the category SET.

In this paper we deal mostly with general properties of approximation theorems in categories and in the following paper [4] we investigate some applications of this model-theoretic point of view.

2. Approximation theorems in $\mathcal{A}$

In this section we deal with an abstract form of an approximation theorem (in abbreviation A.T.) from the point of view of model theory. To motivate the exact definition of an A.T. for the convenience of the reader we recall the main example of an A.T. in valuation theory.

Let K be a field and let w_i , $1 \leq i \leq n$, be valuations of K with valuation rings R_i and value groups G_i . If $R_i \subseteq R_j$ we write $w_i \geq w_j$ and this is equivalent to the existence of an order epimorphism δ which completes the following diagram.

$$\begin{array}{ccc} K^* & \xrightarrow{w_i} & G_i \\ w_j \downarrow & \nearrow \delta & \\ & G_j & \end{array}$$

(Here $K^* = K - \{0\}$.) In this case we say that w_j is coarser than w_i and the relation $\leq$ is an ordering of the set of valuations of a field K . Since valuations coarser than w_i and w_j are totally ordered in $\leq$ there exists a valuation w_{ij} such that it is an infimum of w_i and w_j , $w_{ij} = w_i \wedge w_j$ and there are ordered epimorphisms δ_{ij}, δ_{ji} which complete the following diagram.

$$\begin{array}{ccc}
 K^* & \xrightarrow{w_i} & G_i \\
 w_j \downarrow & & \downarrow \delta_{ij} \\
 G_j & \xrightarrow{\delta_{ji}} & G_{ij}
 \end{array}$$

Moreover, if $R = \bigcap R_i$ ($1 \leq i \leq n$) and $G(R)$ is a group of divisibility of R , i.e. $G(R) = K^*/U(R)$ is a multiplicative group where $U(R)$ is a group of units of R , we obtain $\delta_{ij} \cdot w_i = w_j \cdot \delta_{ji}$, where $w_i(xU(R)) = w_i(x)$ and this equality represents a pushout diagram in a category $\mathcal{K}$ of partially ordered directed groups with order homomorphisms.

Now if C is an inf-semilattice generated by the set $\{w_i: 1 \leq i \leq n\} = J$ in the set of all valuations of K together with an element $*$ such that $* \geq w, w \in C$, we may consider a functor

$$G: C^{op} \longrightarrow \mathcal{K}$$

(where C is a category with unique morphisms $\delta_{w', w}: w \rightarrow w'$ iff $w \leq w'$ and $\delta_w: w \rightarrow *$) such that $Gw = G_w$, $G = G(R)$, $G\delta_w = w$ and $G(\delta_{w', w})$ is from the definition of $\leq$.

We now turn our attention to the A.T. of Ribenboim, who has proved the truth value of interpretation of the following proposition which uses symbols from the category C :

$$\bigwedge_{1 \leq i < j \leq n} (\delta_{ij}(x_i) = \delta_{ji}(x_j)) \Leftrightarrow \exists x \left(\bigwedge_{i \leq i \leq n} \delta_i(x) = x_i \right).$$

Hence if we consider G to be a model of a suitable language, G satisfies this formulae.

This example leads us to the following description of A.T.

Let C be an inf-semilattice with the greatest element $*$ and let J be the set of dual atoms in C . We assume that C is generated by J . C may be considered as a category with unique morphisms $\delta_{x, y}: y \rightarrow x$ iff $x \geq y$. Instead of $\delta_{*, x}$ we write δ_x and for $i, j \in J$ we let $\delta_{ij} := \delta_{i, i \wedge j}$. With this category we associate a language L_c of a type $L_{\infty\omega}$ (but in a many-sorted version) as follows: Its sorts are objects of C , the only many sorted operation symbols are morphisms in C^{op} . Now let $\vec{x} = (x_i)_{i \in J}$ be a set of free variables and let

$$\mu(\vec{x}) \equiv \bigwedge_{i, j \in J} (\delta_{ij}(x_i) = \delta_{ji}(x_j))$$

$$\begin{aligned}\varphi(\vec{x}, y) &\equiv \bigwedge_{i \in J} (\delta_i(y) = x_i) \\ \gamma(\vec{x}) &\equiv \mu(\vec{x}) \Leftrightarrow (\exists y)\varphi(\vec{x}, y) \\ \gamma_0(x, y) &\equiv \bigwedge_{i \in J} (\delta_i(x) = \delta_i(y)) \Leftrightarrow x = y,\end{aligned}$$

where the last two expressions are Gentzen sequences (see [2]).

Now let $\mathcal{R}$ be a small-complete category which has images, i.e. for every subobject $A \hookrightarrow X$ and every morphism $X \xrightarrow{f} Y$, $\exists_f(A) \hookrightarrow Y$ exists.

DEFINITION 2.1. An approximation theorem of a type $(C, \mathcal{R})$ is a functor $M: C^{op} \rightarrow \mathcal{R}$ such that $M \models \gamma(\vec{x})$, $M \models \gamma_0(x, y)$.

The interpretation of a formulae φ from L_C in M we denote by $M[\varphi]$ (sometimes only $[\varphi]$). Every x_i from $\vec{x}$ is of a type i and the interpretation of $\mu(\vec{x})$ in M is a subobject of $M\vec{x} = \prod_{i \in J} Mi$ in $\mathcal{R}$. This subobject is called the *subobject of compatible elements of M* and it is denoted by $\alpha_M: K(M) \hookrightarrow M\vec{x}$. Analogously the interpretation of $\varphi(\vec{x}, y)$ in M is a subobject of $M\vec{x} \times M*$ and the interpretation of $(\exists y)\varphi(\vec{x}, y)$ is the smallest subobject E of $M\vec{x}$ for which the following diagram may be completed, i.e. $E = \exists_{\text{projection}}([\varphi(\vec{x}, y)])$.

$$\begin{array}{ccc} M[\varphi(\vec{x}, y)] & \hookrightarrow & M\vec{x} \times M* \\ \downarrow & & \downarrow \text{projection} \\ E & \hookrightarrow & M\vec{x} \end{array}$$

Then $M \models \gamma(\vec{x})$ iff $M[\mu] = M[(\exists y)\varphi]$ as subobjects. More details about interpretation logic in categories may be found in [2].

We deal at first with an analogy of set theoretic properties of A.T. as they are known for classical A. T. A classical A.T. we may identify with a map which assigns to a compatible family of elements its solution. For example, if G is an A.T. for valuations, for every $\mathbf{a} = (a_i)_{1 \leq i \leq n} \in K(G)$, $\xi(\mathbf{a}) = x$ is an element of $G(R) (= G*)$ such that $w_i(x) = a_i$, $1 \leq i \leq n$, i.e. $\xi(\mathbf{a})$ is a solution of $\mathbf{a}$. Our principal result is that analogous situation occurs for any general approximation theorem.

In the sequel we frequently identify for simplicity a functional symbols in L_C with their interpretation in a given category.

LEMMA 2.2. Let $M: C^{op} \rightarrow \mathcal{R}$ be a functor and let $\delta = \prod_{i \in J} \delta_i: M* \rightarrow M\vec{x}$. Then $M \models \gamma_0(x, y)$ iff δ is a monomorphism in $\mathcal{R}$.

Proof. Clearly δ is a monomorphism if and only if from $\delta_i \alpha = \delta_i \beta$ for every $i \in J$ it follows $\alpha = \beta$. Let $M \models \gamma_0$. We note at first that the interpretation of a formulae $\delta_i(x) = \delta_i(y)$ is a subobject $R_i \xrightarrow{u_i} M* \times M*$ such that the diagram

$$\begin{array}{ccc}
 M* & \xrightarrow{\delta_i} & Mi \\
 \rho_2 u_i \uparrow & & \uparrow \delta_i \\
 R_i & \xrightarrow{\rho_1 u_i} & M*
 \end{array}$$

is a pullback, where ρ_i are canonical projections. Let $\delta_i \alpha = \delta_i \beta$ for every $i \in J$. Then there exist morphisms $\omega_i: M* \rightarrow R_i$ such that $\rho_2 u_i \omega_i = \alpha$, $\rho_1 u_i \omega_i = \beta$. Since $M \models \gamma_0$, the interpretation of $\bigwedge \delta_i(x) = \delta_i(y)$ is a subobject

$$1 \times 1: M* \hookrightarrow M* \times M*$$

(i.e. the interpretation of $x=y$) which must be at the same time a pullback of u_i , i.e. for some v_i we have $1 \times 1 = u_i v_i$. Then

$$\rho_1 u_i \omega_i = \beta = \rho_1 u_j \omega_j, \quad \rho_2 u_i \omega_i = \alpha = \rho_2 u_j \omega_j$$

and for every $i, j \in J$ we have $u_i \omega_i = u_j \omega_j$. Hence there exists a morphism $\gamma: M* \rightarrow M*$ such that $\omega_i = v_i \gamma$ and we have

$$\alpha = \rho_2 u_i \omega_i = \rho_2 u_i v_i \gamma = \gamma = \rho_1 u_i v_i \gamma = \rho_1 u_i \omega_i = \beta.$$

Conversely, let δ be a monomorphism. We show that the interpretation I of $\bigwedge \delta_i(x) = \delta_i(y)$ is the equalizer of ρ_1 and ρ_2 . Let $u: I \hookrightarrow M* \times M*$. At first, $\delta_i(\rho_1 u_i v_i) = \delta_i(\rho_2 u_i v_i)$ for every $i \in J$ where u_i, v_i have the same meaning as in the first part of the proof. Hence $\rho_1 u_i v_i = \rho_2 u_i v_i$. Further, let $f: X \rightarrow M* \times M*$ be such that $\rho_1 f = \rho_2 f$. Then there is the unique morphism $g_i: X \rightarrow R_i$ such that $\rho_1 u_i g_i = \rho_1 f = \rho_2 u_i g_i$ and it follows $u_i g_i = f$. Analogously, since I is a pullback of u_i , there exists the unique morphism g such that $v_i g = g_i$ and it follows $u_i v_i g = u_i g_i = f$. Therefore, I is an equalizer of ρ_1 and ρ_2 and $M \models \gamma_0$.

We may prove our principal result.

THEOREM 2.3. *Let $M: C^{op} \rightarrow \mathcal{R}$ be a functor. Then M is an approximation theorem of a type $(C, \mathcal{R})$ iff there exists an isomorphism (in $\mathcal{R}$) $\xi (= \xi_M)$ such that the following diagram commutes.*

$$\begin{array}{ccc}
 K(M) & \xrightarrow{\xi_M} & M* \\
 \alpha_M \searrow & & \swarrow \delta \\
 & M\vec{x} &
 \end{array}$$

Proof. We assume at first that M is an A.T. of a type $(C, \mathcal{R})$. Let $R(\vec{x}, y) = \mu(\vec{x}) \wedge \varphi(\vec{x}, y)$ and let $R = [R(x, y)] \hookrightarrow M\vec{x} \times M*$. Since $[\mu(\vec{x})]$ (with free variables $\{\vec{x}, y\}$) is a subobject

$$K(M) \times M* \hookrightarrow M\vec{x} \times M*$$

we have $R = K(M) \times M^* \wedge [\varphi(\vec{x}, y)] \leq K(M) \times M^*$, i.e. R is a subobject of $K(M) \times M^*$. We show that this subobject is functorial. Let

$$\eta \equiv R(\vec{x}, y_1) \wedge R(\vec{x}, y_2) \Leftrightarrow y_1 = y_2.$$

We show that $\gamma_0 \models \eta$. Since η is a Horn-sequence, we turn our attention to a SET-value models of L_C and in this case the proof is trivial. Hence, $\gamma_0 \models \eta$ is SET and according [2], 3.1.1, we have $\gamma_0 \models \eta$ in $\mathcal{D}$. Since $M \models \gamma_0$, we obtain $M \models \eta$.

Further we show that the sequence

$$\Rightarrow (\exists y)R(\vec{x}, y)$$

holds in M . For this purpose it suffices to show that the interpretation of $(\exists y)R(\vec{x}, y)$ equals $K(M)$. We have

$$\begin{aligned} (\exists y)R(\vec{x}, y) &\Leftrightarrow (\exists y)(\mu(\vec{x}) \wedge \varphi(\vec{x}, y)) \Leftrightarrow \\ &\Leftrightarrow \mu(\vec{x}) \wedge (\exists y)\varphi(\vec{x}, y) \end{aligned}$$

and since in M the sequence

$$\mu(\vec{x}) \Leftrightarrow (\exists y)\varphi(\vec{x}, y)$$

holds, we obtain

$$[(\exists y)R(\vec{x}, y)] = [\mu(\vec{x})] = K(M).$$

Therefore, the relation R is functorial and according [2], 2.4.4, there exists the unique morphism $\xi: K(M) \rightarrow M^*$ such that the diagram

$$\begin{array}{ccc} K(M) & \xrightarrow{1 \times \xi} & K(M) \times M^* \\ & \searrow & \nearrow \\ & [R(\vec{x}, y)] & \end{array}$$

commutes. Hence, $1 \times \xi$ may be considered as the interpretation of $R(\vec{x}, y)$ in M . On the other hand, $[R(\vec{x}, y)]$ is the pullback of subobjects $[\varphi(\vec{x}, y)]$ and $[\mu(\vec{x})]$. Since $\alpha_i: [\delta_i(y) = x_i] \hookrightarrow M\vec{x} \times M^*$ is a pullback of π_i and δ_i , we have especially $\pi_i \rho_1 \alpha_i = \delta_i \rho_2 \alpha_i$, where ρ_i are the canonical projection of $M\vec{x} \times M^*$ and π_i are canonical projections of $M\vec{x}$. Then there exist morphisms β_i such that $\alpha_i \beta_i = \alpha_j \beta_j$ is a pullback of α_i , $i \in J$. Hence, if $\varepsilon_1, \varepsilon_2$ are canonical projections of $K(M) \times M^*$, the diagram

$$\begin{array}{ccc} K(M) \times M^* & \xrightarrow{\alpha_M \varepsilon_1 \times \varepsilon_2} & M\vec{x} \times M^* \\ \uparrow & & \uparrow \alpha_i \beta_i \\ K(M) & \xrightarrow{u} & [\varphi(\vec{x}, y)] \end{array}$$

serves as the interpretation of $R(\vec{x}, y)$ in M and we obtain

$$\pi_i \alpha_M = \pi_i \rho_1 \alpha_i \beta_i u = \delta_i \rho_2 \alpha_i \beta_i u = \delta_i \xi, \quad i \in J.$$

We now construct a morphism $M^* \rightarrow K(M)$. The interpretation of $\delta_{ij}(x_i) = \delta_{ji}(x_j)$ is an equalizer f_{ij} of morphisms $\delta_{ij}\pi_i$ and $\delta_{ji}\pi_j$. Since for $\delta = \prod \delta_i$ we have

$$\delta_{ij}\pi_i\delta = \delta_{ij}\delta_i = \delta_{i \wedge j} = \delta_{ji}\delta_j = \delta_{ji}\pi_j\delta,$$

there exists a morphism $\alpha_{ij}: M^* \rightarrow [\delta_{ij}(x_i) = \delta_{ji}(x_j)]$ such that $\delta = f_{ij}\alpha_{ij}$ for every $i, j \in J$. Since the interpretation of μ is a pullback $f'_{ij}: K(M) \rightarrow [\delta_{ij} = \delta_{ji}]$ of morphisms f_{ij} , there exists the unique morphism $\eta: M^* \rightarrow K(M)$ such that $f'_{ij}\eta = \alpha_{ij}$. Then

$$\pi_i \alpha_M \eta = \pi_i f_{ij} f'_{ij} \eta = \pi_i f_{ij} \alpha_{ij} = \rho_i \delta = \delta_i, \quad i \in J.$$

We show that ξ and η are mutually inverse. In fact, for every $i \in J$ we have

$$\pi_i \alpha_M = \delta_i \xi, \quad \pi_i \alpha_M \eta = \delta_i$$

and it follows $\pi_i \alpha_M = \pi_i \alpha_M (\eta \xi)$, $\delta_i (\xi \eta) = \delta_i$ for every i . Since α_M is a monomorphism, we obtain $1 = \eta \xi$ and according 2.2, $\xi \eta = 1$.

Conversely, let us assume that ξ_M is a required isomorphism. Let $R(\vec{x}, y)$ be the same as in the first part. We show that the interpretation of $\exists y R(\vec{x}, y)$ is $K(M)$.

As we have observed, the interpretation of $\wedge \delta_{ij} = \delta_{ji}$ with free variables $\{\vec{x}, y\}$ is a subobject

$$\alpha_M \varepsilon_1 \times \varepsilon_2: K(M) \times M^* \hookrightarrow M\vec{x} \times M^*.$$

Let $A = [\wedge \delta_i(y) = x_i] \hookrightarrow M\vec{x} \times M^*$. Then the interpretation R of $R(\vec{x}, y)$ is a pullback of A and $K(M) \times M^*$. Since $[\delta_i(y) = x_i]$ is an equalizer of $\pi_i \rho_1$ and $\delta_i \rho_2$ and since for morphism $(\alpha_M \varepsilon_1 \times \varepsilon_2)(1 \times \xi_M)$ we have

$$\pi_i \rho_1 (\alpha_M \varepsilon_1 \times \varepsilon_2)(1 \times \xi_M) = \delta_i \rho_2 (\alpha_M \varepsilon_1 \times \varepsilon_2)(1 \times \xi_M),$$

there exists a morphism $v_i: K(M) \rightarrow [\delta_i(y) = x_i]$. Analogously, since A is a pullback, there exists a morphism $v: K(M) \rightarrow A$. Let us consider the following diagram.

$$\begin{array}{ccccc} A & \hookrightarrow & K(M) \times M^* & \xleftarrow{1 \times \xi_M} & K(M) \\ \downarrow & & \downarrow \varepsilon_1 & & \downarrow 1 \\ [(\exists y)\varphi(\vec{x}, y)] & \hookrightarrow & K(M) & \xleftarrow{1} & K(M) \end{array}$$

Since $K(M)$ is the image of $K(M)$ under ε_1 and $K(M)$ is a subobject of A , according [2], 3.1.8, $K(M)$ is a subobject of $[(\exists y)\varphi(\vec{x}, y)]$. Hence, we have

$$K(M) \geq [(\exists y)R(\vec{x}, y)] = [\mu(\vec{x})] \wedge [(\exists y)\varphi(\vec{x}, y)] \geq K(M)$$

and it follows $[\mu(\vec{x})] = K(M) = [(\exists y)\varphi(\vec{x}, y)]$. Therefore, $M \models \gamma(\vec{x})$. Since $\delta = \alpha_M \xi_M^{-1}$ is

a monomorphism, $M \models \gamma_0(x, y)$ according 2.2.

If M is a functor $C^{op} \rightarrow \mathcal{R}$ such that $\delta = \prod \delta_i$ is a monomorphism, then M may seem to be a trivial approximation theorem since the diagram

$$\begin{array}{ccc} M* & \xrightarrow{1} & M* \\ & \searrow \delta & \swarrow \delta \\ & M\tilde{x} & \end{array}$$

commutes. That this is not true shows the following example. Let C be a lattice with the Hasse diagram

and let $M: C^{op} \rightarrow \text{SET}$ be a functor with a diagram

$$\begin{array}{ccc} & \{0\} & \\ \delta_1 \swarrow & & \searrow \delta_2 \\ \{0, 1\} & & \{0, 1\} \\ \searrow & & \swarrow \\ = & \{0, 1\} & = \end{array}$$

Then $\delta = \delta_1 \times \delta_2$ is a monomorphism but $K(M) = \{(0, 0), (1, 1)\} \not\cong M*$, i.e. M is not an approximation theorem of a type (C, SET) .

On the other hand, the following corollary of 2.3 holds.

COROLLARY. *Let $M: C^{op} \rightarrow \mathcal{R}$ be a functor such that $\delta = \prod \delta_i$ ($i \in J$) is an isomorphism. Then M is an approximation theorem of a type $(C, \mathcal{R})$.*

Proof. We show that $M*$ is an interpretation of $\mu(\tilde{x})$, i.e. we show that $M*$ is a pullback of f_{ij} (we use a notation of the proof of 2.3). Since

$$\delta_{ij}\pi_i\delta = \delta_{i \wedge j} = \delta_{ji}\pi_j\delta$$

there exist morphisms g_{ij} such that $f_{ij}g_{ij} = \delta$ for every $i, j \in J$. Let $t_{ij}: X \rightarrow [\delta_{ij} = \delta_{ji}]$ be such that $f_{ij}t_{ij} (= g)$ does not depend on i, j . Then for $t = \delta^{-1}g$ we have $g_{ij}t = t_{ij}$ and t is clearly unique. Hence, $M* \cong K(M)$ and the rest follows from 2.3.

It is clear that the approximation theorem from this corollary is rather trivial.

The isomorphism ξ_M from 2.3 will be called a *solution of M* . As it was mentioned above, the Theorem 2.3 has a strong intuitive meaning. Roughly speaking it says

that for every compatible system we may find its solution. On the other hand, from the existence of a morphism ξ_M^{-1} it follows that every "element" of M^* may be considered as a compatible system. For example if we consider the approximation theorem for valuations, the map ξ_M^{-1} assigns to every element $x \in G(R)$ a compatible system $(w_i(x))_{1 \leq i \leq n} \in K(M)$. An analogical situation occurs in a case of lattice-ordered groups. Let G be a lattice-ordered group. We let

$$J = \{G/H_i : 1 \leq i \leq n\}, \quad C = \left\{ G / \sum_{i \in I} H_i : I \subseteq \{1, \dots, n\} \right\} \cup \left\{ G / \bigcap_i H_i \right\}.$$

Assume that there is no containment relation between distinct elements of J . The element $G_0 = G / \bigcap H_i$ we denote by $*$ and for $c_1, c_2 \in C^{op}$ we put $c_1 \leq c_2$ iff there exists an order homomorphism which completes the following diagram, where δ_c are canonical maps:

$$\begin{array}{ccc} & * & \\ \delta_{c_1} \swarrow & & \searrow \delta_{c_2} \\ c_1 & \dashrightarrow & c_2 \end{array}$$

Moreover we set $* \geq c, c \in C$. Let $\mathcal{R}$ be the category of partially ordered groups and let $M: C^{op} \rightarrow \mathcal{R}$ be the inclusion functor. Then M is an approximation theorem of a type $(C, \mathcal{R})$, $K(M) = \{(h_i + H_i)_i \in \prod G/H_i : h_i - h_j \in H_i + H_j, 1 \leq i, j \leq n\}$, $\xi_M((h_i + H_i)_i) = h + \bigcap H_i$ is such that $h + H_i = h_i + H_i, 1 \leq i \leq n$, and $\xi_M^{-1}(h + \bigcap H_i) = (h + H_i)_i$.

The diagram in 2.3 expresses naturally the original idea of an approximation theorem. This original idea may be described in a functorial way as follows.

Let J be a full subcategory of C such that $|J| = J \cup \{*\}$. For an approximation theorem M of a type $(C, \mathcal{R})$ we may define a functor

$$K^M: J^{op} \longrightarrow \mathcal{R}$$

such that $K^M i = M i, i \in J, K^M * = K(M), K^M \delta_i = \pi_i \alpha_M$. A diagram from 2.3 may be then extended onto a natural transformation

$$\xi_M: K^M \longrightarrow M$$

where M we identify with the restriction of M onto J . This natural transformation represents a classical version of an approximation theorem in $\mathcal{R}$, and moreover it allows us to compare various approximation theorems and, namely, it enables us to specify the notion "an approximation theorem M_1 may be derived from an approximation theorem M_2 ". Intuitively this notion has a following interpretation: When we want to find a solution of a compatible system from M_1 , we may create a new compatible system from M_2 , find its solution in M_2 and after some modification this solution may serve as a solution of an original compatible system. The following definition gives an exact description.

DEFINITION 2.4. Let $M_1(M_2)$ be an approximation theorem of a type $(C_1, \mathcal{R})$

$((C_2, \mathcal{R}))$. Then M_1 may be derived from M_2 if there exists a functor $\rho: J_2 \rightarrow J_1$ and natural transformations u, v such that the diagram

$$\begin{array}{ccc} K^{M_1} \rho & \xrightarrow{\xi_1 \circ 1_\rho} & M_1 \rho \\ u \downarrow & & \uparrow v \\ K^{M_2} & \xrightarrow{\xi_2} & M_2 \end{array}$$

commutes.

Here the composition “ $\circ$ ” is the horizontal composition of natural transformations ([1]). In several cases it will be more convenient to translate this definition using components of natural transformations. The conditions from 2.4 are then equivalent to the following system of commutative diagrams:

- (1) $\xi_{1,*} = v_* \cdot \xi_{2,*} \cdot u_*$
- (2) $v_j \cdot u_j = 1 \quad (= 1_{M_1(\rho(j))}), j \in J_2$
- (3) $u_j \cdot \pi_{\rho(j)}^1 \cdot \alpha_{M_1} = \pi_j^2 \cdot \alpha_{M_2} \cdot u_*, j \in J_2$
- (4) $v_j \cdot M_2(\delta_j) = M_1(\delta_{\rho(j)}) \cdot v_*, j \in J_2$,

where π are canonical projections of $M\tilde{x}$.

THEOREM 2.5. *Let M_1 (M_2) be an approximation theorem of a type $(C_1, \mathcal{R})$ ($(C_2, \mathcal{R})$) and let the following conditions are satisfied:*

- (1) *There exists a functor $S: C_1 \rightarrow C_2$ such that*
 - (a) *S is continuous,*
 - (b) *$S(J_1) = J_2$,*
- (2) *there exist natural transformations $\alpha: M_2 S \rightarrow M_1$, $\beta: M_1 \rightarrow M_2 S$ such that $\alpha \cdot \beta = 1$.*

Then M_1 may be derived from M_2 .

For the proof of 2.5 we need the following simple lemma, the proof of which will be omitted.

LEMMA 2.6. *Let M be an approximation theorem of a type $(C, \mathcal{R})$. Then for every $i, j \in J$,*

$$M(\delta_{ij}) \cdot \pi_i \cdot \alpha_M = M(\delta_{ji}) \cdot \pi_j \cdot \alpha_M.$$

Now, we prove the following simple rule:

$$(2.7) \quad \text{If } Si_1 = Si_2, \quad i_1, i_2 \in J_1, \quad \text{then } \beta_{i_1} \pi_{i_1}^1 \alpha_{M_1} = \beta_{i_2} \pi_{i_2}^1 \alpha_{M_1}.$$

In fact, since S is continuous, we have $S(i_1 \wedge i_2) = Si_1 = Si_2$ and the diagram

$$\begin{array}{ccccc}
 M_1 i_1 & \xrightarrow{M_1(\delta_{i_1 i_2})} & M_1(i_1 \wedge i_2) & \xleftarrow{M_1(\delta_{i_2 i_1})} & M_1 i_2 \\
 & \searrow \beta_{i_1} & \downarrow \beta_{i_1 \wedge i_2} & \swarrow \beta_{i_2} & \\
 & & M_2 S i_1 & &
 \end{array}$$

commutates. Then

$$\beta_{i_1} \pi_{i_1}^1 \alpha_{M_1} = \beta_{i_1 \wedge i_2} M_1(\delta_{i_1 i_2}) \pi_{i_1}^1 \alpha_{M_1} = \beta_{i_1 \wedge i_2} M_1(\delta_{i_2 i_1}) \pi_{i_2}^1 \alpha_{M_1} = \beta_{i_2} \pi_{i_2}^1 \alpha_{M_1}$$

according 2.6.

Since $S(J_1) = J_2$, there exists a map $\rho: J_2 \rightarrow J_1$ such that $S\rho(j) = j$ for every $j \in J_2$. We shall now construct a morphism $u_*: K(M_1) \rightarrow K(M_2)$ (in $\mathcal{R}$) as follows. The interpretation of $\delta_{ij} = \delta_{ji}$ in M_2 is an equalizer $\alpha_{ij}: [\delta_{ij} = \delta_{ji}] \hookrightarrow M_2 \tilde{x}$ of morphisms $M_2(\delta_{ij})\pi_i^2, M_2(\delta_{ji})\pi_j^2$. Let

$$z = \prod_{j \in J_2} \beta_{\rho(j)} \cdot \pi_{\rho(j)}^1 \cdot \alpha_{M_1}$$

be a morphism $K(M_1) \rightarrow M_2 \tilde{x}$. Then for every $i, j \in J_2$ we have

$$\begin{aligned}
 M_2(\delta_{ji})\pi_j^2 z &= M_2(\delta_{ji})\beta_{\rho(j)}\pi_{\rho(j)}^1 \alpha_{M_1} \\
 &= \beta_{\rho(i) \wedge \rho(j)} M_1(\delta_{\rho(j)\rho(i)}) \pi_{\rho(j)}^1 \alpha_{M_1} \\
 &= \beta_{\rho(i) \wedge \rho(j)} M_1(\delta_{\rho(i)\rho(j)}) \pi_{\rho(i)}^1 \alpha_{M_1} \\
 &= M_2(\delta_{ij})\beta_{\rho(i)}\pi_{\rho(i)}^1 \alpha_{M_1} \\
 &= M_2(\delta_{ij})\pi_i^2 z
 \end{aligned}$$

according (2.7) and 2.6. Hence, for every $i, j \in J_2$ there exists the unique morphism $\varepsilon_{ij}: K(M_1) \rightarrow M_2[\delta_{ij} = \delta_{ji}]$ such that $\alpha_{ij}\varepsilon_{ij} = z$. Since $K(M_2)$ is a pullback of α_{ij} and z does not depend on i, j , there exists the unique morphism $u_*: K(M_1) \rightarrow K(M_2)$ which completes the following pullback diagram

$$\begin{array}{ccccc}
 & & M_2[\delta_{ij} = \delta_{ji}] & & \\
 & \nearrow \varepsilon_{ij} & \uparrow \mu_{ij} & \searrow \alpha_{ij} & \\
 K(M_1) & \xrightarrow{\quad u_* \quad} & K(M_2) & & M_2 \tilde{x}
 \end{array}$$

It should be observed that u_* does not depend on the choice of a function ρ . In fact, it suffices to show that x does not depend on ρ . Let $\gamma: J_2 \rightarrow J_1$ be such that $S\gamma(j) = j$, $j \in J_2$. Let $j \in J_2$, $i_1 = \rho(j)$, $i_2 = \gamma(j)$. Since $S i_1 = S i_2 = j$, according (2.7) we obtain that x

is the same for ρ and γ .

Now, let $u_j = \beta_{\rho(j)}$, $j \in J_2$. We show that the relation (3) holds. In fact,

$$\pi_j^2 \alpha_{M_1} u_* = \pi_j^2 \alpha_{ij} \mu_{ij} u_* = \pi_j^2 \alpha_{ij} \varepsilon_{ij} = \pi_j^2 x = \beta_{\rho(j)} \pi_{\rho(j)}^1 \alpha_{M_1} = u_j \pi_{\rho(j)}^1 \alpha_{M_1}.$$

Further, let $v_* = \alpha_*$, $v_j = \alpha_{\rho(j)}$, $j \in J_2$. Then v is a natural transformation and the relations (4), (2) hold. We show finally that the relation (1) holds. For $i \in J_1$ there exists $j \in J_2$ such that $Si = S\rho(j) = j$. Then

$$\begin{aligned} M_1(\delta_i) v_* \xi_2 u_* &= M_1(\delta_i) \xi_2 u_* = \alpha_i M_2(\delta_{Si}) \xi_2 u_* \\ &= \alpha_i \pi_j^2 \alpha_{M_2} u_* = \alpha_i u_j \pi_{\rho(j)}^1 \alpha_{M_1} = \alpha_i \beta_{\rho(j)} \pi_{\rho(j)}^1 \alpha_{M_1} \\ &= \alpha_i \beta_i \pi_i^1 \alpha_{M_1} = \pi_i^1 \alpha_{M_1} = M_1(\delta_i) \xi_1 \end{aligned}$$

according (2.7). Therefore, according 2.2 we obtain the relation (1).

COROLLARY. Let M_1, M_2 be approximation theorems of a type $(C, \mathcal{R})$ and let $\alpha: M_2 \rightarrow M_1$, $\beta: M_1 \rightarrow M_2$ be natural transformations such that $\alpha \cdot \beta = 1$. Then M_1 may be derived from M_2 .

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