Huber's theorem on analytical mappings of a ring domain in a ring domain.

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Let \( \mathfrak{M} \) be a family of \( f(z) \), which satisfy the following condition. \( w=f(z) \) is one-valued and regular in a ring domain \( \Delta: r<|z|<1 \) \((r>0)\) and the range of \( w \) is contained in a ring domain \( \Delta_1: r_1<|w|<1 \) \((r_1>0)\), such that

\[
r_1<|f(z)|<1 \quad \text{in} \quad \Delta.
\]

Then

\[
U(f)=\frac{1}{2\pi} \int_{|z|=r} d \text{arg} \ f(z) \quad (r<\rho<1)
\]

is an integer \( n \), independent of \( \rho \), where we integrate in the positive sense on \(|z|=\rho\).

Let \( \mathfrak{M}_n \) be the subset of \( \mathfrak{M} \), which consists of \( f(z) \), such that \( U(f)=n \). If \( f_1 \in \mathfrak{M}_n , f_2 \in \mathfrak{M}_n \), then for any \( \lambda \) \((0<\lambda<1)\),

\[
f(z; \lambda)=f_1(z)^{1-\lambda} f_2(z)^{\lambda} \in \mathfrak{M}_n
\]

and \( f(z; 0)=f_1(z), f(z; 1)=f_2(z) \), so that \( f_1 \sim f_2 \) in \( \mathfrak{M}_n \). Hence if \( U(f)=0 \), then \( f \sim \text{const.}=a \) \((r_1<a<1)\). Huber proved the following theorem.

**Theorem.** If \( f \in \mathfrak{M} \) and \( U(f)\neq 0 \), then

\[
|U(f)| \leq \frac{\log 1/r_1}{\log 1/r}.
\]

If the equality holds, then \( \frac{\log 1/r_1}{\log 1/r} = n \) is a positive integer.

If \( U(f)=n \), then \( w=e^{\alpha z^2} \) \((\alpha: \text{real})\), so that \(|z|=r, |z|=1 \) correspond to \(|w|=r_1 , |w|=1 \) respectively.

If \( U(f)=-n \), then \( w=r e^{\alpha z^2} \), so that \(|z|=r, |z|=1 \) correspond to \(|w|=1 , |w|=r_1 \) respectively.

In each case, \( w=f(z) \) maps \( \Delta \) conformally on an \( n \)-sheeted unramified covering Riemann surface of \( \Delta_1 \).

We shall give a simple proof as follows.

**Proof.** We may assume that \( U(f)>0 \), since if \( U(f)<0 \), then we consider \( f_1(z)=r_1 f(z) \) instead of \( f(z) \).

First suppose that \( \log f(z) \) is schlicht in \( \Delta \), then by considering on the log \( w \)-plane, we have

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\[ \int_{r_0}^{r_1} \left| \frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right|^2 \rho d\rho d\theta \leq 2\pi U(f) \log 1/r, \quad (1) \]

\[ 0 < 2\pi U(f) \leq \int_{|z|=\rho} |d \arg f(z)| \leq \int_{|z|=\rho} |d \log f(z)| = \int_0^{2\pi} \left| \frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right| \rho d\theta. \quad (2) \]

Hence by Schwarz's inequality,

\[ 4\pi^2 U(f)^2 \leq 2\pi \rho \int_0^{2\pi} \left| \frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right|^2 \rho d\theta, \]

or

\[ 2\pi U(f)^2 \frac{1}{\rho} \int_0^{2\pi} \left| \frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right| \rho d\theta, \]

hence by (1),

\[ 2\pi U(f)^2 \log 1/r \leq \int_{r_0}^{r_1} \int_{C_{\rho}} \left| \frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right|^2 \rho d\rho d\theta \leq 2\pi U(f) \log 1/r_1, \quad (3) \]

\[ 0 < U(f) \leq \frac{\log 1/r}{\log 1/r}. \]

Next suppose that \( \log f(z) \) is not schlicht in \( \Delta \). Let a circle \( C_{\rho} : |z|=\rho \ (r<\rho<1) \) be mapped on a curve \( \Gamma_{\rho} \) on the log \( w=u+iv \)-plane, then \( \Gamma_{\rho} \) has double points in general. If \( \Gamma_{\rho} \) has a double point, then there exist two points \( z_1, z_2 \) on \( C_{\rho} \), such that \( \log f(z_1)=\log f(z_2) \) and the arc \( z_1z_2 \) of \( C_{\rho} \) is mapped on a closed curve on the log \( w \)-plane. We take off the arc \( z_1z_2 \) from \( C_{\rho} \), then \( C_{\rho}-z_1z_2 \) is mapped on a continuous curve \( \Gamma'_{\rho} \) on the log \( w=u+iv \)-plane, such that \( \int_{\Gamma'_{\rho}} dv=2\pi U(f) \). If \( \Gamma'_{\rho} \) has a double point, then we make a similar modification and after a finite number of steps, we get a finite number of arcs \( z^{(i)}_{1}z^{(i)}_{2}(i=1, 2, \cdots, n(\rho)) \) on \( C_{\rho} \), such that \( C'_{\rho}=C_{\rho}-\sum z^{(i)}_{1}z^{(i)}_{2} \) is mapped on a Jordan arc \( \Gamma'_{\rho} \) on the log \( w=u+iv \)-plane, which has no double points and \( \int_{\Gamma'_{\rho}} dv=2\pi U(f) \).

Let \( \Delta' \) be the set of \( z \), swept by \( C_{\rho} \), when \( \rho \) varies in \((r, 1)\), then \( \Delta' \) consists of connected domains. \( \log f(z) \) is schlicht in \( \Delta' \), for if \( \log f(z) \) is not schlicht in \( \Delta' \), then for a suitable \( \rho \), \( C_{\rho} \) is mapped on a curve on the log \( w \)-plane, which has a double point, which is impossible from the definition of \( C_{\rho} \).

Hence \( \log f(z) \) is schlicht in \( \Delta' \), so that similarly as (1), (2),

\[ \int_{z \in \Delta'} \left| \frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right|^2 \rho d\rho d\theta = \int_{C_{\rho}} \left| \frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right|^2 \rho d\theta \leq 2\pi U(f) \log 1/r_1, \quad (4) \]

and by (2)

\[ 0 < 2\pi U(f) \leq \int_{C_{\rho}} \left| \frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})} \right| \rho d\theta. \quad (5) \]
By means of (4), (5), we can prove (3) similarly as above, hence (3) holds in general.

Next we study the case, where the equality holds in (3).

Then \( U(f) = \log \frac{1}{r_1} = n \) is a positive integer. Since the equality holds in (4), the range of \( w \) coincides with \( \Delta \), and since \( U(f) > 0 \), \( |z| = r \), \( |z| = 1 \) corresponds to \( |w| = r_n \), \( |w| = 1 \) respectively.

Let \( \delta \) be a rectangle, contained in the image of \( \Delta' \) on the log \( z = \xi + i\eta \)-plane, then since the equality holds in (5), a segment \( \in \delta \) on a line \( \xi = \text{const.} \) is mapped on a segment on a line \( u = \text{const.} \) on the log \( w = u + iv \)-plane, hence being an orthogonal trajectory of \( \xi = \text{const.}, \) a segment \( \in \delta \) on a line \( \eta = \text{const.}, \) is mapped on a segment on a line \( v = \text{const.}, \) so that \( \log w \) is a linear function of \( \log z \), \( \log w = a \log z + b \), \( w = e^b z^a \). Since \( U(f) = n \) and \( |z| = 1 \) corresponds to \( |w| = 1 \), we have

\[
w = f(z) = e^{ia} z^n \quad (\alpha: \text{real}) .
\]

(6)

If \( U(f) = -n \), then \( U(r_1, f(z)) = n \), so that \( r_1, f(z) = e^{ia} z^n \), \( f(z) = r e^{-ia} z^{-n} \).

Hence the theorem is proved.

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