Note on the effective choice of a point in the complement of an analytic set

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In their paper [1], N. Lusin and P. Novikoff gave a method of the effective choice of a point in the complement of an analytic set. In the present paper, we shall note that, modifying the process of N. Lusin and P. Novikoff by imposing various requirements, we can obtain those processes which enable us to choose a point, in the complement of an analytic set, satisfying some supplementary conditions.

1. **A generalization of Lusin-Novikoff's theorem.** Let E be the complement of an analytic set in the space of all irrational numbers x , $0 < x < 1$. Assume that the set E is defined by means of a sieve

$$(1) \quad C = \sum_{n=1}^{\infty} \delta_n \times \{r_n\},$$

where

$$(2) \quad \delta_1, \delta_2, \dots, \delta_n, \dots$$

are Baire's intervals and r_n , $n=1, 2, \dots$, are rational numbers. Assume moreover that the sieve C satisfies the condition:

1° If any two of Baire's intervals $\delta_{n_1}, \delta_{n_2}, \dots, \delta_{n_k}, \dots$ are not disjoint, then the set $\prod_{i=1}^{\infty} \delta_{n_i}$ consists of exactly one point.

This condition is obviously satisfied, if no Baire's interval appears infinitely many times in (2).

Providing for the necessity in sections 2 and 3, we assume moreover:

2° $P_x C$ is countable for every x , where P_x is the perpendicular to the x -axis at x .

Let C_n denote the sieve, consisting of all those points $x \in C$ which lie strictly below the set $\delta_n \times \{r_n\}$. We shall denote order types of $P_x C$ and $P_x C_n$ by $\nu(x)$ and $\nu_n(x)$ respectively.

We have

LEMMA. Let $E^{(k-1)}$, n_i and γ_i , $i=1, 2, \dots$, be respectively a sequence of subsets of the interval $0 < x < 1$, that of natural numbers and that of ordinal numbers. Assume

- 1) $E^{(k-1)} \neq \emptyset$,
- 2) $E \supseteq E^{(k-1)} \supseteq E^{(k)}$,

- 3) n_i is the smallest n such that $n > n_{i-1}$ and $\delta_n E^{(i-1)} \neq 0$,
 4) if $x \in E^{(i)}$ then $\nu_{n_i}(x) = \gamma_i$,
 5) $E^{(i)} \subseteq \delta_{n_i}$,

for every $i=1, 2, \dots$.

Then, the intervals $\delta_{n_1}, \delta_{n_2}, \dots, \delta_{n_k}, \dots$ have exactly one point $\xi = \prod_{i=1}^{\infty} \delta_{n_i}$ in common. Among the intervals in (2), $\delta_{n_1}, \delta_{n_2}, \dots, \delta_{n_k}, \dots$ are only intervals containing the point ξ . Finally, ξ is contained in E .

PROOF. It follows from 2) that $E^{(i)} \supseteq E^{(j)}$ when $i < j$. Then, 5) and 1) imply that $\delta_{n_i} \delta_{n_j} \neq 0$. Hence, on account of 1°, it follows that the intervals $\delta_{n_k}, k=1, 2, \dots$, have exactly one common point $\xi = \prod_{i=1}^{\infty} \delta_{n_i}$.

Suppose that $n_{i-1} < n \leq n_i$ and $\xi \in \delta_n$. On account of the fact that $\xi = \prod_{i=1}^{\infty} \delta_{n_i}$, we have a k such that $k \geq i$ and $\delta_n \supseteq \delta_{n_k}$. Then, from 5), 3) and 2), it follows that $\delta_n E^{(i-1)} \neq 0$. Hence $n = n_i$.

Suppose that $i < j$. Take a point $x \in E^{(j)}$. Then, it follows from 2) that $x \in E^{(i)}$. Hence $x \in \delta_{n_i}$ and $x \in \delta_{n_j}$. Therefore,

$$r_{n_i} > r_{n_j} \text{ if and only if } \nu_{n_i}(x) > \nu_{n_j}(x),$$

or, on account of 4),

$$r_{n_i} > r_{n_j} \text{ if and only if } \gamma_i > \gamma_j.$$

This shows that the set $P_\xi C$ is well-ordered. Hence $\xi \in E$.

2. Effective choice of a point in a set of class A_p . Let A be an analytic set given as a nucleus of a regular system $\{F_{m_1 m_2 \dots m_k}\}$ of Souslin, of closed sets. Put

$$A^{m_1 m_2 \dots m_k} = \sum_{(n_1, n_2, \dots, n_l, \dots)} \prod_{i=1}^{\infty} F_{m_1 m_2 \dots m_k n_1 n_2 \dots n_l}$$

Let E be the complement of an analytic set, defined by means of a sieve (1) with the properties 1° and 2°.

Assuming that $AE \neq 0$, we shall choose a point ξ in AE .

The choice-process consists of the following steps.

FIRST-STEP. Let E_γ be the first constituent of E , defined by C , such that $AE_\gamma \neq 0$. Put $E^{(0)} = AE_\gamma$.

We define $n_1, H_1, \gamma_1, K_1, m_1$ and $E^{(1)}$ in the following way:

$$n_1 \text{ is the smallest } n \text{ such that } \delta_n E^{(0)} \neq 0,$$

$$H_1 = \delta_{n_1} E^{(0)},$$

γ_1 is the minimum of all $\nu_{n_1}(x)$, $x \in H_1$,
 K_1 is the set of all $x \in H_1$ such that $\nu_{n_1}(x) = \gamma_1$,
 m_1 is the smallest m such that $A^m K_1 \neq 0$,
 $E^{(1)} = A^{m_1} K_1$.

Obviously, $E^{(1)} \subseteq A^{m_1}$.

k-TH STEP. Suppose that we have already constructed those $n_i, H_i, \gamma_i, K_i, m_i$ and $E^{(i)}$ ($i=1, 2, \dots, k-1$) which, besides 1)–5), satisfy the condition:

$$6) \quad E^{(i)} \subseteq A^{m_1 m_2 \dots m_i},$$

for every $i=1, 2, \dots, k-1$.

We define $n_k, H_k, \gamma_k, K_k, m_k$ and $E^{(k)}$ in the following way:

n_k is the smallest n such that $\delta_n E^{(k-1)} \neq 0$ and $n > n_{k-1}$,
 $H_k = \delta_{n_k} E^{(k-1)}$,
 γ_k is the minimum of all $\nu_{n_k}(x)$, $x \in H_k$,
 K_k is the set of all $x \in H_k$ such that $\nu_{n_k}(x) = \gamma_k$,
 m_k is the smallest m such that $A^{m_1 m_2 \dots m_{k-1} m} K_k \neq 0$,
 $E^{(k)} = A^{m_1 m_2 \dots m_{k-1} m_k} K_k$.

Then, the conditions 1)–6) are obviously satisfied even when $i=k$.

Because of 2°, we can progress these steps indefinitely. Thus, we have infinite sequences $n_i, H_i, \gamma_i, K_i, m_i$ and $E^{(i)}$, which satisfy conditions 1)–6) for every $i=1, 2, \dots$. Hence, by Lemma, we have

$$\xi = \prod_{i=1}^{\infty} \delta_{n_i} \in E.$$

On the other hand, 1), 5) and 6) imply that

$$\delta_{n_i} A^{m_1 m_2 \dots m_i} \neq 0.$$

Taking into account that diameters of sets δ_{n_i} and $A^{m_1 m_2 \dots m_i}$ tend to zero as $i \rightarrow \infty$, we have

$$\xi = \prod_{i=1}^{\infty} A^{m_1 m_2 \dots m_i} \in A.$$

Moreover, taking a point $x \in E^{(i)} \subseteq E^{(0)} \subseteq E_\gamma$, we have $\gamma_i = \nu_{n_i}(x) < \nu(x) = \gamma$. Hence $\xi \in A E_\gamma = E^{(0)}$.

3. A choice of a point in a constituent of the complement of an analytic set. Let E be the complement of an analytic set, given by means of a sieve (1) with the properties 1° and 2°. Let

$$E = E_\omega + E_{\omega+1} + \dots + E_\alpha + \dots$$

be the development of E into a transfinite sequence of the constituents E_α , defined by C .

Under the assumption that an effective enumeration

$$\alpha_1, \alpha_2, \dots, \alpha_n, \dots$$

of the non-zero ordinal numbers less than α_0 is given, we shall choose a point ξ in E_{α_0} . Of course, it is assumed that $E_{\alpha_0} \neq 0$.

The process of choice is as follows:

FIRST-STEP. Put $E^{(0)} = E_{\alpha_0}$. We define n_1, H_1, γ_1 and K_1 in the same way as in 2. Moreover, we define

m_1 as the smallest m such that $\nu_m(x) = \alpha_1$ for some $x \in K_1$,

$E^{(1)}$ as the set of all $x \in K_1$ such that $\nu_{m_1}(x) = \alpha_1$.

Besides 1)–5), the conditions

7) $E^{(i)} \subseteq \delta_{m_i}$,

8) if $x \in E^{(i)}$, then $\nu_{m_i}(x) = \alpha_i$,

are satisfied for $i=1$.

k -TH STEP. Suppose that we have already constructed those $n_i, H_i, \gamma_i, K_i, m_i$ and $E^{(i)}$ ($i=1, 2, \dots, k-1$) which satisfy the conditions 1)–5) and 7), 8) for $i=1, 2, \dots, k-1$.

The definition of n_k, H_k, γ_k and K_k are the same as in 2. We define

m_k as the smallest m such that $\nu_m(x) = \alpha_k$ for some $x \in K_k$,

$E^{(k)}$ as the set of all $x \in K_k$ such that $\nu_{m_k}(x) = \alpha_k$.

The conditions 1)–5) and 7), 8) are satisfied even when $i=k$.

Progressing these steps indefinitely, we have infinite sequences $n_i, H_i, \gamma_i, K_i, m_i$ and $E^{(i)}$ ($i=1, 2, \dots$), satisfying the conditions 1)–5) and 7), 8) for every $i=1, 2, \dots$.

It follows from Lemma that:

$$\xi = \prod_{i=1}^{\infty} \delta_{n_i} \in E.$$

Taking $x \in E^{(i)} \subseteq E^{(0)}$, we have

$$\gamma_i = \nu_{n_i}(x) < \nu(x) = \alpha_0.$$

On the other hand, we have, by 5) and 7), that

$$\xi = \prod_{i=1}^{\infty} \delta_{n_i} \in \delta_{m_j}, \quad j=1, 2, \dots$$

Hence, by Lemma, each of $\delta_{m_1}, \delta_{m_2}, \dots, \delta_{m_j}, \dots$ coincides with some δ_{n_i} . Suppose that $\delta_{m_j} = \delta_{n_i}$. Then, taking a point x in some $E^{(k)}$, $k \geq n_i$, $k \geq m_j$, we have

$$\alpha_j = \nu_{m_j}(x) = \nu_{n_i}(x) = \gamma_i.$$

Therefore, every α_j coincides with some γ_i .

Thus, we conclude that the sequence $\gamma_1, \gamma_2, \dots, \gamma_i, \dots$ consists of all ordinal numbers less than α_0 . Hence $\xi \in E^{(0)}$.

Summing up, we have:

Given an effective enumeration of the ordinal numbers less than α_0 , we can choose effectively a point in the α_0 -th constituent (if it is non-void) of the complement of an analytic set.

The converse is also true. For an effectively chosen point $\xi \in E_{\alpha_0}$, the sequence

$$0, \overline{\xi^{[n_1]}}, \overline{\xi^{[n_2]}}, \dots, \overline{\xi^{[n_i]}}, \dots$$

gives an effective enumeration of the ordinal numbers less than α_0 . Where n_i is first n such that $n > n_{i-1}$ and $\overline{\xi^{[n]}} \neq 0$. The notation $\overline{\xi^{[n]}}$ is borrowed from C. Kuratowski [2]¹⁾.

References

- [1] N. Lusin and P. Novikoff, Choix effectif d'un point dans un complémentaire analytique arbitraire, donné par un crible, *Fund. Math.*, Vol. 25 (1935)
- 2] C. Kuratowski, Topologie, Vol. 1, 3rd ed., Warszawa, 1952.

1) See, p. 11 and p. 271.