On the cluster set of a meromorphic function.

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1. Let $\mathcal{D}$ be a domain on the $z$-plane and $\Gamma$ be its boundary and $z_0$ be a non-isolated boundary point. We denote the part of $\mathcal{D}$, contained in $|z-z_0|<r$ by $\mathcal{D}_r$, and that of $\Gamma$ in $|z-z_0|\leq r$ by $\Gamma_r$.

Let $w=f(z)$ be one-valued and meromorphic in $\mathcal{D}$ and $\overline{W}_r$ be the set of values taken by $w=f(z)$ in $\mathcal{D}_r$, and $\overline{W}_r$ be its closure, then

$$\lim_{r \to 0} \overline{W}_r = H_\Delta(z_0)$$

(1)

is called the cluster set of $f(z)$ in $\mathcal{D}$ at $z_0$.

In this paper, “capacity” means “logarithmic capacity” and $\gamma(E)$ denotes the capacity of $E$. Let $e$ be a set of capacity zero on $\Gamma$, such that $z_0 \in e$ and let

$$V_e(\Gamma-e) = \sum_{\zeta \in \Gamma_r-e_r} H_\Delta(\zeta), \text{ added for all } \zeta \in \Gamma_r-e_r$$

(2)

and $\overline{V}_e(\Gamma-e)$ be its closure, then

$$\lim_{r \to 0} \overline{V}_e(\Gamma-e) = H_{\Gamma-e}(z_0)$$

(3)

is called the cluster set of $f(z)$ on $\Gamma-e$ at $z_0$. $H_\Delta(z_0)$ and $H_{\Gamma-e}(z_0)$ are closed sets and $H_{\Gamma-e}(z_0) \subset H_\Delta(z_0)$.

**Theorem 1**$^3$. Every boundary point of $H_\Delta(z_0)$ belongs to $H_{\Gamma-e}(z_0)$.

If $e$ consists of only one point $z_0$, the theorem is proved by Iversen$^3$.

Hence if $H_\Delta(z_0)-H_{\Gamma-e}(z_0)$ is not an empty set, it consists of at most a countable number of connected domains (components).

**Theorem 2.** Let $D$ be one of components of $H_\Delta(z_0)-H_{\Gamma-e}(z_0)$, then in any small neighbourhood of $z_0$, $f(z)$ takes any value of $D$ infinitely often, except a set of capacity zero$^3$. If $e$ consists of only one point $z_0$, then $f(z)$ takes any value of $D$ infinitely often, with two possible exceptions$^3$.

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3) M. Tsuji: I.c. 1 (i) and (ii).
In this paper, we shall prove a more general theorem as follows.

**Theorem 3.** Let $D$ be one of components of $H_{\Delta}(z_0)-H_{\Delta}(z_0)$ and $K_\rho: |w-w_0|<\rho$ and $K_{\rho_0}: |w-w_0|<\rho_0$ ($0<\rho<\rho_0$) be discs, contained in $D$. We choose $r_0>0$, so small that $V_{r_0} (I'-e)$ lies outside of $|w-w_0|=2\rho_0$ and the circle $|z-z_0|=r_0$ does not contain points of $e$ and let $F$ be the Riemann surface, generated by $w=f(z)$, $|z-z_0|\leq r_0$ on the $w$-plane. Let $F_\rho \subseteq F_{\rho_0}$ be non-compact connected pieces of $F$, which lie above $K_\rho$ and $K_{\rho_0}$ respectively. Then $F_{\rho_0}$ is regularly exhaustible in Ahlors' sense$^5$ and covers any point of $K_{\rho_0}$ infinitely often, except a set of capacity zero. If $e$ consists of only one point $z_0$, then $F_{\rho_0}$ covers any point of $K_{\rho_0}$ infinitely often, with two possible exceptions.

For the proof, we use the following lemma.

**Lemma.** Let $K_0: |w-a|<\delta$ be a disc, contained in $D$, and let $F_0$ be a connected piece of $F_{\rho_0}$, which lies above $K_0$. Then $F_0$ covers any point of $K_0$ at least once, except a set of capacity zero.

**Proof.** Let $\Delta_0$ be the image of $F_0$ on the $z$-plane, then by the choice of $r_0$, the common part $e_0$ of the boundary of $\Delta_0$ with $I'$ is a sub-set of $e$, so that it is of capacity zero. Hence if we map the universal covering surface of $\Delta_0$ conformally on $|\zeta|<1$ by $z=\varphi(\zeta)$, then $e_0$ is mapped on a null set on $|\zeta|=1$, so that if we put $w=f(\varphi(\zeta))=F(\zeta)$, then $(F(\zeta)-a)/\delta$ belongs to $U$-class in Seidel's sense, so that $F_0$ covers any point of $K_0$ at least once, except a set of capacity zero$^6$.

2. **Proof of Theorem 3.**

Let $D_i$ ($i=1, 2, \cdots, N$) be the same as the lemma. First we shall prove that any disc, contained in $D_i$ is not covered exactly a finite number of times by $F_{\rho_0}$. Let $K_i$ be a disc in $D_i$ and suppose that $K_i$ is covered exactly $n_i$-times ($1\leq n_i<\infty$) by $F_{\rho_0}$. Let $\Sigma_i$ be a connected domain, which contains $K_i$ and is contained in $D_i$ and every point of which is covered $n_i$-times by $F_{\rho_0}$. If $\Sigma_i$ does not coincide with $D_i$, let $A_i$ be the part of the boundary of $\Sigma_i$, which lies in $D_i$, then by the lemma, we can prove as in my former paper$^7$, that $A_i$ is of capacity zero. Let $z_i(w)$ ($i=1, 2, \cdots, n_i$) be $n_i$ branches of the inverse function $z=f(w)$ in $D_i-A_i$ and consider

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5) i.e., there exists an exhaustion of $F_{\rho_0}$ by compact Riemann surfaces $F^{(1)}(w) \subseteq F^{(2)}(w) \subseteq \cdots \subseteq F^{(n)}(w)$, such that the relative boundary $\Gamma^{(n)}$ of $F^{(n)}$ consists of a finite number of analytic curves, and $|\Gamma^{(n)}|/|F^{(n)}| \to 0$, where $|\Gamma^{(n)}|$ is the length of $\Gamma^{(n)}$ and $|F^{(n)}|$ is the area of $F^{(n)}$.

6) M. Tsuji: loc. 1) (ii).

7) M. Tsuji: loc. 1) (ii).
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\[ \prod_{i=1}^{n_1} (z - z_i(w)) = z^{n_1} + a_1(w)z^{n_1 - 1} + \cdots + a_{n_1}(w) = 0, \]

then \( a_i(w) \) is one-valued, regular and bounded in a neighbourhood of \( A_i \). Since \( \gamma(A_i) = 0 \), \( a_i(w) \) is regular on \( A_i \), so that every point of \( A_i \) is covered \( n_1 \)-times by \( F_{\rho_0} \), which is absurd. Hence \( \Sigma_i \) coincides with \( D_1 \), so that every point of \( D_1 \) is covered \( n_1 \)-times by \( F_{\rho_0} \). Let \( D_2 \) abuts on \( D_1 \) along an image curve \( \lambda \) of \( |z - z_0| = r_0 \). We choose \( r_0' (> r_0) \) so near to \( r_0 \), that \( |z - z_0| = r_0' \) does not contain points of \( e \) and \( \overline{D}_{r_0'}(G - e) \) lies outside of \( |w - w_0| = 2\rho_0 \) and let \( F' \) be the Riemann surface, generated by \( w = f(z) \), \( |z - z_0| \leq r_0' \) and \( (F_{\rho_0} \subset)F'_{\rho_0} \) be a connected piece of \( F' \), which lies above \( K_{r_0} \), then the boundary of \( F'_{\rho_0} \), which lies above \( \lambda \) is contained entirely in \( F''_{\rho_0} \), so that we see that there exists a disc \( K_5 \) in \( D_2 \), which lies in a small neighbourhood of \( \lambda \) and is covered exactly \( n_2 \)-times by \( F'_{\rho_0} \), where \( n_2 = n_1 + 1 \), or \( n_1 - 1 \). Hence every point of \( D_2 \) is covered \( n_2 \)-times by \( F'_{\rho_0} \). Similarly we conclude that every point of \( K_{r_0} \) is covered at most \( n_1 + N \)-times by \( F'_{\rho_0} \), so that \( F'_{\rho_0} \) is a compact piece, which contradicts the hypothesis. Hence any disc in \( D_2 \) is not covered exactly a finite number of times by \( F'_{\rho_0} \).

Next we shall prove that \( F'_{\rho_0} \) covers any point of \( K_{r_0} \) infinitely often, except a set of capacity zero. First we shall prove that \( F'_{\rho_0} \) covers any point of \( D_2 \) infinitely often, except a set of capacity zero. For, if otherwise, then since by the lemma, \( F'_{\rho_0} \) covers any point of \( D_2 \) at least once, except a set of capacity zero, there exists a closed set \( E \) of positive capacity in \( D_2 \), which is covered \( n \)-times \((1 \leq n < \infty) \) by \( F'_{\rho_0} \). There exists a point \( a \in E \), such that \( \gamma(E, K) > 0 \) for any small disc \( K : |w - a| < \delta \), contained in \( D_2 \). Since \( a \in E \), \( a \) is covered \( n \)-times by \( F'_{\rho_0} \), so that there are \( n \) discs \( F^{(1)}, \ldots, F^{(n)} \) above \( K \), consisting of inner points of \( F'_{\rho_0} \). Since any disc in \( D_2 \) is not covered exactly a finite number of times by \( F'_{\rho_0} \), there is another connected piece \( F^{(n)}(\supset K) \) above \( K \), other than \( F^{(1)}, \ldots, F^{(n)} \). Since \( E.K \) is covered \( n \)-times in \( F^{(1)}, \ldots, F^{(n)}, F^{(\supset K)} \) does not cover \( E.K \), which contradicts the lemma, since \( \gamma(E,K) > 0 \). Hence \( F'_{\rho_0} \) covers any point of \( \sum_{\rho_1} \mathbb{D}_\rho \) infinitely often, except a set of capacity zero. By considering \( F' \) as before, we see that \( F'_{\rho_0} \) covers any point of \( K_{r_0} \) infinitely often, except a set of capacity zero. Next we shall prove that \( F'_{\rho_0} \) is regulary exhaustible in Ahlfors' sense. Let \( A_0 \) be the image of \( F'_{\rho_0} \) on the \( z \)-plane, then since \( F'_{\rho_0} \) is non-compact, \( A_0 \) has boundary points on \( \Gamma \). Let \( e_0 \) be the set of such boundary points, then by the choice of \( r_0, e_0 \) is a closed sub-set of \( e \), so that it is of capacity zero. Hence by Evans' theorem\(^8\),

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there exists a positive mass distribution $d\mu(a)$ on $e_0$, such that
\[
u(z) = \int \log \frac{1}{|z-a|} d\mu(a), \quad \int d\mu(a) = 1, \tag{1}\]
tends to $+\infty$, when $z$ tends to any point of $e_0$. Let $\theta(z)$ be the conjugate harmonic function of $\nu(z)$ and put
\[
\zeta = e^{\nu + i\theta} = \tau e^{i\theta}, \quad \tau = \tau(z), \quad \theta = \theta(z), \quad f(z) = F(\zeta). \tag{2}\]

Let $C_\tau$ be the niveau curve $\tau(z) = \text{const} = \tau$ and $C_\tau^0$ be its part, contained in $A_0$. Then since
\[
\int_{C_\tau} d\tau = \int_{C_\tau} \frac{\partial u}{\partial \nu} ds = 2\pi
\]
and $d\theta \geq 0$, we have
\[
\int_{C_\tau} d\theta \leq 2\pi. \tag{3}\]

Let $A_\tau(\tau)$ be the part of $A_\tau$, which lies between $C_{\tau_0}$ and $C_{\tau}$ ($\tau_0 < \tau$) and put
\[
A(\tau) = \int_{A_\tau(\tau)} |F'(\zeta)|^2 \tau d\tau d\theta, \quad L(\tau) = \int_{e^\theta} |F''(\zeta)| \tau d\theta. \tag{4}\]

Then by Schwarz's inequality and (3), we have
\[
L(\tau) \leq 2\pi \int \frac{dA(\tau)}{d\tau}. \tag{5}\]
Since $F'_{\tau_0}$ covers any point of $K_{\tau_0}$ infinitely often, except a set of capacity zero, $\lim_{\tau \to \infty} A(\tau) = \infty$. Hence by (5), we can prove by the usual way that there exists $\tau_1 < \tau_2 < \cdots < \tau_n \to \infty$, such that $L(\tau_i)/A(\tau_i) \to 0$, hence $F'_{\tau_0}$ is regularly exhaustible in Ahlfors' sense.

Next suppose that $e$ consists of only one point $z_0$. Then, since $F'_{\tau_0}$ is a covering surface of $K_{\tau_0}$ and is regularly exhaustible in Ahlfors' sense, we can prove easily by Ahlfors' fundamental theorem on covering surfaces, that $F'_{\tau_0}$ covers any point of $K_{\tau_0}$ infinitely often, with two possible exceptions.

3. Proof of Theorem 2.

By means of Theorem 3, we can prove Theorem 2 simply as follows. Let $E$ be a set of positive capacity in $D$ and $f(z)$ takes any value in $E$ finite times in a neighbourhood $U$ of $z_0$. We may assume that $E$ is a closed set and $f(z)$ takes any value in $E$ at most $n$-times ($n < \infty$) in $U$. Then by taking $U$ small, we may assume that $f(z)$ does not take any value in $E$ in $U$. 
Then there exists \( w_0 \in E \), such that \( \gamma(E.K_p) > 0 \) for any small disc \( K_p: |w - w_0| < \rho \), which is contained in \( D \) and let \( F_\rho \subset F_{\rho_0} \) be connected pieces of \( F \), which lie above \( K_p \) and \( K_{\rho_0} (0 < \rho < \rho_0) \) respectively. Since \( F_\rho \) does not cover \( E.K_p \), \( F_\rho \) is non-compact, so that by Theorem 3, \( F_{\rho_0} \) covers any point of \( K_{\rho_0} \) infinitely often, except a set of capacity zero, but this is absurd, since \( F_{\rho_0} \) does not cover \( E.K_p \) and \( \gamma(E.K_p) > 0 \). Hence \( f(z) \) takes any value of \( D \) infinitely often, except a set of capacity zero.

Next suppose that \( e \) consists of only one point \( z_0 \), and suppose that \( f(z) \) takes \( a, b, c \) only a finite number of times in \( U \), where \( a, b, c \) belong to \( D \). Then by taking \( U \) small, we may assume that \( f(z) = a, b, c \) in \( U \). We enclose \( a, b, c \) in a Jordan curve \( C \), which lies in \( D \) with its inside \( K \). Let \( C_0 \) be a Jordan curve, which contains \( C \) and is contained in \( D \) with its inside \( K_0 \). Let \( F(K) \subset F(K_0) \) be connected pieces of \( F \), which lie above \( K \) and \( K_0 \) respectively, then \( F(K) \) is non-compact, so that by Theorem 3, \( F(K_0) \) covers any point of \( K_0 \) infinitely often, with two possible exceptions, which is absurd, since \( F(K_0) \) does not cover \( a, b, c \). Hence \( f(z) \) takes any value of \( D \) infinitely often, with two possible exceptions.

4. From the proof of Theorem 3, we have

**Theorem 4.** Let \( E \) be a bounded closed set of capacity zero on the \( z \)-plane and \( w = f(z) \) be one-valued and meromorphic in a neighbourhood \( U \) of \( E \), every point of which is an essential singularity of \( f(z) \). Let \( F \) be the Riemann surface, generated by \( w = f(z) \) on the \( w \)-sphere. Let \( K_p: |w - w_0| < \rho \) and \( K_{\rho_0}: |w - w_0| < \rho_0 (0 < \rho < \rho_0) \) be discs and \( F_\rho \subset F_{\rho_0} \) be non-compact connected pieces of \( F \), which lie above \( K_p \) and \( K_{\rho_0} \) respectively.

Then \( F_{\rho_0} \) is regularly exhaustible in Ahlfors' sense and covers any point of \( K_{\rho_0} \) infinitely often, except a set of capacity zero.

**Theorem 5.** Let \( w = f(z) \) be one-valued and meromorphic for \( r_0 \leq |z| < \infty \) and \( z = \infty \) be an essential singularity of \( f(z) \) and \( F \) be the Riemann surface, generated by \( w = f(z) \) on the \( w \)-sphere. With the same notation as theorem 4, let \( F_\rho \subset F_{\rho_0} \) be non-compact pieces of \( F \), which lies above \( K_p \) and \( K_{\rho_0} \) respectively. Then \( F_{\rho_0} \) is regularly exhaustible in Ahlfors' sense and (i) \( F_{\rho_0} \) covers any point of \( K_{\rho_0} \) infinitely often, with two possible exceptions. (ii) If \( F_{\rho_0} \) is simply connected, then \( F_{\rho_0} \) covers any point of \( K_{\rho_0} \) infinitely often, with one possible exception.

Under the hypothesis, that \( F_{\rho_0} \) contains an accessible transcendental boundary point, (i) is proved by Kunugui\(^9\) and (ii) by Noshiro\(^10\).

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