A note on nearly C-compact spaces

by

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Abstract

A topological space is said to be nearly C-compact if every open cover of every regularly closed set of it has a finite subfamily whose union is dense in the set. The object of the present note is to obtain the various results including the related results of nearly C-compact spaces with almost regular and almost completely regular spaces.

Introduction

In 1969, G. Viglino [8] has introduced a new class of topological spaces which properly contains the class of compact spaces, called C-compact spaces. A space is said to be C-compact if every open cover of every closed set has a finite subfamily, the closures of whose members cover the set. In [4] we have introduced the concept of nearly C-compact spaces as a generalization of C-compact spaces where various results including filter characterization of nearly C-compact spaces together with the concept of nearly C-compact spaces coincides with almost compact spaces, have been obtained.

Throughout this note X and Y will always mean topological spaces. The interior of the set U in X and the closure of the set U in X will be denoted by \( U^o \) and \( U \) respectively. A set U in X is said to be regularly open if \( \overline{U^o} = U \) and regularly closed if \( \overline{U} = U \). A set regularly open iff its complement is regularly closed.

Definition 1. [Sharma and Namdeo, 4]. A space X is said to be nearly C-compact if given a regularly closed set A and an open cover \( \mathcal{U} \) of A there exists a finite subfamily \( \{O_i: i = 1, 2, \ldots, n\} \) of \( \mathcal{U} \) such that \( A \subseteq \bigcup \{\overline{O_i}: i = 1, 2, \ldots, n\} \).

Theorem 1. In a space X, the following are equivalent:

(a) X is nearly C-compact.

(b) For each regularly closed set A of X and each regular open cover \( \mathcal{U} \) of A there exists a finite subfamily \( \{O_i: i = 1, 2, \ldots, n\} \) of \( \mathcal{U} \) such that

\[
A \subseteq \bigcup \{\overline{O_i}: i = 1, 2, \ldots, n\}.
\]
(c) For each regularly closed set $A$ of $X$ and for each family $\mathcal{F} = \{F_i\}$ of non empty regularly closed sets such that $\cap \mathcal{F} \cap A = \emptyset$, there exists a finite sub-family $\{F_i: i=1, 2, \ldots, n\}$ of $\mathcal{F}$ such that $\cap \{F_i: i=1, 2, \ldots, n\} \cap A = \emptyset$.

(d) For each regularly closed set $A$ of $X$ and each family $\mathcal{F} = \{F_i\}$ of regularly closed sets, if each finite subfamily $\{F_i: i=1, 2, \ldots, n\}$ of $\mathcal{F}$ we have $\cap \{F_i: i=1, 2, \ldots, n\} \cap A \neq \emptyset$ then $\cap \mathcal{F} \cap A \neq \emptyset$.

Proof. (a)$\Rightarrow$(b). If $X$ is nearly $C$-compact then the condition (b) is obviously follows. Suppose the condition (b) holds. Let $\mathcal{U} = \{O_i\}$ be an open cover of $A$ then $\overline{O_i}$ will be a regular open cover of $A$ and therefore there exists a finite subfamily $\{\overline{O_i: i=1, 2, \ldots, n}\}$ such that $A \subset \cap \{\overline{O_i: i=1, 2, \ldots, n}\}$. For each $i$, $\overline{O_i} = \overline{O_i}$, therefore we have $A \subset \cap \{\overline{O_i: i=1, 2, \ldots, n}\}$ and hence $X$ is nearly $C$-compact.

(b)$\Rightarrow$(c). Let $\mathcal{F} = \{F_i\}$ be a family of regularly closed sets of the space $X$ such that $\cap \mathcal{F} \cap A = \emptyset$ for each regularly closed set $A$ of $X$. Then $\mathcal{U} = \{X - F_i: F_i \in \mathcal{F}\}$ will be a family of regularly open sets of $X$ covering the regularly closed set $A$. Therefore there exists a finite subfamily $\{O_i = X - F_i: i=1, 2, \ldots, n\}$ of $\mathcal{U}$ such that $A \subset \cap \{O_i: i=1, 2, \ldots, n\}$. Now for each $i$, $F_i^\circ = (X - O_i)^\circ = X - (X - (X - O_i)) = X - \overline{O_i}$. Therefore $\cap \{F_i^\circ: i=1, 2, \ldots, n\} = X - \cap \{O_i: i=1, 2, \ldots, n\} \subset X - A$. This shows that $\cap \{F_i^\circ: i=1, 2, \ldots, n\} \cap A = \emptyset$.

(c)$\Rightarrow$(b). Let $\mathcal{U} = \{O_i\}$ be a regular open cover of the regularly closed set $A$ of the space $X$. $A \subset \cup O_i$ shows that $\cap (X - O_i) \cap A = \emptyset$. $(X - O_i)$ is a family of regularly closed sets satisfying (c) then there exists a finite subfamily $\{X - O_i: i=1, 2, \ldots, n\}$ such that $\cap \{X - O_i^\circ: i=1, 2, \ldots, n\} \cap A = \emptyset$. This gives $A \subset \cup \{X - (X - O_i^\circ): i=1, 2, \ldots, n\}$. Now for each $i$, $(X - O_i^\circ) = X - (X - (X - O_i)) = X - \overline{O_i}$. Therefore it follows that $A \subset \cup \{\overline{O_i: i=1, 2, \ldots, n}\}$.

(c)$\Rightarrow$(d). Obviously follows.

Theorem 2. Every regularly closed subset of a nearly $C$-compact space $X$ is nearly $C$-compact.

Proof. It follows easily in view of the fact that if $B$ is a regularly closed subset of a regularly closed subset $A$ of $X$, then $B$ is also a regularly closed subset of $X$.

Definition 2. [Singal and Singal, 5] A mapping is said to be almost continuous if the inverse image of every regularly open (closed) set is open (closed). A mapping is said to be almost open (closed) if the image of every regularly open (closed) set is open (closed).

Lemma 1. Let $f: X \to Y$ be an almost continuous almost open map. Then the inverse image of every regularly open (regularly
closed) set in \( Y \) is a regularly open (regularly closed) set, respectively in \( X \) [2, Theorem 1.2].

**Theorem 3.** The image of a nearly \( C \)-compact space under an almost continuous almost open mapping is nearly \( C \)-compact.

**Proof.** Let \( f : X \to Y \) be an almost continuous almost open mapping from a nearly \( C \)-compact space \( X \) onto \( Y \). Let \( B \) be any regularly closed subset of \( Y \) and let \( \mathcal{U} = \{ O_i \} \) be a regular open covering of \( B \). Since \( f \) is almost continuous almost open therefore by the Lemma 1, \( f^{-1}(B) \) is a regularly closed set of \( X \) and \( \{ f^{-1}(O_i) \} \) a regular open covering of \( f^{-1}(B) \). Since \( X \) is nearly \( C \)-compact therefore there exists a finite subfamily \( \{ f^{-1}(O_{i,j}) : j=1, 2, \ldots, n \} \) such that \( f^{-1}(B) \subseteq \bigcup \{ f^{-1}(O_{i,j}) : j=1, 2, \ldots, n \} \) and consequently \( B \subseteq \bigcup \{ O_{i,j} : j=1, 2, \ldots, n \} \). Therefore \( Y \) is nearly \( C \)-compact.

**Theorem 4.** Almost continuous image of a \( C \)-compact space is nearly \( C \)-compact.

**Proof.** Let \( f : X \to Y \) be an almost continuous mapping from the \( C \)-compact space onto \( Y \). Let \( \mathcal{U} = \{ O_i \} \) be any regular open cover of a regularly closed set \( A \) of \( Y \). Since \( f \) is almost continuous therefore \( \{ f^{-1}(O_i) \} \) is an open cover of the closed set \( f^{-1}(A) \). Now, since \( X \) is \( C \)-compact therefore there exists a finite subfamily \( \{ f^{-1}(O_{i,j}) : j=1, 2, \ldots, n \} \) such that \( f^{-1}(A) \subseteq \bigcup \{ f^{-1}(O_{i,j}) : j=1, 2, \ldots, n \} \) and therefore \( A \subseteq \bigcup \{ O_{i,j} : j=1, 2, \ldots, n \} \). Hence the result.

**Definition 3.** [Singal and Arya, 6]. A space \( X \) is said to be almost regular if for every regularly closed set \( A \) and a point \( x \in A \), there exist open sets \( U \) and \( V \) such that \( A \subseteq U \), \( x \in V \) and \( U \cap V = \emptyset \). Or equivalently, for every regularly closed set \( A \) and each point \( x \) not belonging to \( A \), there exist open sets \( U \) and \( V \) such that \( x \in U \), \( A \subseteq V \) and \( U \cap V = \emptyset \).

**Definition 4.** A space \( X \) is said to be mildly normal if for every pair of disjoint regularly closed sets \( F_1 \) and \( F_2 \) of \( X \), there exist disjoint open sets \( U \) and \( V \) such that \( F_1 \subseteq U \), \( F_2 \subseteq V \).

**Theorem 5.** Every almost regular, nearly \( C \)-compact space is mildly normal.

**Proof.** Let \( A \) and \( B \) be disjoint regularly closed subsets of an almost regular, nearly \( C \)-compact space \( X \). Since \( X \) is almost regular therefore for each \( x \in A \) there exist open sets \( U_{(x)} \) and \( V_{(x)} \) such that \( x \in U_{(x)} \), \( B \subseteq V_{(x)} \) and \( \bar{U}_{(x)} \cap \bar{V}_{(x)} = \emptyset \). The collection \( \{ U_{(x)} : x \in A \} \) is therefore an open covering of the regularly closed set \( A \). Since \( X \) is nearly \( C \)-compact, there exists a finite subfamily \( \{ U_{(i)} : i=1, 2, \ldots, n \} \)
such that $A \subseteq \bigcup_{i=1}^{n} U_{(x_i)}$. Suppose $M = \cap_{i=1}^{n} V_{B(x_i)}$ and $N = X - \cap_{i=1}^{n} \bar{V}_{B(x_i)}$. Then, $A \subseteq \bigcup_{i=1}^{n} \bar{U}_{(x_i)} \subseteq X - \cap_{i=1}^{n} \bar{V}_{B(x_i)} = N$, $B \subseteq M$ and $M \cap N = \emptyset$. Therefore $X$ is mildly normal.

**Corollary 1.** If $A$ is a regularly closed subset of an almost regular, nearly $C$-compact spaces $X$ and $B$ is a regularly open set containing $A$. Then there exists a regular open set $V$ such that $A \subseteq V \subseteq \bar{V} \subseteq B$.

**Proof.** $X - B$ is a regularly closed set and $A \cap (X - B) = \emptyset$. Then by the Theorem 5 there exist open sets $U_1$, $U_2$ such that $A \subseteq U_1$, $X - B \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$. Also, $\bar{U}_1 \cap U_2 = \emptyset$ and hence $\bar{U}_1 \subseteq X - U_2 \subseteq B$. Therefore $A \subseteq U_1 \subseteq \bar{U}_1 \subseteq B$. Thus $A \subseteq U_1 \subseteq \bar{U}_1 \subseteq \bar{U}_1 \subseteq B$. Suppose $\bar{U}_1 = V$. Then $V$ is regularly open and $\bar{\bar{U}}_1 = \bar{U}_1 = \bar{V}$. Thus $A \subseteq V \subseteq \bar{V} \subseteq B$.

**Corollary 2.** Let $A$ and $B$ be two disjoint regularly closed subsets of an almost regular, nearly $C$-compact space $X$. Then there exist open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $\bar{U} \cap \bar{V} = \emptyset$.

**Proof.** Clearly, $X - B$ is a regularly open set containing the regularly closed set $A$. Then by the Corollary 1, there exists a regularly open set $M$ such that $A \subseteq M \subseteq \bar{M} \subseteq X - B$. Also, since $M$ is a regularly open set containing the regularly closed set $A$, again there exists a regularly open set $N$ such that $A \subseteq N \subseteq \bar{N} \subseteq M$. Suppose $N = U$ and $X - \bar{M} = V$. Then clearly $A \subseteq U$, $B \subseteq V$ and $\bar{U} \cap \bar{V} = \emptyset$.

**Definition 5.** A space $X$ is said to be almost completely regular if for every regularly closed set $A$ and a point $x \notin A$, there is a continuous function $f$ on $X$ into the closed interval $[0, 1]$ such that $f(x) = 1$ and $f(A) = 0$ or equivalently $f(x) = 0$ and $f(A) = 1$.

**Definition 6.** [Gillman and Jerison, 1]. If $f$ is a real valued continuous mapping on a topological space $X$ then the set $f^{-1}(0)$ is said to be zero set of $f$. Clearly, every zero set is a closed set and every set of the form $\{ x : f(x) \geq 0 \}$ or $\{ x : f(x) \leq 0 \}$ is a zero set in $X$.

The complement of a zero set is said to be a cozero set. Clearly, every cozero set is an open set and every set of the form $\{ x : f(x) > 0 \}$ or $\{ x : f(x) < 0 \}$ is a cozero set.

**Definition 7.** If a zero set $B$ is a neighbourhood of a set $A$ then $B$ is said to be a zero set neighbourhood of $A$.

**Theorem 6.** Let $X$ be a nearly $C$-compact almost completely regular space. If $A$ and $B$ be two disjoint regularly closed sets of $X$, then there exists a continuous function $f$ on $X$ into $[0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. 
Proof. Since $X$ is almost completely regular therefore for each $x \in A$ there exists a continuous function $f_x$ on $X$ into $[0, 1]$ such that $f_x(x) = 0$ and $f_x(B) = 1$. Let $F_{\alpha}(x) = \{x : f_x(x) \leq 1/3\}$ and $F_{\beta}(x) = \{x : f_x(x) \geq 2/3\}$. Then $F_{\alpha}(x)$ and $F_{\beta}(x)$ are therefore disjoint zero set neighborhoods of $x$ and $B$ respectively. Therefore, there exist open sets $U_{\alpha}(x)$ and $V_{\beta}(x)$ such that $x \in U_{\alpha}(x) \subset F_{\alpha}(x)$ and $B \subset V_{\beta}(x) \subset F_{\beta}(x)$. The collection $\{U_{\alpha}(x) : x \in A\}$ is therefore an open covering of the regularly closed set $A$. Since $X$ is nearly $C$-compact therefore there exists a finite subfamily $\{U_{\alpha}(x_i) : i = 1, 2, \ldots, n\}$ such that $A \subset \bigcup_{i=1}^{n} U_{\alpha}(x_i)$. Clearly, $B \subset \bigcap_{i=1}^{n} V_{\beta}(x_i)$. Then $A \subset \bigcap_{i=1}^{n} F_{\alpha}(x_i) = F_1$ (say) and $B \subset \bigcap_{i=1}^{n} F_{\beta}(x_i) = F_2$ (say).

Therefore $F_1$ and $F_2$ are disjoint zero sets containing $A$ and $B$ respectively. Suppose $F_1$ and $F_2$ be zero sets of mappings $f_1$ and $f_2$ respectively. $|f_1| + |f_2|$ has no zeros since $F_1 \cap F_2 = \emptyset$. Now we define a mapping $f = f_1/(|f_1| + |f_2|)$. Therefore $f$ is a continuous function on $X$ into $[0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

**Corollary 3.** Let $X$ be a nearly $C$-compact almost completely regular space. Let $A$ be a regularly closed set contained in a regular open set $B$. Then $B$ contains a zero set and a cozero set containing $A$.

Proof. $B$ is regularly open therefore $X - B$ is regularly closed and $A \cap (X - B) = \emptyset$. As in the proof of Theorem 6 we can prove that there exist disjoint zero sets $F_1$ and $F_2$ such that $A \subset F_1$ and $X - B \subset F_2$. Therefore $A \subset F_1 \subset X - F_2 \subset X - (X - B) = B$. Hence it follows that $F_1$ is a zero set and $X - F_2$ is a cozero set containing $A$ and contained in $B$.

**Corollary 4.** Let $X$ be a nearly $C$-compact almost completely regular space. Every set which is a countable intersection of regularly open sets and which contains a regularly closed set $A$ contains a zero set containing $A$.

Proof. Let $O = \bigcap_{n \in \mathbb{N}} O_n$ where each $O_n$ is regularly open. If $A$ is a regularly closed set contained in $O$, then $A$ is also contained in a regularly open set $O_n$ for each $n$. By the Corollary 3 there exists a zero set $F_n$ for each $n$ such that $A \subset F_n \subset O_n$. Therefore $A \subset \bigcap_{n \in \mathbb{N}} F_n \subset \bigcap_{n \in \mathbb{N}} O_n$. Thus $F = \bigcap_{n \in \mathbb{N}} F_n$ is a zero set containing the regularly closed set $A$ and contained in $O$.

**Definition 8.** [Velicko, 7]. A set $P$ is said to be $\delta$-closed if for each point $x \notin P$, there exists an open set $G$ containing $x$ such that $G \cap P = \emptyset$, or equivalently, for each point $x \notin P$, there exists a regular open set containing $x$ which has empty intersection with $P$. A set $G$ is $\delta$-open iff its complement is $\delta$-closed.

**Theorem 7.** Let $X$ be a nearly $C$-compact almost completely
regular space. If $A$ and $B$ be two disjoint subset of $X$ such that $A$ is regularly closed and $B$ is $\delta$-closed, then there exists a continuous function $f$ on $X$ into $[0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

Proof. Since every regularly closed set is $\delta$-closed. Therefore theorem follows from Theorem 6.

References


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