COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 66, No. 1&2 2017

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

On a Congruence of Fourier Coefficients of an Eta Product

by

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(Received August 2, 2017)
(Revised August 27, 2017)

Abstract. In [10], Martin classified modular forms that are multiplicative eta quotients. We will show that weight one eta quotients of Martin's classification are in fact theta functions associated to imaginary quadratic fields. As an application, a congruence relation of the coefficients of some eta products is obtained. This may be regarded as a generalization of Wilton's congruence of Ramanujan's τ-function. We will also determine the Galois representation which corresponds to each of the eta quotients of weight one.

1. Introduction

Ramanujan's τ-function is defined to be the Fourier coefficients of the cuspidal elliptic modular form Δ of weight 12

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n .$$

After Ramanujan has showed the congruence relation ([14])

$$\tau(p) \equiv 1 + p^{11} \pmod{691}, \quad p \text{ is a prime},$$

various congruence relations of the τ-function have been found (eg. [2], [9], [13], [16]). In particular, Wilton obtained the following congruence relation

$$\tau(p) \pmod{23} = \begin{cases} 0 & \text{if } (\frac{-23}{p}) = -1\,; \\ 2 & \text{if } (\frac{-23}{p}) = 1 \text{ and } p \text{ is represented by } x^2+xy+6y^2\,; \\ -1 & \text{if } (\frac{-23}{p}) = 1 \text{ and } p \text{ is represented by } 2x^2+xy+3y^2 \end{cases}$$

for a prime p. Here we mention that n is *represented* by an integral quadratic form $Q(x, y) = ax^2+bxy+cy^2$ if the equation $Q(x, y) = n$ has an integral solution. Since the conditions of Wilton's results are described by quadratic forms of discriminant -23, it is

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2010 Mathematics Subject Classification. Primary 11E12, 11E16, 11F33, 11F80, 14H42, 14K25, 32N20.
Key words and phrases. eta products, Ramanujan's τ-function, congruence, theta functions, binary quadratic forms.

natural to expect that the congruence relations may be related to theta functions for the congruence group of level 23. In fact, in [3], Zagier clarified the relationship. Let us briefly recall his argument. Using the bijective correspondence between quadratic forms of discriminant -23 and the ideal class group $\mathrm{Pic}(\mathcal{O}_K)$ of the maximal order $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{-23})$, one has $\mathrm{Pic}(\mathcal{O}_K) = \{Q_0,\ Q_1,\ Q_2\}$, where

$$Q_0 := x^2 + xy + 6y^2, \quad Q_1 := 2x^2 + xy + 3y^2, \quad Q_2 := 2x^2 - xy + 3y^2 .$$

In particular $\mathrm{Pic}(\mathcal{O}_K)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ by the map

$$\mathbb{Z}/3\mathbb{Z} \to \mathrm{Pic}(\mathcal{O}_K), \quad i \mapsto Q_i, \quad (i = 0, 1, 2) .$$

Let χ be one of the non-trivial characters of $\mathbb{Z}/3\mathbb{Z}$, and set

$$f = \frac{1}{2}\sum_{i=0}^{2} \chi(Q_i)\theta_{Q_i} ,$$

where θ_{Q_i} is the theta function associated to Q_i (cf. **Definition 3.1**). We remark that f is independent of the choice of χ because $\theta_{Q_1} = \theta_{Q_2}$ (see §4.1) and

$$f = \frac{1}{2}(\theta_{Q_0} - \theta_{Q_1}) .$$

Since the level of the Q_is are all 23, f is a cusp form on $\Gamma_0(23)$ of weight one with Nebentypus character ϵ_{-23} (see **Fact 3.1**). Moreover, one sees that it is a Hecke eigenform (see §4) and that it has an infinite product

$$f = q\prod_{n=1}^{\infty}(1-q^n)(1-q^{23n}) . \tag{1}$$

Now the obvious congruence

$$(1-x)^p \equiv 1 - x^p \pmod{p} , \tag{2}$$

implies that

$$\sum_{i=1}^{2} \chi(Q_i)\theta_{Q_i} \equiv q\prod_{n=1}^{\infty}(1-q^n)^{24} = \Delta \qquad \pmod{23} ,$$

which yields Wilton's congruence. In this paper we will generalize his results along the Zagier's arguments.

DEFINITION 1.1. Let d be a positive divisor of 24. For a positive integer n we define $T_d(n)$ by

$$q\prod_{n=1}^{\infty}(1-q^{dn})^{\frac{24}{d}} = \sum_{n=1}^{\infty} T_d(n)q^n$$

Note that $T_1(n)$ is nothing but the Ramanujan's tau function, $\tau(n)$. Here are our results.

THEOREM 1.1. (1)

$$T_2(p) \pmod{11} = \begin{cases} 0 & \textit{if } (\frac{-44}{p}) = -1 , \\ 2 & \textit{if } (\frac{-44}{p}) = 1 \textit{ and } p \textit{ is represented by } x^2 + 11y^2 , \\ -1 & \textit{if } (\frac{-44}{p}) = 1 \textit{ and } p \textit{ is represented by } 3x^2 + 2xy + 4y^2 . \end{cases}$$

(2) *Suppose that the prime* p *satisfies one of the following conditions ;*

- $(\frac{-63}{p}) = -1$
- $(\frac{-63}{p}) = 1$ *and* p *is represented by* $2x^2 + xy + 8y^2$.

Then

$$T_3(p) \equiv 0 \pmod 7 .$$

Moreover,

$$T_3(p) \pmod 7 = \begin{cases} 2 & \textit{if } (\frac{-63}{p}) = 1 \textit{ and } p \textit{ is represented by } x^2 + xy + 16y^2 , \\ -2 & \textit{if } (\frac{-63}{p}) = 1 \textit{ and } p \textit{ is represented by } 4x^2 + xy + 4y^2 . \end{cases}$$

(3) *Suppose that the prime* p *satisfies one of the following conditions ;*

- $(\frac{-80}{p}) = -1$
- $(\frac{-80}{p}) = 1$ *and* p *is represented by* $3x^2 + 2xy + 7y^2$.

Then

$$T_4(p) \equiv 0 \pmod 5 .$$

Moreover,

$$T_4(p) \pmod 5 = \begin{cases} 2 & \textit{if } (\frac{-80}{p}) = 1 \textit{ and } p \textit{ is represented by } x^2 + 20y^2 , \\ -2 & \textit{if } (\frac{-80}{p}) = 1 \textit{ and } p \textit{ is represented by } 4x^2 + 5y^2 . \end{cases}$$

(4)

$$T_2(p) \pmod 3 = \begin{cases} 0 & \textit{if } (\frac{-108}{p}) = -1 , \\ -1 & \textit{if } (\frac{-108}{p}) = 1 . \end{cases}$$

(5) $T_1(p)(= \tau(p)) \pmod 2 = 0$ *for every odd prime* p.

(6) *Suppose that the prime* p *satisfies one of the following conditions ;*

- $p \equiv -1 \pmod 4$
- $p \equiv 1 \pmod 4$ *and* p *is represented by* $5x^2 + 4xy + 8y^2$.

Then

$$T_4(p) \equiv 0 \pmod 3 .$$

Moreover

$$T_4(p) \pmod 3 = \begin{cases} -1 & \textit{if } p \equiv 1 \pmod 4 \textit{ and } p \textit{ is represented by } x^2 + 36y^2 , \\ 1 & \textit{if } p \equiv 1 \pmod 4 \textit{ and } p \textit{ is represented by } 4x^2 + 9y^2 . \end{cases}$$

The assertion (5) is fairly well known. Let us explain how the theorem will be proved. For a pair of positive integers (a, b) whose sum is equal to 24, we define

$$\eta_{a \cdot b}(q) = \eta(az)\eta(bz) = q \prod_{n=1}^{\infty} (1 - q^{an})(1 - q^{bn})$$

where $\eta(z)$ is the eta function

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad q = \exp(2\pi i z), \quad (\mathrm{Im} z > 0) .$$

Since $\eta(z)$ is a modular form of weight $1/2$, $\eta_{a\cdot b}$ is a modular form of weight one. Martin ([10]) has shown that $\eta_{a\cdot b}$ is a cusp form if and only if

$$(a,b) = (1,23),\ (2,22),\ (3,21),\ (4,20),\ (6,18),\ (8,16),\ (12,12)\,. \tag{3}$$

Moreover he has shown that $\eta_{a\cdot b}$ is a Hecke eigenform on a congruence subgroup of level ab for these pairs. We are interested in the coefficient of the Fourier expansion

$$\eta_{a\cdot b}(q) = \sum_{n=1}^{\infty} c_{\eta_{a\cdot b}}(n) q^n\,.$$

Wilton's congruence is the case $(a,b) = (1,23)$. We will show a similar equation as (1), namely that $\eta_{a\cdot b}(q)$ can be written by a linear combination of theta functions associated to integral binary quadratic forms of level ab. Although in order to obtain the relation of Wilton the maximal order of $\mathbb{Q}(\sqrt{-23})$ plays an important role, we will use a *non-maximal* order of the quadratic field $\mathbb{Q}(\sqrt{-ab})$. Using **Proposition 2.1** we will compute the Fourier coefficients of a linear combination of theta functions which are cuspidal Hecke eigenforms. The results are summarized in **Proposition 4.1** and **Proposition 4.3**. On the other hand, the Fourier coefficients of $\eta_{a\cdot b}(q)$ can be obtained by machine calculation. Comparing these coefficients, we will show that Martin's eta products are linear combinations of theta functions. These calculations will be carried out in **Theorem 4.1** and **Theorem 4.2**. Taking the equation (2) into account, **Theorem 1.1** is an immediate consequence of these theorems and will be proved at the end of the paper. Since $\eta_{a\cdot b}$ is a cuspidal Hecke eigenform of weight one for (a,b) listed by (3), it corresponds to a Galois representation ([5]). We will identify these Galois representations in **Proposition 4.2** and **Proposition 4.4**.

Acknowledgements. The author thanks Prof. Yokoyama who has kindly informed us of the results of numerical experiments and Prof. Geisser for careful reading of the manuscript. He is also grateful to the referee for corrections and valuable comments.

2. Quadratic forms and orders

In this section, we recall basic facts of binary quadratic forms and the ideal class group of orders of imaginary quadratic fields.

Let K be an imaginary quadratic field of discriminant d_K and $\mathcal{O}$ its order of conductor f. Thus $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers and the discriminant of $\mathcal{O}$ is $d_K f^2$. The number of units of $\mathcal{O}$ is denoted by w. Let $\mathrm{Pic}(\mathcal{O})$ be the ideal class group of $\mathcal{O}$. This is a finite abelian group of the order the number $h(\mathcal{O})$. In the next section we will associate a theta function with every element of $\mathrm{Pic}(\mathcal{O})$. So we explicitly describe the relation between binary quadratic forms and the ideal classes. A binary quadratic form with integral coefficients

$$f(x,y) = ax^2 + bxy + cy^2, \quad a,b,c \in \mathbb{Z}$$

is called *a positive definite primitive quadratic form* if the common divisor of $a,\ b,\ c$ is one and if

$$a > 0, \quad \Delta_f = b^2 - 4ac < 0\,.$$

We call Δ_f *the discriminant* of f. We say that two forms f and g are *equivalent* if there are integers $p,\ q,\ r$ and s such that

$$g(x, y) = f(px + qy, rx + sy), \qquad ps - qr = 1\,.$$

Then $\Delta_f = \Delta_g$, and this defines an equivalence relation between quadratic forms of discriminant D. Let $C(D)$ be the set of equivalence classes, a finite set of cardinality $h(D)$. A primitive positive definite quadratic form $f(x, y) = ax^2 + bxy + cy^2$ is said to be *reduced* if

$$|b| \le a \le c, \quad \text{and} \quad b \ge 0 \quad \text{if either} \quad |b| = a \quad \text{or} \quad a = c\,.$$

It is known that every primitive positive definite form is equivalent to a unique reduced form ([4], **Theorem 2.8**). Suppose that we are given two positive definite quadratic forms $f(x, y) = ax^2 + bxy + cy^2$ and $g(x, y) = a'x^2 + b'xy + c'y^2$ of discriminant D satisfying

$$\gcd\Bigl(a, a', \frac{b + b'}{2}\Bigr) = 1\,.$$

Then *the Dirichlet composition* of $f(x, y)$ and $g(x, y)$ is defined to be

$$F(x, y) = aa'x^2 + Bxy + \frac{B^2 - D}{4aa'}y^2\,,$$

where B is the unique integer modulo $2aa'$ such that

$$\begin{array}{ccll} B & \equiv & b & \bmod 2a \\ B & \equiv & b' & \bmod 2a' \\ B^2 & \equiv & D & \bmod 4aa'\,. \end{array}$$

(see [4] p.139). Dirichlet's composition preserves the equivalence relation and makes $C(D)$ a finite abelian group of order $h(D)$([4], **Theorem 3.9**). In particular, the identity element of $C(D)$ is the class containing the form

$$x^2 - \frac{D}{4}y^2, \quad \text{if} \quad D \equiv 0 \bmod 4\,,$$

$$x^2 + xy + \frac{1 - D}{4}y^2, \quad \text{if} \quad D \equiv 1 \bmod 4\,,$$

and the inverse of the class of $ax^2 + bxy + cy^2$ is the class of $ax^2 - bxy + cy^2$.

FACT 2.1. ([4], **Theorem 7.7**) Let $\mathcal{O}$ be the order of discriminant D in an imaginary quadratic field K.

(1) If $f(x, y) = ax^2 + bxy + cy^2$ is a primitive positive definite quadratic form of discriminant D, then

$$\left[a, \frac{-b + \sqrt{D}}{2}\right]$$

is a proper ideal of $\mathcal{O}$. Here $[\alpha, \beta]$ denotes the free abelian group of rank two in K generated by α and β.

(2) The map sending $f(x, y)$ to $[a, \frac{-b+\sqrt{D}}{2}]$ induces an isomorphism between $C(D)$ and $\mathrm{Pic}(\mathcal{O})$. In particular,

$$h(D) = h(\mathcal{O}).$$

(3) A positive integer n is represented by the form $Q(x, y)$ (i.e. there is a pair of integers (x, y) such that $Q(x, y) = n$) if and only if n is the norm $N(\mathfrak{A})$ of some integral ideal $\mathfrak{A}$ in the class $A \in \mathrm{Pic}(\mathcal{O})$ which corresponds to Q.

Here is a remark to **Fact 2.1 (3)**. Let $r_A(n)$ be the number of integral solutions of $Q(x, y) = n$ and $\rho_A(n)$ the number of integral ideals with norm n in the ideal class A. Then $r_A(n) = w\rho_A(n)$.

Let $\mathcal{O}$ be the order of conductor f of an imaginary quadratic field K of discriminant d_K. The class number is given by the following formula.

FACT 2.2. ([4], **Theorem 7.24**)

$$h(\mathcal{O}) = \frac{h(\mathcal{O}_K)f}{[\mathcal{O}_K^\times : \mathcal{O}^\times]} \prod_{p|f} \Big(1 - \Big(\frac{d_K}{p}\Big)\frac{1}{p}\Big).$$

LEMMA 2.1.

$$\mathcal{O}\Big[\frac{1}{f}\Big] = \mathcal{O}_K\Big[\frac{1}{f}\Big].$$

Proof. Obviously $\mathcal{O}[\frac{1}{f}]$ is contained in $\mathcal{O}_K[\frac{1}{f}]$. It is sufficient to show that $\mathcal{O}_K$ is contained in $\mathcal{O}[\frac{1}{f}]$. Let x be an element of $\mathcal{O}_K$. Since the conductor of $\mathcal{O}$ is f, there is an integer m such that $y = m + fx$ is an element of $\mathcal{O}$. Therefore

$$x = \frac{y - m}{f} \in \mathcal{O}\Big[\frac{1}{f}\Big].$$

□

PROPOSITION 2.1. *Let n be a positive integer coprime to fd_K. Then*

$$\frac{1}{w} \sum_{A \in \mathrm{Pic}(\mathcal{O})} r_A(n) = \sum_{m|n} \Big(\frac{d_K}{m}\Big).$$

Proof. We define the zeta function

$$\zeta_{\mathcal{O}}^{\dagger}(s) = \sum_{\mathfrak{A}, (N\mathfrak{A}, fd_K)=1} (N\mathfrak{A})^{-s},$$

where $\mathfrak{A}$ runs through the integral ideals of $\mathcal{O}$ whose norm is coprime to fd_K. Computing this function in two ways, we will show the claim. By the bijective correspondence between quadratic forms of discriminant $f^2 d_K$ and ideal classes of $\mathcal{O}$ (in particular the remark after **Fact 2.1**),

$$\zeta_{\mathcal{O}}^{\dagger}(s) = \sum_{n=1, (fd_K, n)=1}^{\infty} n^{-s} \sum_{A \in \mathrm{Pic}\mathcal{O}} \rho_A(n) = \frac{1}{w} \sum_{n=1, (fd_K, n)=1}^{\infty} n^{-s} \sum_{A \in \mathrm{Pic}\mathcal{O}} r_A(n).$$

On the other hand, since

$$\{\mathfrak{P} \in \mathrm{Spec}\mathcal{O} \mid f \notin \mathfrak{P}\} = \mathrm{Spec}\mathcal{O}\Big[\frac{1}{f}\Big]$$

and

$$\{\mathfrak{P} \in \mathrm{Spec}\mathcal{O}_K \mid f \notin \mathfrak{P}\} = \mathrm{Spec}\mathcal{O}_K\Big[\frac{1}{f}\Big],$$

the lemma shows that

$$\{\mathfrak{P} \in \mathrm{Spec}\mathcal{O} \mid f \notin \mathfrak{P}\} = \{\mathfrak{P} \in \mathrm{Spec}\mathcal{O}_K \mid f \notin \mathfrak{P}\}.$$

Therefore

$$\begin{aligned}
\zeta_{\mathcal{O}}^{\dagger}(s) &= \sum_{\mathfrak{A}\in \mathrm{Pic}\mathcal{O}, (N\mathfrak{A}, f d_K)=1} (N\mathfrak{A})^{-s} \\
&= \sum_{\mathfrak{A}\in \mathrm{Pic}\mathcal{O}_K, (N\mathfrak{A}, f d_K)=1} (N\mathfrak{A})^{-s} \\
&= \prod_{\mathfrak{P}, (N\mathfrak{P}, f d_K)=1} \sum_{k=0}^{\infty} (N\mathfrak{P})^{-ks}
\end{aligned}$$

where $\mathfrak{P}$ runs through the non-zero prime ideals of $\mathcal{O}_K$ whose norms are coprime to $f d_K$. Thus

$$\begin{aligned}
\zeta_{\mathcal{O}}^{\dagger}(s) &= \prod_{(p, f d_K)=1, (\frac{d_K}{p})=1} (1-p^{-s})^{-2} \prod_{(p, f d_K)=1, (\frac{d_K}{p})=-1} (1-p^{-2s})^{-1} \\
&= \prod_{(p, f d_K)=1} (1-p^{-s})^{-1} \prod_{(p, f d_K)=1} \Big(1-\Big(\frac{d_K}{p}\Big)p^{-s}\Big)^{-1} \\
&= \zeta^{\dagger}(s) L^{\dagger}(s, \epsilon_{d_K}),
\end{aligned}$$

where $\zeta^{\dagger}(s)$ is the Riemann zeta function with the Euler factors at the primes dividing $f d_K$ removed :

$$\zeta^{\dagger}(s) = \sum_{n=1, (n, f d_K)=1}^{\infty} n^{-s},$$

and

$$L^{\dagger}(s, \epsilon_{d_K}) = \sum_{n=1, (n, f d_K)=1}^{\infty} \Big(\frac{d_K}{n}\Big) n^{-s}.$$

Therefore

$$\sum_{n=1, (f d_K, n)=1}^{\infty} n^{-s} \Big(\frac{1}{w} \sum_{A\in \mathrm{Pic}\mathcal{O}} r_A(n)\Big) = \sum_{n=1, (n, f d_K)=1}^{\infty} n^{-ks} \sum_{m|n} \Big(\frac{d_K}{m}\Big).$$

Comparing the coefficients of n^{-s}, the desired equation is obtained. □

3. Theta functions

Following [12], we recall basic facts of modular forms. Let k and N be a nonnegative integer and a positive integer, respectively. In this report we will consider modular forms for the following congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : a \equiv d \equiv 1,\ c \equiv 0 \pmod N \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : c \equiv 0 \pmod N \right\}.$$

$\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ with quotient is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$. Let $\mathbb{H}$ be the upper half plane, $\mathbb{H} := \{z \in \mathbb{C} \ : \ \Im z > 0\}$, and $\mathcal{O}(\mathbb{H})$ the set of holomorphic functions on $\mathbb{H}$. A modular form on $\Gamma_i(N)(i = 0, 1)$ of weight k is a function $f \in \mathcal{O}(\mathbb{H})$ satisfying the functional equation

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^k f(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_i(N).$$

Since f admits the Fourier expansion at the cusp ν,

$$f = \sum_n a_\nu(n) q_\nu^n$$

where q_ν is the local parameter at ν defined by

$$q_\nu = \exp(2\pi\alpha_\nu z), \quad \exists \alpha_\nu \in \mathbb{Q}.$$

We say that f is *regular* at ν if $a_\nu(n) = 0$ for $n < 0$. The set of modular forms on $\Gamma_i(N)$ of weight k which are regular at any cusp is denoted by $M_k(\Gamma_i(N))$. Moreover, we define the set of cusp forms on $\Gamma_i(N)$ of weight k to be

$$S_k(\Gamma_i(N)) = \{f \in M_k(\Gamma_i(N)) \,|\, a_\nu(0) = 0 \quad \forall \nu\}.$$

Since $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ whose quotient is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$, $S_k(\Gamma_1(N))$ is decomposed by the characters of $(\mathbb{Z}/N\mathbb{Z})^\times$,

$$S_k(\Gamma_1(N)) = \oplus_\chi S_k(\Gamma_0(N), \chi),$$

where

$$S_k(\Gamma_0(N), \chi) = \left\{ f \in S_k(\Gamma_1(N)) \,\Big|\, f\Big(\frac{az+b}{cz+d}\Big) = \chi(a)(cz+d)^k f(z),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.$$

By definition $S_k(\Gamma_0(N), \mathbf{1}) = S_k(\Gamma_0(N))$ where $\mathbf{1}$ is the trivial character. It is well-known that $M_k(\Gamma_i(N))$ (and hence $S_k(\Gamma_i(N))$) is a finite dimensional vector space over $\mathbb{C}$ for $i = 0, 1$.

Now let $Q(x, y) = ax^2 + bxy + cy^2$ be a positive definite primitive integral quadratic form.

DEFINITION 3.1.

$$\theta_Q(z) = \sum_{n=0}^{\infty} r(Q,n)q^n, \quad q = \exp(2\pi i z)\,,$$

where

$$r(Q,n) = |\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid Q(x,y) = n\}|\,.$$

The quadratic form $Q(x, y)$ can be written in the form

$$Q(x,y) = \frac{1}{2}(x,y)A(x,y)^t, \quad A = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

Since Q is positive definite, A is invertible.

DEFINITION 3.2. The *level* of Q is the smallest positive integer N such that NA^{-1} is again an integral matrix whose diagonal entries are all even. The *discriminant* Δ of Q is defined by

$$\Delta := b^2 - 4ac\,.$$

For any odd prime p not dividing N let

$$\epsilon_\Delta(p) = \left(\frac{\Delta}{p}\right), \quad \text{(Legendre symbol)}$$

(see [11], p. 303).

FACT 3.1. ([1], **Theorem 2.2**) Let Q be a positive definite binary quadratic form of level N and discriminant Δ. Then θ_Q is a modular form on $\Gamma_0(N)$ of weight one with Nebentypus character ϵ_Δ which is regular at any cusp. Namely

$$\theta_Q\left(\frac{az+b}{cz+d}\right) = \epsilon_\Delta(a)(cz+d)\theta_Q(z)$$

for $z \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

4. Results

4.1. The cases of class number 3; $\eta_{2\cdot 22}$ and $\eta_{6\cdot 18}$

Let $\mathcal{O}_{-d}$ be the order of the quadratic fields K in the following table;

d	44	108
K	$\mathbb{Q}(\sqrt{-11})$	$\mathbb{Q}(\sqrt{-3})$
$\mathcal{O}_{-d}$	$\mathbb{Z} + 2\mathcal{O}_K$	$\mathbb{Z} + 6\mathcal{O}_K$
f	2	6

Here f and $-d$ are the conductor and the discriminant of $\mathcal{O}_{-d}$, respectively. Let us identify $C(-d)$ with $\mathrm{Pic}(\mathcal{O}_{-d})$ by **Fact 2.1**. Then **Fact 2.2** tells us that the class numbers of these orders are 3. Hence $C(-d) \simeq \mathrm{Pic}(\mathcal{O}_{-d})$ are isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and we write the isomorphism as,

$$\phi_{-d} : \mathrm{Pic}(\mathcal{O}_{-d}) \to \mathbb{Z}/3\mathbb{Z}, \quad \phi_{-d}(Q_i^{(-d)}) = [i],\ (i = 0, 1, 2)\,,$$

where $\{Q_i^{(-d)}\}_{i=0,1,2}$ are defined by the following table;

d	$Q_0^{(-d)}$	$Q_1^{(-d)}$	$Q_2^{(-d)}$
44	$x^2 + 11y^2$	$3x^2 + 2xy + 4y^2$	$3x^2 - 2xy + 4y^2$
108	$x^2 + 27y^2$	$4x^2 + 2xy + 7y^2$	$4x^2 - 2xy + 7y^2$

Let χ_3 be the character of $\mathrm{Pic}(\mathcal{O}_{-d})$ defined by

$$\chi_3(Q_i^{(-d)}) = \zeta_3^i, \quad \zeta_3 = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right), \tag{4}$$

and set

$$f_{d,\chi_3}(q) = \frac{1}{2}\left\{\chi_3(Q_0^{(-d)})\theta_{Q_0^{(-d)}}(q) + \chi_3(Q_1^{(-d)})\theta_{Q_1^{(-d)}}(q) + \chi_3(Q_2^{(-d)})\theta_{Q_2^{(-d)}}(q)\right\}.$$

It is easy to check that the level of the quadratic form $Q_i^{(-d)}$ is d for $i = 0, 1, 2$.

Therefore, $f_{d,\chi_3}(q) \in S_1(\Gamma_0(d), \epsilon_{-d})$ by **Fact 3.1**. Moreover it is a Hecke eigenform since the L-function has an Euler product ;

$$L(f_{d,\chi_3}(q), s) = \sum_{\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_{-d})} \chi_3(\mathcal{A}) \sum_{\mathfrak{A}\in\mathcal{A}:\text{integral}} N\mathfrak{A}^{-s} = \prod_{\mathfrak{P}:\text{prime}} \frac{1}{1 - \chi_3(\mathfrak{P})N\mathfrak{P}^{-s}}\,. \tag{5}$$

Here we regard χ_3 as a character on the ideal group of $\mathcal{O}_{-d}$ via the projection. Since the solutions of $Q_1^{(-d)}(x, y) = n$ and $Q_2^{(-d)}(x, y) = n$ are exchanged by the involution

$$(x, y) \to (x, -y)\,,$$

we see that $r(Q_1^{(-d)}, n) = r(Q_2^{(-d)}, n)$ for every nonnegative integer n. Therefore

$$\theta_{Q_1^{(-d)}}(q) = \theta_{Q_2^{(-d)}}(q)\,, \tag{6}$$

and

$$f_{d,\chi_3}(q) = \frac{1}{2}\left\{\theta_{Q_0^{(-d)}}(q) - \theta_{Q_1^{(-d)}}(q)\right\}.$$

PROPOSITION 4.1. *Suppose that $d = 44$ or 108. Let $c(n)$ be the n-th Fourier coefficient of $\frac{1}{2}\{\theta_{Q_0^{(-d)}}(q) - \theta_{Q_1^{(-d)}}(q)\}$ and p a prime satisfying $(p, d) = 1$. Then*

$$c(p) = 0 \quad \text{if} \left(\frac{-d}{p}\right) = -1\,.$$

Moreover if $(\frac{-d}{p}) = 1$,

$$c(p) = \begin{cases} 2 & \text{if } p \text{ is represented by } Q_0^{(-d)}, \\ -1 & \text{if } p \text{ is represented by } Q_1^{(-d)}. \end{cases}$$

Proof. By definition,

$$c(p) = \frac{1}{2}\left(r(Q_0^{(-d)}, p) - r(Q_1^{(-d)}, p)\right).$$

On the other hand, **Proposition 2.1** and (6) show that

$$r\left(Q_0^{(-d)}, p\right) + 2r\left(Q_1^{(-d)}, p\right) = 2\left(1 + \left(\frac{-d}{p}\right)\right). \tag{7}$$

Thus if $(\frac{-d}{p}) = -1$, then $r(Q_0^{(-d)}, p) = r(Q_1^{(-d)}, p) = 0$ and this implies half of our claim. Suppose that $(\frac{-d}{p}) = 1$. Then (7) yields

$$r\left(Q_0^{(-d)}, p\right) + 2r\left(Q_1^{(-d)}, p\right) = 4.$$

Here note that if (x, y) is an integral solution of $Q_i^{(-d)}(x, y) = n$ so is $(-x, -y)$. Therefore $\{r(Q_i^{(-d)}, p)\}_{i=0,1}$ are always nonnegative even integers and there are two possibilities,

$$\left(r\left(Q_0^{(-d)}, p\right), r\left(Q_1^{(-d)}, p\right)\right) = (4, 0), \quad \text{or} \quad (0, 2),$$

which prove the rest of the claim. □

As we have seen in (5), since $f_{d,\chi_3}(q) \in S_1(\Gamma_0(d), \epsilon_{-d})$ $(d = 44, 108)$ is a cuspidal Hecke eigenform of weight one, it corresponds to a Galois representation ([5]),

$$\rho_d : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

unramified outside d, such that

$$\mathrm{Tr}\rho_d(\phi_p) = c_p(f_{d,\chi_3}(q)), \quad \det\rho_d(\phi_p) = \epsilon_{-d}(p) = \left(\frac{-d}{p}\right).$$

Here ϕ_p is the Frobenius at a prime p with $(p, d) = 1$. Let us identify the conjugacy class of $\rho_d(\phi_p)$. By **Proposition 4.1** we see that

$$\mathrm{Tr}\rho_d(\phi_p) = \begin{cases} 0 & \text{if} \quad (\frac{-d}{p}) = -1, \\ 2 & \text{if} \quad (\frac{-d}{p}) = 1 \text{ and } p \text{ is representable by } Q_0^{(-d)}, \\ -1 & \text{if} \quad (\frac{-d}{p}) = 1 \text{ and } p \text{ is representable by } Q_1^{(-d)}. \end{cases}$$

Since $\det\rho_d(\phi_p) = (\frac{-d}{p})$, $\rho_d(\phi_p)$ satisfies

$$\rho_d(\phi_p) \sim \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if} \quad (\frac{-d}{p}) = -1, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad (\frac{-d}{p}) = 1 \text{ and } p \text{ is representable by } Q_0^{(-d)}, \\ \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix} & \text{if} \quad (\frac{-d}{p}) = 1 \text{ and } p \text{ is representable by } Q_1^{(-d)}, \end{cases}$$

where $\sim$ means "conjugate". By class field theory, the character (4) induces a homomorphism

$$\chi_{3,K} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times, \quad K = \mathbb{Q}(\sqrt{-d}),$$

and the previous computation shows that ρ_d is the induced representation of $\chi_{3,K}$. Thus we have proved the following proposition.

PROPOSITION 4.2. *Suppose that* $d = 44$ *or* 108. *Then the Galois representation* ρ_d *is the induced representation of* $\chi_{3,K}$.

THEOREM 4.1. *Denote* $\eta_{d_1 d_2} = \eta_{d_1 \cdot d_2}$. *If* $(d_1, d_2) = (2, 22)$ *or* $(6, 18)$,

$$\eta_d(q) = f_{d,\chi_3}(q) = \frac{1}{2}\left\{\theta_{Q_0^{(-d)}}(q) - \theta_{Q_1^{(-d)}}(q)\right\}.$$

REMARK 4.1. In [6] [7], Hiramatsu determined the Fourier coefficients of $\eta_{2\cdot 22}$ and $\eta_{6\cdot 18}$.

Proof. The results of Martin ([10]) says that $\eta_d(q)$ is a cuspidal Hecke eigenform on $\Gamma_1(d)$ of weight one. In (5) we have seen that $f_{d,\chi_3}(q) \in S_1(\Gamma_0(d), \epsilon_{-d})$ is also a cupsidal Hecke eigenform. Since the Fourier coefficients $\{c_n\}_{n\geq 1}$ of a Hecke eigenform are determined by $\{c_p\}_{p:\text{prime}}$, let us compare some of them. Let p be a prime. If $(p, d) = 1$, the p-th Fourier coefficient of

$$f_{d,\chi_3}(q) = \frac{1}{2}\left\{\theta_{Q_0^{(-d)}}(q) - \theta_{Q_2^{(-d)}}(q)\right\}$$

can be computed by **Proposition 4.1**. If p divides d, we can compute c_p by hand. On the other hand we obtain the coefficients of $\eta_d(q)$ by computer. Here are the results which suggests that $f_{d,\chi_3}(q) = \eta_d(q)$.

p	2	3	5	7	11	13	17	19	23	29
$c_p(\eta_{44})$	0	-1	-1	0	1	0	0	0	-1	0
$c_p(f_{44,\chi_3})$	0	-1	-1	0	1	0	0	0	-1	0

p	31	37	41	43	47	53	59	61	67	71
$c_p(\eta_{44})$	-1	-1	0	0	2	2	-1	0	-1	-1
$c_p(f_{44,\chi_3})$	-1	-1	0	0	2	2	-1	0	-1	-1

p	2	3	5	7	11	13	17	19	23	29
$c_p(\eta_{108})$	0	0	0	-1	0	-1	0	-1	0	0
$c_p(f_{108,\chi_3})$	0	0	0	-1	0	-1	0	-1	0	0

p	31	37	41	43	47	53	59	61	67	71
$c_p(\eta_{108})$	2	-1	0	2	0	0	0	-1	-1	0
$c_p(f_{108,\chi_3})$	2	-1	0	2	0	0	0	-1	-1	0

In order to show the desired identity we will compare the Fourier coefficients of $f_{d,\chi_3}(q)$ and $\eta_d(q)$ (not only for primes). According to [8]**Theorem 1.12**, it is enough to check $c_n(f_{d,\chi_3}) = c_n(\eta_d)$ for $n \leq \frac{\mu(d)}{12}$, where

$$\mu(d) := d \prod_{p|d,\, p:\text{prime}} \left(1 + \frac{1}{p}\right).$$

The coefficients of $\eta_d(q)$ are obtained by machine calculations. On the other hand, the coefficients of $f_{d,\chi_3}(q)$ are computed by the associated L-function. In fact, since $f_{d,\chi_3}(q) \in S_1(\Gamma_0(d), \epsilon_{-d})$ is a Hecke eigenform, the L-function has an Euler product

$$L(f_{d,\chi_3}(q), s) = \sum_{n=1}^{\infty} c_n(f_{d,\chi_3}) n^{-s} = \prod_p \frac{1}{1 - c_p(f_{d,\chi_3}) p^{-s} + \epsilon_{-d}(p) p^{-2s}} .$$

Knowing $c_p(f_{d,\chi_3})$ for the primes p less than or equal to $\mu(d)/12$, we expand the right hand side and $c_n(f_{d,\chi_3})$ for $n \leq \frac{\mu(d)}{12}$ are determined. For example, let us consider the case of $d = 44$. Since $\mu(44) = 72$, it is sufficient to compare the Fourier coefficients up to order 6. Using the table, we see that the Euler factors of $L(f_{44,\chi_3}(q), s)$ at 2 and 11 are 1 and $1 - 11^{-s}$, respectively. Hence we have

$$\sum_{n=1}^{\infty} c_n(f_{44,\chi_3}) n^{-s} = \frac{1}{1 - 11^{-s}} \prod_{p \neq 2,11} \frac{1}{1 - c_p(f_{44,\chi_3}) p^{-s} + (\frac{-11}{p}) p^{-2s}} .$$

In order to compute $c_n(f_{44,\chi_3})$ for $n \leq 6$, it is sufficient to know $c_p(f_{44,\chi_3})$ for the primes $p = 2, 3$ and 5, which are given by the table. On the other hand, the coefficients of $\eta_{44}(q)$ can be determined by computer from the q-expansion of the definition,

$$\eta_{44}(q) = q \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{22n}) .$$

We compare these coefficients and conclude that

$$f_{44,\chi_3}(q) = \eta_{44}(q) .$$

The case of $d = 108$ is similar. $\square$

4.2. The cases of class number 4 **;** $\eta_{3\cdot 21}$**,** $\eta_{4\cdot 20}$**,** $\eta_{8\cdot 16}$ **and** $\eta_{12\cdot 12}$

Let $\mathcal{O}_{-d}$ be the order of the quadratic fields K in the following table;

d	63	80	128	144
K	$\mathbb{Q}(\sqrt{-7})$	$\mathbb{Q}(\sqrt{-5})$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-1})$
$\mathcal{O}_{-d}$	$\mathbb{Z} + 3\mathcal{O}_K$	$\mathbb{Z} + 2\mathcal{O}_K$	$\mathbb{Z} + 4\mathcal{O}_K$	$\mathbb{Z} + 6\mathcal{O}_K$
f	3	2	4	6

As before f and $-d$ are the conductor and the discriminant of $\mathcal{O}_{-d}$, respectively. Using **Fact 2.2**, one sees that the class numbers of these orders are 4. Let us identify $\mathrm{Pic}(\mathcal{O}_{-d})$ with $C(-d)$ by **Fact 2.1**. A simple computation shows that $C(-d)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, and we write the isomorphism as

$$\phi_{-d} : \mathrm{Pic}(\mathcal{O}_{-d}) \to \mathbb{Z}/4\mathbb{Z}, \quad \phi_{-d}(Q_i^{(-d)}) = [i], \ (i = 0, 1, 2, 3) .$$

Here is a table of $\{Q_i^{(-d)}\}_{i=0,1,2,3}$.

d	$Q_0^{(-d)}$	$Q_1^{(-d)}$	$Q_2^{(-d)}$	$Q_3^{(-d)}$
63	$x^2+xy+16y^2$	$2x^2+xy+8y^2$	$4x^2+xy+4y^2$	$2x^2-xy+8y^2$
80	x^2+20y^2	$3x^2+2xy+7y^2$	$4x^2+5y^2$	$3x^2-2xy+7y^2$
128	x^2+32y^2	$3x^2+2xy+11y^2$	$4x^2+4xy+9y^2$	$3x^2-2xy+11y^2$
144	x^2+36y^2	$5x^2+4xy+8y^2$	$4x^2+9y^2$	$5x^2-4xy+8y^2$

It is easy to check that the level of $Q_i^{(-d)}$ is d. Let χ_4 be the character of $\mathrm{Pic}(\mathcal{O}_{-d})$ defined by

$$\chi_4\big(Q_i^{(-d)}\big) = (\sqrt{-1})^i\,, \tag{8}$$

and set

$$f_{d,\chi_4}(q) = \frac{1}{2}\sum_{i=0}^{3}\chi_4\big(Q_i^{(-d)}\big)\theta_{Q_i^{(-d)}}(q) = \frac{1}{2}\big\{\theta_{Q_0^{(-d)}}(q) - \theta_{Q_2^{(-d)}}(q)\big\}\,.$$

Here the right equality is due to

$$Q_1^{(-d)}(x,-y) = Q_3^{(-d)}(x,y)\,,$$

which yields

$$\theta_{Q_1^{(-d)}}(q) = \theta_{Q_3^{(-d)}}(q)\,,$$

as (6). By the same reason as $f_{d,\chi_3}(q)$, $f_{d,\chi_4}(q)$ is a Hecke eigenform in $S_1(\Gamma_0(d),\epsilon_{-d})$.

PROPOSITION 4.3. *Suppose that* $d = 63, 80, 128$ *or* 144. *Let* $c(n)$ *be the* n*-th Fourier coefficient of* $\frac{1}{2}\{\theta_{Q_0^{(-d)}}(q) - \theta_{Q_2^{(-d)}}(q)\}$ *and* p *a prime satisfying* $(p,d)=1$. *Then*

$$c(p) = 0 \quad \text{if}\left(\frac{d}{p}\right) = -1\,.$$

Moreover, if $(\frac{d}{p}) = 1$,

$$c(p) = \begin{cases} 0 & \text{if } p \text{ is represented by } Q_1^{(-d)}\,,\\ 2 & \text{if } p \text{ is represented by } Q_0^{(-d)}\,,\\ -2 & \text{if } p \text{ is represented by } Q_2^{(-d)}\,. \end{cases}$$

Proof. By definition,

$$c(p) = \frac{1}{2}\Big(r\big(Q_0^{(-d)},p\big) - r\big(Q_2^{(-d)},p\big)\Big)\,,$$

and **Proposition 2.1** shows that

$$r\big(Q_0^{(-d)},p\big) + 2r\big(Q_1^{(-d)},p\big) + r\big(Q_2^{(-d)},p\big) = 2\Big(1+\Big(\frac{-d}{p}\Big)\Big)\,. \tag{9}$$

Thus if $(\frac{-d}{p}) = -1$,

$$r\big(Q_0^{(-d)},p\big) = r\big(Q_1^{(-d)},p\big) = r\big(Q_2^{(-d)},p\big) = 0\,,$$

and half of the claim is proved. Suppose that $(\frac{-d}{p}) = 1$. Then (9) yields

$$r\big(Q_0^{(-d)}, p\big) + 2r\big(Q_1^{(-d)}, p\big) + r\big(Q_2^{(-d)}, p\big) = 4\,.$$

As we have seen, $\{r(Q_i^{(-d)}, p)\}_{i=0,1,2}$ are always nonnegative even integers. We claim that $r(Q_i^{(-d)}, p)$ is a multiple of 4 for $i = 0, 2$. In fact, the space of solutions

$$S = \big\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \,\big|\, Q_i^{(-d)}(x, y) = p\big\}, \quad (i = 0, 2)\,,$$

admits the action of involutions σ and τ defined by

$Q_i^{(-d)}$	$Q_0^{(-63)}$	$Q_2^{(-63)}$	$Q_2^{(-128)}$	the remainings
$\sigma(x, y)$	$(-x, -y)$	$(-x, -y)$	$(-x, -y)$	$(-x, y)$
$\tau(x, y)$	$(x + y, -y)$	(y, x)	$(x + y, -y)$	$(x, -y)$

It is easy to check that the actions have no fixed point. Hence we see that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by the involutions σ and τ freely acts on S and the claim is proved. Therefore there are three possibilities,

$$\big(r(Q_0^{(-d)}, p),\ r\big(Q_1^{(-d)}, p\big),\ r\big(Q_2^{(-d)}, p\big)\big) = (4, 0, 0), \quad (0, 2, 0) \quad \text{or} \quad (0, 0, 4)\,.$$

which proves the rest of our claim. □

$f_{d,\chi_4}(q) \in S_1(\Gamma_0(d), \epsilon_{-d})$ $(d = 63, 80, 128, 144)$ is a cuspidal Hecke eigenform of weight one since the L-function has an Euler product :

$$(10)\quad L(f_{d,\chi_4}(q), s) = \sum_{\mathcal{A}\in \mathrm{Pic}(\mathcal{O}_{-d})} \chi_4(\mathcal{A}) \sum_{\mathfrak{A}\in\mathcal{A}:\text{integral}} N\mathfrak{A}^{-s} = \prod_{\mathfrak{P}:\text{prime}} \frac{1}{1 - \chi_4(\mathfrak{P})N\mathfrak{P}^{-s}}\,.$$

Hence it corresponds to a Galois representation

$$\rho_d : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

satisfying

$$\mathrm{Tr}\rho_d(\phi_p) = c_p(f_{d,\chi_4}), \quad \det\rho_d(\phi_p) = \epsilon_{-d}(p) = \Big(\frac{-d}{p}\Big)$$

for any prime p with $(p, d) = 1$ as before. By class field theory the character defined by (8) induces a homomorphism

$$\chi_{4,K} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times, \quad K = \mathbb{Q}(\sqrt{-d})$$

The same argument as in the previous subsection implies the following theorem.

PROPOSITION 4.4. *Let* $d = 63, 80, 128$ *or* 144. *Then the Galois representation* ρ_d *is the induced representation of* $\chi_{4,K}$.

THEOREM 4.2. *Set* $\eta_{d_1 d_2} = \eta_{d_1\cdot d_2}$, *where* $(d_1, d_2) = (3, 21), (4, 20), (8, 16)$ *or* $(12, 12)$. *Then*

$$\eta_d(q) = f_{d,\chi_4}(q) = \frac{1}{2}\big\{\theta_{Q_0^{(-d)}}(q) - \theta_{Q_2^{(-d)}}(q)\big\}\,.$$

In fact, the following tables suggest the theorem. Since the proof of the theorem is similar to that of **Theorem 4.1**, we omit it.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61
$a_p(\eta_{63})$	0	0	0	-1	0	0	0	0	0	0	0	-2	0	-2	0	0	0	0
$a_p(f_{63,\chi_4})$	0	0	0	-1	0	0	0	0	0	0	0	-2	0	-2	0	0	0	0

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61
$a_p(\eta_{80})$	0	0	-1	0	0	0	0	0	0	2	0	0	-2	0	0	0	0	-2
$a_p(f_{80,\chi_4})$	0	0	-1	0	0	0	0	0	0	2	0	0	-2	0	0	0	0	-2

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61
$a_p(\eta_{128})$	0	0	0	0	0	0	-2	0	0	0	0	0	2	0	0	0	0	0
$a_p(f_{128,\chi_4})$	0	0	0	0	0	0	-2	0	0	0	0	0	2	0	0	0	0	0

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61
$a_p(\eta_{144})$	0	0	0	0	0	-2	0	0	0	0	0	2	0	0	0	0	0	2
$a_p(f_{144,\chi_4})$	0	0	0	0	0	-2	0	0	0	0	0	2	0	0	0	0	0	2

Proof of Theorem 1.1.. Since the proofs of the congruences are similar, we will only show the congruence of $T_2(p)$ modulo 11. Putting $p = 11$ and $x = q^{2n}$, the equation (2) yields

$$(1 - q^{2n})^{11} \equiv 1 - q^{22n} \pmod{11}.$$

Thus we have

$$\eta_{2\cdot 22}(q) = q\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{22n}) \equiv q\prod_{n=1}^{\infty}(1-q^{2n})^{12} = \sum_{n=1}^{\infty} T_2(n)q^n \pmod{11}$$

and **Theorem 4.1** yields

$$\sum_{n=1}^{\infty} T_2(n)q^n \equiv \frac{1}{2}\left\{\theta_{Q_0^{(-44)}}(q) - \theta_{Q_2^{(-44)}}(q)\right\} \pmod{11}.$$

Now the desired congruence follows from **Proposition 4.1**. □

References

[1] A. N. Andrianov and V. G. Zhuravlev, Modular Forms and Hecke Operators, AMS, 1989. ISBN 978-1-4704-1868-7.

[2] R. P. Bambah, Two congruence properties of Ramanujan's function $\tau(n)$, Proc. London Math. Soc., 21: 91–93, 1946.

[3] J. H. Bruiner, G. van der Geer, G. Harder and D. Zagier, The $1-2-3$ of modular forms, Springer, 2008. ISBN 978-3-540-74117-6.

[4] D. Cox, Primes of The Form $x^2 + ny^2$, Wiley Interscience, 1989. ISBN 0-471-50654-0.

[5] P. Deligne and J. P. Serre, Formes modulaires de poids 1, Annales. Sci. de l'E.N.S., tome 7, nombre 4: 507–530, 1974.

[6] T. Hiramatsu, Higher reciprocity laws and modular forms of weight one, Comment. Math. Univ. St. Pauli, 31, no. 1: 75–85, 1982.

[7] T. Hiramatsu, Theory of automorphic forms of weight 1, Investigations in number theory, Adv. Stud. Pure Math. 13: 503–584, 1988.

[8] G. Köhler, Eta Products and Theta Series Identities, Springer, 2010. ISBN 978-3-642-16152-0.

[9] D. H. Lehmer, The vanishing of Ramanujan's function $\tau(n)$, Duke Math. J., 14: 429–433, 1947.

[10] Y. Martin, Multiplicative η-quotients, Trans. of A.M.S., 348: 4825–4856, 1996.
[11] Y.I. Manin and A. A. Panchishkin, Introduction to Modern Number Theory: Fundamental problems, Ideas and Theories, Springer-Verlag, 2005. ISBN 978-3540203643.
[12] T. Miyake, Modular Forms, Springer-Verlag, 1989. ISBN 978-3540295921.
[13] K. G. Ramanathan, Congruence properties of Ramanujan's function $\tau(n)$ (II), J. Indian Math. Soc., 9: 55–59, 1945.
[14] S. Ramanujan, On certain arithmetic functions, Trans. Cambridge Phil. Soc., 22: 159–184, 1916.
[15] J. P. Serre, Une interprétation des congruences relatives à la fonction de Ramanujan, Sém. Delange-Pisot-Poitou. Théorie des nombres., tome 9, nombre 1: 1–17, 1968.
[16] J. R. Wilton, Congruence properties of Ramanujan's function $\tau(n)$, Proc. London Math. Soc., 31: 1–10, 1930.

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