Axiom of strong infinity and analytic hierarchy of ordinal numbers

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Introduction

Axiom of strong infinity in set theory is one of the most important problems of modern mathematics, and therefore many problems and conjectures concerning to axiom of strong infinity are proposed by many mathematicians, among them are, existence of strongly inaccessible cardinals, that of hyper-inaccessible cardinals of arbitrary type (cf. [5]), arithmetical and both-1 transcendency of cardinals (cf. [14]), existence of Ramsey cardinals, existence of measurable cardinals (cf. [8]), etc.

To consider such kind of axiom of strong infinity, it seems to be adequate to consider analytic hierarchy of ordinal numbers. It is very interesting that the axiom of strong infinity, which seems to be somewhat constructible, such that there exists a strongly inaccessible cardinals, there exists hyper-inaccessible of arbitrary type and etc., are expressed by the form $\exists x P(x)$ where $P(a)$ is a Π_1^1 -formula, and the axiom of strong infinity which seems to be somewhat non-constructible, for example, the existence of Ramsey cardinal is expressed by the form $\exists x P(x)$ where $P(a)$ is Δ_2^1 but not Σ_1^1 , the existence of measurable cardinals, is expressible by the form $\exists x P(x)$ where $P(a)$ is Δ_2^1 and not Σ_1^1 , Π_1^1 . So in the first case we may call it the axioms of strong infinity of Π_1^1 quantifier form or that of one quantifier form, and in the last case we may call it as that of Δ_2^1 or that of two quantifier form. And it is interesting to state, of course it is known easily, that the concept, of set theory, on ordinal numbers such as, α is a cardinal, regular cardinal, strongly inaccessible, and so on, is Π_1^1 -predicate. And there is also axiom of strong infinity of the form $\exists x P(x)$ where $P(a)$ is a Σ_1^1 -formula. Existence of certain kind of model, for example existence of model of set theory Σ^* on some ordinal number have this form. There are very interesting contrast between Σ_1^1 -infinity and Π_1^1 -infinity, and this will be discussed later.

Let α be an ordinal number. If there is non-trivial 2-valued measure on α , or equivalently, there is non-principal $\aleph_0$ -complete ultrafilter over α , then α is called 2-valued measurable or $\aleph_0$ -complete (in this paper, we use ' $\aleph_0$ -complete'). If there exists an $\aleph_0$ -complete cardinal, then there is an ordinal α with the property that there is a non-principal

ultrafilter $\mathcal{F}_a$ with the character $a > \aleph_0$. The smallest $\aleph_0$ -complete cardinal number have this property, and in such a case a is said to be normally measurable.

We shall give the definition of arithmetical transcendency of cardinals which is abbreviated as *ATC*.

ATC: For every arithmetical function f , $\aleph_0 \leq \max(a_1, \dots, a_n)$ implies

$$\overline{f(a_1, \dots, a_n)} \leq \overline{\max(a_1, \dots, a_n)}.$$

Both-1 transcendency, abbreviated as $\mathcal{A}_1^1$ -TC, is stated as follows:

$\mathcal{A}_1^1$ -TC: For every $\mathcal{A}_1^1$ -function f , $\aleph_0 \leq \max(a_1, \dots, a_n)$ implies

$$\overline{f(a_1, \dots, a_n)} \leq \overline{\max(a_1, \dots, a_n)}.$$

As a generalization of these, we have f -arithmetical transcendency of cardinals, which is abbreviated as f -ATC.

f -ATC: Let $f_i (i=1, \dots, n)$ be functions such that

- 1) $f_i(a) \leq c_0$ for all a .
- 2) $f_i(a) = 0$ for all $a \geq c_0$.

And let g be a function arithmetical in $f_1, \dots, f_n$. Then we have

$$\overline{g(f_1, \dots, f_n, a_1, \dots, a_s)} \leq \overline{\max(\aleph_0, c_0, a_1, \dots, a_s)}.$$

f - $\mathcal{A}_1^1$ -TC: Let g be a $\mathcal{A}_1^1$ -function in $f_1, \dots, f_n$. And let $f_i (i=1, \dots, n)$ satisfy the above conditions 1) and 2). Then we have

$$\overline{g(f_1, \dots, f_n, a_1, \dots, a_s)} \leq \overline{\max(\aleph_0, c_0, a_1, \dots, a_s)}.$$

Now we consider the cardinal number $\aleph_2$ and analytic hierarchy considering $\aleph_2$ as universe. Then the predicate $\mathcal{C}ard(a)$ which means a is a cardinal number ($< \aleph_2$) is expressible by Π_1^1 -formula. And $b > \omega \wedge \mathcal{C}ard(b)$ have exactly one solution $\aleph_1$, and therefore the function f defined by $f(a) = b \Leftrightarrow b > \omega \wedge \mathcal{C}ard(b)$ is both-1 function. So $\mathcal{A}_1^1$ -TC does not hold for this function. But it should be noted that such kind of predicate, for example, $a > \omega \wedge \mathcal{C}ard(a)$, do not define a $\mathcal{A}_1^1$ -function, if we consider a suitable cardinal number $\aleph_\tau$ as universe.

Let us assume that there is a normally measurable cardinal number. And let $\aleph_\tau$ be any normally measurable cardinal number. Then $\mathcal{A}_1^1$ -TC and f - $\mathcal{A}_1^1$ -TC hold if we consider $\aleph_\tau$ as universe. From this we obtain that the predicates 'a is a cardinal', 'a is regular', 'a is strongly inaccessible' and etc., are Π_1^1 but not Σ_1^1 on $\aleph_\tau$. For if it is Σ_1^1 , then $\mathcal{A}_1^1$ -TC do not hold on $\aleph_\tau$. The following result means that the uniformity of $\mathcal{A}_1^1$ -function, namely $\mathcal{A}_1^1$ -function depends essentially on its form of defining postulate and not essentially on its universe.

Let $\aleph_\tau$ be a normally measurable cardinal number and $\aleph_\theta$ be an arbitrary cardinal number less than $\aleph_\tau$. And let f' and f be two

$\mathcal{A}_1^1$ -functions such that they are defined by the formula of the same form considering $\aleph_\theta$ and $\aleph_\tau$ as universe, respectively. Then we have $f'(a_1, \dots, a_n) = f(a_1, \dots, a_n)$ for all $a_1, \dots, a_n < \aleph_\theta$. Let $f_i (i=1, \dots, m)$ be a function such that $\forall x (f_i(x) < c_0) \wedge \forall x (x \geq c_0 \rightarrow f_i(x) = 0)$, and $\bar{c}_0 < \aleph_\theta$. Then we have $f'(f'_1, \dots, f'_m, a_1, \dots, a_n) = f(f_1, \dots, f_m, a_1, \dots, a_n)$ for all $a_1, \dots, a_n$ where f'_i in $f'(f'_1, \dots, f'_m, a_1, \dots, a_n)$ is the restriction of f_i to $\aleph_\theta$. This theorem may be called the uniformity of defining postulate of $\mathcal{A}_1^1$ -function.

Let $\aleph_\tau$ be a normally measurable cardinal number. And let $A(a_1, \dots, a_n)$ be a $\mathcal{A}_1^1$ -formula considering $\aleph_\tau$ as universe. We consider two sequence of cardinals $\aleph_0 < \aleph_{c_1} < \dots < \aleph_{c_n} < \aleph_\tau$ and $\aleph_0 < \aleph_{a_1} < \dots < \aleph_{a_n} < \aleph_\tau$. Then we have the following

$$\forall x (x < \min(\aleph_{c_1}, \aleph_{a_1}) \rightarrow (A(x, \aleph_{c_1}, \dots, \aleph_{c_n}) \leftrightarrow A(x, \aleph_{a_1}, \dots, \aleph_{a_n}))) .$$

Let f_0 be a function such that $\forall x (f_0(x) \leq c_0) \wedge \forall x (x \geq c_0 \rightarrow f_0(x) = 0)$. And let $A(f_0, a, a_1, \dots, a_n)$ be a $\mathcal{A}_1^1$ -function in f_0 considering $\aleph_\tau$ as universe. We consider two sequence of cardinals $c_0, \aleph_0 < \aleph_{c_1} < \dots < \aleph_{c_n} < \aleph_\tau$ and $c_0, \aleph_0 < \aleph_{a_1} < \dots < \aleph_{a_n} < \aleph_\tau$. Then we have

$$\forall x (x < \min(\aleph_{c_1}, \aleph_{a_1}) \rightarrow (A(f_0, x, \aleph_{c_1}, \dots, \aleph_{c_n}) \leftrightarrow A(f_0, x, \aleph_{a_1}, \dots, \aleph_{a_n}))) .$$

These theorems show that in the case of $\mathcal{A}_1^1$ -formula it depends only on order of cardinality in the formula and not on its magnitude. So this theorem may be called the uniformity of cardinals under $\mathcal{A}_1^1$ -predicate.

Let $\mathfrak{A}$ be a set of functions and let D be a set of ordinal numbers. Then a function f is called a generative function with respect to $\mathfrak{A}$ and D , if the following conditions are satisfied:

- 1) f is strictly increasing.
- 2) $f(a) \notin [D \cup F_a]_{\mathfrak{A}}$ where $F_a = \{f(b) : b < a\}$.
- 3) For any subset B of F_a , we have

$$[D \cup B \cup \{f(a)\}]_{\mathfrak{A}} \cap f(a) \subset [D \cup B]_{\mathfrak{A}} .$$

A regular cardinal number a is said to be a generative cardinal if for any countable set of functions $\mathfrak{A}$ and any set of ordinals B such that $\sup(B) < a$ there is a countable set of functions $\mathfrak{B} \supseteq \mathfrak{A}$ and a set of ordinal numbers $D \supseteq B$ with the cardinality $\bar{D} \leq \bar{B} \cdot \aleph_0$. And there is a generative function f defined on a and values in a such that $f(0) \geq \sup(B)$. By $G(a)$ we denote that there is a generative cardinal number (it is expressible by $\mathcal{A}_2^1$ form). And by G we mean the strong infinity that there is a generative cardinal number, namely $\exists x G(x)$.

We have the following implications, where MC is the existence of $\aleph_0$ -complete cardinals.

$$MC \rightarrow G \rightarrow \mathcal{A}_1^1\text{-}TC \rightarrow ATC \rightarrow \mathfrak{P}(\omega) - L \neq \phi \rightarrow V \neq L .$$

Now we consider some properties of Π_1^1 -formulas. Let $\aleph_\tau$ be a normally measurable cardinal number, and let $\mathcal{F}_{\aleph_\tau}$ be a non-principal $\aleph_0$ -complete ultrafilter over $\aleph_\tau$ with character $\aleph_\tau$. By $\text{On}^{\aleph_\tau}/\mathcal{F}_{\aleph_\tau}$ we denote the ultraproduct of On determined by $\mathcal{F}_{\aleph_\tau}$. And by a^* we denote the representative element of a -th equivalence class of $\text{On}^{\aleph_\tau}/\mathcal{F}_{\aleph_\tau}$. Let a^* be $(a_1, \dots, a_\nu, \dots)$ ($\nu < \aleph_\tau$). Then by $(a)_\nu$ we denote the ν -th component a_ν of a^* . According to this notation, for every Π_1^1 -formula $P(a)$ we have $P(a) \rightarrow \{\nu : P((a)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$ for all a . So if we substitute $\aleph_\tau$ for a in the formula, we have $P(\aleph_\tau) \rightarrow \{\nu : P((\aleph_\tau)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$. And of course we have $\{\nu : (\aleph_\tau)_\nu < \aleph_\tau\} \in \mathcal{F}_{\aleph_\tau}$. Therefore we obtain that the first element of Π_1^1 -class of ordinals can not be a normally measurable cardinal number. Let $P_1(a), P_2(a), \dots, P_n(a), \dots$ be an enumeration of all Π_1^1 -formulas such that $P_n(\aleph_\tau)$. Then clearly we have $\{\nu : P_n((\aleph_\tau)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$ for all n , and $\{\nu : (\aleph_\tau)_\nu < \aleph_\tau\} \in \mathcal{F}_{\aleph_\tau}$. Since $\mathcal{F}_{\aleph_\tau}$ is $\aleph_0$ -complete, we obtain that $(\bigcap_{n \in \omega} \{\nu : P_n((\aleph_\tau)_\nu)\}) \cap \{\nu : (\aleph_\tau)_\nu < \aleph_\tau\} \in \mathcal{F}_{\aleph_\tau}$. By this we have a ν_0 such that $P_n((\aleph_\tau)_{\nu_0})$ for all n , and $(\aleph_\tau)_{\nu_0} < \aleph_\tau$. By putting $a = (\aleph_\tau)_{\nu_0}$, we obtain that there is a cardinal number $a < \aleph_\tau$ such that a has all Π_1^1 -properties of $\aleph_\tau$.

The following theorem concerning to Π_1^1 or Σ_1^1 -formulas interests us considerably. Let $\aleph_\tau$ and $\aleph_\sigma$ be two normally measurable cardinal number such that $\aleph_\tau < \aleph_\sigma$. Let $P(a)$ be a Σ_1^1 -formula considering $\aleph_\sigma$ as universe. Then $P(\aleph_\tau)$ implies $P(\aleph_\omega)$ for all $\aleph_\omega$ such that $\aleph_0 < \aleph_\omega \leq \aleph_\tau$, of course $\aleph_\omega$ may be a singular cardinal. Especially putting $\aleph_\omega$ as $\aleph_1$, we have $P(\aleph_\tau) \rightarrow P(\aleph_1)$. If we consider a Π_1^1 -formula $P(a)$, then we have $P(\aleph_\omega) \rightarrow P(\aleph_\tau)$ for all $\aleph_0 < \aleph_\omega \leq \aleph_\tau$. Namely we have, if there is a cardinal number $\aleph_\omega$ less than $\aleph_\tau$ such that $P(\aleph_\omega)$ then the normally measurable cardinal number $\aleph_\tau$ has the property $P(\aleph_\cdot)$, in other words, $\aleph_\tau$ has all Π_1^1 -properties which is a property of a cardinal less than $\aleph_\tau$. Now we remind the fact that there is a cardinal number $\aleph_\omega < \aleph_\tau$ such that $\aleph_\omega$ has all Π_1^1 -properties of $\aleph_\tau$. Combining this with the above, we have that $\aleph_\omega$ has exactly those Π_1^1 -properties of $\aleph_\tau$. Namely there is neither Σ_1^1 - nor Π_1^1 -formula which distinguish two different cardinals $\aleph_\omega$ and $\aleph_\tau$, one of which is normally measurable and the other is not. Let $\aleph_{\tau_1}, \aleph_{\tau_2} < \aleph_\sigma$ be two normally measurable cardinal such that $\aleph_{\tau_1} < \aleph_{\tau_2}$. Then $P(\aleph_{\tau_1}) \rightarrow P(\aleph_{\tau_2})$. This means that greater normally measurable cardinal has more Π_1^1 -properties.

We now consider a normally measurable cardinal $\aleph_\sigma$ such that for every $a < \aleph_\sigma$ there is a normally measurable cardinal $\aleph_\tau$ such that $a < \aleph_\tau < \aleph_\sigma$. Under this assumption we consider the fact that the existence of a model for first-order formulas on certain ordinal number is expressible by the form $\exists x P(x)$ where $P(a)$ is Σ_1^1 -formula. We consider a formula defined by $Q(a) \equiv \exists x (P(x) \wedge x < a)$. Clearly this is a Σ_1^1 -formula

and by the assumption there is a normally measurable cardinal number such that $Q(\aleph_\tau)$ where $\aleph_\tau < \aleph_\sigma$. By the above theorem, we have $Q(\aleph_1)$. Namely we have $\exists x(x < \aleph_1 \wedge P(x))$. This means that for every Σ_1^1 -formula, we have $\exists x P(x) \rightarrow \exists x(x < \aleph_1 \wedge P(x))$. This may be considered as a generalization of Löwenheim-Skolem's theorem. And this also indicates that how large $\aleph_1$ is!: It is also very powerful tool to prove certain kind of reflection principle of set theory.

Now we shall state further conjectures on strong infinity.

Conjecture A. Transcendancy of $\aleph_0$ -complete cardinals under Π_1^1 -formulas, which is abbreviated as Π_1^1 -TMC.

For every Π_1^1 -formula $P(a)$, we have $\exists x P(x) \rightarrow \exists x(x < \aleph_\tau \wedge P(x))$, where $\aleph_\tau$ is a normally measurable cardinal number.

Conjecture B. Uniformity of normally measurable cardinals under Π_1^1 - or Σ_1^1 -formulas.

Let $x < \aleph_{\tau_1} < \dots < \aleph_{\tau_n}$ and $x < \aleph_{\sigma_1} < \dots < \aleph_{\sigma_n}$ where $\aleph_{\tau_i}, \aleph_{\sigma_i}$ are normally measurable cardinals. Then for every Π_1^1 - or Σ_1^1 -formula $A(a, a_1, \dots, a_n)$ we have $A(x, \aleph_{\tau_1}, \dots, \aleph_{\tau_n}) \leftrightarrow A(x, \aleph_{\sigma_1}, \dots, \aleph_{\sigma_n})$.

Conjecture C. For any formula, $P(a)$ which is arithmetical in Π_1^1 -, Σ_1^1 -formulas, we have

$$P(\aleph_\tau) \rightarrow \{\nu : P((\aleph_\tau)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Let $P(a)$ be the formula expressing that the continuum hypothesis does not hold at a which is expressed as

$$Q(a) \equiv \exists y (\neg \exists x (\bar{a} < \bar{x} < \bar{y}) \leftrightarrow 2^{\bar{a}} < \bar{y})$$

where $\bar{a} < \bar{b}$ and $2^{\bar{a}} < \bar{b}$ are Π_1^1 -formulas. It is known by Vopěnka that $Q(\aleph_\tau) \rightarrow \{\nu : Q((\aleph_\tau)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$.

Conjecture D. Let $f(a_1, \dots, a_n)$ be a function arithmetical in Π_1^1 -, Σ_1^1 -formulas. And let $\aleph_\tau$ be a normally measurable cardinal number. Then we have

$$\forall x_1 \dots \forall x_n (x_1 < \aleph_\tau \wedge \dots \wedge x_n < \aleph_\tau \rightarrow f(x_1, \dots, x_n) < \aleph_\tau).$$

Since the formula expressing that a is normally measurable belongs to the class $\mathcal{A}_2^1$, Conjecture 1 to 4 seems to be very interesting. Recently I know that J. Silver obtained that if the existence of normally measurable cardinal number (NMC) is consistent, then generalized continuum hypothesis (GCH) and the existence of normally measurable cardinals is consistent. Let $\mathfrak{M}$ be a countable model in which GCH and NMC is satisfied. And we consider the least normally measurable cardinal $\aleph_\tau$. By R. Solovay's method we obtain a model $\mathfrak{N}$ in which

$$1) \ 2^{\aleph_{\tau+1}} = \aleph_{\tau+3}, \quad \forall a (a \neq \tau + 1 \rightarrow 2^{\aleph_a} = \aleph_{a+1}).$$

2) the class of normally measurable cardinals in $\mathfrak{N}$ is exactly that of $\mathfrak{M}$. Now we consider a predicate that a is the least cardinal such that the

continuum hypothesis does not hold at a . In the model $\mathfrak{N}$ it consists only one cardinal number $\aleph_{\tau+1}$. And we obtain that the class of cardinals a such that the continuum hypothesis fails to hold at a is arithmetical in Π_1^1 -formulas. So in $\mathfrak{N}$, Conjectures D does not hold.

The following conjecture seems to be very strong.

Conjecture E. There is a formula $P(a)$ in Σ_n^1 or Π_n^1 such that $P(a)$ is transcendental for every Δ_n^1 -functions. Namely for every Δ_n^1 -function f , if $a < b \wedge P(b)$ then $f(a) < b$.

§ 1. Functions on ordinal numbers

In this paper we shall use the following notations:

Number-variables:

$a, b, \dots, a_1, a_2, \dots$ (range over all ordinal numbers).

Number-constants: $0, 1, \dots, \omega, \dots$

Function-variables:

$\alpha, \beta, \dots, \alpha_1, \alpha_2, \dots$ (range over all functions of ordinal numbers).

Function-constants: $', \max, Iq, g^1, g^2, j, \dots$

Predicate-constants: $=, <$.

Logical symbols:

$\neg$ (not), $\vee$ (or), $\wedge$ (and), $\rightarrow$ (implies), $\leftrightarrow$ (equivalent to),

$\forall$ (for all), $\exists$ (there exist).

The function constants: $', \max, Iq, g^1, g^2$, and j are defined by

a' is the successor of a .

$\max(a, b)$ is the maximum of a and b .

$$Iq(a, b) = \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{otherwise.} \end{cases}$$

$j(g^1(a), g^2(a)) = a, g^1(j(a, b)) = a, g^2(j(a, b)) = b,$

$j(a, b) < j(c, d) \leftrightarrow (\max(a, b) < \max(c, d))$

$\vee (\max(a, b) = \max(c, d) \wedge (b < d \vee (b = d \wedge a < c)))$.

Let $f(a, a_1, \dots, a_n)$ be a function. Then the operations

$\mu x f(x, a_1, \dots, a_n)$ and $f^a(b, a_1, \dots, a_n)$ are defined by the followings:

$$\mu x f(x, a_1, \dots, a_n) = \begin{cases} \text{the least } x \text{ such that } f(x, a_1, \dots, a_n) = 0 & \text{if} \\ \exists x (f(x, a_1, \dots, a_n) = 0), & \\ 0 & \text{otherwise.} \end{cases}$$

$$f^a(b, a_1, \dots, a_n) = \begin{cases} f(b, a_1, \dots, a_n) & \text{if } b < a, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h_1, \dots, h_m$ ($m=0, 1, \dots$) be functions (which may be function-variables). Then a function f is called arithmetical in $h_1, \dots, h_m$ (in the case where $m=0$, we say f is arithmetical), if it can be defined by a series of applications of the following schemata, provided that $g, g_1, \dots, g_i$ are given functions arithmetical in $h_1, \dots, h_m$. And the function f

is often denoted as $f(h_1, \dots, h_m, a_1, \dots, a_n)$ (in the case where $m=0$, f is denoted as $f(a_1, \dots, a_n)$).

- (I) $f(a) = a'$.
- (II) $f(a) = 0$,
 $f(a) = \omega$.
- (III) $f(a) = a$.
- (IV) $f(a, b) = Iq(a, b)$.
- (V) $f(a, b) = \max(a, b)$.
- (VI) $f(a, b) = j(a, b)$.
- (VII) $f(a) = g^1(a)$,
 $f(a) = g^2(a)$.
- (VIII) $f(h, a_1, \dots, a_n) = h(a_1, \dots, a_n)$.
- (IX) $f(h_1, \dots, h_m, a_1, \dots, a_n) = g(h_1, \dots, h_m, g_1(h_1, \dots, h_m, a_1, \dots, a_n), \dots, g_i(h_1, \dots, h_m, a_1, \dots, a_n))$.
- (X) $f(h_1, \dots, h_m, a, a_1, \dots, a_n) = g(h_1, \dots, h_m, a_i, \dots, a_n)$,
 $f(h_1, \dots, h_m, a_1, \dots, a_n, a) = g(h_1, \dots, h_m, a_1, \dots, a_n)$.
- (XI) $f(h, h_1, \dots, h_m, a_1, \dots, a_n) = g(h_1, \dots, h_m, a_1, \dots, a_n)$,
 $f(h_1, \dots, h_m, h, a_1, \dots, a_n) = g(h_1, \dots, h_m, a_1, \dots, a_n)$.
- (XII) $f(h_1, \dots, h_m, a_1, \dots, a_n) = \mu x g(h_1, \dots, h_m, x, a_1, \dots, a_n)$.
- (XIII) $f(h_1, \dots, h_m, a, a_1, \dots, a_n) = g(f^a, h_1, \dots, h_m, a, a_1, \dots, a_n)$.

A function f which is arithmetical in $h_1, \dots, h_m$ is called recursive in $h_1, \dots, h_m$ if the schema (XII) is applied only to the case where the following condition is satisfied:

$$\forall x_1 \dots \forall x_n \exists x (g(h_1, \dots, h_m, x, x_1, \dots, x_n) = 0).$$

A formula $A(h_1, \dots, h_m, a_1, \dots, a_n)$ is called recursive (arithmetical) in $h_1, \dots, h_m$ if there is a function $f(h_1, \dots, h_m, a_1, \dots, a_n)$ which is recursive (arithmetical) in $h_1, \dots, h_m$ such that

$$\forall x_1 \dots \forall x_n (A(h_1, \dots, h_m, x_1, \dots, x_n) \leftrightarrow f(h_1, \dots, h_m, x_1, \dots, x_n) = 0).$$

In the above definition, if $m=0$ then $A(a_1, \dots, a_n)$ is called simply recursive (arithmetical).

A formula $A(h_1, \dots, h_m, a_1, \dots, a_n)$ is called analytic in $h_1, \dots, h_m$ if it is equivalent to a formula obtained from an arithmetical formula by prefixing a sequence of number-quantifiers and function-quantifiers and has no function symbols except $h_1, \dots, h_m$. An analytic formula $A(h_1, \dots, h_m, a_1, \dots, a_n)$ is called Π_1^1 -formula in $h_1, \dots, h_m$ if it is equivalent to a formula of the form $\forall \alpha B(\alpha, h_1, \dots, h_m, a_1, \dots, a_n)$ where $B(\alpha, h_1, \dots, h_m, a_1, \dots, a_n)$ is arithmetical in $\alpha, h_1, \dots, h_m$. A formula is called Σ_1^1 -formula in $h_1, \dots, h_m$ if its negation is Π_1^1 -formula in $h_1, \dots, h_m$. And a formula is called Δ_1^1 -formula in $h_1, \dots, h_m$ if it is both Π_1^1 - and Σ_1^1 -formula in $h_1, \dots, h_m$. In the above definition if $m=0$, then the formula $A(a_1, \dots, a_n)$ is called simply analytic (Π_1^1 -, Σ_1^1 - or Δ_1^1 -) formula.

The concept of $\Pi_2^1, \Sigma_2^1, \mathcal{A}_2^1, \dots$ are defined similarly.

Let $f(h_1, \dots, h_m, a_1, \dots, a_n)$ be a function such that

$$\forall x \forall x_1 \dots \forall x_n (f(h_1, \dots, h_m, x_1, \dots, x_n) = x \leftrightarrow A(h_1, \dots, h_m, x_1, \dots, x_n, x)).$$

Then the function $f(h_1, \dots, h_m, a_1, \dots, a_n)$ is called analytic (Π_i^1 -, Σ_i^1 -, $\mathcal{A}_i^1$ -) function in $h_1, \dots, h_m$ if $A(h_1, \dots, h_m, a_1, \dots, a_n, a)$ is analytic (Π_i^1 -, Σ_i^1 -, $\mathcal{A}_i^1$ -) formula in $h_1, \dots, h_m$.

Let $\aleph_\alpha$ be a cardinal number. We can define analytic hierarchy of ordinal numbers by considering $\aleph_\alpha$ as universe. For the convenience of description we use, for example, $A^{(\aleph_\alpha)}$ to stand for the relativization of notion A on $\aleph_\alpha$. And $A^{(\aleph_\alpha)}$ is sometimes referred as A in $(\aleph_\alpha)$.

§ 2. Ultraproducts of ordinal numbers

Definition. A subset $\mathcal{F}$ of the power set $\mathfrak{P}(A)$ of a set A is called a filter over the set A if the following conditions are satisfied:

- 1) If $B \in \mathcal{F}$ and $A \supset C \supset B$ then $C \in \mathcal{F}$.
- 2) If $B, C \in \mathcal{F}$ then $B \cap C \in \mathcal{F}$.
- 3) $\emptyset \notin \mathcal{F}$.

A filter $\mathcal{F}$ over a set A is called an ultrafilter if for any subset B of A we have $B \in \mathcal{F}$ or $A - B \in \mathcal{F}$. An ultrafilter $\mathcal{F}$ over a set A is called principal if $(\bigcap_{B \in \mathcal{F}} B) \in \mathcal{F}$.

A filter $\mathcal{F}$ is called $\aleph_\alpha$ -complete if $\bar{I} \leq \aleph_\alpha$ and $A_\nu \in \mathcal{F}$ for all $\nu \in I$ imply $\bigcap_{\nu \in I} A_\nu \in \mathcal{F}$.

A cardinal number θ is called $\aleph_\alpha$ -complete if there is a non-principal $\aleph_\alpha$ -complete ultrafilter over θ . The character of a filter $\mathcal{F}$ is defined to be the smallest cardinal number $\aleph_\alpha$ such that $\mathcal{F}$ is not $\aleph_\alpha$ -complete if there is such cardinal number. The character of $\mathcal{F}$ is denoted by $ch(\mathcal{F})$.

A cardinal number $\aleph_\tau$ is called normally measurable if there is a non-principal ultrafilter $\mathcal{F}_{\aleph_\tau}$ over $\aleph_\tau$ with the character $\aleph_\tau > \aleph_0$. It is well-known that the smallest $\aleph_0$ -complete (2-valued measurable) cardinal number is normally measurable. Therefore if there is an $\aleph_0$ -complete cardinal number then there is a normally measurable cardinal number. Let $\aleph_\tau$ be a normally measurable cardinal number. Then we have easily the following properties:

- 1) $\aleph_\tau$ is a strongly inaccessible cardinal number.
- 2) $\mathcal{F}_{\aleph_\tau}$ is $\aleph_\alpha$ -complete for all cardinal number $\aleph_\alpha$ less than $\aleph_\tau$.

Let On be the class of all ordinal numbers and we consider the ultraproduct of On determined by $\mathcal{F}_{\aleph_\tau}$, which is denoted by $\text{On}^{\aleph_\tau} / \mathcal{F}_{\aleph_\tau}$. The equivalence-relation $=^*$ and the order-relation $<^*$ between two elements u and v of $\text{On}^{\aleph_\tau}$ are defined as follows:

$u =^* v$ is defined to be $\{\nu : u(\nu) = v(\nu)\} \in \mathcal{F}_{\aleph_\tau}$,

$u <^* v$ is defined to be $\{\nu : u(\nu) < v(\nu)\} \in \mathcal{F}_{\aleph_\tau}$.

Since $=^*$ is an equivalence-relation, we can classify $\text{On}^{\aleph_\tau}$ into equivalence classes. Let $u \in \text{On}^{\aleph_\tau}$. Then by $\hat{u}$ we denote the equivalence class which includes the function u . We now choose a representative element from each class $\hat{u}$ such a way that if $(a, a, \dots, a, \dots) \in \hat{u}$ then the representative element of $\hat{u}$ is $(a, a, \dots, a, \dots)$, otherwise the representative element of $\hat{u}$ is an arbitrary but fixed element of $\hat{u}$. By T we denote the class of all representative elements defined above. Clearly T is well-ordered by $<^*$, because $\mathcal{F}_{\aleph_\tau}$ is $\aleph_0$ -complete. Let the α -th element of T by the well-ordering $<^*$ be a^* . Then we have $T = \{a^* : a \in \text{On}\}$. And moreover if $a < \aleph_\tau$ then $a^* = (a, a, \dots, a, \dots)$.

Let a be an ordinal number and let $a^* = (a_1, \dots, a_\nu, \dots)$. Then by $(a^*)_\nu$ we denote the ν -th component a_ν of a^* . If no confusion is feared, then $(a^*)_\nu$ is sometimes written as $(a)_\nu$.

An ordinal number a is said to be U -invariant if $a^* = (a, \dots, a, \dots)$ or in other words $\{\nu : (a)_\nu = a\} \in \mathcal{F}_{\aleph_\tau}$. A function $f(a_1, \dots, a_n)$ is said to be U -invariant if the following condition is satisfied

$$\forall x_1 \dots \forall x_n \forall x (f(x_1, \dots, x_n) = x \leftrightarrow \{\nu : f((x_1)_\nu, \dots, (x_n)_\nu) = (x)_\nu\} \in \mathcal{F}_{\aleph_\tau}).$$

or equivalently

$$\forall x_1 \dots \forall x_n (\{\nu : (f(x_1, \dots, x_n))_\nu = f((x_1)_\nu, \dots, (x_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}).$$

Theorem A. *Let $\Phi(h_1, \dots, h_m, a_1, \dots, a_n)$ be an analytic formula in $h_1, \dots, h_m$. And the following condition is satisfied: For some sequence of functions $(h_{i1}, h_{i2}, \dots, h_{i\nu}, \dots)$,*

$$\{\nu : (h_i(a_1, \dots, a_n))_\nu = h_{i\nu}((a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$$

for all $i=1, \dots, m$. Then we have that

$$\Phi^*(h_1, \dots, h_m, a_1, \dots, a_n) \leftrightarrow \{\nu : \Phi(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$$

where Φ^* is the formula obtained from Φ restricting all function quantifiers to the functions such that

$$\beta(a_1, \dots, a_s) = b \leftrightarrow \{\nu : \beta_\nu((a_1)_\nu, \dots, (a_s)_\nu) = (b)_\nu\} \in \mathcal{F}_{\aleph_\tau}$$

for some sequence of functions $(\beta_1, \dots, \beta_\nu, \dots)$.

In order to prove this theorem, we shall first state the following lemma.

Lemma 1. *Let $f(h_1, \dots, h_m, a_1, \dots, a_n)$ be arithmetical in $h_1, \dots, h_m$. And let*

$$\{\nu : (h_i(a_1, \dots, a_n))_\nu = h_{i\nu}((a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$$

for all $i=1, \dots, m$. Then we have

$$\{\nu : (f(h_1, \dots, h_m, a_1, \dots, a_n))_\nu = f(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\mathbf{N}_\tau}.$$

Proof. In the case where the defining schema is one of (I) to (VII), we have the followings:

$$\begin{aligned} \{\nu : (a')_\nu = (a_\nu)'\} &\in \mathcal{F}_{\mathbf{N}_\tau}, \\ \{\nu : (0)_\nu = 0\} &\in \mathcal{F}_{\mathbf{N}_\tau}, \\ \{\nu : (\omega)_\nu = \omega\} &\in \mathcal{F}_{\mathbf{N}_\tau}, \\ \{\nu : (a)_\nu = (a_\nu)\} &\in \mathcal{F}_{\mathbf{N}_\tau}, \\ \{\nu : (Iq(a, b))_\nu = Iq((a)_\nu, (b)_\nu)\} &\in \mathcal{F}_{\mathbf{N}_\tau}, \\ \{\nu : (\max(a, b))_\nu = \max((a)_\nu, (b)_\nu)\} &\in \mathcal{F}_{\mathbf{N}_\tau}, \\ \{\nu : (j(a, b))_\nu = j((a)_\nu, (b)_\nu)\} &\in \mathcal{F}_{\mathbf{N}_\tau}, \\ \{\nu : (g^1(a))_\nu = g^1((a)_\nu)\} &\in \mathcal{F}_{\mathbf{N}_\tau}, \\ \{\nu : (g^2(a))_\nu = g^2((a)_\nu)\} &\in \mathcal{F}_{\mathbf{N}_\tau}. \end{aligned}$$

In the case where the last schema is (VIII), the function f is defined by

$$f(h_i, a_1, \dots, a_n) = h_i(a_1, \dots, a_n).$$

By the assumption of the lemma, we have

$$\{\nu : (h_i(a_1, \dots, a_n))_\nu = h_{i\nu}((a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\mathbf{N}_\tau}.$$

Namely we have

$$\{\nu : (f(h_1, \dots, h_m, a_1, \dots, a_n))_\nu = f(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\mathbf{N}_\tau}.$$

The cases (IX) to (XI) are treated as usual.

In the case where the last schema is (XII), the function f is defined by

$$f(h_1, \dots, h_m, a_1, \dots, a_n) = \mu x g(h_1, \dots, h_m, x, a_1, \dots, a_n).$$

By the assumption of the induction we have

$$\{\nu : (g(h_1, \dots, h_m, a, a_1, \dots, a_n))_\nu = g(h_{1\nu}, \dots, h_{m\nu}, (a)_\nu, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\mathbf{N}_\tau}.$$

It is easily proved that $\exists x(g(h_1, \dots, h_m, x, a_1, \dots, a_n) = 0)$ if and only if

$$\{\nu : \exists x(g(h_{1\nu}, \dots, h_{m\nu}, x, (a_1)_\nu, \dots, (a_n)_\nu) = 0)\} \in \mathcal{F}_{\mathbf{N}_\tau}.$$

Now let $d = \mu x g(h_1, \dots, h_m, x, a_1, \dots, a_n)$. In the case where

$$\exists x(g(h_1, \dots, h_m, x, a_1, \dots, a_n) = 0),$$

we have $\{\nu : g(h_1, \dots, h_m, d, a_1, \dots, a_n)_\nu = 0\} \in \mathcal{F}_{\mathbf{N}_\tau}$.

From this we obtain

$$\{\nu : g(h_{1\nu}, \dots, h_{m\nu}, (d)_\nu, (a_1)_\nu, \dots, (a_n)_\nu) = 0\} \in \mathcal{F}_{\aleph_\tau}.$$

Therefore we have

$$\{\nu : f(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu) \leq (d)_\nu\} \in \mathcal{F}_{\aleph_\tau}.$$

If

$$\{\nu : f(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu) < (d)_\nu\} \in \mathcal{F}_{\aleph_\tau},$$

then we obtain that $f(h_1, \dots, h_m, a_1, \dots, a_n) < d$. This contradicts to the definition of d .

Namely we have

$$\{\nu : (f(h_1, \dots, h_m, a_1, \dots, a_n))_\nu = f(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

In the case where the last schema is (XIII), the function f is defined by

$$f(h_1, \dots, h_m, a, a_1, \dots, a_n) = g(f^a, h_1, \dots, h_m, a, a_1, \dots, a_n).$$

Now we consider the case where there is an ordinal number a such that

$$\{\nu : (f(h_1, \dots, h_m, a, a_1, \dots, a_n))_\nu \neq f(h_{1\nu}, \dots, h_{m\nu}, (a)_\nu, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Let c be the smallest such ordinal number. Then by the definition of c , we have

$$\{\nu : (f^c(h_1, \dots, h_m, a, a_1, \dots, a_n))_\nu = f^{(c)_\nu}_\nu(h_{1\nu}, \dots, h_{m\nu}, (a)_\nu, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Hence by the assumption of the induction, we obtain

$$\{\nu : (g(f^c, h_1, \dots, h_m, c, a_1, \dots, a_n))_\nu = g(f^{(c)_\nu}_\nu, h_{1\nu}, \dots, h_{m\nu}, (c)_\nu, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

This means that

$$\{\nu : (f(h_1, \dots, h_m, c_1, \dots, a_n))_\nu = f(h_{1\nu}, \dots, h_{m\nu}, (c)_\nu, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

This contradicts to the assumption on c . From this we obtain that

$$\{\nu : (f(h_1, \dots, h_m, a, a_1, \dots, a_n))_\nu = f(h_{1\nu}, \dots, h_{m\nu}, (a)_\nu, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Thus we complete the proof of the lemma.

Proof of Theorem A.

In the case where $\phi(h_1, \dots, h_m, a_1, \dots, a_n)$ is of the form

$$f(h_1, \dots, h_m, a_1, \dots, a_n) = g(h_1, \dots, h_m, a_1, \dots, a_n).$$

We have the following equivalence by Lemma 1.

$$f(h_1, \dots, h_m, a_1, \dots, a_n) = g(h_1, \dots, h_m, a_1, \dots, a_n)$$

$$\leftrightarrow \{\nu : (f(h_1, \dots, h_m, a_1, \dots, a_n))_\nu = (g(h_1, \dots, h_m, a_1, \dots, a_n))_\nu\} \in \mathcal{F}_{\aleph_\tau}$$

$$\leftrightarrow \{\nu : f(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu) = g(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Namely we obtain that

$$\phi^*(h_1, \dots, h_m, a_1, \dots, a_n) \leftrightarrow \{\nu : \phi(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

The case where the outermost symbol is one of $<$, $>$, $\vee$, $\wedge$, $\rightarrow$ or $\leftrightarrow$ is treated as usual.

In the case where $\phi(h_1, \dots, h_m, a_1, \dots, a_n)$ is of the form $\exists x \Psi(h_1, \dots, h_m, x, a_1, \dots, a_n)$, first we assume $\exists x \Psi^*(h_1, \dots, h_m, x, a_1, \dots, a_n)$. Then there is an ordinal number a such that $\Psi^*(h_1, \dots, h_m, a, a_1, \dots, a_n)$. By the hypothesis of the induction, we have

$$\{\nu : \Psi(h_{1\nu}, \dots, h_{m\nu}, (a)_\nu, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Hence we have

$$\{\nu : \exists x \Psi(h_{1\nu}, \dots, h_{m\nu}, x, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Conversely we assume that

$$\{\nu : \exists x \Psi(h_{1\nu}, \dots, h_{m\nu}, x, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Then there is an ordinal number b such that

$$\{\nu : \Psi(h_{1\nu}, \dots, h_{m\nu}, (b)_\nu, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

By the hypothesis of the induction, we have $\Psi^*(h_1, \dots, h_m, b, a_1, \dots, a_n)$. Hence we obtain $\exists x \Psi^*(h_1, \dots, h_m, x, a_1, \dots, a_n)$.

In the case where $\phi(h_1, \dots, h_m, a_1, \dots, a_n)$ is of the form $\exists \alpha \Psi(\alpha, h_1, \dots, h_m, a_1, \dots, a_n)$, first we assume that $\exists \alpha^* \Psi^*(\alpha^*, h_1, \dots, h_m, a_1, \dots, a_n)$ where the range of α^* is restricted to the functions such that

$$\{\nu : (\alpha(a_1, \dots, a_s))_\nu = \alpha_\nu((a_1)_\nu, \dots, (a_s)_\nu)\} \in \mathcal{F}_{\aleph_\tau},$$

for some sequence of functions $(\alpha_1, \dots, \alpha_s, \dots)$.

Then there is a function h and a sequence of functions such that

$$\Psi^*(h, h_1, \dots, h_m, a_1, \dots, a_n),$$

$$\{\nu : (h(a_1, \dots, a_s))_\nu = h_\nu((a_1)_\nu, \dots, (a_s)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

By the hypothesis of the induction, we have

$$\{\nu : \Psi(h_\nu, h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Hence we obtain that

$$\{\nu : \exists \alpha \Psi(\alpha, h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Conversely we assume that

$$\{\nu : \exists \alpha \Psi(\alpha, h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Then there is a sequence of functions $(h_1, \dots, h_\nu, \dots)$ and a function h such that

$$\{\nu : \Psi(h_\nu, h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

$$\{\nu : (h(a_1, \dots, a_n))_\nu = h_\nu((a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

By the hypothesis of the induction, we have $\Psi^*(h, h_1, \dots, h_m, a_1, \dots, a_n)$. Therefore we have $\exists \alpha^* \Psi^*(\alpha^*, h_1, \dots, h_m, a_1, \dots, a_n)$. Thus we complete the proof of the theorem.

As a simple application of this theorem we obtain that if $\Phi(h_1, \dots, h_m, a_1, \dots, a_n)$ is a Π_1^1 -formula in $h_1, \dots, h_m$ where $h_1, \dots, h_m$ are U -invariant functions then

$$\Phi(h_1, \dots, h_m, a_1, \dots, a_n) \rightarrow \{\nu : \Phi(h_1, \dots, h_m, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

And if $\Phi(h_1, \dots, h_m, a_1, \dots, a_n)$ is a Σ_1^1 -formula in $h_1, \dots, h_m$ where $h_1, \dots, h_m$ are U -invariant functions then

$$\{\nu : \Phi(h_1, \dots, h_m, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau} \rightarrow \Phi(h_1, \dots, h_m, a_1, \dots, a_n).$$

Therefore if $\Phi(h_1, \dots, h_m, a_1, \dots, a_n)$ is a Δ_1^1 -formula in $h_1, \dots, h_m$ where $h_1, \dots, h_m$ are U -invariant functions then

$$\Phi(h_1, \dots, h_m, a_1, \dots, a_n) \leftrightarrow \{\nu : \Phi(h_1, \dots, h_m, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Let $f(h_1, \dots, h_m, a_1, \dots, a_n)$ be a Δ_1^1 -function in U -invariant functions $h_1, \dots, h_m$. Then we have

$$f(h_1, \dots, h_m, a_1, \dots, a_n) = b \leftrightarrow \{\nu : f(h_1, \dots, h_m, (a_1)_\nu, \dots, (a_n)_\nu) = (b)_\nu\} \in \mathcal{F}_{\aleph_\tau}.$$

Namely we have

$$\{\nu : (f(h_1, \dots, h_m, a_1, \dots, a_n))_\nu = f(h_1, \dots, h_m, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

From this we obtain that every function which is Δ_1^1 in U -invariant functions is U -invariant. Especially every function which is arithmetical in U -invariant functions is U -invariant.

Now we shall give some example of Π_1^1 -formulas, and it should be noted that some well-known axiom of strong infinity have the form $\exists x P(x)$ where $P(a)$ is a Π_1^1 -formula.

$$\bar{a} < \bar{b} \equiv \forall \alpha \exists c (c < b \wedge \forall d (d < a \rightarrow \alpha(d) \neq c)).$$

$$a < cf(b) \equiv \forall \alpha \exists c (c < b \wedge \forall d (d < a \wedge \alpha(d) < b \rightarrow \alpha(d) < c)) .$$

$$2^{\bar{a}} < \bar{b} \equiv \forall \alpha \forall c (\forall d (\alpha(d) \leq 1) \wedge \forall d \forall e (d < c \wedge e < c \wedge d \neq e \rightarrow \exists f (f < a \wedge \alpha(j(d, f)) \neq \alpha(j(e, f)))) \rightarrow c < b) .$$

' a is a cardinal number', denoted by $\mathbb{C}ard(a)$, expressed by

$$\mathbb{C}ard(a) \equiv \forall b (b < a \rightarrow \bar{b} < \bar{a}) .$$

' a is a regular cardinal number', denoted by $\mathbb{R}eg(a)$, is expressed by

$$\mathbb{R}eg(a) \equiv \forall b (b < a \rightarrow b < cf(a)) .$$

' a is weakly inaccessible cardinal', denoted by $\mathbb{W}(a)$, is expressed by

$$\mathbb{W}(a) \equiv \mathbb{R}eg(a) \wedge \forall b (b < a \rightarrow \exists c (\bar{b} < \bar{c} \wedge \bar{c} < \bar{a})) .$$

' a is strongly inaccessible cardinal', denoted by $\mathbb{S}(a)$, is expressed by

$$\mathbb{S}(a) \equiv \mathbb{R}eg(a) \wedge \forall b (b < a \rightarrow 2^{\bar{b}} < \bar{a}) .$$

' α is a normal function', denoted by $\mathbb{N}orm(\alpha)$, is expressed by

$$\begin{aligned} \mathbb{N}orm(\alpha) \equiv & \forall a \forall b (a < b \rightarrow \alpha(a) < \alpha(b)) \\ & \wedge \forall a (\forall c (a \neq c + 1) \rightarrow \forall b (b < \alpha(a) \rightarrow \exists c (c < a \wedge b < \alpha(c)))) . \end{aligned}$$

' a is R -inaccessible', denoted by $\mathbb{I}n(a, R)$, is expressed by

$$\mathbb{I}n(a, R) \equiv \forall \alpha (\mathbb{N}orm(\alpha) \wedge \forall b (b < a \rightarrow \alpha(b) < a) \rightarrow \exists c (c < a \wedge R(c) \wedge \alpha(c) = c)) .$$

' a is hyper-inaccessible cardinal of type 1, 2, $\dots$ ', denoted by $\mathbb{R}_1(a)$, $\mathbb{R}_2(a)$, $\dots$ are expressed by $\mathbb{R}_1(a) \equiv \mathbb{I}n(a, \mathbb{S})$, $\mathbb{R}_2(a) \equiv \mathbb{I}n(a, \mathbb{R}_1)$, $\dots$

' b is hyper-inaccessible cardinal of type a with respect to T ', denoted by $\mathbb{R}(a, b, T)$, is expressed by

$$\begin{aligned} \mathbb{R}(a, b, T) \equiv & \forall \beta [\{\forall c (T(c) \rightarrow \beta(0, c) = 0) \\ & \wedge \forall c \forall d (\mathbb{I}n(d, \lambda x \beta(c, x) = 0) \rightarrow \beta(d + 1, c) = 0) \\ & \wedge \forall c \forall d (\forall e (d \neq e + 1) \wedge \forall e (e < d \rightarrow \beta(e, c) = 0) \\ & \rightarrow \beta(d, c) = 0)\} \rightarrow \beta(a, b) = 0] . \end{aligned}$$

' b is hyper-inaccessible cardinal of type a ', is expressed by $\mathbb{R}(a, b, \mathbb{S})$.

' a is hyper-inaccessible cardinal of type a ', is expressed by $\mathbb{R}(a, a, \mathbb{S})$.

We can also define Π_1^1 -formulas such as $\mathbb{R}(a, a, \lambda x \mathbb{R}(x, x, \mathbb{S}))$ or even $\mathbb{R}(a, a, \lambda x \mathbb{R}(x, x, \lambda y \mathbb{R}(y, y, \mathbb{S})))$, $\dots$, the first cardinal satisfying such formula is tremendously large.

Let $P(a)$ be a Π_1^1 -formula. Then we obtain that

$$P(a) \rightarrow \{\nu : P((a)_\nu)\} \in \mathcal{F}_{\aleph^*} .$$

Let c_0 be an ordinal number such that $c_0 \leq 2^{\aleph^*}$. Then we have

$$\{\nu : (c_0)_\nu < \aleph_\tau\} \in \mathcal{F}_{\aleph_\tau}.$$

Therefore if $\exists x(x < 2^{\aleph_\tau} \wedge P(x))$, then for some ordinal number $c < \aleph_\tau$, $P(c)$ is satisfied. Namely we get

$$\{\nu : P((c)_\nu)\} \in \mathcal{F}_{\aleph_\tau} \quad \text{and} \quad \{\nu : (c)_\nu < \aleph_\tau\} \in \mathcal{F}_{\aleph_\tau}.$$

Therefore $\{\nu : P((c)_\nu) \wedge (c)_\nu < \aleph_\tau\} \in \mathcal{F}_{\aleph_\tau}$.

Hence we have $\exists x(x < \aleph_\tau \wedge P(x))$.

Let $P(\aleph_\tau)$ be satisfied. Then $\exists x(x < \aleph_\tau \wedge P(x))$. So $\aleph_\tau$ is not the first element of Π_1^1 -class of ordinal numbers. Namely every normally measurable cardinal number is not the first element of Π_1^1 -class of ordinals.

Let $P_1(a), \dots, P_n(a), \dots$ be an enumeration of all Π_1^1 -formulas such that $P_n(\aleph_\tau)$. Then we obtain $\{\nu : P_n((\aleph_\tau)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$ for all n . Since $\mathcal{F}_{\aleph_\tau}$ is $\aleph_0$ -complete, we have

$$\{\nu : (\aleph_\tau)_\nu < \aleph_\tau \wedge P_n((\aleph_\tau)_\nu) \text{ for all } n\} \in \mathcal{F}_{\aleph_\tau}.$$

Namely we obtain

$$\exists x(x < \aleph_\tau \wedge \forall n(n < \omega \rightarrow P_n(x))).$$

This means that there is a cardinal number α_0 less than $\aleph_\tau$ which has all Π_1^1 -properties of $\aleph_\tau$.

Let $\aleph_\omega$ be a cardinal number, namely $\text{Card}(\aleph_\tau)$. Then we obtain $\{\nu : \text{Card}((\aleph_\tau)_\nu)\} \in \mathcal{F}_{\aleph_\tau}$. Let $\phi(\alpha_1, \dots, \alpha_m, a_1, \dots, a_n)$ be an analytic formula. As mentioned before by $\phi^{(\aleph_\omega)}(\alpha_1, \dots, \alpha_m, a_1, \dots, a_n)$, we denote the formula obtained from ϕ by considering $\aleph_\omega$ as universe.

By the similar method as the proof of Theorem A, we obtain.

Theorem A'. Let $h_1, \dots, h_m$ be functions on $\aleph_\omega$, and $h_{1\nu}, \dots, h_{m\nu}$ be functions on $(\aleph_\omega)_\nu$. And let

$$\{\nu : (h_i(a_1, \dots, a_n))_\nu = h_{i\nu}((a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau},$$

for all $i=1, \dots, m$. Then we have

$$\begin{aligned} \phi^{(\aleph_\omega)*}(h_1, \dots, h_m, a_1, \dots, a_n) \leftrightarrow \{\nu : \phi^{((\aleph_\omega)_\nu)}(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, \\ \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}, \end{aligned}$$

where $\phi^{(\aleph_\omega)*}$ is the formula obtained from $\phi^{(\aleph_\omega)}$ restricting all function quantifiers to the functions such that

$$\{\nu : (\beta(a_1, \dots, a_n))_\nu = \beta_\nu((a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau},$$

for some sequence of functions $(\beta_1, \dots, \beta_\nu, \dots)$ where β_ν is a function on $(\aleph_\omega)_\nu$.

Let $\phi^{(\aleph_\omega)}(h_1, \dots, h_m, a_1, \dots, a_n)$ be a $\mathcal{A}_1$ -formula on $\aleph_\omega$ and

$\phi^{(\aleph_\alpha)^\nu}(h_{1\nu}, \dots, h_{m\nu}, a_1, \dots, a_n)$ be a $\mathcal{A}_1^1$ -formula on $(\aleph_\alpha)_\nu$ and let

$$\{\nu : (h_i(a_1, \dots, a_n))_\nu = h_{i\nu}((a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau},$$

Then we have

$$\begin{aligned} \phi^{(\aleph_\alpha)}(h_1, \dots, h_m, a_1, \dots, a_n) \leftrightarrow \{\nu : \phi^{(\aleph_\alpha)^\nu}(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \\ \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}. \end{aligned}$$

Let $f(h_1, \dots, h_m, a_1, \dots, a_n)$ be a $\mathcal{A}_1^1$ -function considering $\aleph_\alpha$ as universe defined by $\phi^{(\aleph_\alpha)}(h_1, \dots, h_m, a_1, \dots, a_n, a)$, and $f_\nu(h_{1\nu}, \dots, h_{m\nu}, a_1, \dots, a_n)$ be a $\mathcal{A}_1^1$ -function considering $(\aleph_\alpha)_\nu$ as universe defined by $\phi^{(\aleph_\alpha)^\nu}(h_{1\nu}, \dots, h_{m\nu}, a_1, \dots, a_n, a)$. And let

$$\{\nu : (h_i(a_1, \dots, a_n))_\nu = h_{i\nu}((a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

Then we have

$$\{\nu : (f(h_1, \dots, h_m, a_1, \dots, a_n))_\nu = f_\nu(h_{1\nu}, \dots, h_{m\nu}, (a_1)_\nu, \dots, (a_n)_\nu)\} \in \mathcal{F}_{\aleph_\tau}.$$

§ 3. φ -commutative functions

Definition. Let $\mathfrak{A}$ be a set of functions and let D be a set of ordinal numbers. Then D is called $\mathfrak{A}$ -closed if for any function f of $\mathfrak{A}$ and for any ordinal numbers $a_1, \dots, a_n \in D$, we have $f(a_1, \dots, a_n) \in D$. Let B be a set of ordinal numbers. Then by $[B]_{\mathfrak{A}}$ we denote the smallest set which is $\mathfrak{A}$ -closed and includes B . And $[B]_{\mathfrak{A}}$ is called the $\mathfrak{A}$ -closure of the set B .

A set of functions $\mathfrak{A}$ is called arithmetically (analytically, $\mathcal{A}_1^1$ -) closed if every function which is arithmetical (analytic, $\mathcal{A}_1^1$ -) in the functions of $\mathfrak{A}$ belongs to $\mathfrak{A}$. And a set B of ordinal numbers is called $\mathfrak{A}$ -arithmetically (-analytically, $\mathcal{A}_1^1$ -) closed if B is closed under every function arithmetical (analytic, $\mathcal{A}_1^1$) in the functions of $\mathfrak{A}$.

Let B be a set of ordinals which is $\mathfrak{A}$ -arithmetically (-analytically, $\mathcal{A}_1^1$ -) closed. Then a function f is called $\mathfrak{A}$ -arithmetically (-analytically, $\mathcal{A}_1^1$ -) closed on B , if for every function g which is arithmetical (analytic, $\mathcal{A}_1^1$) in f and the functions of $\mathfrak{A}$, B is g closed.

Let B be a set of ordinal numbers and let $\mathfrak{A}$ be a set of functions such that

- 1) $\mathfrak{A}$ is arithmetically closed.
 - 2) B is $\mathfrak{A}$ -closed.
 - 3) There is an order-preserving onto mapping $\varphi: \text{On} \rightarrow B$.
- Then a function g and a function f which is arithmetically closed on B are said to be φ -commutative, if

$$\forall x_1 \dots \forall x_n (\varphi(g(x_1, \dots, x_n)) = f(\varphi(x_1), \dots, \varphi(x_n))).$$

An ordinal number α is said to be φ -invariant if $\varphi(\alpha) = \alpha$. A function f is said to be φ -invariant if

$$\forall x_1 \cdots \forall x_n (\varphi(f(x_1, \dots, x_n)) = f(\varphi(x_1), \dots, \varphi(x_n))).$$

Let $A(\alpha, h_1, \dots, h_m, a_1, \dots, a_n)$ be an analytic formula containing only one function variable. Then a function g such that

$$\exists \alpha A(\alpha, h_1, \dots, h_m, a_1, \dots, a_n) \leftrightarrow A(\lambda x_1 \cdots x_s g(a_1, \dots, a_n, x_1, \dots, x_s), h_1, \dots, h_m, a_1, \dots, a_n),$$

is called an R -function for the formula.

Let $\mathfrak{A}$ be a set of functions such that for every arithmetical (analytic) formula of the form $A(\alpha, h_1, \dots, h_m, a_1, \dots, a_n)$ where $h_1, \dots, h_m \in \mathfrak{B}$, there is an R -function for the formula in $\mathfrak{A}$. Then the set of functions $\mathfrak{A}$ is called an arithmetical (analytic) basis for the set of functions $\mathfrak{B}$. It is clear that if $\mathfrak{A}$ is an arithmetical (analytic) basis for $\mathfrak{B}$, then $\mathfrak{A}$ includes $\mathfrak{B}$.

Lemma 2. *Let $\mathfrak{B}$ be a countable set of functions. Then there is an arithmetical (analytic) basis for the set $\mathfrak{B}$, which is arithmetically (analytically) closed and countable.*

In order to show this we consider any countable set of functions $\mathfrak{C}$, and we consider the set A of all arithmetical (analytic) formula of the form $A(\alpha, h_1, \dots, h_m, a_1, \dots, a_n)$ where $h_1, \dots, h_m \in \mathfrak{B}$. Then the set A is countable. For any formula of A we take a representative R -function for the formula and consider the set $\mathfrak{B}'$ of all such representative R -functions. Let $g(\mathfrak{B})$ be the set of all functions arithmetical (analytic) in the functions of $\mathfrak{B}'$. Then clearly $g(\mathfrak{B})$ is countable. We define $g^{(n)}(\mathfrak{B})$ as $g^{(0)}(\mathfrak{B}) = \mathfrak{B}$, $g^{(n+1)}(\mathfrak{B}) = g(g^{(n)}(\mathfrak{B}))$.

Let $\mathfrak{A} = \bigcup_{n < \omega} g^{(n)}(\mathfrak{B})$. Then $\mathfrak{A}$ is a countable arithmetical (analytic) basis for the set $\mathfrak{B}$. And $\mathfrak{A}$ is arithmetically (analytically) closed.

Theorem B. *Let $\mathfrak{A}$ be an analytic basis which is analytically closed. And let B be a class of ordinal numbers such that there is an order-preserving onto mapping $\varphi: \text{On} \rightarrow B$. And let the following conditions be satisfied:*

- 1) B is $\mathfrak{A}$ -closed.
- 2) $h_1, \dots, h_m \in \mathfrak{A}^\#$ where $\mathfrak{A}^\# = \{\lambda x_1 \cdots x_s g(b_1, \dots, b_i, x_1, \dots, x_s) : g \in \mathfrak{A}, b_1, \dots, b_i \in B\}$.
- 3) $\tilde{h}_i$ is a function such that $\varphi(\tilde{h}_i(a_1, \dots, a_s)) = h_i(\varphi(a_1), \dots, \varphi(a_s))$,
($i = 1, \dots, m$).

Then we have

$$\tilde{\Phi}(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n) \leftarrow \Phi(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

where $\tilde{\Phi}$ is the formula obtained from Φ by restricting all function quantifiers to the functions such that $\beta(a_1, \dots, a_s) = \varphi^{-1}(\gamma(\varphi(a_1), \dots, \varphi(a_s)))$

for some function γ in $\mathfrak{A}^\#$.

In order to prove the theorem, we shall first prove the following lemma.

Lemma 3. *Let $\mathfrak{A}$ be a set of functions which is arithmetically closed. And let B be a class of ordinal numbers such that there is an order-preserving onto mapping $\varphi: \text{On} \rightarrow B$. And let the following conditions be satisfied:*

- 1) B is $\mathfrak{A}$ -closed.
- 2) $h_1, \dots, h_m \in \mathfrak{A}^\#$ where $\mathfrak{A}^\# = \{\lambda x_1 \dots x_s g(b_1, \dots, b_t, x_1, \dots, x_s) : g \in \mathfrak{A}, b_1, \dots, b_t \in B\}$.
- 3) $\varphi(\tilde{h}_i(a_1, \dots, a_s)) = h_i(\varphi(a_1), \dots, \varphi(a_s)), \quad (i=1, \dots, m).$

Then we have

$$\varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)) = f(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

for every arithmetical function.

Proof. Let $f(h_1, \dots, h_m, a_1, \dots, a_n)$ be an arithmetical function in $h_1, \dots, h_m$ where $h_1, \dots, h_m \in \mathfrak{A}^\#$. Since $\mathfrak{A}^\#$ is arithmetically closed, we have

$$f(h_1, \dots, h_m, a_1, \dots, a_n) \in \mathfrak{A}^\#.$$

The lemma is proved by the induction on the number of stages to construct f .

The following properties are easily verified:

$$\begin{aligned} \varphi(a') &= \varphi(a)', \quad \varphi(0) = 0, \quad \varphi(\omega) = \omega, \quad \varphi(\max(a, b)) = \max(\varphi(a), \varphi(b)), \\ \varphi(Iq(a, b)) &= Iq(\varphi(a), \varphi(b)), \quad \varphi(j(a, b)) = j(\varphi(a), \varphi(b)), \quad \varphi(g_1(a)) \\ &= g_1(\varphi(a)), \quad \varphi(g_2(a)) = g_2(\varphi(a)). \end{aligned}$$

Therefore in the case where the last schema used in the definition of f is one of (I) to (VII), the functions $f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)$ and $f(h_1, \dots, h_m, a_1, \dots, a_n)$ are φ -commutative by the properties given above. In the case of (VIII), the functions are defined by

$$f(\tilde{h}_i, a_1, \dots, a_s) = \tilde{h}_i(a_1, \dots, a_s),$$

$$f(h_i, a_1, \dots, a_s) = h_i(a_1, \dots, a_s),$$

respectively where $\tilde{h}_i$ and h_i are φ -commutative. Hence $f(\tilde{h}_i, a_1, \dots, a_s)$ and $f(h_i, a_1, \dots, a_s)$ are φ -commutative. The cases (IX) to (XI) are treated as usual.

In the case of (XII), the functions are defined by

$$f(\tilde{h}_i, \dots, \tilde{h}_m, a_1, \dots, a_n) = \mu x g(\tilde{h}_1, \dots, \tilde{h}_m, x, a_1, \dots, a_n),$$

$$f(h_1, \dots, h_m, a_1, \dots, a_n) = \mu x g(h_1, \dots, h_m, x, a_1, \dots, a_n).$$

By the hypothesis of induction, we obtain

$$\varphi(g(\tilde{h}_1, \dots, \tilde{h}_m, a, a_1, \dots, a_n)) = g(h_1, \dots, h_m, \varphi(a), \varphi(a_1), \dots, \varphi(a_n)).$$

We shall first show that

$$\begin{aligned} \exists x(g(\tilde{h}_1, \dots, \tilde{h}_m, x, a_1, \dots, a_n) = 0) \\ \leftrightarrow \exists x(g(h_1, \dots, h_m, x, \varphi(a_1), \dots, \varphi(a_n)) = 0). \end{aligned}$$

To show this it is enough to prove

$$\begin{aligned} \exists x(g(h_1, \dots, h_m, x, \varphi(a_1), \dots, \varphi(a_n)) = 0) \\ \rightarrow \exists x(g(\tilde{h}_1, \dots, \tilde{h}_m, x, a_1, \dots, a_n) = 0). \end{aligned}$$

Now we assume that

$$\exists x(g(h_1, \dots, h_m, x, \varphi(a_1), \dots, \varphi(a_n)) = 0).$$

And we consider the ordinal number defined by

$$b = \mu x g(h_1, \dots, h_m, x, \varphi(a_1), \dots, \varphi(a_n)).$$

By the fact that $\varphi(a_1), \dots, \varphi(a_n) \in B$ and B is closed under the function arithmetical in $h_1, \dots, h_m$, we obtain that $b \in B$.

Hence there is an ordinal number c such that $b = \varphi(c)$. From this we obtain that

$$\varphi(g(\tilde{h}_1, \dots, \tilde{h}_m, c, a_1, \dots, a_n)) = g(h_1, \dots, h_m, \varphi(c), \varphi(a_1), \dots, \varphi(a_n)) = 0.$$

Therefore we obtain

$$\exists x(g(\tilde{h}_1, \dots, \tilde{h}_m, x, a_1, \dots, a_n) = 0).$$

If $\neg \exists x(g(\tilde{h}_1, \dots, \tilde{h}_m, x, a_1, \dots, a_n) = 0)$,

then we have

$$\varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)) = 0 = f(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

If $\exists x(g(\tilde{h}_1, \dots, \tilde{h}_m, x, a_1, \dots, a_n) = 0)$,

then we have

$$g(\tilde{h}_1, \dots, \tilde{h}_m, f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n), a_1, \dots, a_n) = 0.$$

Hence we have

$$g(h_1, \dots, h_m, \varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)), \varphi(a_1), \dots, \varphi(a_n)) = 0.$$

Therefore we obtain that

$$f(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)) \leq \varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)).$$

Now we assume that

$$f(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)) < \varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)).$$

Since f is arithmetical in $h_1, \dots, h_m$ and $\varphi(a_1), \dots, \varphi(a_n) \in B$, we have that

$$f(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)) = \varphi(c),$$

for some c such that $c < f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)$.

Therefore we obtain that

$$\begin{aligned} \varphi(g(\tilde{h}_1, \dots, \tilde{h}_m, c, a_1, \dots, a_n)) \\ = g(h_1, \dots, h_m, \varphi(c), \varphi(a_1), \dots, \varphi(a_n)) = 0. \end{aligned}$$

This contradicts to the definition of $f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)$.

Namely we have

$$\varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)) = f(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

In the case of (XIII), the functions are defined by

$$\begin{aligned} f(\tilde{h}_1, \dots, \tilde{h}_m, a, a_1, \dots, a_n) &= g(f^a, \tilde{h}_1, \dots, \tilde{h}_m, a, a_1, \dots, a_n), \\ f(h_1, \dots, h_m, a, a_1, \dots, a_n) &= g(f^a, h_1, \dots, h_m, a, a_1, \dots, a_n). \end{aligned}$$

Now we assume that

$$\varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a, a_1, \dots, a_n)) \neq f(h_1, \dots, h_m, \varphi(a), \varphi(a_1), \dots, \varphi(a_n)).$$

for some a . Let x_0 be the smallest ordinal number such that

$$\varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, x_0, a_1, \dots, a_n)) \neq f(h_1, \dots, h_m, \varphi(x_0), \varphi(a_1), \dots, \varphi(a_n)).$$

Then the functions $f^{x_0}(\tilde{h}_1, \dots, \tilde{h}_m, a, a_1, \dots, a_n)$ and $f^{\varphi(x_0)}(h_1, \dots, h_m, a, a_1, \dots, a_n)$ are φ -commutative. By the hypothesis of the induction, we obtain.

$$\begin{aligned} \varphi(g(f^{x_0}, \tilde{h}_1, \dots, \tilde{h}_m, x_0, a_1, \dots, a_n)) \\ = g(f^{\varphi(x_0)}, h_1, \dots, h_m, \varphi(x_0), \varphi(a_1), \dots, \varphi(a_n)). \end{aligned}$$

This means that

$$\varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, x_0, a_1, \dots, a_n)) = f(h_1, \dots, h_m, \varphi(x_0), \varphi(a_1), \dots, \varphi(a_n)).$$

This is a contradiction. Namely we have

$$\varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a, a_1, \dots, a_n)) = f(h_1, \dots, h_m, \varphi(a), \varphi(a_1), \dots, \varphi(a_n)).$$

Thus we complete the proof of the lemma.

Proof of Theorem B.

In the case where $\phi(h_1, \dots, h_m, a_1, \dots, a_n)$ is of the form

$$f(h_1, \dots, h_m, a_1, \dots, a_n) = g(h_1, \dots, h_m, a_1, \dots, a_n).$$

We have the following equivalence by Lemma 3.

$$\begin{aligned} f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n) &= g(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n) \\ \leftrightarrow \varphi(f(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)) &= \varphi(g(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)) \\ \leftrightarrow f(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)) &= g(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)). \end{aligned}$$

The case where the outermost symbol of ϕ is one of $<$, $>$, $\vee$, $\wedge$, $\rightarrow$ or $\leftrightarrow$ is treated as usual.

In the case where the outermost symbol is $\exists x$, first we assume

$$\exists x \tilde{\psi}(\tilde{h}_1, \dots, \tilde{h}_m, x, a_1, \dots, a_n).$$

Then we obtain $\tilde{\psi}(\tilde{h}_1, \dots, \tilde{h}_m, a, a_1, \dots, a_n)$,

for some a . By the assumption of induction we have

$$\Psi(h_1, \dots, h_m, \varphi(a), \varphi(a_1), \dots, \varphi(a_n)),$$

so $\exists x \Psi(h_1, \dots, h_m, x, \varphi(a_1), \dots, \varphi(a_n))$.

Conversely we assume

$$\exists x \Psi(h_1, \dots, h_m, x, \varphi(a_1), \dots, \varphi(a_n)).$$

And we consider a function f defined by

$$f(a_1, \dots, a_n) = \begin{cases} \text{the least } a \text{ such that } \Psi(h_1, \dots, h_m, a, a_1, \dots, a_n) \\ \quad \text{if } \exists x \Psi(h_1, \dots, h_m, x, a_1, \dots, a_n). \\ 0 & \text{otherwise} \end{cases}$$

Then clearly f is analytic in $h_1, \dots, h_m$. Now we consider the ordinal defined by $f(\varphi(a_1), \dots, \varphi(a_n)) = d$. Since B is closed under the function f , we have $d \in B$. Hence there is an ordinal c such that $\varphi(c) = d$. Namely we have

$$\Psi(h_1, \dots, h_m, \varphi(c), \varphi(a_1), \dots, \varphi(a_n)).$$

So we have

$$\tilde{\psi}(\tilde{h}_1, \dots, \tilde{h}_m, c, a_1, \dots, a_n), \text{ and so } \exists x \tilde{\psi}(\tilde{h}_1, \dots, \tilde{h}_m, x, a_1, \dots, a_n).$$

In the case where the outermost symbol is $\exists \alpha$, first we assume

$$\exists \tilde{\alpha} \tilde{\psi}(\tilde{\alpha}, \tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n).$$

Then we obtain a $\tilde{h}$ such that $\tilde{\Psi}(\tilde{h}, \tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)$ and

$$\varphi(\tilde{h}(a_1, \dots, a_s)) = h(\varphi(a_1), \dots, \varphi(a_s)),$$

for some function $h \in \mathfrak{A}^\#$. By induction hypothesis, we have

$$\Psi(h, h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

Namely we obtain

$$\exists \alpha \Psi(\alpha, h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

Conversely we assume

$$\exists \alpha \Psi(\alpha, h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

Then by $h_1, \dots, h_m \in \mathfrak{A}^\#$ and by the fact $\mathfrak{A}$ is an analytic basis, there is a function h in $\mathfrak{A}^\#$ such that

$$\begin{aligned} & \exists \alpha \Psi(\alpha, h_1, \dots, h_m, a_1, \dots, a_n) \\ & \leftrightarrow \Psi(\lambda x_1 \dots x_s h(a_1, \dots, a_n, x_1, \dots, x_s), h_1, \dots, h_m, a_1, \dots, a_n). \end{aligned}$$

Now we consider the function $\tilde{h}$ defined by

$$\tilde{h}(a_1, \dots, a_n, b_1, \dots, b_s) = \varphi^{-1}(h(\varphi(a_1), \dots, \varphi(a_n), \varphi(b_1), \dots, \varphi(b_s))).$$

This function $\tilde{h}$ is totally defined because $h \in \mathfrak{A}^\#$ and B is $\mathfrak{A}_\#$ -closed. Clearly we have

$$\varphi(\tilde{h}(a_1, \dots, a_n, b_1, \dots, b_s)) = h(\varphi(a_1), \dots, \varphi(a_n), \varphi(b_1), \dots, \varphi(b_s)).$$

By the induction hypothesis, we have

$$\tilde{\Psi}(\lambda x_1 \dots x_s \tilde{h}(a_1, \dots, a_n, x_1, \dots, x_s), \tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n).$$

Hence we obtain $\exists \tilde{\alpha} \tilde{\Psi}(\tilde{\alpha}, \tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n)$.

Thus we complete the proof of the theorem.

As a simple application of this theorem, we obtain that if $\Phi(h_1, \dots, h_m, a_1, \dots, a_n)$ is a Π_1^1 -formula in $h_1, \dots, h_m$ where $h_1, \dots, h_m$ are φ -invariant functions, then we have

$$\Phi(h_1, \dots, h_m, a_1, \dots, a_n) \rightarrow \Phi(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

If the formula is Σ_1^1 , we have

$$\Phi(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)) \rightarrow \Phi(h_1, \dots, h_m, a_1, \dots, a_n).$$

If the formula is Δ_1^1 , we have

$$\Phi((h_1, \dots, h_m, a_1, \dots, a_n) \leftrightarrow \Phi(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n))).$$

Let $f(h_1, \dots, h_m, a_1, \dots, a_n)$ be a $\mathcal{A}_1^1$ -function in φ -invariant functions $h_1, \dots, h_m$.

Then we have

$$\varphi(f(h_1, \dots, h_m, a_1, \dots, a_n)) = f(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)).$$

From this we obtain that every function which is $\mathcal{A}_1^1$ in φ -invariant functions is φ -invariant. Especially every function which is arithmetical in φ -invariant function is φ -invariant.

By the similar method as the proof of theorem B, we obtain.

Theorem B'. *Let $\mathfrak{A}$ be an analytic basis considering $\aleph_\alpha$ as universe which is analytically closed. And let B be a subset of $\aleph_\alpha$ such that there is an order-preserving onto mapping $\varphi: \aleph_0 \rightarrow B$. And let the following conditions be satisfied:*

- 1) B is $\mathfrak{A}$ -closed.
- 2) $h_1, \dots, h_m \in \mathfrak{A}^\#$ where $\mathfrak{A}^\# = \{\lambda x_1 \dots x_s g(b_1, \dots, b_i, x_1, \dots, x_s) : g \in \mathfrak{A}, b_1, \dots, b_i \in B\}$.
- 3) $\varphi(\tilde{h}_i(a_1, \dots, a_s)) = h_i(\varphi(a_1), \dots, \varphi(a_s))$, for every $a_1, \dots, a_s < \aleph_0$.

Then we have

$$\tilde{\varphi}^{(\aleph_0)}(\tilde{h}_1, \dots, \tilde{h}_m, a_1, \dots, a_n) \leftrightarrow \varphi^{(\aleph_\alpha)}(h_1, \dots, h_m, \varphi(a_1), \dots, \varphi(a_n)),$$

where $\tilde{\varphi}^{(\aleph_0)}$ is the formula obtained from $\varphi^{(\aleph_\alpha)}$ by restricting all function quantifiers to the functions such that

$$\beta(a_1, \dots, a_s) = \varphi^{-1}(\gamma(\varphi(a_1), \dots, \varphi(a_s))),$$

for some function γ in $\mathfrak{A}^\#$.

§ 4. Generative functions

Let $\mathfrak{A}$ be a set of functions and let D be a set of ordinal members. Then a function f is called a generative function with respect to $\mathfrak{A}$ and D , if the following conditions are satisfied:

- 1) f is strictly increasing.
- 2) $f(a) \notin [D \cup F_a]_{\mathfrak{A}}$ where $F_a = \{f(b) : b < a\}$.
- 3) For any subset B of F_a , we have

$$[D \cup B \cup \{f(a)\}]_{\mathfrak{A}} \cap f(a) \subset [D \cup B]_{\mathfrak{A}}.$$

A regular cardinal number a is said to be a generative cardinal if for any countable set of functions $\mathfrak{A}$ and any set of ordinals B such that $\sup(B) < a$ there is a countable set of functions $\mathfrak{B} \supseteq \mathfrak{A}$ and a set of ordinal numbers $D \supseteq B$ with the cardinality $\bar{D} \leq \bar{B} \cdot \aleph_0$. And there is a generative function f defined on a and having values in a such that $f(0) \geq \sup(B)$.

By $G(a)$ we denote that there is a generative cardinal number. And by G we mean the axiom of strong infinity that there is a generative cardinal number, namely $\exists x G(x)$.

Let f be a function. Then an ordinal number a is called a node of the function f if

$$\forall x_1 \cdots \forall x_n (x_1 < a \wedge \cdots \wedge x_n < a \rightarrow f(x_1, \dots, x_n) < a).$$

Using this terminology, $\mathcal{A}_1$ -TC means that every cardinal greater than $\aleph_0$ is a node for every $\mathcal{A}_1$ -functions.

Theorem C. *Let $\aleph_{\tau_1} < \cdots < \aleph_{\tau_\beta} < \cdots$ ($\beta < \delta_0 < \aleph_{\tau_1}$) be a sequence of normally measurable cardinals. And let $\mathfrak{A}$ be a set of functions such that $\bar{\mathfrak{A}} < \aleph_{\tau_1}$ and B be a set of ordinals such that $\bar{B} < \aleph_{\tau_1}$. Then there is a set of functions $\mathfrak{B} \supset \mathfrak{A}$ such that $\bar{\mathfrak{B}} \leq \max(\bar{\mathfrak{A}}, \bar{\delta}_0, \aleph_0)$ and $\mathfrak{B}$ is closed under substitution and there is a set of ordinal numbers $D \supset B$ such that $\bar{D} \leq \max(\bar{B}, \bar{\delta}_0, \aleph_0)$ and there is a generative function f defined on $\sup_{\beta < \delta_0} \aleph_{\tau_\beta}$ and having values in it such that every ordinal $\aleph_{\tau_\beta}$ ($\beta < \delta_0$) is a node for the function f .*

In order to prove the theorem, we need some definitions and lemmas.

Let $\aleph_\tau$ be a normally measurable cardinal number. We consider a non-principal ultrafilter $\mathcal{F}_{\aleph_\tau}$ with the character $\aleph_\tau > \aleph_0$. And we consider $\text{On}^{\aleph_\tau} / \mathcal{F}_{\aleph_\tau}$. By $a^{(\tau)}$ we denote a representative element of a -th equivalence class of $\text{On}^{\aleph_\tau} / \mathcal{F}_{\aleph_\tau}$. And by $(a)_\nu^{(\tau)}$, we denote the ν -th component of $a^{(\tau)}$.

The function p_τ is defined by

$$p_\tau(a) = b \quad \text{if and only if} \quad \{\nu : (b)_\nu^{(\tau)} = a\} \in \mathcal{F}_{\aleph_\tau}.$$

Namely $p_\tau(a)$ is the ordinal number such that $(a, \dots, a, \dots)$ belongs to $p_\tau(a)$ -th equivalence class of $\text{On}^{\aleph_\tau} / \mathcal{F}_{\aleph_\tau}$.

Let $f(a_1, \dots, a_n)$ be a function. For any fixed ordinal numbers $a_1, \dots, a_n$ we consider a function defined on $\aleph_\tau$:

$$u = (f((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)}), \dots, f((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)}), \dots).$$

Since $u \in \text{On}^{\aleph_\tau}$, there is unique b such that u belongs to $b^{(\tau)}$. Now the function $\sigma_\tau f$ is defined by $\sigma_\tau f(a_1, \dots, a_n) = b$. According to this definition we have

$$\{\nu : (\sigma_\tau f(a_1, \dots, a_n))_\nu^{(\tau)} = f((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)})\} \in \mathcal{F}_{\aleph_\tau}.$$

Now let

$$f(a_1, \dots, a_n) = g(h_1(a_1, \dots, a_n), \dots, h_m(a_1, \dots, a_n)).$$

Then we have

$$\{\nu : (\sigma_\tau f(a_1, \dots, a_n))_\nu^{(\tau)} = g(h_1((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)}), \dots, h_m((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)}))\} \in \mathcal{F}_{\aleph_\tau}.$$

By the definition of $\sigma_\tau g$, $\sigma_\tau h_1$, $\dots$, $\sigma_\tau h_m$, we have

$$\{\nu : (\sigma_\tau f(a_1, \dots, a_n))_\nu^{(\tau)} = (\sigma_\tau g_1(a_1, \dots, a_n), \dots, \sigma_\tau h_m(a_1, \dots, a_n))_\nu^{(\tau)}\} \in \mathcal{F}_{\aleph_\tau}.$$

This mean that

$$\sigma_\tau f(a_1, \dots, a_n) = \sigma_\tau g(\sigma_\tau h_1(a_1, \dots, a_n), \dots, \sigma_\tau h_m(a_1, \dots, a_n)).$$

Lemma 4. *Let $\aleph_\tau$ be a normlly measurable cardinal and $\mathcal{F}_{\aleph_\tau}$ be a non-principal ultrafilter $\mathcal{F}_{\aleph_\tau}$ with the character $\aleph_\tau > \aleph_0$. Let $\mathcal{A}$ be a set of functions and B be a set of ordinal numbers such that*

- 1) $\overline{\mathcal{A}}, \overline{B} < \aleph_\tau$.
- 2) $\aleph_\tau \in B$ and if $a \in B$ then $p_\tau(a) \in B$.
- 3) $f(a_1, \dots, a_n) \in \mathcal{A}$ then $\sigma_\tau f(a, \dots, a_n) \in \mathcal{A}$.
- 4) $\mathcal{A}$ contains the identity function, and if $f, f_1, \dots, f_m \in \mathcal{A}$, then we have $f(f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)) \in \mathcal{A}$.

Then there is a strongly inaccessible cardinal number a_0 such that

- 1) $[B \cup \{a_0\}]_{\mathcal{A}} \cap a_0 \subset [B]_{\mathcal{A}}$.
- 2) $\sup \{[B]_{\mathcal{A}} \cap \aleph_\tau\} \leq a_0 < \aleph_\tau$ and $a_0 \notin [B]_{\mathcal{A}}$.

Proof. Let $f_1, \dots, f_\delta, \dots$ ($\delta \leq \overline{\mathcal{A}}$) be an enumeration of functions in $\mathcal{A}$. We shall now define a set $A_{\delta a_1 \dots a_{n+1}}$ where $\delta < \overline{\mathcal{A}}$ and $a_1, \dots, a_n \in [B]_{\mathcal{A}}$ as follows:

$$A_{\delta a_1 \dots a_{n+1}} = \begin{cases} \{\nu : f_\delta((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)}) < (a_{n+1})_\nu^{(\tau)}\} \\ \quad \text{if } \sigma_\tau f_\delta(a_1, \dots, a_n) < a_{n+1}, \\ \{\nu : f_\delta((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)}) = (a_{n+1})_\nu^{(\tau)}\} \\ \quad \text{if } \sigma_\tau f_\delta(a_1, \dots, a_n) = a_{n+1}, \\ \{\nu : f_\delta((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)}) > (a_{n+1})_\nu^{(\tau)}\} \\ \quad \text{if } \sigma_\tau f_\delta(a_1, \dots, a_n) > a_{n+1}. \end{cases}$$

By the definition of $\sigma_\tau f_\delta(a_1, \dots, a_n)$ we obtain that $A_{\delta a_1 \dots a_{n+1}} \in \mathcal{F}_{\aleph_\tau}$ for every $\delta < \overline{\mathcal{A}}$ and $a_1, \dots, a_{n+1} \in [B]_{\mathcal{A}}$. Since the cardinality of $\mathcal{A}$ and $[B]_{\mathcal{A}}$ is less than $\aleph_\tau$, the cardinality of the set

$$\{<\delta, a_1, \dots, a_{n+1}> : n < \omega, \delta < \overline{\mathcal{A}}, a_1, \dots, a_{n+1} \in [B]_{\mathcal{A}}\},$$

is less than $\aleph_\tau$. Let C be a subset of $\aleph_\tau$ defined by

$$C = \bigcap_{\substack{\delta < \overline{\mathcal{A}} \\ a_1, \dots, a_{n+1} \in [B]_{\mathcal{A}}}} A_{\delta a_1 \dots a_{n+1}}.$$

Since the character of the ultrafilter $\mathcal{F}_{\aleph_\tau}$ is $\aleph_\tau$, we obtain that $C \in \mathcal{F}_{\aleph_\tau}$. Hence there is an ordinal number ν_0 such that $\nu_0 \in C$.

Let f_δ be the identity function, namely $f_\delta(a) = a$,

$$\{\nu : (\aleph_\tau)_{\nu_0}^{(\tau)} < \aleph_\tau\} \supset A_{\delta \aleph_\tau p_\tau(\aleph_\tau)}.$$

Therefore $(\aleph_\tau)_{\nu_0}^{(\tau)} < \aleph_\tau$. Now we shall show that

$$[B \cup \{(\aleph_\tau)_{\nu_0}^{(\tau)}\}]_{\mathfrak{A}} \cap (\aleph_\tau)_{\nu_0}^{(\tau)} \subset [B]_{\mathfrak{A}}.$$

Assume that this inclusion is not true. Then there is a function f_β of $\mathfrak{A}$ such that

$$f_\beta((\aleph_\tau)_{\nu_0}^{(\tau)}, b_1, \dots, b_n) = c, \quad c \notin [B]_{\mathfrak{A}} \text{ and } c < (\aleph_\tau)_{\nu_0}^{(\tau)},$$

where $b_1, \dots, b_n \in B$.

By the assumption of the lemma that B is closed under the function p_τ we have

$$\{\nu : (p_\tau(b_i))_{\nu}^{(\tau)} = b_i\} \in \mathcal{F}_{\aleph_\tau} \text{ and } p_\tau(b_i) \in B \text{ for all } i=1, \dots, n.$$

We shall now consider the ordinal number defined by

$$\sigma_\tau f_\beta(\aleph_\tau, p_\tau(b_1), \dots, p_\tau(b_n)).$$

By $\aleph_\tau, p_\tau(b_1), \dots, p_\tau(b_n) \in B$ and $\sigma_\tau f_\beta \in \mathfrak{A}$, we obtain that

$$\sigma_\tau f_\beta(\aleph_\tau, p_\tau(b_1), \dots, p_\tau(b_n)) \in [B]_{\mathfrak{A}}.$$

We assume that $\sigma_\tau f_\beta(\aleph_\tau, p_\tau(b_1), \dots, p_\tau(b_n)) \geq \aleph_\tau$. Then we obtain that

$$\{\nu : f_\beta((\aleph_\tau)_{\nu}^{(\tau)}, b_1, \dots, b_n) \geq (\aleph_\tau)_{\nu}^{(\tau)}\} \supset A_{\beta \aleph_\tau p_\tau(b_1) \dots p_\tau(b_n) \aleph_\tau} \supset C \ni \nu_0.$$

Hence we have $f_\beta((\aleph_\tau)_{\nu_0}^{(\tau)}, b_1, \dots, b_n) \geq (\aleph_\tau)_{\nu_0}^{(\tau)}$. This contradicts to our assumption. Namely we obtain

$$e = \sigma_\tau f_\beta(\aleph_\tau, p_\tau(b_1), \dots, p_\tau(b_n)) < \aleph_\tau \text{ and } e \in [B]_{\mathfrak{A}}.$$

Therefore, we obtain that

$$\{\nu : f_\beta((\aleph_\tau)_{\nu}^{(\tau)}, b_1, \dots, b_n) = e\} \supset A_{\beta \aleph_\tau p_\tau(b_1) \dots p_\tau(b_n) e} \supset C \ni \nu_0.$$

Namely $f_\beta((\aleph_\tau)_{\nu_0}^{(\tau)}, b_1, \dots, b_n) = e \in [B]_{\mathfrak{A}}$. This is a contradiction, because we assumed that $f_\beta((\aleph_\tau)_{\nu_0}^{(\tau)}, b_1, \dots, b_n) \notin [B]_{\mathfrak{A}}$. We have

$$[B \cup \{(\aleph_\tau)_{\nu_0}^{(\tau)}\}]_{\mathfrak{A}} \cap (\aleph_\tau)_{\nu_0}^{(\tau)} \subset [B]_{\mathfrak{A}}.$$

Next we shall show that

$$\sup \{[B]_{\mathfrak{A}} \cap \aleph_\tau\} \leq (\aleph_\tau)_{\nu_0}^{(\tau)}, \quad (\aleph_\tau)_{\nu_0}^{(\tau)} \notin [B]_{\mathfrak{A}}.$$

We consider that identity function f_β of $\mathfrak{A}$. And let d be any element of $[B]_{\mathfrak{A}} \cap \aleph_\tau$. We obtain

$$\{\nu : f_\beta(d) < (\aleph_\tau)_\nu^{(\tau)}\} = \{\nu : d < (\aleph_\tau)_\nu^{(\tau)}\} \supset A_{fd\aleph_\tau} \supset C \ni \nu_0.$$

Hence $d < (\aleph_\tau)_{\nu_0}^{(\tau)}$. From this we obtain that

$$\sup \{[B]_{\mathfrak{A}} \cap \aleph_\tau\} \leq (\aleph_\tau)_{\nu_0}^{(\tau)} \text{ and } (\aleph_\tau)_{\nu_0}^{(\tau)} \notin [B]_{\mathfrak{A}}.$$

Thus we complete the proof of the lemma.

Lemma 5. *Let $\mathfrak{A}$ be a set of functions and D be a set of ordinal numbers such that*

- 1) *If $a \in D$ then $p_\tau(a) \in D$.*
- 2) *If $f(a_1, \dots, a_n) \in \mathfrak{A}$ then $\sigma_\tau f(a_1, \dots, a_n) \in \mathfrak{A}$.*
- 3) *$\mathfrak{A}$ contains the identity function, and if $f, f_1, \dots, f_m \in \mathfrak{A}$ then*

$$f(f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)) \in \mathfrak{A}.$$

Then $[D]_{\mathfrak{A}}$ is closed under the function p_τ .

Proof. Let d be any element of $[D]_{\mathfrak{A}}$. Then by the definition of $[D]_{\mathfrak{A}}$, there is a function f of $\mathfrak{A}$ such that

$$d = f(a_1, \dots, a_n) \text{ where } a_1, \dots, a_n \in D.$$

We have

$$\{\nu : (p_\tau(f(a_1, \dots, a_n)))_\nu^{(\tau)} = f((p_\tau(a_1))_\nu^{(\tau)}, \dots, (p_\tau(a_n))_\nu^{(\tau)})\} \in \mathcal{F}_{\aleph_\tau}.$$

And by the definition of $\sigma_\tau f$, we obtain

$$\{\nu : (\sigma_\tau f(a_1, \dots, a_n))_\nu^{(\tau)} = f((a_1)_\nu^{(\tau)}, \dots, (a_n)_\nu^{(\tau)})\} \in \mathcal{F}_{\aleph_\tau}.$$

Hence we have

$$p_\tau(f(a_1, \dots, a_n)) = \sigma_\tau f(p_\tau(a_1), \dots, p_\tau(a_n)).$$

Since $p_\tau(a_1), \dots, p_\tau(a_n) \in D$ and $\sigma_\tau f \in \mathfrak{A}$, we have $p_\tau(d) \in [D]_{\mathfrak{A}}$. Thus we complete the proof of the lemma.

Proof of Theorem C. Let $\mathcal{F}_{\aleph_\tau}(\beta < \delta_0)$ an ultrafilter over $\aleph_{\tau_\beta}$ with the character $\aleph_{\tau_\beta} > \aleph_0$. $\mathfrak{B}$ we denote the smallest set of functions such that

- 1) $\mathfrak{A} \subset \mathfrak{B}$.
 - 2) $f(a_1, \dots, a_n) \in \mathfrak{B}$ then $\sigma_\tau f(a_1, \dots, a_n) \in \mathfrak{B}$.
 - 3) $\mathfrak{B}$ contains the identity function and $\mathfrak{B}$ is closed under substitution.
- Then we have $\overline{\mathfrak{B}} \leq \max(\overline{\mathfrak{A}}, \overline{\delta_0}, \aleph_0)$.

Let D be the set $[B \cup \{\aleph_{\tau_\beta} : \beta < \delta_0\}]_{[p_{\tau_\beta} : \beta < \delta_0]}$. Then we have

- 1) $\aleph_{\tau_\beta} \in D$ and if $a \in D$ then $p_{\tau_\beta}(a) \in D$.

2) $\bar{D} \leq \max(\bar{B}, \bar{\delta}_0, \aleph_0)$.

Since $\bar{A}, \bar{B}, \bar{\delta}_0 < \aleph_{\tau_1}$, we have $\bar{B}, \bar{D} < \aleph_{\tau_1}$.

Now we shall define the generative function f_0 on $\sup\{\aleph_{\tau_\beta} : \beta < \delta_0\}$. By Lemma 4 we have an ordinal number a_0 such that

- 1) $[D \cup \{a_0\}]_{\mathfrak{B}} \cap a_0 \subset [D]_{\mathfrak{B}}$.
- 2) $\sup\{[D]_{\mathfrak{B}} \cap \aleph_{\tau_\beta} \mid \beta < \delta_0\} \leq a_0 < \aleph_{\tau_1}$ and $a_0 \notin [D]_{\mathfrak{B}}$.

$f_0(0)$ is defined by $f_0(0) = a_0$. Now we assume that for an ordinal number $b_0 < \sup\{\aleph_{\tau_\beta} : \beta < \delta_0\}$ the values of the function f_0 for ordinals less than b_0 are already defined so that

- 1) $f_0(a) < f_0(b)$ if $a < b < b_0$.
- 2) $f_0(a) \notin [D \cup F_a]_{\mathfrak{B}}$ where $F_a = \{f(b) : b < a\}$ for $a < b_0$.
- 3) For every subset B of F_a we have

$$[D \cup B \cup \{f(a)\}]_{\mathfrak{B}} \cap f(a) \subset [D \cup B]_{\mathfrak{B}}.$$

- 4) if $a < \aleph_{\tau_\beta}$ then $f_0(a) < \aleph_{\tau_\beta}$.

Since $b_0 < \sup\{\aleph_{\tau_\beta} : \beta < \delta_0\}$, there is the least β_0 such that $b_0 < \aleph_{\tau_{\beta_0}}$. We consider the set $D_1 = D \cup \{f_0(a) : a < b_0\}$. Then we have

- 1) $\bar{A}, \bar{D}_1 < \aleph_{\tau_{\beta_0}}$.
 - 2) $\aleph_{\tau_{\beta_0}} \in D_1$ and if $a \in D_1$ then $p_{\tau_{\beta_0}}(a) \in D_1$.
 - 3) $f(a_1, \dots, a_n) \in \mathfrak{B}$ then $\sigma_{\tau_{\beta_0}} f(a_1, \dots, a_n) \in \mathfrak{B}$.
 - 4) $\mathfrak{B}$ contains the identity function and $\mathfrak{B}$ is closed under substitution.
- By the same method as in the proof of Lemma 4, we have an ordinal number a_0 such that

- 1) For every subset B of $F_{b_0} = \{f_0(a) : a < b_0\}$, we have

$$[D \cup B \cup \{a_0\}]_{\mathfrak{B}} \cap a_0 \subset [D \cup B]_{\mathfrak{B}}.$$

- 2) $\sup\{[D_1]_{\mathfrak{B}} \cap \aleph_{\tau_{\beta_0}} \mid \beta < \delta_0\} \leq a_0 < \aleph_{\tau_{\beta_0}}$ and $a_0 \notin [D_1]_{\mathfrak{B}}$.

Now $f_0(b_0)$ is defined to be this a_0 . By the definition of f_0 , we have f_0 is a generative function defined on $\sup\{\aleph_{\tau_\beta} : \beta < \delta_0\}$. Since $f_0(a) < \aleph_{\tau_\beta}$ for all $a < \aleph_{\tau_\beta}$ we have $\aleph_{\tau_\beta}$ is a node for the function f_0 . Thus the theorem is proved.

§ 5. Applications

Let $\aleph_\tau$ be a normally measurable cardinal number.

Theorem 1. *Let $g_1, \dots, g_m$ be functions defined on $\aleph_\tau$ and having values in $\aleph_\tau$ such that*

- 1) $g_i(a_1, \dots, a_n) \leq c_0 < \aleph_\tau$ for all $a_1, \dots, a_n < \aleph_\tau$, $i = 1, \dots, m$.
- 2) $g_i(a_1, \dots, a_n) = 0$ if $\max(a_1, \dots, a_n) \geq c_0$.

And let $f(g_1, \dots, g_m, a_1, \dots, a_s)$ be a A_1^1 -function in $g_1, \dots, g_m$ considering $\aleph_\tau$ as universe. Then we have

$$\forall x_1 \dots \forall x_s (x_1 < \aleph_\tau \wedge \dots \wedge x_s < \aleph_\tau \rightarrow \overline{\overline{f(g_1, \dots, g_m, x_1, \dots, x_s)}} \leq \max(\aleph_0, \bar{c}_0, \bar{x}_1, \dots, \bar{x}_s)).$$

Proof, Let $\mathfrak{U}$ be a set of functions defined on $\aleph_\tau$ and having values in $\aleph_\tau$ satisfying the following conditions:

- 1) $\bar{\mathfrak{U}} \leq \aleph_0$.
- 2) $\mathfrak{U}$ is $\mathcal{A}_1^1$ -closed.
- 3) $\mathfrak{U}$ is a basis for the formulas of the form $A(\alpha, g_1, \dots, g_m, a_1, \dots, a_s)$ where $A(\alpha, g_1, \dots, g_m, a_1, \dots, a_s)$ is arithmetical in $\alpha, g_1, \dots, g_m$.
- 4) $g_1, \dots, g_m \in \mathfrak{U}$.

Let g be a function defined on $\aleph_\tau$ and having values in $\aleph_\tau$. Then by g^* we denote the function defined by

$$g^*(a_1, \dots, a_n) = \begin{cases} g(a_1, \dots, a_n) & \text{if } a_1 < \aleph_\tau \wedge \dots \wedge a_n < \aleph_\tau, \\ 0 & \text{otherwise.} \end{cases}$$

By $\mathfrak{U}^*$ we denote the set of functions defined by $\mathfrak{U}^* = \{g^* : g \in \mathfrak{U}\}$. Clearly $\bar{\mathfrak{U}}^* \leq \aleph_0$.

Let $a_1, \dots, a_s$ be ordinals less than $\aleph_\tau$. And let B be the set

$$\{a : a \leq c_0 \vee a \leq a_1 \vee \dots \vee a \leq a_s\}. \text{ Put } B^* = [B]_{\mathfrak{U}}.$$

Then we have

$$B^* \subset \aleph_\tau, \bar{B}^* \leq \max(\aleph_0, \bar{c}_0, \bar{a}_1, \dots, \bar{a}_s) \text{ and } f(g_1, \dots, g_m, a_1, \dots, a_s) \in B^*.$$

By Theorem C there is a set of functions $\mathfrak{B}$ such that $\mathfrak{B} \supset \mathfrak{U}^*$, $\bar{\mathfrak{B}} \leq \aleph_0$ and a set of ordinal numbers $\bar{D} \supset B^*$ such that

$$\bar{D} \leq \max(\aleph_0, \bar{c}_0, \bar{a}_1, \dots, \bar{a}_s).$$

And there is a function f_0 which is generative with respect to $\mathfrak{B}$ and D defined on $\aleph_\tau$ and having values in it such that

$$f_0(0) \geq \sup \{[D]_{\mathfrak{B}} \cap \aleph_\tau\}.$$

Let $E = [D \cup \{f_0(a) : a < \aleph_\tau\}]_{\mathfrak{B}} \cap \aleph_\tau$. Then we have

- 1) E is $\mathfrak{U}$ -closed.
- 2) $\bar{E} = \aleph_\tau$.

Now we shall show that $E \cap f_0(0) \subset [D]_{\mathfrak{B}} \cap \aleph_\tau$. We consider an ordinal number $d = h(b_1, \dots, b_k, f_0(c_1), \dots, f_0(c_l)) < f_0(0)$ such that

$$b_1, \dots, b_k \in D, c_1, \dots, c_l < \aleph_\tau, h \in \mathfrak{B}.$$

By the fact that f_0 is generative, we have

$$d = h(b_1, \dots, b_k, f_0(c_1), \dots, f_0(c_l)) \subset [D \cup \{f_0(c) : c < c_l\}]_{\mathfrak{B}}.$$

Hence $d = h'(b'_1, \dots, b'_{k'}, f_0(c'_i), \dots, f_0(c'_{i'}))$ where $c'_1 < \dots < c'_{i'} < c_i$.

By repeating this method we have

$$d \in [D]_{\mathfrak{S}} \cap f_0(0) \subset [D]_{\mathfrak{S}} \cap \mathfrak{N}_\tau.$$

Namely we have $E \cap f_0(0) \subset [D]_{\mathfrak{S}} \cap \mathfrak{N}_\tau$. By $f_0(0) \geq \sup \{[D]_{\mathfrak{S}} \cap \mathfrak{N}_\tau\}$, we have $E \cap f_0(0) \supset [D]_{\mathfrak{S}} \cap \mathfrak{N}_\tau$.

Let $\varphi: \mathfrak{N}_\tau \rightarrow E$ be an order preserving onto mapping. Assume that

$$\varphi(a) = f(g_1, \dots, g_m, a_1, \dots, a_s) \text{ and } \bar{a} > \max(\mathfrak{N}_0, \bar{c}_0, \bar{a}_1, \dots, \bar{a}_s).$$

Then we have

$$\overline{E \cap f_0(0)} = \overline{[D]_{\mathfrak{S}} \cap \mathfrak{N}_\tau} > \max(\mathfrak{N}_0, \bar{c}_0, \bar{a}_1, \dots, \bar{a}_s),$$

and

$$\overline{[D]_{\mathfrak{S}}} \leq \max(\mathfrak{N}_0, \bar{c}_0, \bar{a}_1, \dots, \bar{a}_s).$$

This is a contradiction. Namely $\bar{a} \leq \max(\mathfrak{N}_0, \bar{c}_0, \bar{a}_1, \dots, \bar{a}_s)$. By $E \supset a_0$ we have

$$\varphi(g_i(b_1, \dots, b_n)) = g_i(\varphi(b_1), \dots, \varphi(b_n)) \text{ for } i=1, \dots, m.$$

By Theorem B, we have

$$\begin{aligned} \varphi(f(g_1, \dots, g_m, a_1, \dots, a_s)) &= f(g_1, \dots, g_m, \varphi(a_1), \dots, \varphi(a_s)) \\ &= f(g_1, \dots, g_m, a_1, \dots, a_s). \end{aligned}$$

And so $\overline{f(g_1, \dots, g_m, a_1, \dots, a_s)} \leq \max(\mathfrak{N}_0, \bar{c}_0, \bar{a}_1, \dots, \bar{a}_s)$.

Theorem 2 (Uniformity of $\mathcal{A}_1$ -defining schemata). Let $g_1, \dots, g_m$ be functions defined on $\mathfrak{N}_\tau$ and having values in it such that

- 1) $g_i(a_1, \dots, a_n) \leq c_0$ for all $a_1, \dots, a_n < \mathfrak{N}_\tau$, $i=1, \dots, m$.
- 2) $g_i(a_1, \dots, a_n) = 0$ if $\max(a_1, \dots, a_n) \geq c_0$.

And let $f(g_1, \dots, g_m, a_1, \dots, a_n)$ be a $\mathcal{A}_1$ -function in $g_1, \dots, g_m$ defined by $\phi^{(\mathfrak{N}_\tau)}(g_1, \dots, g_m, a_1, \dots, a_n, a)$. And let $\mathfrak{N}_\omega$ be any cardinal number (which may be singular) such that $c_0 < \mathfrak{N}_\omega < \mathfrak{N}_\tau$. Now we consider the function $f'(g_1, \dots, g_m, a_1, \dots, a_n)$ defined on $\mathfrak{N}_\omega$ and having values in it defined by $\phi^{(\mathfrak{N}_\omega)}(g_1, \dots, g_m, a_1, \dots, a_n, a)$. Then we have

$$\begin{aligned} \forall x_1 \dots \forall x_s (x_1 < \mathfrak{N}_\omega \wedge \dots \wedge x_s < \mathfrak{N}_\omega \rightarrow f'(g_1, \dots, g_m, x_1, \dots, x_s) \\ = f(g_1, \dots, g_m, x_1, \dots, x_s)). \end{aligned}$$

Proof. The sets and the functions in the proof of Theorem 1 are used here without changing their meaning.

Now let $\max(c_0, a_1, \dots, a_s) < \mathfrak{N}_\omega$. We define the set E' by

$$E' = [D \cup \{f_0(a) : a < \mathfrak{N}_\omega\}]_{\mathfrak{S}} \cap \mathfrak{N}_\tau.$$

Then we have

- 1) E is $\aleph$ -closed.
- 2) The order-type of E is exactly the same as $\aleph_\omega$.

And moreover $E \cap f_0(\aleph_\tau) = E'$.

Let $\phi: \aleph_\omega \rightarrow E'$ be an order-preserving onto mapping. Then $\phi(a) = \phi(b) \in E'$ implies $a = b$. By Theorem B' we have

$$\begin{aligned} \phi(f'(g_1, \dots, g_m, a_1, \dots, a_s)) &= g(g_1, \dots, g_m, \phi(a_1), \dots, \phi(a_s)) \\ &= f(g_1, \dots, g_m, \varphi(a_1), \dots, \varphi(a_s)) = \varphi(f(g_1, \dots, g_m, a_1, \dots, a_s)). \end{aligned}$$

And so $\phi(f'(g_1, \dots, g_m, a_1, \dots, a_s)) = \varphi(f(g_1, \dots, g_m, a_1, \dots, a_s)) \in E'$.

This means $f'(g_1, \dots, g_m, a_1, \dots, a_s) = f(g_1, \dots, g_m, a_1, \dots, a_s)$.

Thus we complete the proof of Theorem 2.

Theorem 3 (Uniformity of cardinals under Δ_1^1 -formulas). Let $g_1, \dots, g_m$ be functions defined on $\aleph_\tau$ and having values in it such that

- 1) $g_i(a_1, \dots, a_n) \leq c_0$ for all $a_1, \dots, a_n < \aleph_\tau$, $i = 1, \dots, m$.
- 2) $g_i(a_1, \dots, a_n) = 0$ if $\max(a_1, \dots, a_n) \geq c_0$.

And let $\aleph_{c_1}, \dots, \aleph_{c_n}$ and $\aleph_{a_1}, \dots, \aleph_{a_n}$ be cardinals such that

$$\aleph_0, \aleph_{c_0} < \aleph_{c_1} < \dots < \aleph_{c_n} < \aleph_\tau \text{ and } \aleph_0, \aleph_{c_0} < \aleph_{a_1} < \dots < \aleph_{a_n} < \aleph_\tau.$$

Then we have

$$\begin{aligned} \forall x (x < \aleph_\tau \rightarrow (A(g_1, \dots, g_m, x, \aleph_{c_1}, \dots, \aleph_{c_n}) \\ \leftrightarrow A(g_1, \dots, g_m, x, \aleph_{a_1}, \dots, \aleph_{a_n}))) , \end{aligned}$$

where A is Δ_1^1 in $g_1, \dots, g_m$.

Proof. The sets and the functions in the proof of Theorem 1 are used here without changing their meaning.

Let $\aleph_e = \max(\aleph_{c_n}, \aleph_{a_n}) < \aleph_\tau$. The sets D_1 and D_2 are defined by

$$\begin{aligned} D_1 = \{f_0(a) : a < \aleph_{c_1} \vee \aleph_e \cdot 2 \leq a < \aleph_e \cdot 2 + \aleph_{c_2} \vee \dots \vee \aleph_e \cdot 2(n-1) \\ \leq a < \aleph_e \cdot 2(n-1) + \aleph_{c_n} \vee \aleph_e \cdot 2n \leq a < \aleph_\tau\} . \end{aligned}$$

$$\begin{aligned} D_2 = \{f_0(a) : \aleph_e \leq a < \aleph_e + \aleph_{a_1} \vee a = \aleph_e \cdot 2 \vee \aleph_e \cdot 3 \leq a < \aleph_e \cdot 3 + \aleph_{a_2} \\ a = \aleph_e \cdot 4 \vee \dots \vee \aleph_e \cdot (2n-1) \leq a < \aleph_e \cdot (2n-1) + \aleph_{a_n} \\ \vee \aleph_e \cdot 2n \leq a < \aleph_\tau\} , \end{aligned}$$

where the sum and the product means that of ordertypes. Clearly we have

$$D_1 \cap D_2 = \{f_0(a) : a = \aleph_e \cdot 2 \vee \dots \vee a = \aleph_e \cdot 2(n-1) \vee \aleph_e \cdot 2n \leq a < \aleph_\tau\} .$$

Let ϕ and φ be two order-preserving onto mappings such that

$$\psi: \aleph_\tau \rightarrow [D \cup D_1]_{\aleph} \cap \aleph_\tau,$$

$$\varphi: \aleph_\tau \rightarrow [D \cup D_2]_{\aleph} \cap \aleph_\tau.$$

By the properties of f_0 , D_1 and D_2 , we obtain

$$\psi(\aleph_{c_i}) = \varphi(\aleph_{a_i}) = \aleph_e \cdot 2i \text{ for every } i=1, \dots, n.$$

Now we consider a $\mathcal{A}_1$ -function defined by

$$\begin{aligned} f(g_1, \dots, g_m, a, a_1, \dots, a_n) &= b \\ \leftrightarrow ((A(g_1, \dots, g_m, a, a_1, \dots, a_n) \wedge b=0) \vee (A(g_1, \dots, g_m, a, a_1, \dots, a_n) \wedge b=1)). \end{aligned}$$

By Theorem B we have

$$\begin{aligned} \psi(f(g_1, \dots, g_m, a_0, \aleph_{c_1}, \dots, \aleph_{c_n})) &= f(g_1, \dots, g_m, \psi(a_0), \psi(\aleph_{c_1}), \dots, \psi(\aleph_{c_n})) \\ &= f(g_1, \dots, g_m, \varphi(a_0), \varphi(\aleph_{a_1}), \dots, \varphi(\aleph_{a_n})) = \varphi(f(g_1, \dots, g_m, a_0, \aleph_{a_1}, \dots, \aleph_{a_n})). \end{aligned}$$

$$\text{But } f(g_1, \dots, g_m, a_0, \aleph_{c_1}, \dots, \aleph_{c_n}), f(g_1, \dots, g_m, a_0, \aleph_{a_1}, \dots, \aleph_{a_n}) < 2.$$

$$\text{We have } f(g_1, \dots, g_m, a_0, \aleph_{c_1}, \dots, \aleph_{c_n}) = f(g_1, \dots, g_m, a_0, \aleph_{a_1}, \dots, \aleph_{a_n}).$$

$$\text{Namely } A(g_1, \dots, g_m, a_0, \aleph_{c_1}, \dots, \aleph_{c_n}) \leftrightarrow A(g_1, \dots, g_m, a_0, \aleph_{a_1}, \dots, \aleph_{a_n}).$$

Thus we complete the proof of the theorem.

Theorem 4 (Σ_1^1 -transcendancy of cardinals). *Let $P(a, a_1, \dots, a_n)$ be a Σ_1^1 -formula considering $\aleph_\tau$ as universe. Then we have for any $\aleph_c(\aleph_0 < \aleph_c < \aleph_\tau)$.*

$$\begin{aligned} \forall x_1 \dots \forall x_n (x_1 < \aleph_c \wedge \dots \wedge x_n < \aleph_c \rightarrow (\exists x P(x, x_1, \dots, x_n) \\ \rightarrow \exists x (x < \aleph_c \wedge P(x, x_1, \dots, x_n))))). \end{aligned}$$

Proof. Let $a_1, \dots, a_n$ be ordinals less than $\aleph_\omega$. And assume that $\exists x \exists \alpha A(\alpha, x, a_1, \dots, a_n)$. Now let a_0 be an ordinal number such that $\exists \alpha A(\alpha, a_0, a_1, \dots, a_n)$. Let $\mathfrak{A}$ be a set of functions defined on $\aleph_\tau$ and having values in $\aleph_\tau$ such that

- 1) $\overline{\mathfrak{A}} \leq \aleph_0$.
 - 2) $\mathfrak{A}$ is arithmetically closed.
 - 3) $\mathfrak{A}$ is a basis for the formulas of the form $A(\alpha, a_1, \dots, a_n)$ where $A(\alpha, a_1, \dots, a_n)$ is arithmetical in α .
- And $\mathfrak{A}^* = \{g^*: g \in \mathfrak{A}\}$. And let B be the set defined by

$$B = \{a: a \leq \max(a_1, \dots, a_n) \vee a = a_0\}. \text{ Put } B^* = [B]_{\mathfrak{A}}.$$

Then we have $B^* \subset \aleph_\tau$ and $\overline{B^*} \leq \max(\aleph_0, a_1, \dots, a_n)$. By Theorem C there is a set of functions $\mathfrak{B} \supset \mathfrak{A}^*$ such that $\overline{\mathfrak{B}} \leq \aleph_0$ and a set of ordinal numbers $D \supset B^*$ such that $\overline{D} < \max(\aleph_0, a_1, \dots, a_n)$. And there is a function f_0 which is generative with respect to $\mathfrak{B}$ and D defined on $\aleph_\tau$ such

that $f_0(0) \geq \sup \{[D]_{\aleph} \cap \aleph_\tau\}$. By the property on $\aleph$, there is a function g in $\aleph$ such that

$$\begin{aligned} & \exists \alpha A(\alpha, a_0, a_1, \dots, a_n) \\ & \leftrightarrow A(\lambda x_1 \dots x_s g(a_0, a_1, \dots, a_n, x_1, \dots, x_s), a_0, a_1, \dots, a_n). \end{aligned}$$

Let E be the set $[D \cup \{f_0(a) : a < \aleph_\tau\}]_{\aleph} \cap \aleph_\tau$. Let $\varphi : \aleph_\tau \rightarrow E$ be the order-preserving onto mapping. As before we have if $a_0 = \varphi(b)$ then

$$\bar{b} \leq \max(\aleph_0, a_1, \dots, a_n) < \aleph_c. \text{ Namely } b < \aleph_c.$$

Since E is closed under the functions in $\aleph$ we can define the function g by the following schema:

$$\begin{aligned} & \tilde{g}(c_0, c_1, \dots, c_n, d_1, \dots, d_s) \\ & = \varphi^{-1}(g(\varphi(c_0), \varphi(c_1), \dots, \varphi(c_n), \varphi(d_1), \dots, \varphi(d_s))). \end{aligned}$$

Clearly $\tilde{g}$ is defined for all ordinal numbers less than $\aleph_\tau$. Moreover $\tilde{g}$ and g are φ -commutative. Hence by Theorem B, we have

$$\begin{aligned} & A(\lambda x_1 \dots x_s g(a_0, a_1, \dots, a_n, x_1, \dots, x_s), a_0, a_1, \dots, a_n) \\ & \leftrightarrow A(\lambda x_1 \dots x_s g(\varphi(b), \varphi(a_1), \dots, \varphi(a_n), x_1, \dots, x_s), \varphi(b), \varphi(a_1), \dots, \varphi(a_n)) \text{ where } \varphi(b) = a_0 \\ & \leftrightarrow A(\lambda x_1 \dots x_s \tilde{g}(b, a_1, \dots, a_n, x_1, \dots, x_s), b, a_1, \dots, a_n) \\ & \rightarrow \exists \alpha A(\alpha, b, a_1, \dots, a_n). \end{aligned}$$

Therefore by $b < \aleph_c$ we have

$$\exists x \exists \alpha A(\alpha, x, a_1, \dots, a_n) \rightarrow \exists x (x < \aleph_c \wedge \exists \alpha A(\alpha, x, a_1, \dots, a_n)).$$

Thus we complete the proof of Theorem 4.

Let $P(a)$ be a Σ_1^1 -formula in $(\aleph_\tau)$. Then by Theorem 4, we have

$$\exists x P(x) \rightarrow \exists x (x < \aleph_1 \wedge P(x)).$$

Since every ordinal definable in L (cf. Gödel [3]) is arithmetically definable, we see that every ordinal definable in L is countable ordinal.

The concept of transcendency is now stated as follows. Namely a predicate $P(a, a_1, \dots, a_n)$ is $Q(a)$ transcendental if $\exists x Q(x)$ and

$$\begin{aligned} & \forall x (Q(x) \rightarrow \forall y_1 \dots \forall y_n (y_1 < x \wedge \dots \wedge y_n < x \rightarrow (\exists z P(z, y_1, \dots, y_n) \\ & \rightarrow \exists z (z < x \wedge P(z, y_1, \dots, y_n))))). \end{aligned}$$

Using this terminology we have that there is a Π_1^1 -formula, for example $\aleph_{\text{ord}}(a) \wedge a > \omega$, which is transcendental for any Σ_1^1 -formula. So from

this point of view we can say that some Π_1^1 -formula can express the class of greater ordinals than any Σ_1^1 -formula.

Let A and B be two classes of predicates, and assume that there is a predicate $Q(a)$ in B such that for any predicate $P(a, a_1, \dots, a_n)$ of A , $P(a, a_1, \dots, a_n)$ is $Q(a)$ transcendental. In this case we write as $A < B$.

Clearly we have, assuming the existence of normally measurable cardinals, $\Sigma_1^1 < \Pi_1^1$. It is interesting whether $\Pi_2^1 < \Sigma_2^1$, $\Sigma_2^1 < \Pi_2^1$ or they are incomparable.

The following theorem seems to be one of the most typical properties of normally measurable cardinals.

Theorem 5. *Let $\aleph_\sigma$ and $\aleph_\tau$ be two normally measurable cardinals such that $\aleph_\tau < \aleph_\sigma$. Let $P(a)$ be a Σ_1^1 -formula considering $\aleph_\sigma$ as universe. Then $P(\aleph_\tau)$ implies $P(\aleph_c)$ for all $\aleph_c$ such that $\aleph_0 < \aleph_c < \aleph_\tau$.*

Proof Let $\mathfrak{A}$ be a set of functions defined on $\aleph_\sigma$ and having values in $\aleph_\sigma$ such that

- 1) $\overline{\mathfrak{A}} \leq \aleph_0$.
- 2) $\mathfrak{A}$ is arithmetically closed.
- 3) $\mathfrak{A}$ is a basis for the formulas of the form $A(\alpha, a_1, \dots, a_n)$ where $A(\alpha, a_1, \dots, a_n)$ is arithmetical in α .

Let g be a function defined on $\aleph_\sigma$ and having values in $\aleph_\sigma$. Then by g^* we denote the function defined by

$$g^*(a_1, \dots, a_n) = \begin{cases} g(a_1, \dots, a_n) & \text{if } a_1 < \aleph_\sigma \wedge \dots \wedge a_n < \aleph_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

By $\mathfrak{A}^*$ we denote the set of functions defined by $\mathfrak{A}^* = \{g^* : g \in \mathfrak{A}\}$. Clearly $\overline{\mathfrak{A}^*} \leq \aleph_0$.

By Theorem C, there is a set of functions $\mathfrak{B}$ such that $\mathfrak{B} \supset \mathfrak{A}^*$, $\overline{\mathfrak{B}} \leq \aleph_0$ and a set of ordinal numbers $D \supset \{\aleph_\tau, \aleph_\sigma\}$ such that $\overline{D} \leq \aleph_0$. And there is a function f_0 which is generative with respect to $\mathfrak{B}$ and D such that

- 1) f_0 is defined on $\aleph_\sigma$ and having values in $\aleph_\sigma$.
- 2) f_0 is strictly increasing.
- 3) $\aleph_\tau$ is a node for the function f_0 .

Let $d_0 = \sup \{D \cap \aleph_\tau\} < \aleph_\tau$. Then there is an ordinal number $d_1 < \aleph_\tau$ such that $d_0 < f_0(d_1)$.

Now we consider the set defined by

$$E = [\{\aleph_\tau, \aleph_\sigma\} \cup \{f_0(a) : d_1 \leq a < d_1 + \aleph_c \vee \aleph_\tau \leq a < \aleph_\sigma\}]_{\mathfrak{B}} \cap \aleph_\sigma.$$

By this definition we have that E is $\mathfrak{A}$ closed. And there is an order-preserving onto mapping φ such that $\varphi : \aleph_\sigma \rightarrow E$ and $\varphi(\aleph_c) = \aleph_\tau$.

Since E is $\mathfrak{A}$ closed, we have by Theorem B' that for any Σ_1^1 -formula $P(a)$ in $(\aleph_\sigma)$, $P(\varphi(a))$ implies $P(a)$. Namely by $\varphi(\aleph_c) = \aleph_\tau$ it follows

that $P(\aleph_r)$ implies $P(\aleph_e)$. Thus we complete the proof of the theorem.

Note that Theorem 5 means that for any Π_1^1 -formula $P(a)$, $P(\aleph_e)$ implies $P(\aleph_r)$ for any $\aleph_e$ such that $\aleph_0 < \aleph_e < \aleph_r$. Namely a normally measurable cardinal number has all Π_1^1 -properties any of which is a property of smaller cardinals (except $\aleph_0$).

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