COMMENTARII MATHEMATICI UNIVERSITATIS SANCTI PAULI
Vol. 65, No. 2 2016

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

Compositions of Automorphic Differential Operators on Siegel Modular Forms

by

Tomoyoshi IBUKIYAMA

(Received August 2, 2016)

Abstract. First, we characterize differential operators on Siegel modular forms of degree nm such that the restrictions of the images of the operation to the $n \times n$ diagonal blocks which consist of the same matrices are again Siegel modular forms of degree n of some different weight. The characterization is given in terms of pluri-harmonic polynomials. Then we show that when $n = 1$, all such differential operators are obtained by composing two kinds of operators, one which preserves automorphy for the restriction of H_m to the diagonals (the product of the upper half planes), and one which preserves automorphy for the restriction from the product of m pieces of upper half planes to the upper half plane embedded diagonally.

1. Introduction

If we apply any holomorphic differential operators on holomorphic Siegel modular forms, in most cases the images are not modular at all. But we can give a theory of good differential operators such that the restriction of the image to some smaller domain is again modular for several fixed pairs of the domains. We would like to call such operators *automorphic differential operators*. For example, if we denote by H_N the Siegel upper half space of degree N, well studied pairs of the domains are

(i) The restriction from H_n to $H_{n_1} \times \cdots \times H_{n_1}$ with $n = n_1 + \cdots + n_r$, where the latter is embedded to diagonal blocks of H_n.

(ii) The restriction from $H_n^m = H_n \times \cdots \times H_n$ to H_n, where the latter is embedded diagonally to H_n^m.

Automorphic differential operators for (ii) are called Rankin-Cohen operators. Automorphic differential operators for (i) are important in various stages of number theory, including the pullback formula of Eisenstein series and calculation of special values of the standard L function. General characterization for these two cases has been given in [3] and there are several related deeper results such as [1], [5], [2], [4], [6], [8].

In this paper, we consider the following pair of domains.

This work was supported by JSPS KAKENHI Grant Number JP25247001.
2010 Mathematics Subject Classification. Primary 11F46; Secondly 11F11, 11F60.

(iii) The restriction from H_{nm} to H_n, where H_n is embedded in H_{nm} by

$$H_n \ni \tau \to \begin{pmatrix} \tau & 0 & \cdots & 0 \\ 0 & \tau & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \tau \end{pmatrix} \in H_{nm}\,.$$

The characterization of such operators will be given in Theorem 2.3. The proof is more or less similar to those in [3]. The second problem of this paper is to answer the question posed by D. Zagier. He asked if all the differential operators for (iii) for $n = 1$ is obtained from compositions of operators for (i) for $n = r = m$, $n_1 = \cdots = n_m = 1$ and for (ii) with $n = 1$. We give an affirmative answer to this in Theorem 4.1. The same question for the case $n > 1$ involves the case starting from vector valued forms in (ii) and seems more complicated. We would like to thank Don Zagier for asking the author an interesting question.

2. Characterization of the case (iii).

In this section, we solve the first problem, that is, a characterization of automorphic differential operators for the case (iii). For any positive integer N, we denote by H_N the Siegel upper half space of degree N, and by $Sp(N,\mathbb{R}) \subset GL_{2N}(\mathbb{R})$ the symplectic group of matrix size $2N$. For any irreducible representations (ρ, V) of $GL_N(\mathbb{C})$, any V-valued functions f on H_N, and any elements $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(N,\mathbb{R})$, we write

$$(f|_\rho[g])(Z) = \rho(c\tau + d)^{-1} f(gZ)\,. \tag{1}$$

When $\rho = det^k$, we write $f|_\rho = f|_k$. Let m, n be positive integers. For any $n \times n$ matrix A and $m \times m$ matrix $B = (b_{ij})$, we denote by $A \otimes B$ the Kronecker product defined by

$$A \otimes B = \begin{pmatrix} Ab_{11} & \cdots & Ab_{1m} \\ \vdots & \cdots & \vdots \\ Ab_{m1} & \cdots & Ab_{mm} \end{pmatrix}.$$

We consider an embedding

$$\iota : H_n \ni \tau \to \tau \otimes 1_m = \begin{pmatrix} \tau & 0 & \cdots & 0 \\ 0 & \tau & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \tau \end{pmatrix} \in H_{nm}\,.$$

We embed $Sp(n,\mathbb{R})$ into $Sp(nm,\mathbb{R})$ by

$$\iota : Sp(n,\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \otimes 1_m & b \otimes 1_m \\ c \otimes 1_m & d \otimes 1_m \end{pmatrix} \in Sp(nm,\mathbb{R})\,.$$

We sometimes identify $Sp(n,\mathbb{R})$ with the image of this embedding. Then the action of $Sp(n,\mathbb{R})$ on H_n and the action on $H_n \otimes 1_m \subset H_{nm}$ are equivariant.

We fix a positive integer k and an irreducible polynomial representation (ρ, V) of $GL_n(\mathbb{C})$. Let $\mathbb{D}$ be a V-valued linear holomorphic differential operator on holomorphic functions on H_{nm} with constant coefficients and consider the following condition.

CONDITION 2.1. *For any holomorphic functions F on H_{nm}, any elements $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathbb{R}) \subset Sp(nm, \mathbb{R})$, and $\tau \in H_n$, we have*

$$(\mathbb{D}(F|_k[\iota(g)]))(\tau \otimes 1_m) = ((\mathbb{D}F)(\tau \otimes 1_m))|_{det^{mk}\otimes\rho}[g]\,. \tag{2}$$

We would like to characterize such differential operators. Let T be an $nm \times nm$ symmetric matrix of variable components. Let V be any vector space over $\mathbb{C}$. We consider a V-valued polynomial $P(T)$ in the components of T. For such P, we define $\mathbb{D}_P$ by

$$\mathbb{D}_P = P\left(\frac{\partial}{\partial Z}\right),$$

where for $Z = (z_{ij}) \in H_{nm}$, we put

$$\frac{\partial}{\partial Z} = \left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}}\right)_{1\le i,j\le mn}.$$

This $\mathbb{D}_P$ is a V-valued differential operator on holomorphic functions on H_{nm}. If we consider a V-valued linear partial differential operator $\mathbb{D}$ on functions on H_{nm} with constant coefficients, then of course there exists some V valued polynomial $P(T)$ such that $\mathbb{D}_P = \mathbb{D}$. So a characterization of $\mathbb{D}$ which satisfies Condition 2.1 is given by characterizing P.

We prepare several definitions. For any positive integers N and d, we consider an $N \times d$ matrix Y of variable components and a polynomial $\widetilde{P}(Y)$ in the components of Y. We define mixed Laplacians $\Delta_{ij}(Y)$ by

$$\Delta_{ij}(Y) = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial y_{i\nu}\partial y_{j\nu}} \quad (1 \le i, j \le N)\,,$$

and we say that $\widetilde{P}$ is pluri-harmonic if $\widetilde{P}$ satisfies

$$\Delta_{ij}(Y)\widetilde{P}(Y) = 0 \quad \text{for all } 1 \le i \le j \le N\,.$$

This is equivalent to say that $\widetilde{P}(AY)$ is harmonic with respect to Nd variables of components of Y for any $A \in GL_N(\mathbb{R})$. We denote by $\mathcal{H}_{N,d}$ the space of pluri-harmonic polynomials $\widetilde{P}(Y)$ where Y is an $N \times d$ matrix. Assume $N = mn$. For integers i with $1 \le i \le m$, let Y_i be a $n \times d$ matrix, and put

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{pmatrix}.$$

Denote by $O(d)$ the orthogonal group. Assume that $d \ge N$ and $\widetilde{P}(Yh) = \widetilde{P}(Y)$ for all $h \in O(d)$. Then by the fundamental theorem of classical invariant theory ([9]), there exists a polynomial $P(T)$ such that $\widetilde{P}(Y) = P(Y\,{}^tY)$. We can rewrite the mixed Laplacians

$\Delta_{ij}(Y)$ in terms of t_{ij} (see [5]). For each i, j with $1 \le i, j \le N$, we write $\partial_{ij} = (1+\delta_{ij})\frac{\partial}{\partial t_{ij}}$ and we put

$$D_{ij}(d) = d\partial_{ij} + \sum_{k,l=1}^{N} t_{kl}\partial_{ik}\partial_{jl}\,.$$

Then we have

$$(D_{ij}(d)P)(Y\,{}^tY) = (\Delta_{ij}(Y)\widetilde{P})(Y)\,.$$

We also define $n \times dm$ matrix Y_0 by $Y_0 = (Y_1, Y_2, \dots, Y_m)$. For a V-valued polynomial $P(T)$ for $nm \times nm$ symmetric matrix T, we put $P^*(Y_0) = P(Y\,{}^tY)$. (We use here a different notation P^* instead of $\widetilde{P}$ to emphasize that the argument is an $n \times md$ matrix. This is only by a psychological reason.) We consider the following two conditions.

CONDITION 2.2. (1) *Every component of $P^*(Y_0)$ is a pluri-harmonic polynomial with respect to Y_0, that is, an element of $\mathcal{H}_{n,md}$.*

(2) *For any $A \in GL_n(\mathbb{C})$, we have $P^*(AY_0) = \rho(A)P^*(Y)$.*

Here the pluri-harmonicity of $P^*(Y_0) = P(Y\,{}^tY)$ is written for P by

$$\sum_{l=0}^{m-1} D_{i+nl,j+nl}(d)P(T) = 0 \quad \text{for all } i, j \text{ with } 1 \le i,\ \ j \le n\,,$$

where $D_{ij}(d)$ is defined for $N = mn$, i.e. for $nm \times nm$ symmetric matrix T.

THEOREM 2.3. *We put $d = 2k$ and we assume that $d \ge nm$. A V-valued differential operator $\mathbb{D}_P$ on functions on H_{nm} satisfies the condition* 2.1 *if and only if $P^*(Y_0) = P(Y\,{}^tY)$ satisfies the condition* 2.2.

REMARK. The representation ρ cannot be taken arbitrary if we demand existence of $P \ne 0$. For example, when $n = 1$ where the representation ρ is a representation of GL_1 and $\rho(x) = x^\kappa$, this κ should be obviously an even integer if $P \ne 0$, since $P(c^2T) = c^\kappa P(T)$.

Proof. First we prove if-part for a special function. We put $d = 2k$. We define a function $F_0(Z)$ of $Z \in H_{nm}$ by $F_0(Z) = \exp\left(\frac{i}{2}Tr({}^tYZY)\right) = \exp\left(\frac{i}{2}Tr(Y\,{}^tYZ)\right)$ where $Y = (y_{i\nu})$ is an $nm \times d$ matrix of variable components. For any integer N, we write $J_N = \begin{pmatrix} 0 & -1_N \\ 1_N & 0 \end{pmatrix} \in Sp(N,\mathbb{R})$. Then $\iota(J_n) = J_{nm}$. We prove that the condition 2.1 is satisfied for F_0 and J_n for P^* satisfying condition 2.2. It is well known that we have

$$F_0|_k[J_{nm}] = (2\pi i)^{-mnk}\int_{M_{nm,d}(\mathbb{R})} \exp(iTr({}^tXY))\exp\left(\frac{i}{2}Tr({}^tXZX)\right)dX.$$

(See [3] Lemma 1 for $P = 1$.) For any $c \in \mathbb{C}$, we have $P^*(cY_0) = \rho(c1_n)P^*(Y_0) = c^{l'}P^*(Y_0)$ for some positive integer l' by Schur's lemma. So $P(c^2T) = c^{l'}P(T)$ and l' is even, so we write $l' = 2l$. Then differentiating under the integral by $\mathbb{D}_P$, we have

$$\begin{aligned}&(2\pi i)^{nmk}\mathbb{D}_P(F_0|_k[J_{nm}])\\ &\quad = \int_{M_{nm,d}(\mathbb{R})} \exp(iTr({}^tXY))\mathbb{D}_P\left(\exp\left(\frac{i}{2}Tr({}^tXZX)\right)\right)dX\,.\end{aligned}$$

$$= (i/2)^{2l} \int_{M_{n,d}(\mathbb{R})} \exp(iTr({}^tXY)) \exp\left(\frac{i}{2}Tr({}^tXZX)\right) P(X\,{}^tX)dX\,.$$

We write $X = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}$ by $n \times d$ matrices X_i and put $X_0 = (X_1, \ldots, X_m)$, then the polynomial $P(X\,{}^tX) = P^*(X_0)$ is pluri-harmonic with respect to X_0 by assumption. We see also $Tr(X\,{}^tX) = \sum_{i=1}^m Tr(X_i\,{}^tX_i) = Tr(X_0\,{}^tX_0)$ and $Tr(X\,{}^tY) = \sum_{i=1}^m Tr(X_i\,{}^tY_i) = Tr(X_0\,{}^tY_0)$. In the last expression of the above integral, we restrict Z to $\tau \otimes 1_m$ where $\tau \in H_n$. Then the integral part becomes

$$\int_{M_{n,md}(\mathbb{R})} e(iTr({}^tX_0Y_0)) \exp\left(\frac{i}{2}Tr({}^tX_0\tau X_0)\right) P^*(X_0)dX_0\,.$$

Again applying [3] Lemma 1 for the pluri-harmonic polynomial P^*, we see that this integral is equal to

$$\begin{aligned}
&(2\pi)^{mnd/2}\det(\tau/i)^{-dm/2}\exp\left(\frac{-i}{2}Tr({}^tY_0(\tau^{-1}\otimes 1_m)Y_0)\right)P^*(-\tau^{-1}Y_0)\\
&= (2\pi i)^{mnd/2}\det(\tau)^{-km}\rho(\tau)^{-1}P(Y\,{}^tY)F_0(-\tau^{-1}\otimes 1_m)\\
&= (2\pi i)^{mnk}(i/2)^{-2l}\det(\tau)^{-km}\rho(\tau)^{-1}(\mathbb{D}F_0)(-\tau^{-1}\otimes 1_m)\,.
\end{aligned}$$

So as a whole we have

$$(\mathbb{D}(F_0|_k[J_{nm}]))(\tau\otimes 1_m) = ((\mathbb{D}F_0)(\tau\otimes 1_m))|_{det^{km}\otimes\rho}|[J_n]\,.$$

So the condition 2.1 is satisfied for F_0 and J_n. We see that the same holds for any holomorphic functions $F(Z)$ on H_{nm}. We denote by $\boldsymbol{\nu} = (\nu_{ij})$ a multi-index with $\nu_{ij} \in \mathbb{Z}_{\geq 0}$ with $1 \leq i \leq j \leq nm$. We put

$$D^{\boldsymbol{\nu}} = \prod_{1\leq i\leq j\leq n}\left(\frac{\partial}{\partial z_{ij}}\right)^{\nu_{ij}}\,.$$

Since $\mathbb{D}_P$ is a differential operator with constant coefficients, it is clear by the chain rule that there are V valued holomorphic functions $Q_{\boldsymbol{\nu}}(Z)$ on H_{nm} such that

$$\mathbb{D}_P(\det(Z)^{-k}F(-Z^{-1})) = \sum_{\boldsymbol{\nu}} Q_{\boldsymbol{\nu}}(Z)(D^{\boldsymbol{\nu}}F)(-Z^{-1})\,,$$

where $\boldsymbol{\nu}$ runs over a finite number of indices. So the restriction is

$$\sum_{\boldsymbol{\nu}} Q_{\boldsymbol{\nu}}(\tau\otimes 1_m)(D^{\boldsymbol{\nu}}F)(-\tau^{-1}\otimes 1_m)\,.$$

On the other hand, we also have V-valued functions $R_{\boldsymbol{\nu}}(\tau)$ such that

$$(\mathbb{D}_PF)(\tau\otimes 1_m)|_{det^{km}\rho}[J_n] = \sum_{\boldsymbol{\nu}} R_{\boldsymbol{\nu}}(\tau)(D^{\boldsymbol{\nu}}F)(-\tau^{-1}\otimes 1_m)\,.$$

So Condition 2.1 is satisfied if $Q_{\boldsymbol{\nu}}(\tau \otimes 1_m) = R_{\boldsymbol{\nu}}(\tau)$ for all indices $\boldsymbol{\nu}$. Now if we write i-th row of Y by y_i, then we have $Tr(Y\,{}^tYZ) = \sum_{1\le i,j\le nm}(y_i, y_j)z_{ij}$. So

$$(D^{\boldsymbol{\nu}}F_0)(Z) = \left(\frac{i}{2}\right)^{\sum_{i\le j}\nu_{ij}} \prod_{1\le i\le j\le nm}(y_i, y_j)^{\nu_{ij}}F_0(Z).$$

By the assumption that $d \ge nm$ and the fundamental theorem of invariant theory [9], the polynomials (y_i, y_j) for $1 \le i \le j \le nm$ are algebraically independent. This means that $D^{\boldsymbol{\nu}}F_0(Z)$ are linearly independent for all indices for any Z. Since the condition 2.1 is satisfied for F_0 and J_n, the relation $Q_{\boldsymbol{\nu}}(\tau \otimes 1_m) = R_{\boldsymbol{\nu}}(\tau)$ should be satisfied, and this means the condition is satisfied for general F. Next we see other elements of $Sp(n,\mathbb{R})$. For $u(S) = \begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix} \in Sp(n,\mathbb{R})$, we have $(F|_k[\iota(u(S))])(\tau \otimes 1_m) = F((\tau + S)\otimes 1_m)$ and $F(\tau)|_{det^{km}\rho}[u(S)] = F((\tau+S)\otimes 1_m)$. Since $\mathbb{D}_P(F(Z+S)) = (\mathbb{D}_P F)(Z+S)$, the condition is obvious. For $t(U) = \begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix} \in Sp(n,\mathbb{R})$ where $U \in GL_n(\mathbb{R})$, put $W = (U\otimes 1_m)Z({}^tU \otimes 1_m)$. Then we have

$$\frac{\partial}{\partial Z} = ({}^tU \otimes 1_m)\frac{\partial}{\partial W}(U\otimes 1_m).$$

$$P\left(\frac{\partial}{\partial Z}\right) = \rho({}^tU)P\left(\frac{\partial}{\partial W}\right).$$

Since

$$F|_k[\iota(t(U))] = \det(U)^{km}F((U\otimes 1_m)Z({}^tU\otimes 1_m))\,,$$

we have

$$\mathbb{D}_P(F|_k[\iota(t(U))] = \det(U)^{km}\rho({}^tU^{-1})^{-1}(\mathbb{D}_P F)((U\otimes 1_m)Z({}^tU\otimes 1_m))\,.$$

Restricting to $\tau \otimes 1_m$, we have the condition 2.1. Since $Sp(n,\mathbb{R})$ are generated by these elements J_n, $u(S)$ and $t(U)$, the condition 2.1 is satisfied for any F and any element of $Sp(n,\mathbb{R})$. Now we prove the converse, that is, if $\mathbb{D} = Q\left(\frac{\partial}{\partial Z}\right)$ satisfies the condition 2.1, then we may take the polynomial Q such that Q satisfies the condition 2.2. If we write $Q^*(Y_0) = Q(Y\,{}^tY)$, it is clear from behaviour under $t(U)$ for $U \in GL_n(\mathbb{C})$ on F_0 that $Q^*(UY_0) = \rho(U)Q^*(Y)$. So it is sufficient to prove that $Q^*(Y_0)$ is pluri-harmonic. By previous calculation, it is obvious that $\mathbb{D}_Q$ satisfies the condition only if

$$\int_{M_{n,md}(\mathbb{R})} e(iTr({}^tX_0Y_0))\exp\left(\frac{i}{2}Tr({}^tX_0\tau X_0)\right)Q^*(X_0)dX_0$$
$$= (2\pi)^{mnd/2}\det(\tau/i)^{-dm/2}\exp\left(-\frac{i}{2}Tr({}^tY_0\tau^{-1}Y_0)\right)Q^*(-\tau^{-1}Y_0)$$

Here we put $\tau = \sqrt{-1}\,{}^t\alpha\alpha$ for $\alpha \in GL_n(\mathbb{R})$. In the above relation, we replace X_0 by $\alpha^{-1}X_0$ and Y_0 by ${}^t\alpha Y_0$. Then since $d(\alpha^{-1}X_0) = \det(\alpha)^{-md}dX_0$, we have

$$\int_{M_{n,md}(\mathbb{R})} e(iTr({}^tX_0Y_0)))\exp\left(\frac{-1}{2}Tr({}^tX_0X_0)\right)Q^*(\alpha^{-1}X_0)dX_0\,.$$

$$= (2\pi)^{mnd/2} \exp\left(\frac{-1}{2} Tr({}^tY_0 Y_0)\right) Q^*(i\alpha^{-1}Y_0)$$

So by [3] Lemma 2, we see that $Q^*(\alpha^{-1}Y_0)$ is harmonic for all $\alpha \in GL_n(\mathbb{R})$. This means that $Q^*(Y_0)$ is pluri-harmonic. □

3. Other embeddings

In the next section, we will see that when $n = 1$, the differential operators treated in section 2 are all obtained by composing two kinds of differential operators for (i) and (ii). So to prepare for that, in this section we review the cases (i) and (ii) more precisely, without assuming that $n = 1$. First we explain the case (i). We fix an ordered partition $\mathbf{n} = (n_1, \ldots, n_r)$ of n with $n_1 + \cdots + n_r = n$. We put

$$Sp(\mathbf{n}, \mathbb{R}) = Sp(n_1, \mathbb{R}) \times \cdots \times Sp(n_r, \mathbb{R}) .$$

According to the embedding $H_{n_1} \times \cdots \times H_{n_r} \subset H_n$, we have the natural embedding $Sp(\mathbf{n}, \mathbb{R}) \to Sp(n, \mathbb{R})$. Let k be a fixed positive integer and (ρ_i, V_i) be fixed irreducible polynomial representations of $GL_{n_i}(\mathbb{C})$ for i with $1 \le i \le r$. Put $V = V_1 \otimes \cdots \otimes V_r$. For a V-valued linear holomorphic partial differential operator $\mathbb{D}$ with constant coefficients on holomorphic functions on H_n, we consider the following condition.

CONDITION 3.1. *For any elements $g = (g_1, \ldots, g_r) \in Sp(\mathbf{n}, \mathbb{R}) \subset Sp(n, \mathbb{R})$ and for any holomorphic functions F on H_n, we have*

$$\mathbb{D}(F(Z)|_k[g]) \begin{pmatrix} \tau_1 & 0 & \cdots & 0 \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \tau_r \end{pmatrix}$$
$$= (\mathbb{D}F) \begin{pmatrix} \tau_1 & 0 & \cdots & 0 \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \tau_r \end{pmatrix} \Big|_{det^k\rho_1}^{\tau_1} [g_1] \Big|_{det^k\rho_2}^{\tau_2} [g_2] \cdots \Big|_{det^k\rho_r}^{\tau_r} [g_r] .$$

We put $d = 2k$ and we define $D_{ij}(d)$ for $n \times n$ symmetric matrix T as before. For the partition $\mathbf{n}$ and integer p with $1 \le p \le r$, we define

$$I(p) = \left\{ (i, j) \in \mathbb{Z}^2; 1 + \sum_{q=1}^{p-1} n_q \le i, j \le \sum_{q=1}^{p} n_q \right\} .$$
$$I(\mathbf{n}) = \bigcup_{p=1}^{r} I(p) .$$

So $I(\mathbf{n})$ is the set of the row and column numbers which appear in the diagonal blocks for the partition. We denote by $\mathbb{C}[T]$ the ring of polynomials in the components of T. We put

$$\mathcal{P}_n^{\mathbf{n}}(d) = \{P(T) \in \mathbb{C}[T]; D_{ij}(d)P = 0 \text{ for all } (i, j) \in I(\mathbf{n})\} .$$

We put $GL_{\mathbf{n}}(\mathbb{C}) = GL_{n_1}(\mathbb{C}) \times \cdots \times GL_{n_r}(\mathbb{C})$ and embed this to $GL_n(\mathbb{C})$ by

$$(A_1, \dots, A_r) \to A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_r \end{pmatrix} \in GL_n(\mathbb{C})$$

and sometimes identify $GL_{\mathbf{n}}(\mathbb{C})$ with the image of this embedding.

THEOREM 3.2 ([3]). *Assume that $d \geq n$. Then for any V-valued polynomial P, $\mathbb{D}_P = P\left(\frac{\partial}{\partial Z}\right)$ satisfies the condition* 3.1 *if and only if P satisfies the following two conditions.*

(1) *All the components of $P(T)$ are in $\mathcal{P}_n^{\mathbf{n}}(d)$.*

(2) *For any $A \in GL_{\mathbf{n}}(\mathbb{C}) \subset GL_n(\mathbb{C})$, we have*

$$P(AT\,{}^tA) = \rho(A)P(T)\,.$$

Next we explain the case (ii). We consider the embedding

$$H_n \ni \tau \to (\tau, \dots, \tau) \in H_n \times \cdots \times H_n = H_n^m\,.$$

We embed $Sp(n, \mathbb{R})$ diagonally to $Sp(n, \mathbb{R})^m$ by $g \to (g, \dots, g)$. We fix positive integers $k_1, \dots, k_m$ and an irreducible polynomial representation (ρ, V) of $GL_n(\mathbb{C})$. For a V-valued holomorphic linear partial differential operator $\mathbb{D}$ with constant coefficients on holomorphic functions on H_n^m, we consider the following condition.

CONDITION 3.3. *For any holomorphic function $F(\tau_1, \dots, \tau_m) \in H_n^m$ and any elements $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathbb{R})$, we have*

$$\left(\mathbb{D}\left(F(g\tau_1 \dots, g\tau_m)\prod_{i=1}^{m}\det(c\tau_i + d)^{-k_i}\right)\right)(\tau, \tau, \dots, \tau)$$
$$= \det(c\tau + d)^{-k_1 - \dots - k_m}\rho(c\tau + d)^{-1}(\mathbb{D}F)(g\tau, \dots, g\tau)\,.$$

For a characterization of such $\mathbb{D}$, we prepare some notation. Since $\mathbb{D}$ is a differential operator on H_n^m, if we write $(\tau_1, \dots, \tau_m) \in H_n^m$, then there is a $n \times n$ symmetric matrices T_1, $T_2, \dots, T_m$ and a V-valued polynomial $P(T_1, \dots, T_m)$ such that $\mathbb{D} = P\left(\frac{\partial}{\partial \tau_1}, \dots, \frac{\partial}{\partial \tau_m}\right)$. We put $d_p = 2k_p$ for $1 \leq p \leq m$ and we would like to write down a condition that for $n \times d_p$ matrices Y_p with $1 \leq p \leq m$, a function $P(Y_1\,{}^tY_1, \dots, Y_m\,{}^tY_m)$ is pluri-harmonic w.r.t. $Y_0 = (Y_1, \dots, Y_m)$. Pluri-harmonicity condition is written in terms of T_p as before. But to make notation consistent, we write this as follows. We rewrite T_p by T_{pp} and regard these as diagonal blocks of $nm \times nm$ matrix T, that is, we regard that the specialization of

T to $T_{pq} = 0$ for all $p \neq q$ is given by

$$\begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & T_m \end{pmatrix}.$$

The components of each $T_p (= T_{pp})$ is given by t_{ij} for $(i, j) \in I(p)$, where $I(p)$ is defined as before for nm, m instead of n and r, respectively, and for $n_1 = \cdots = n_m = n$. But in this formulation, T_{pq} with $p \neq q$ does not appear, so the shape of the mixed Laplace operators is slightly different from the one we define before. For $nm \times nm$ matrix T and (i, j) with $1 \leq i, j \leq n$, we put

$$\begin{aligned} D_{ij}^{(p)}(d_p) &= d_p \partial_{i+n(p-1),j+n(p-1)} \\ &\quad + \sum_{k,l=1}^{n} t_{k+n(p-1),l+n(p-1)} \partial_{i+n(p-1),k+n(p-1)} \partial_{j+n(p-1),l+n(p-1)}. \end{aligned}$$

Then the pluri-harmonicity condition in this case is written by

$$\sum_{p=1}^{m} D_{ij}^{(p)}(d_p) P(T_{11}, T_{22}, \ldots, T_{mm}) = 0 \quad \text{for all } (i, j) \text{ with } 1 \leq i, j \leq n.$$

THEOREM 3.4 ([3]). *We assume that $d_p = 2k_p \geq n$ for all p with $1 \leq p \leq m$ and let (ρ, V) be an irreducible polynomial representation of $GL_n(\mathbb{C})$. Then any V-valued linear holomorphic partial differential operator $\mathbb{D}_P = P(\partial\tau_1, \ldots, \partial\tau_m)$ satisfies the condition* 3.3 *if and only if P satisfies the following two conditions.*

(1) $\sum_{p=1}^{m} D_{ij}^{(p)}(d_p) P = 0$ *for all* (i, j) *with* $1 \leq i, j \leq n$.

(2) *For any $A \in GL_n(\mathbb{C})$, we have*

$$P(AT_{11}\,{}^tA, AT_{22}\,{}^tA, \ldots, AT_{mm}\,{}^tA) = \rho(A) P(T_{11}, T_{22}, \ldots, T_{mm}).$$

We note that some more explicit description of differential operators in Theorems 3.2, 3.4 are given for example in [2], [5], [6].

4. Composition of operators

Before stating next theorem, we prepare notation. We say that a polynomial $P(T)$ for $m \times m$ symmetric matrix $T = (t_{ij})$ is of multidegree $\mathbf{a} = (a_1, \ldots, a_m)$ if $P((c_i c_j t_{ij})) = \left(\prod_{i=1}^{m} c_i^{a_i}\right) P(T)$. If $P \neq 0$, we have $\sum_{i=1}^{m} a_i$ is even. For a partition $\mathbf{m} = (1, \ldots, 1)$, we put $\mathcal{P}_m^{(1,\ldots,1)}(d) = \mathcal{P}_m(d)$ and we denote by $\mathcal{P}_{\mathbf{a}}(d)$ the subspace of $\mathcal{P}_m(d)$ of polynomials of multidegree $\mathbf{a}$. We have

$$\mathcal{P}_m(d) = \sum_{\mathbf{a}} \mathcal{P}_{\mathbf{a}}(d).$$

For any integer l, we denote by ρ_l the representation of $GL(1, \mathbb{C})$ given by $\rho_l(x) = x^l$. In this section, we prove the following second main theorem.

THEOREM 4.1. *We fix an integer l with $l \geq 0$ and k with $2k \geq m$. Then any linear holomorphic differential operator $\mathbb{D}$ with constant coefficients which satisfies Condition* 2.1 *for $n = 1$, k above and $\rho = \rho_l$, is a linear combination of compositions of a differential operator which satisfies Condition* 3.1 *for the same k above, both n and r being replaced by m, all $n_i = 1$, and representations $\rho_i(x) = x^{a_i}$ for some non-negative integers a_i $(1 \leq i \leq m)$ with $a_1 + \cdots + a_m \leq l$, and a differential operator which satisfies Condition* 3.3 *for $k_1 = k + a_1, \ldots, k_m = k + a_m$ and $\rho(x) = x^{k_0}$ with $k_0 = l - (a_1 + \cdots + a_m)$.*

If we say this in more down to earth fashion, all the differential operators which map Siegel modular forms F of weight k on H_n to elliptic modular forms of weight $mk + l$ by the restriction to H_1 are obtained by composing differential operators which map F into the space of products of elliptic modular forms of weight $k + a_i$ and differential operators which maps m pieces of elliptic modular forms of weight $k + a_i$ $(1 \leq i \leq m)$ to an elliptic modular form of weight $mk + l = mk + a_1 + \cdots + a_m + k_0$.

Proof. We prove this by using characterization by pluri-harmonic polynomials. We start from a differential operator which satisfies Condition 2.1 for $n = 1$. Then the representation ρ is one dimensional and we have a scalar valued polynomial $P(T)$ for $m \times m$ matrix T which safisfies the condition of Theorem 2.3. For any $c \in \mathbb{C}^\times$, we have $P(c^2T) = c^l P(T)$ by the assumption. (So l is even, but this does not matter.) By assumption, we have

$$\sum_{i=1}^{m} D_{ii}(d)P(T) = 0 \tag{3}$$

where $d = 2k$. Now we would like to decompose $P(T)$ into a linear combination of products of two parts. The polynomial $P(T)$ itself does not vanish under a single $D_{ii}(d)$ in general but we have the following harmonic decomposition of $P(T)$. For $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{Z}_{\geq 0}^m$, we define $\delta(T)^{\mathbf{b}} = \prod_{i=1}^m t_{ii}^{b_i}$. By [5] Corollary to Theorem 1, we have

$$\mathbb{C}[T] = \oplus_{\mathbf{b}} \delta(T)^{\mathbf{b}} \mathcal{P}_m(d)\,,$$

where $\mathcal{P}_m(d) = \mathcal{P}_m^{(1,\ldots,1)}(d)$, that is the space of elements $Q(T)$ such that $D_{ii}(d)Q = 0$ for all i with $1 \leq i \leq m$. So since $P(c^2T) = c^l P(T)$, this is contained in the space

$$\sum_{\mathbf{a},\mathbf{b}} \delta(T)^{\mathbf{b}} \mathcal{P}_{\mathbf{a}}(d)$$

where the sum is taken over multidegrees $\mathbf{a}$, $\mathbf{b}$ such that $2\sum_{i=1}^m b_i + \sum_{i=1}^m a_i = l$. We note that any multidegree $\mathbf{a}$ of non-zero polynomial always satisfy that $\sum_i a_i$ is even, so this is consistent with the fact that l is even. Since we have $\partial_{iq}(\delta(T)^{\mathbf{b}}) = 0$ for any (i, q) with $i \neq q$, we have

$$\begin{aligned} D_{ii}(d)(\delta(T)^{\mathbf{b}}) &= (d\partial_{ii} + t_{ii}\partial_{ii}^2)(\delta(T)^{\mathbf{b}}) \\ &= 2b_i(d + 2b_i - 2)\delta(T)^{\mathbf{b}-\mathbf{e}_i}\,, \end{aligned}$$

where $\mathbf{e}_i$ is a m dimensional vector whose i-th component is one and the other components are zero. If we take any $P_0 \in \mathcal{P}_{\mathbf{a}}(d)$, then we have

$$D_{ii}(d)(\delta(T)^{\mathbf{b}} P_0(T)) = 2b_i(d + 2b_i - 2)\delta(T)^{\mathbf{b}-e_i} P_0(T)$$

$$+\, 2 \sum_{q \neq i} t_{iq} \partial_{ii} (\delta(T)^{\mathbf{b}}) \partial_{iq} P_0(T) + 2 t_{ii} \partial_{ii} (\delta(T)^{\mathbf{b}}) \partial_{ii} P_0(T)\,.$$

By homogeneity of $P_0(T)$, we have

$$\sum_{q=1}^{m} t_{iq} \partial_{iq} P_0(T) = a_i P_0(T)\,.$$

So we have

$$D_{ii}(d)(\delta(T)^{\mathbf{b}} P_0(T)) = 2b_i(d + 2a_i + 2b_i - 2)\delta(T)^{\mathbf{b}-e_i} P_0(T)\,.$$

On the other hand, we have

$$D_{ii}^{(i)}(d + 2a_i) = (d + 2a_i)\partial_i + t_{ii}\partial_{ii}^2\,,$$

so

$$D_{ii}^{(i)}(d + 2a_i)\delta(T)^{\mathbf{b}} = 2b_i(d + 2a_i + 2b_i - 2)\delta(T)^{\mathbf{b}-e_i}\,.$$

Hence

$$\sum_{i=1}^{m} D_{ii}(d)(\delta(T)^{\mathbf{b}} P_0(T)) = \left(\sum_{i=1}^{m} D_{ii}^{(i)}(d + 2k_i)\delta(T)^{\mathbf{b}} \right) P_0(T)\,.$$

For any multidegrees $\mathbf{a} = (a_1, \ldots, a_m)$, fix a basis $\{P_{\mathbf{a},\lambda}(T);\, \lambda \in \Lambda(\mathbf{a})\}$ of $\mathcal{P}_{\mathbf{a}}(d)$. For any polynomial $R(T) \in \mathbb{C}[T]$, we can write

$$R(T) = \sum_{\mathbf{b},\mathbf{a},\lambda \in \Lambda(\mathbf{a})} c_\lambda(\mathbf{b}, \mathbf{a})\delta(T)^{\mathbf{b}} P_{\mathbf{a},\lambda}(T)\,,$$

where $\mathbf{b}$ and $\mathbf{a}$ are suitable multidegrees. If $R(T) = 0$, then obviously $c_\lambda(\mathbf{b}, \mathbf{a}) = 0$ since the harmonic decomposition is a direct sum. For our $P(T)$, write

$$P(T) = \sum_{\mathbf{b},\mathbf{a},\lambda} c_\lambda(\mathbf{b}, \mathbf{a})\delta(T)^{\mathbf{b}} P_{\mathbf{a},\lambda}(T)\,.$$

In this sum, for a fixed $\mathbf{a}$ and λ, we write

$$Q_{\mathbf{a},\lambda}(t_{11}, t_{22}, \ldots, t_{mm}) = \sum_{\mathbf{b}} c_\lambda(\mathbf{b}, \mathbf{a})\delta(T)^{\mathbf{b}}$$

where $\mathbf{b}$ runs over those multidegrees such that $2\sum_{i=1}^{m} b_i = l - \sum_{i=1}^{m} a_i$. Then we have

$$\begin{aligned} 0 &= \sum_{i=1}^{m} D_{ii}(d) \left(\sum_{\mathbf{a},\lambda} Q_{\mathbf{a},\lambda}(t_{11}, t_{22}, \ldots, t_{mm}) P_{\mathbf{a},\lambda}(T) \right) \\ &= \sum_{\mathbf{a},\lambda} \left(\sum_{i=1}^{m} D_{ii}^{(i)}(d + 2a_i) Q_{\mathbf{a},\lambda}(t_{11}, \ldots, t_{mm}) \right) P_{\mathbf{a},\lambda}(T) \end{aligned}$$

From the last expression and by the previous remark, we have

$$\sum_{i=1}^{m} D_{ii}^{(i)}(2(k + a_i)) Q_{\mathbf{a},\lambda}(t_{11}, \ldots, t_{mm}) = 0\,.$$

So $Q_{\mathbf{a},\lambda}$ corresponds to a differential operator which satisfies Condition 3.3 for $n = 1$, weights $k_1 = k + a_1, \ldots, k_m = k + a_m$ and $\rho(x) = x^{\sum_{i=1}^m 2b_i}$. The polynomial $P_{\mathbf{a},\lambda}(T)$ of course corresponds to a differential operator which satisfies Condition 3.1 for the same k, taking n and r to be m, $n_i = 1$ for all i, and $\rho_i(x) = x^{a_i}$ $(1 \le i \le m)$ there. So $P(T)$ is a linear combination of the products of these. □

Finally, we give a following easy remark.

PROPOSITION 4.2. *Rankin-Cohen differential operators for the restriction of H_1^m to H_1 is obtained by iterate use of Rankin-Cohen differential operators for the restriction of H_1^2 to H_1.*

Proof. We prove this by induction on m. We take a polynomial $P(t_1, \ldots, t_m)$ which gives a Rankin-Cohen operator for $H_1^m \to H_1$ from weights $k_1, \ldots, k_r$ to weight $k_1 + \cdots + k_m + 2l$. By condition 3.3, this means that $P(t_1, \ldots, t_m)$ is homogeneous of degree l and satisfies the harmonic condition

$$\sum_{i=1}^m \left(k_i \frac{\partial P}{\partial t_i} + t_i \frac{\partial^2 P}{\partial t_i^2}\right) = 0\,. \tag{4}$$

Now we expand P by a product of powers of t_1 and polynomials in $t_2, \ldots, t_m$ and apply the harmonic decomposition for the latter. That is, we may write

$$P(t_1, t_2, \ldots, t_m) = \sum_{a+b+c=l} c(a,b) t_1^a (t_2 + \cdots + t_m)^b P_c(t_2, \ldots, t_m),$$

where $c(a,b)$ are constants, P_c are harmonic of homogeneous degree c for weight $k_2, \ldots, k_m$ (i.e. $P_c(n(y_2), \ldots, n(y_m))$ are harmonic for $(y_2, \ldots, y_m)$ of degree $2c$ where y_i are vectors of length $2k_i$). Using the condition (4) on P, we have

$$\begin{aligned}
&\sum_{a+b+c=l} t_1^{a-1}(t_2 + \cdots + t_m)^b c(a,b)(k_1 a + a(a-1)) P_c(t_2, \ldots, t_m) \\
&\quad + \sum_{a+b+c=l} c(a,b) t_1^a \sum_{i=2}^m k_i b (t_2 + \cdots + t_m)^{b-1} P_c(t_2, \ldots, t_m) \\
&\quad + \sum_{a+b+c=l} c(a,b) t_1^a \sum_{i=2}^m b(b-1) t_i (t_2 + \cdots + t_m)^{b-2} P_c(t_2, \ldots, t_m) \\
&\quad + 2 \sum_{a+b+c=l} c(a,b) t_1^a b (t_2 + \cdots + t_m)^{b-1} \sum_{i=2}^m t_i \frac{\partial P_c}{\partial t_i}(t_2, \ldots, t_m) = 0\,.
\end{aligned}$$

By homogeneity of P_c, the last term is equal to $2cP_c(t_2, \ldots, t_m)$. So if we put

$$Q_c(t_1, x) = \sum_{a+b=l-c} c(a,b) t_1^a x^b$$

then

$$\left(k_1 \frac{\partial}{\partial t_1} + t_1 \frac{\partial^2}{\partial t_1}\right) Q_c(t_1, x) + \left(\left(2c + \sum_{i=2}^m k_i\right) \frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2}\right) Q_c(t_1, x) = 0\,.$$

So Q_c defines a Rankin-Cohen operator from weight k_1, $2c + \sum_{i=2}^{m} k_i$ to $2l + \sum_{i=1}^{m} k_i$. Since each P_c define a Rankin-Cohen operator from weight $k_2, \ldots, k_m$ to $2c + \sum_{i=2}^{m} k_i$, we are done by induction. $\square$

We note that, in general for $n > 1$, it is *not true at all* that a scalar valued Rankin-Cohen operator for the restriction of H_n^m to H_n is obtained from scalar valued Rankin-Cohen operators for the restriction of H_n^2 to H_n.

References

[1] S. Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen. II. Math. Z. **189** (1985), 81–110.
[2] W. Eholzer and T. Ibukiyama, Rankin-Cohen type differential operators for Siegel modular forms, International J. Math. **9** (1998), 443–463.
[3] T. Ibukiyama, On differential operators on automorphic forms and invariant pluri-harmonic polynomials, Commentarii Math. Univ. St. Pauli **48** (1999), 103–118.
[4] T. Ibukiyama, T. Kuzumaki and H. Ochiai, Holonomic systems of Gegenbauer type polynomials of matrix arguments related with Siegel modular forms, J. Math. Soc. Japan **64** (2012), 273–316.
[5] T. Ibukiyama and D. Zagier, Higher spherical polynomials, MPI preprint No. 2014–41.
[6] T. Ibukiyama, Universal automorphic differential operators on Siegel modular forms and applications, in preparation.
[7] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, Invent. Math. **44** (1978), 1–47.
[8] M. Pevzner, Rankin-Cohen brackets and representations of conformal Lie groups. Ann. Math. Blaise Pascal **19** (2012), 455–484.
[9] H. Weyl, *The Classical Groups. Their Invariants and Representations.* Princeton University Press, Princeton, N.J., (1939). xii+302 pp.

Department of Mathematics,
Graduate School of Science,
Osaka University,
Machikaneyama 1–1, Toyonaka,
Osaka, 560–0043 Japan
e-mail: ibukiyam@math.sci.osaka-u.ac.jp