COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 65, No. 2 2016

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

On Analogues of the Arakawa-Kaneko Zeta Functions of Mordell-Tornheim Type

by

Takuma ITO

(Received March 29, 2016)
(Revised October 10, 2016)

Abstract. In this paper, we construct certain analogues of the Arakawa-Kaneko zeta functions. We prove functional relations between these functions and the Mordell-Tornheim multiple zeta functions. Furthermore we give some formulas among Mordell-Tornheim multiple zeta values as their applications.

1. Introduction

Let $\mathbb{Z}$ be the rational integer ring, $\mathbb{N}$ the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Q}$ the rational number field and $\mathbb{C}$ the complex number field. We denote n repetitions of m by $\{m\}^n$ for $m, n \in \mathbb{N}$ and r-tuple unordered pair of $\{k_1, k_2, \ldots, k_r\}$ by $[k_1; k_2; \ldots; k_r]$ for $k_1, k_2, \ldots, k_r \in \mathbb{N}_0$.

Arakawa and Kaneko [1] introduced

$$\xi(k_1, k_2, \ldots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \mathrm{Li}_{k_1,k_2,\ldots,k_r}(1 - e^{-t}) dt \,, \tag{1}$$

so called "Arakawa-Kaneko zeta function", for $(k_1, k_2, \ldots, k_r) \in \mathbb{N}^r$ and $s \in \mathbb{C}$ with $\Re(s) > 0$, where $\mathrm{Li}_{k_1,k_2,\ldots,k_r}(z)$ is the polylogarithm defined by

$$\mathrm{Li}_{k_1,k_2,\ldots,k_r}(z) = \sum_{0<m_1<m_2<\cdots<m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \quad (z \in \mathbb{C},\ |z| < 1)\,.$$

When $r = 1$, $\xi(k; s)$ is also denoted by $\xi_k(s)$. They proved that for $m \in \mathbb{N}_0$, $\xi(\{1\}^{r-1}, k; m + 1)$ can be written in terms of multiple zeta values (MZVs) in [1, Theorem 9].

On the other hand, Matsumoto defined the "Mordell-Tornheim r-ple zeta function" by

$$\zeta_{MT,r}(s_1, s_2, \ldots, s_r; s_{r+1}) = \sum_{m_1,m_2,\ldots,m_r=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r} \left(\sum_{j=1}^{r} m_j\right)^{s_{r+1}}}\,. \tag{2}$$

The sum on the right-hand side of (2) converges absolutely when

$$\sum_{l=1}^{j} \Re(s_{k_l}) + \Re(s_{r+1}) > j$$

with $1 \leq k_1 < k_2 < \cdots < k_j \leq r$ for any $j = 1, 2, \ldots, r$ (see [8, Lemma 2.1]). Matsumoto proved that this function can be continued meromorphically to the whole $\mathbb{C}^{r+1}$-space in [4] and [5]. This zeta function in the double sum case was first studied by Tornheim [10] for the values at positive integers in 1950s. He gave some evaluation formulas for $\zeta_{MT,2}(k_1, k_2; k_3)$ when $k_1, k_2, k_3 \in \mathbb{N}$. Mordell [6] independently proved that $\zeta_{MT,2}(k, k; k)\pi^{-3k} \in \mathbb{Q}$ for all even $k \geq 2$. Tsumura [11, Theorem 4.5] and Nakamura [7, Theorem 1] showed certain functional relations among the Mordell-Tornheim double zeta functions and the Riemann zeta functions.

In this paper, for $\Bbbk \in \mathbb{N}^r$, we first define the function

$$\xi([\Bbbk]; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \prod_{j=1}^{r} \mathrm{Li}_{k_j}(1 - e^{-t}) dt \quad (s \in \mathbb{C},\ \Re(s) > 0)$$

which can be regarded as an analogue of the Arakawa-Kaneko zeta function of Mordell-Tornheim type (see Definition 1). We construct functional relations between $\xi([\Bbbk]; s)$ and the Mordell-Tornheim multiple zeta functions (see Theorem 8). For example,

$$\begin{aligned} &\zeta(2)^2\zeta(s) - 2\zeta(2)\xi([2]; s) + \xi([2; 2]; s) \\ &= \zeta_{MT,3}(2, 2, 0; s) + 2s\zeta_{MT,3}(2, 1, 0; s + 1) + s(s + 1)\zeta_{MT,3}(1, 1, 0; s + 2)\,. \end{aligned}$$

This can be proved by the method similar to the proof of [1, Theorem 8].

Secondly, we show certain relation formulas among Mordell-Tornheim multiple zeta values (see Corollary 9). For example,

$$\begin{aligned} &\zeta(2)^2\zeta(m + 1) - 2\zeta(2)\frac{1}{m!}\zeta_{MT,m+1}(2, \{1\}^m; 1) + \frac{1}{m!}\zeta_{MT,m+2}(2, 2, \{1\}^m; 1) \\ &= \zeta_{MT,3}(2, 2, 0; m + 1) + 2(m + 1)\zeta_{MT,3}(2, 1, 0; m + 2) \\ &\quad + (m + 1)(m + 2)\zeta_{MT,3}(1, 1, 0; m + 3) \quad (m \in \mathbb{N})\,, \end{aligned}$$

and

$$\zeta_{MT,3}(2, 1, 1; 1) = 2\zeta(2)\zeta(3) - \zeta(5)\,. \tag{3}$$

(3) had been obtained in [9, Theorem 5]. Lastly, we consider a generalization of main results (see Theorem 15).

2. Preliminaries

We first construct a Mordell-Tornheim type analogue of $\xi(k_1, k_2, \ldots, k_r; s)$ and continue it analytically to an entire function. We define $\{C^{[\Bbbk]}_{m,MT}\}$ by

$$\frac{\prod_{j=1}^{r} \mathrm{Li}_{k_j}(1 - e^{-t})}{e^t - 1} = \sum_{m=0}^{\infty} C^{[\Bbbk]}_{m,MT} \frac{t^m}{m!} \tag{4}$$

for $[\mathbb{k}] = [k_1; k_2; \dots; k_r]$ and $\mathbb{k} = (k_1, k_2, \dots, k_r) \in \mathbb{Z}^r$. These are generalizations of poly-Bernoulli numbers $\{C_m^{(k)}\}$ defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} = \sum_{m=0}^{\infty} C_m^{(k)} \frac{t^m}{m!}$$

for $k \in \mathbb{Z}$ (see [1]). Since $\mathrm{Li}_k(1-e^{-t}) = O(t)$ $(t \to 0)$ and $\mathrm{Li}_k(1-e^{-t}) = O(t)$ $(t \to \infty)$ for all $k \in \mathbb{N}$, we can define the following function.

DEFINITION 1. For $\mathbb{k} \in \mathbb{N}^r$ and $s \in \mathbb{C}$ with $\Re(s) > 1 - r$, let

$$\xi([\mathbb{k}]; s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t-1} \prod_{j=1}^{r} \mathrm{Li}_{k_j}(1-e^{-t})dt\,, \tag{5}$$

where $\Gamma(s)$ is the gamma function.

The integral on the right-hand side of (5) converges absolutely uniformly in the region $\Re(s) > 1 - r$. We can see that $\xi([\mathbb{k}]; s) = \xi_k(s)$ holds for $\mathbb{k} = (k) \in \mathbb{N}$ when $r = 1$. Further, we define $\xi(\emptyset; s) = \zeta(s)$ for $r = 0$.

THEOREM 2. *For* $\mathbb{k} = (k_1, k_2, \dots, k_r) \in \mathbb{N}^r$, *the function* $\xi([\mathbb{k}]; s)$ *can be continued analytically to an entire function, and satisfies*

$$\xi([\mathbb{k}]; -m) = (-1)^m C_{m,MT}^{[\mathbb{k}]} \quad (m \in \mathbb{N}_0)\,. \tag{6}$$

Proof. Let

$$\begin{aligned} A([\mathbb{k}]; s) &= \int_C \frac{t^{s-1}}{e^t-1} \prod_{j=1}^{r} \mathrm{Li}_{k_j}(1-e^{-t})dt \\ &= \left(e^{2\pi\sqrt{-1}s} - 1\right) \int_{\varepsilon}^{\infty} \frac{t^{s-1}}{e^t-1} \prod_{j=1}^{r} \mathrm{Li}_{k_j}(1-e^{-t})dt \\ &\quad + \int_{C_\varepsilon} \frac{t^{s-1}}{e^t-1} \prod_{j=1}^{r} \mathrm{Li}_{k_j}(1-e^{-t})dt \quad (s \in \mathbb{C})\,, \end{aligned}$$

where C is the contour which goes from $+\infty$ to a efficiently small $\varepsilon > 0$ along the positive real axis (top side), goes round counterclockwise along the circle C_ε centered at 0 with radius ε, and goes back to $+\infty$ along the positive real axis (bottom side). We note that t^s means $e^{s \log t}$, where $\Im(t)$ varies 0 (on the top side of the positive real axis) to 2π (on the bottom side). Since the integrand has no singularity on C and the contour integral converges absolutely for all $s \in \mathbb{C}$, we can see that $A([\mathbb{k}]; s)$ is entire. Suppose $\Re(s) > 1 - r$, then the second integral tends to 0 as $\varepsilon \to 0$. Therefore we have

$$\xi([\mathbb{k}]; s) = \frac{1}{\left(e^{2\pi\sqrt{-1}s} - 1\right)\Gamma(s)} A([\mathbb{k}]; s)\,.$$

Since $\xi([\mathbb{k}]; s)$ is holomorphic for $\Re(s) > 1 - r$, this function has no singularity at any positive integer. Therefore this gives the analytic continuation of $\xi([\mathbb{k}]; s)$ to an entire function. Let $s = -m$ for $m \in \mathbb{N}_0$. Using (4), we have

$$\begin{aligned}\xi([\mathbb{k}]; -m) &= \frac{(-1)^m m!}{2\pi\sqrt{-1}} A([\mathbb{k}]; -m) \\ &= \frac{(-1)^m m!}{2\pi\sqrt{-1}} \int_{C_\varepsilon} t^{-m-1} \sum_{n=0}^{\infty} C_{n,MT}^{[\mathbb{k}]} \frac{t^n}{n!} dt \\ &= (-1)^m C_{m,MT}^{[\mathbb{k}]} .\end{aligned}$$

This completes the proof. □

Secondly, we show a relation between the Mordell-Tornheim multiple zeta values and $\xi([\mathbb{k}]; m+1)$ for $m \in \mathbb{N}_0$. For this aim, we consider the following function and give a lemma.

DEFINITION 3. For $\mathbb{k} = (k_1, k_2, \ldots, k_{r+1}) \in \mathbb{N}^r \times \mathbb{N}_0$ and $z \in \mathbb{C}$ with $|z| < 1$, let

$$\mathcal{L}_{\mathbb{k}}(z) = \sum_{m_1, m_2, \ldots, m_r = 1}^{\infty} \frac{z^{\sum_{j=1}^r m_j}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r} \left(\sum_{j=1}^r m_j\right)^{k_{r+1}}} . \tag{7}$$

Under the above condition, the sum on the right-hand side of (7) converges absolutely uniformly. We note that $\mathcal{L}_{\mathbb{k}}(z) = \mathrm{Li}_{k_1+k_2}(z)$ holds for $r = 1$ and $\mathbb{k} = (k_1, k_2)$. By direct calculation, we have

LEMMA 4. *For* $\mathbb{k} = (k_1, k_2, \ldots, k_r, k_{r+1}) \in \mathbb{N}^{r+1}$ *and* $z \in \mathbb{C}$ *with* $|z| < 1$,

$$\frac{d}{dz} \mathcal{L}_{\mathbb{k}}(z) = \begin{cases} \dfrac{1}{z} \mathcal{L}_{\underline{\mathbb{k}}^{(r+1)}}(z) & (k_{r+1} \geq 2), \\ \dfrac{1}{z} \displaystyle\prod_{j=1}^{r} \mathrm{Li}_{k_j}(z) & (k_{r+1} = 1), \end{cases}$$

where $\underline{\mathbb{k}}^{(r+1)} = (k_1, k_2, \ldots, k_r, k_{r+1} - 1)$.

Using Lemma 4 and calculating directly, we obtain

PROPOSITION 5. *For* $\mathbb{k} = (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r$ *and* $m \in \mathbb{N}_0$,

$$\xi([\mathbb{k}]; m+1) = \frac{1}{m!} \zeta_{MT, m+r}(k_1, k_2, \ldots, k_r, \{1\}^m; 1) .$$

Since $\xi(\emptyset; s) = \zeta(s)$, Proposition 5 still holds for $r = 0$ when $m \geq 1$. We can recover [3, Corollary 4.2 and Theorem 4.4] as follows.

COROLLARY 6. *For* $m \in \mathbb{N}_0$,

$$\zeta_{MT, m+1}(\{1\}^{m+1}; 1) = (m+1)! \zeta(m+2) .$$

Proof. By $\xi_1(s) = s\zeta(s+1)$ and Proposition 5, we obtain the assertion. □

3. Main results

In this section, we give main results. We first prepare the following lemma which is necessary to show the first and second main results.

LEMMA 7. *For* $s_j \in \mathbb{C}$ *with* $\Re(s_j) > 0$ $(2 \le j \le r)$ *and* $\Re(s_{r+1}) > r$,

$$\zeta_{MT,r}(0, s_2, \ldots, s_r; s_{r+1}) = \tag{8}$$

$$\frac{1}{\prod_{j=2}^{r+1}\Gamma(s_j)} \underbrace{\int_0^\infty \int_0^\infty \cdots \int_0^\infty}_{r} \frac{\prod_{j=2}^{r+1} t_j^{s_j-1}}{(e^{t_{r+1}}-1)\prod_{j=2}^{r}(e^{t_j+t_{r+1}}-1)} dt_2 \cdots dt_r dt_{r+1}\,.$$

Proof. Using the well-known relation

$$m^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-mt}dt \quad (m \in \mathbb{N},\ s \in \mathbb{C},\ \Re(s) > 0)$$

for $s_j \in \mathbb{C}$ with $\Re(s_j) > 0$ $(2 \le j \le r)$ and $\Re(s_{r+1}) > r$, we have

$$\begin{aligned}
&\prod_{j=2}^{r+1}\Gamma(s_j) \times \zeta_{MT,r}(0, s_2, \ldots, s_r; s_{r+1}) \\
&= \sum_{m_1, m_2, \ldots, m_r = 1}^{\infty} \prod_{j=2}^{r}\left(\int_0^\infty t_j^{s_j-1} e^{-m_j t} dt_j\right) \\
&\quad \times \left(\int_0^\infty t_{r+1}^{s_{r+1}-1} e^{-\left(\sum_{j=1}^{r} m_j\right) t_{r+1}} dt_{r+1}\right) \\
&= \sum_{m_1, m_2, \ldots, m_r = 1}^{\infty} \underbrace{\int_0^\infty \int_0^\infty \cdots \int_0^\infty}_{r} dt_2 dt_3 \cdots dt_{r+1} \\
&\quad \times \left(\prod_{j=2}^{r+1} t_j^{s_j-1}\right)\left(e^{-m_1 t_{r+1}}\right)\left(\prod_{j=2}^{r} e^{-m_j(t_j+t_{r+1})}\right) \\
&= \underbrace{\int_0^\infty \int_0^\infty \cdots \int_0^\infty}_{r} \frac{\prod_{j=2}^{r+1} t_j^{s_j-1}}{\left(e^{t_{r+1}-1}-1\right)\prod_{j=2}^{r}\left(e^{t_j+t_{r+1}}-1\right)} dt_2 dt_3 \cdots dt_{r+1}\,.
\end{aligned}$$

Changing the order of summation and integration is justified by absolutely convergence. Therefore we complete the proof. □

Using Lemma 7, we have the first main result as follows.

THEOREM 8. *For* $r \in \mathbb{N}$ *and* $s \in \mathbb{C}$,

$$\sum_{j=0}^{r}\binom{r}{j}(-1)^j \zeta(2)^{r-j}\xi([\{2\}^j]; s)$$

$$= \sum_{j=0}^{r} \binom{r}{j} (s)_j \zeta_{MT,r+1}(\{2\}^{r-j}, \{1\}^j, 0; s+j)\,.$$

where $\zeta(s)$ is the Riemann zeta-function and $(s)_j$ is the Pochhammer symbol defined by

$$(s)_j = s(s+1)\cdots(s+j-1)$$

for $j \in \mathbb{N}_0$.

Proof. For $s \in \mathbb{C}$ with $\Re(s) > 0$, we let

$$J_{MT,r+1}(s) = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{r+1} dt_1 dt_2 \cdots dt_{r+1} \frac{t_{r+1}^{s-1}}{e^{t_{r+1}}-1} \prod_{j=1}^{r} \frac{t_j + t_{r+1}}{e^{t_j+t_{r+1}}-1}\,.$$

Using

$$\frac{\partial}{\partial t_j} \operatorname{Li}_2(1 - e^{-t_j - t_{r+1}}) = \frac{t_j + t_{r+1}}{e^{t_j+t_{r+1}}-1} \quad (1 \le j \le r)\,,$$

we have

$$J_{MT,r+1}(s) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \zeta(2)^{r-j} \Gamma(s) \xi([\{2\}^j]; s)\,.$$

On the other hand, by Lemma 7 and

$$\zeta_{MT,r+1}(\ldots, \overset{i}{\check{s_i}}, \ldots, \overset{j}{\check{s_j}}, \ldots; s_{r+2}) = \zeta_{MT,r+1}(\ldots, \overset{i}{\check{s_j}}, \ldots, \overset{j}{\check{s_i}}, \ldots; s_{r+2})$$

for $1 \le i \le j \le r+1$, we have

$$J_{MT,r+1}(s) = \sum_{j=0}^{r} \binom{r}{j} \Gamma(s+j) \zeta_{MT,r+1}(\{2\}^{r-j}, \{1\}^j, 0; s+j)\,.$$

By the analytic continuation, we obtain the desired identity. □

By Theorem 8 and Proposition 5, we immediately obtain the second main result as follows.

COROLLARY 9. *For $r, m \in \mathbb{N}$,*

$$\sum_{j=0}^{r} \binom{r}{j} \frac{(-1)^j \zeta(2)^{r-j}}{m!} \zeta_{MT,j+m}(\{2\}^j, \{1\}^m; 1)$$
$$= \sum_{j=0}^{r} \binom{r}{j} (m+1)_j \zeta_{MT,r+1}(\{2\}^{r-j}, \{1\}^j, 0; m+1+j)\,.$$

Next, in order to evaluate $\zeta_{MT,2k+1}(2, \{1\}^{2k}; 1)$, we quote [2, (75)]:

(9) $$\zeta(a,b) = \frac{1}{2}\left\{\left((-1)^b \binom{M}{a} - 1\right)\zeta(M) + (1 + (-1)^b)\zeta(a)\zeta(b)\right\}$$

$$+ (-1)^{b+1} \sum_{k=1}^{(M-3)/2} \left\{ \binom{2k}{a-1} + \binom{2k}{b-1} \right\} \zeta(2k+1)\zeta(M-2k-1) ,$$

where $a, b \in \mathbb{N}$ with $a, b \geq 2$, $M = a + b \equiv 1 \pmod 2$ and $\zeta(a, b) = \zeta_{MT,2}(0, a; b)$.

REMARK 10. We note that (9) also holds for $a = 1$ providing we remove the term containing $\zeta(1)$.

Combining (9) and Corollary 9 in the case $r = 1$, we have the third main result as follows.

PROPOSITION 11. *For* $k \in \mathbb{N}$,

$$\zeta_{MT,2k+1}(2, \{1\}^{2k}; 1) = (2k)! \Bigg\{ \zeta(2)\zeta(2k+1) - \frac{1}{2}(2k^2 + k - 2)\zeta(2k+3) + \sum_{n=1}^{k-1} (2k + 1 - 2n)\zeta(2n+1)\zeta(2k+2-2n) \Bigg\} .$$

EXAMPLE 12.

$$\zeta_{MT,3}(2, 1, 1; 1) = 2\zeta(2)\zeta(3) - \zeta(5) ,$$
$$\zeta_{MT,5}(2, 1, 1, 1, 1; 1) = 4! \{\zeta(2)\zeta(5) + 3\zeta(3)\zeta(4) - 4\zeta(7)\} .$$

These results correspond to [1, Theorems 6, 8, 9 and Corollary 11]. Results in [1] are relations between $\xi(k_1, k_2, \ldots, k_r; s)$ and multiple zeta functions or MZVs. On the other hand, our results are relations between $\xi([\mathbb{k}]; s)$ and Mordell-Tornheim multiple zeta functions or Mordell-Tornheim multiple zeta values.

4. A generalization of the function $\xi([\mathbb{k}]; s)$

In this section, we consider a certain generalization of the function $\xi([\mathbb{k}]; s)$ and aim to generalize Theorem 8.

By the definition (7), for $\mathbb{k} = (k_1, k_2, \ldots, k_r, k_{r+1}) \in \mathbb{N}^r \times \mathbb{N}_0$, we have

$$\mathcal{L}_{\mathbb{k}}(1 - e^{-t}) = \begin{cases} O(t^l) & \text{if } k_{r+1} = 0 \text{ and } l = \sharp\{j \mid k_j = 1\} \geq 1 , \\ O(1) & \text{otherwise} \quad (t \to \infty) \end{cases} \tag{10}$$

and

$$\mathcal{L}_{\mathbb{k}}(1 - e^{-t}) = O(t^r) \ \ (t \to 0) . \tag{11}$$

Using (10) and (11), we can define the following function.

DEFINITION 13. For $r_1, r_2, \ldots, r_g \in \mathbb{N}$, $\mathbb{k}_i = (k_1^{(i)}, k_2^{(i)}, \ldots, k_{r_i}^{(i)}, k_{r_i+1}^{(i)}) \in \mathbb{N}^{r_i} \times \mathbb{N}_0$ and $s \in \mathbb{C}$ with $\Re(s) > 1 - \sum_{i=1}^{g} r_i$, let

$$\xi_g([\mathbb{k}_1; \mathbb{k}_2; \ldots; \mathbb{k}_g]; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \prod_{i=1}^{g} \mathcal{L}_{\mathbb{k}_i}(1 - e^{-t}) dt . \tag{12}$$

The integral on the right-hand side of (12) converges absolutely uniformly in the region $\Re(s) > 1 - \sum_{i=1}^{g} r_i$. Further we note that

$$\xi_1([\mathbb{k}_1]; s) = \xi([\mathbb{k}]; s)$$

for $\mathbb{k}_1 = (k_1, k_2, \ldots, k_r, 0)$, $\mathbb{k} = (k_1, k_2, \ldots, k_r)$ and $s \in \mathbb{C}$. Therefore we can see that Definition 13 is a generalization of the function $\xi([\mathbb{k}]; s)$. By the same method as in the proof of Theorem 2, we have

THEOREM 14. *For $g, r_1, r_2, \ldots, r_g \in \mathbb{N}$ and $\mathbb{k}_i \in \mathbb{N}^{r_i} \times \mathbb{N}_0$ $(1 \leq i \leq g)$, the function $\xi_g([\mathbb{k}_1; \mathbb{k}_2; \ldots; \mathbb{k}_g]; s)$ can be continued analytically to an entire function.*

By the same method as in the proof of Theorem 8, we obtain

THEOREM 15. *For $g \in \mathbb{N}$, $\mathbf{r} = (r_1, r_2, \ldots, r_g) \in \mathbb{N}^g$ and $s \in \mathbb{C}$,*

$$\begin{aligned} &\sum_{n=0}^{g} (-1)^n \sum_{\substack{J \subset I_g \\ \sharp J = n}} \left\{ \prod_{j \in I_g \setminus J} r_j! \zeta(r_j + 1) \right\} \xi_n([\{1\}^{r_j+1}]_{j \in J}; s) \\ &= \sum_{n=0}^{\mathrm{wt}(\mathbf{r})} (s)_n \sum_{\substack{i_1 + \cdots + i_g = n \\ r_l \geq i_l \geq 0}} \left\{ \prod_{j=1}^{g} \binom{r_j}{i_j} (r_j - i_j)! \right\} \\ &\quad \times \zeta_{MT, g+1} \left(r_1 - i_1 + 1, r_2 - i_2 + 1, \ldots, r_g - i_g + 1, 0; s + n \right), \end{aligned} \tag{13}$$

where $\xi_g(\emptyset; s) = \zeta(s)$, $I_g = \{1, 2, \ldots, g\}$, $[\mathbb{k}_1; \mathbb{k}_2; \ldots; \mathbb{k}_r] = [\mathbb{k}_j]_{j \in I_g}$ and $\mathrm{wt}(\mathbf{r}) = \sum_{i=1}^{g} r_i$.

Proof. We define the function $J_{MT, \mathbf{r}}(s)$ by

$$\begin{aligned} J_{MT, \mathbf{r}}(s) &= \underbrace{\int_0^\infty \int_0^\infty \cdots \int_0^\infty}_{g+1} dt_1 dt_2 \cdots dt_{g+1} \\ &\quad \times \frac{t_{g+1}^{s-1}}{e^{t_{g+1}} - 1} \prod_{j=1}^{g} \frac{(t_j + t_{g+1})^{r_j}}{e^{t_j + t_{g+1}} - 1} \quad (s \in \mathbb{C},\ \Re(s) > 0) \end{aligned}$$

for $\mathbf{r} = (r_1, r_2, \ldots, r_g) \in \mathbb{N}^g$. It follows from Lemma 4 that

$$\frac{\partial}{\partial t_j} \mathcal{L}_{\{1\}^{r_j+1}}(1 - e^{-t_j - t_{g+1}}) = \frac{(t_j + t_{g+1})^{r_j}}{e^{t_j + t_{g+1}} - 1}. \tag{14}$$

Using (14) and Corollary 6, we have

$$\begin{aligned} J_{MT, \mathbf{r}}(s) &= \sum_{n=0}^{g} (-1)^n \sum_{\substack{J \subset I_g \\ \sharp J = n}} \left\{ \prod_{j \in I_g \setminus J} r_j! \zeta(r_j + 1) \right\} \\ &\quad \times \Gamma(s) \xi_n([\{1\}^{r_j+1}]_{j \in J}; s) \end{aligned}$$

for $\Re(s) > 1$. On the other hand, by Lemma 7, we have

$$J_{MT,\mathbf{r}}(s) = \sum_{n=0}^{\mathrm{wt}(\mathbf{r})} \Gamma(s+n) \sum_{\substack{i_1+\cdots+i_g=n \\ r_l \geq i_l \geq 0}} \left\{ \prod_{j=1}^{g} \binom{r_j}{i_j} (r_j - i_j)! \right\}$$
$$\times \zeta_{MT,g+1}\left(r_1 - i_1 + 1, r_2 - i_2 + 1, \ldots, r_g - i_g + 1, 0; s+n\right)$$

for $\Re(s) > g$. By the analytic continuation, we obtain (13) for all $s \in \mathbb{C}$. Therefore the proof is completed. □

REMARK 16. In particular, Theorem 15 in the case $(g, \mathbf{r}) = (r, \{1\}^r)$ coincides with Theorem 8. Hence we can see that Theorem 15 is a generalization of Theorem 8.

We have not obtained the values of $\xi_g([\mathbb{k}_1; \mathbb{k}_2; \ldots; \mathbb{k}_g]; m+1)$ for $m \in \mathbb{N}_0$. But we have obtained a certain proposition as follows.

PROPOSITION 17. *For* $g, r_1, r_2, \ldots, r_g \in \mathbb{N}$,

$$\sum_{j=1}^{g} r_j! \xi_{g-1}([\{1\}^{r_i+1}]_{i \in I_g \setminus \{j\}}; r_j + 1) = \prod_{j=1}^{g} \zeta_{MT,r_j}(\{1\}^{r_j}; 1).$$

Proof. By (14) and the partial integration, we have

$$\begin{aligned}
&r_1! \xi([\{1\}^{r_i+1}]_{i \in I_g \setminus \{1\}}; r_1 + 1) \\
&= \int_0^\infty \frac{t^{r_1}}{e^t - 1} \prod_{j=2}^{g} \mathcal{L}_{\{1\}^{r_j+1}}(1 - e^{-t}) dt \\
&= \prod_{j=1}^{g} \zeta_{MT,r_j}(\{1\}^{r_j}; 1) - \sum_{j=2}^{g} \int_0^\infty \frac{t^{r_j}}{e^t - 1} \prod_{\substack{i=1 \\ i \neq j}}^{g} \mathcal{L}_{\{1\}^{r_i+1}}(1 - e^{-t}) dt \\
&= \prod_{j=1}^{g} \zeta_{MT,r_j}(\{1\}^{r_j}; 1) \\
&\quad - \sum_{j=2}^{g} r_j! \xi_{g-1}([\{1\}^{r_i+1}]_{i \in I_g \setminus \{j\}}; r_j + 1).
\end{aligned}$$

Therefore we complete the proof. □

REMARK 18. In particular, combining Corollary 6, Theorem 15 in the case $g = 2$ and Proposition 17 in the case $g = 2$, we have the Euler decomposition (cf. [1]).

$$\zeta(k+1)\zeta(r+1) = \sum_{m=0}^{k} \binom{r+m}{r} \zeta(k+1-m, r+1+m)$$
$$+ \sum_{n=0}^{r} \binom{k+n}{k} \zeta(r+1-n, k+1+n) \quad (r, k \in \mathbb{N}).$$

Acknowledgments. The author thanks Professor Hirofumi Tsumura and Kohji Matsumoto for useful advice and pointing out some mistakes and unsuitable expressions. The author also thanks Professor Yasushi Komori for his detailed explanations of the analytic theory of multiple zeta functions in the intensive course at Tokyo Metropolitan University.

References

[1] T. Arakawa and M. Kaneko, *Multiple zeta values, poly-Bernoulli numbers, and related zeta functions*, Nagoya Math. J. Vol. **153** (1999), 189–209.

[2] J. M. Borwein, D. M. Bradley and D. J. Broadhurst, *Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k*, Electron. J. Combin. **4** (1997), no. 2, Research Paper 5, approx. 21 pp.

[3] M. Hoffman, *Multiple harmonic series*, Pacific J. Math., **152** (1992), 275–290.

[4] K. Matsumoto, *On the analytic continuation of various multiple zeta-functions*, in "Number Theory for the Millennium II", Proc. Millennial Conf. on Number Theory (Urbana-Champaign, 2000), M. A. Bennett et al. (eds.), A K Peters, 2002, pp.417–440.

[5] K. Matsumoto, *On Mordell-Tornheim and multiple zeta-functions*, in "Proc. Session in Analytic Number Theory and Diophantine Equations", D. R. Heath-Brown and B. Z. Moroz (eds.), Bonner Math. Scriften **360**, Bonn, 2003, n.15, 17pp.

[6] L. J. Mordell, *On the evaluation of some multiple series*, J. London Math. Soc. **33** (1958), 368–371.

[7] T. Nakamura, *A functional relation for the Tornheim double zeta function*, Acta Arith. **125**, no. 3 (2006), 257–263.

[8] T. Okamoto and T. Onozuka, *Functional equation for the Mordell-Tornheim multiple zeta-function*, to appear in Funct. Approx. Comment. Math.

[9] K. Onodera, *Generalized log sine integrals and the Mordell-Tornheim zeta values*, Trans. Amer. Math. Soc., **363** (2011), 1463-1485.

[10] L. Tornheim, *Harmonic double series*, Amer. J. Math. **72** (1950), 303–314.

[11] H. Tsumura, *On functional relations between the Mordell-Tornheim double zeta functions and the Riemann zeta function*, Math. Proc. Cambridge Philos. Soc. **142**, no. 3 (2007), 161–178.

Takuma ITO
St. Ursula Gakuin Eichi High School
1–2, Ipponsugimachi, Wakabayashi-ku,
Sendai 984–0828 Japan
e-mail: sugakunotakuma.ito@gmail.com