Orbital Exponential Sums for Some Quadratic Prehomogeneous Vector Spaces

by

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(Received January 16, 2019)
(Revised May 21, 2019)

Abstract. Let \((G, V)\) be a prehomogeneous vector space over a finite field of odd characteristic. Taniguchi and Thorne [5] developed a method to calculate explicit formulas of the Fourier transforms of any \(G\)-invariant functions over \(V\). By means of their method, we calculate the Fourier transform of any \(G\)-invariant function for several “quadratic” prehomogeneous vector spaces, parametrizing quadratic fields. We use PARI/GP [8] for explicit matrix calculations.

1. Introduction

Let \(K\) be a field and \(\overline{K}\) be its algebraic closure. Let \(V\) be a finite dimensional representation of a reductive algebraic group \(G\) defined over \(K\). When there exists a \(G(\overline{K})\)-orbit of \(V(\overline{K})\) which is Zariski open, we refer to the pair \((G, V)\) as a prehomogeneous vector space. Let us consider a prehomogeneous vector space \((G, V)\) defined over a finite field.

Let \(p\) be an odd prime, and let \(\mathbb{F}_q\) be a finite field of order \(q = p^n\). Let \(V^*\) be the dual space of \(V\). For a function \(\phi : V(\mathbb{F}_q) \rightarrow \mathbb{C}\), its Fourier transform \(\hat{\phi} : V^*(\mathbb{F}_q) \rightarrow \mathbb{C}\) is defined as follows:

\[
\hat{\phi}(y) := |V(\mathbb{F}_q)|^{-1} \sum_{x \in V(\mathbb{F}_q)} \phi(x) \exp \left( \frac{2\pi i \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}([x, y])}{p} \right).
\]

Here, \([x, y] = y(x) \in \mathbb{F}_q\) is the canonical pairing of \(V(\mathbb{F}_q)\) and \(V^*(\mathbb{F}_q)\), and \(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \rightarrow \mathbb{F}_p\) is the trace map. The purpose of this paper is to determine an explicit formula for the Fourier transform of any \(G(\mathbb{F}_q)\)-invariant function \(\phi\) for certain prehomogeneous vector spaces. Taniguchi and Thorne [5] developed a general method to compute this type of Fourier transform and applied it to obtain explicit formulas for the following prehomogeneous vector spaces \((G, V)\) over \(\mathbb{F}_q\):

- \(V = \text{Sym}^3(2)\), the space of binary cubic forms; \(G = \text{GL}_2\),
- \(V = \text{Sym}^2(2),\) the space of binary quadratic forms; \(G = \text{GL}_1 \times \text{GL}_2,\)
- \(V = \text{Sym}^2(3)\), the space of ternary quadratic forms; \(G = \text{GL}_1 \times \text{GL}_3,\)
- \(V = 2 \otimes \text{Sym}^2(2)\), the space of pairs of binary quadratic forms; \(G = \text{GL}_2 \times \text{GL}_2,\)
- \(V = 2 \otimes \text{Sym}^2(3)\), the space of pairs of ternary quadratic forms; \(G = \text{GL}_2 \times \text{GL}_3,\)
There are many prehomogeneous vector spaces for which the Fourier transform is not yet calculated. In this paper, we study the following six more prehomogeneous vector spaces over $\mathbb{F}_q$:

- $V = 2 \otimes 2 \otimes 2$, the space of pairs of 2-by-2 matrices; $G = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_2$,
- $V = 2 \otimes 2 \otimes 3$, the space of triplets of 2-by-2 matrices; $G = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_3$,
- $V = 2 \otimes 2 \otimes 4$, the space of quadruples of 2-by-2 matrices; $G = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_4$,
- $V = 2 \otimes H_2(\mathbb{F}_q^2)$, the space of pairs of Hermitian matrices of order 2; $G = \text{GL}_2 \times \text{GL}_2(\mathbb{F}_q^2)$,
- $V = 2 \otimes \wedge^2(4)$, the space of pairs of alternating matrices of order 4; $G = \text{GL}_2 \times \text{GL}_4$,
- $V$ is the space of binary tri-Hermitian forms over $\mathbb{F}_q^3$; $G = \text{GL}_1 \times \text{GL}_2(\mathbb{F}_q^3)$.

Our main theorem is as follows:

**Theorem 1.1.** Let $(G, V)$ be the prehomogeneous vector space in the above. We have an explicit formula for the Fourier transform $\widehat{e_i}$ of any indicator function $e_i$ of $G(\mathbb{F}_q)$-orbit $O_i$ in $V(\mathbb{F}_q)$.

For the concrete formulas, we refer to Theorems 4.1, 5.1, 6.1, 7.1, 8.1 and 9.1, respectively. As a consequence, we have the Fourier transform $\widehat{\Psi}$ of the indicator function $\Psi$ of the singular set of each space (see Corollaries 4.1, 5.1, 6.1, 7.1, 8.1 and 9.1). As observed in [5], we obtain better than the square root cancellation for $\widehat{\Psi}$ in each space.

The prehomogeneous vector spaces $2 \otimes 2 \otimes 2$ and $2 \otimes \wedge^2(4)$ are appeared in the work of Wright and Yukie [9], where they studied the structure of rational orbits for 8 prehomogeneous vector spaces. $2 \otimes H_2(\mathbb{F}_q^2)$ and the space of binary tri-Hermitian forms are $\mathbb{F}_q$-forms of $2 \otimes 2 \otimes 2$, and were studied by Kable and Yukie [2]. In [9] and [2], those 4 spaces are studied over an infinite field. As stated in [9] and [2], if we consider the prehomogeneous vector space $2 \otimes 2 \otimes 2$, the space of pairs of binary Hermitian forms, $2 \otimes \wedge^2(4)$ or the space of binary tri-Hermitian forms over an infinite field $K$, then the set of non-singular orbits naturally corresponds to the set of isomorphism classes of the separable quadratic algebras of $K$. In this sense we call these spaces as of quadratic cases. (Even though they assume that the base field is infinite, the assumption seems to be irrelevant for their arguments.) In a forthcoming paper [1], we will use the results in this paper for calculations of even more interesting cubic cases.

A classification of reduced irreducible prehomogeneous vector spaces over $\mathbb{C}$ was given by Sato and Kimura [3]. The prehomogeneous vector spaces we study in this paper may be defined over an arbitrary field and in particular over $\mathbb{C}$. If we consider them over $\mathbb{C}$, $2 \otimes 2 \otimes 3$ is a castling transform of a trivial prehomogeneous vector space $2 \otimes 2$, and $2 \otimes 2 \otimes 4$ is another trivial prehomogeneous vector space. $2 \otimes 2 \otimes 3$ and $2 \otimes 2 \otimes 4$ are not of quadratic cases, but we can calculate the Fourier transforms for them by means of the results of $2 \otimes 2 \otimes 2$.

The composition of this paper is as follows. In Section 2, we recall Taniguchi-Thorne’s method of calculating the Fourier transform, and see a simple example of the calculation with the prehomogeneous vector space $(\text{GL}_2, M_2(\mathbb{F}_q))$. 

In Section 3, we review some results for the calculation of the Fourier transform. In Section 3.1, we recall the orbit decomposition of the prehomogeneous vector spaces \((\text{GL}_1 \times \text{GL}_n, \text{Sym}^2(n))\) for \(n = 2, 3, 4\) over \(\mathbb{F}_q\). In Section 3.2, we recall the number of matrices of given rank and the order of the general linear group and the special linear group over \(\mathbb{F}_q\). In Section 3.3, we consider a relationship of orbits and their cardinalities of the prehomogeneous vector spaces \((\text{GL}_2 \times \text{GL}_2 \times \text{GL}_n, 2 \otimes 2 \otimes n)\) for different \(n \in \mathbb{Z}_{\geq 1}\).

In the remaining five sections, we calculate the Fourier transforms for the prehomogeneous vector spaces above by turns. Each section consists of three subsections. In the first subsection, we look into the orbit decomposition of each space. For the orbits \(O_i\) in the tables in Propositions 4.1, 5.1, 6.1, 7.1 and 8.1, we prove \(V = \bigsqcup_i O_i\) by direct calculation. This calculation method can be applied to cases that the base field has only one quadratic extension. In Proposition 9.1, we prove \(V = \bigsqcup_i O_i\) by computing the cardinality of each orbit because the direct calculation is complicated. In the second subsection, we choose appropriate subspaces and count the cardinality of the intersection of each subspace and each orbit. In the third subsection, we obtain an explicit formula for the Fourier transform \(\hat{e}_i\) in Theorem 1.1. As an application, we determine the Fourier transform of the indicator function of the singular set. We used PARI/GP [8] for the matrix calculation which derives the explicit formula from the result in the second subsection. We put these source codes in the author’s website (https://sites.google.com/view/kazukiishimoto).

Throughout this paper, \(p\) is an odd prime and \(\mathbb{F}_q\) is a finite field of order \(q\) with characteristic \(p\). Also, for a matrix \(A\), we write its transpose as \(A^T\).

2. Calculation method of Fourier transform

Let \(V\) be a finite dimensional vector space over \(\mathbb{F}_q\) with a finite group \(G\) linearly acting on \(V\). Suppose the pair \((G, V)\) satisfies the following Assumption 2.1.

**Assumption 2.1.** There exist an automorphism \(\iota : G \ni g \mapsto g^\iota \in G\) of order 2 and a non-degenerate bilinear form \(\beta : V \times V \rightarrow \mathbb{F}_q\) such that

\[
\beta(gx, g^\iota y) = \beta(x, y) \quad (x, y \in V, g \in G).
\]

Then we can identify the dual space \(V^*\) with \(V\) by the linear isomorphism \(V \ni x \mapsto \beta(x, \cdot) \in V^*\) (see [5] for detail). We reformulate the definition of the Fourier transform only in terms of \(V\). For \(\phi : V \rightarrow \mathbb{C}\), we define its Fourier transform \(\hat{\phi} : V \rightarrow \mathbb{C}\) as follows:

\[
\hat{\phi}(y) := |V|^{-1} \sum_{x \in V} \phi(x) \exp \left( \frac{2\pi i \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\beta(x, y))}{p} \right).
\]

Here \(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \rightarrow \mathbb{F}_p\) is the trace map. Let \(\mathcal{F}_V^G\) be the set of all \(G\)-invariant maps from \(V\) to \(\mathbb{C}\), i.e.,

\[
\mathcal{F}_V^G := \{ \phi : V \rightarrow \mathbb{C} \mid \phi(gx) = \phi(x) \quad (g \in G, x \in V) \}.
\]

Note that \(\mathcal{F}_V^G\) is a finite dimensional vector space over \(\mathbb{C}\). We can easily see that if \(\phi\) is a \(G\)-invariant function, \(\hat{\phi}\) is also \(G\)-invariant. In fact, the Fourier transform map \(\mathcal{F}_V^G \ni \phi \mapsto \hat{\phi} \in \mathcal{F}_V^G\) is a linear isomorphism. Let \(O_i(1 \leq i \leq r)\) be all the distinct \(G\)-orbits in \(V\), and for each \(i\) let \(e_i\) be the indicator function of \(O_i\). The functions \(e_1, \ldots, e_r\) are clearly
$G$-invariant, and they form a basis of $\mathcal{F}_V^G$. Thus we only have to calculate the Fourier transform of $e_1, \ldots, e_r$ to calculate that of all $\phi \in \mathcal{F}_V^G$. We use the following proposition for our calculation of $\hat{e}_i$.

**Proposition 2.1.** [5, Proposition 6] Let $W$ be a subspace of $V$, and let $W^\perp := \{y \in V \mid \forall x \in W, \beta(x, y) = 0\}$. Then

$$\sum_{i=1}^r \frac{|O_i \cap W|}{|O_i|} \cdot \hat{e}_i = \frac{|W|}{|V|} \sum_{j=1}^r \frac{|O_j \cap W^\perp|}{|O_j|} \cdot e_j.$$ 

In this paper, we call $W^\perp$ the orthogonal complement of $W$. By Proposition 2.1, when we choose one subspace of $V$, we obtain one equation of linear combinations of $\hat{e}_i$ and $e_j$. Therefore if we choose $r$ different subspaces and the corresponding equations are linearly independent, we obtain an expression of each $\hat{e}_i$ in terms of $e_1, \ldots, e_r$. In other words, we can determine the following $r$-by-$r$ matrix $M$ explicitly:

$$(\hat{e}_1, \ldots, \hat{e}_n) = (e_1, \ldots, e_n)M.$$ 

We calculate the matrix $M$ with this approach.

**Example 2.1.** Now we will look at a simple example for demonstration. Let $G = \text{GL}_2 \times \text{GL}_2$ and $V = M_2(\mathbb{F}_q)$. $G$ acts on $V$ by

$$G \times V \ni ((g_1, g_2), x) \mapsto g_1 x g_2^T \in V.$$ 

We define an automorphism $\iota$ on $G$ by

$$\iota : G \ni (g_1, g_2) \mapsto (g_1^{-1})^T, (g_2^{-1})^T) \in G$$

and a bilinear form $\beta$ on $V$ by

$$\beta : V \times V \ni (x, y) \mapsto \text{Tr}(xy^T) \in \mathbb{F}_q.$$ 

We can easily confirm that these $\iota$ and $\beta$ satisfy Assumption 2.1.

Elements $x, y \in V$ are $G$-invariant if and only if $x$ and $y$ move each other by elementary operation. Therefore the orbit decomposition is given as follows:

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>Cardinality</th>
</tr>
</thead>
</table>
| $O_1$      | \[
\begin{pmatrix}
0 & 0 \\
0 & 0 
\end{pmatrix}
\] | 1            |
| $O_2$      | \[
\begin{pmatrix}
1 & 0 \\
0 & 0 
\end{pmatrix}
\] | $(q - 1)(q + 1)^2$ |
| $O_3$      | \[
\begin{pmatrix}
1 & 0 \\
0 & 1 
\end{pmatrix}
\] | $(q - 1)^2q(q + 1)$ |

Choose three subspaces $\{0\}$, $W_0 = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \in V \right\}$ and $V$. We have $\{0\}^\perp = V$, $V^\perp = \{0\}$ and $W_0^\perp = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \in V \right\}$. $W_0$ and $W_0^\perp$ are different but since $g \in G$ such that $g \cdot W_0 = W_0^\perp$ exists, we have $|W_0^\perp \cap O_i| = |gW_0 \cap O_i| = |gW_0 \cap gO_i| = \ldots$
$|g(W_0 \cap O_i)| = |W_0 \cap O_i|$ for $i = 1, 2, 3$. Thus when we count the cardinalities of the intersections of the orbits and the subspaces, we can identify $W_0$ and $W_0^\perp$. In this sense we write $W_0^\perp = W_0$.

The cardinalities of the intersections are given as follows:

<table>
<thead>
<tr>
<th>$O_i$</th>
<th>$[0]$</th>
<th>$W_0$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$O_2$</td>
<td>0</td>
<td>$(q-1)(q+1)$</td>
<td>$(q-1)(q+1)^2$</td>
</tr>
<tr>
<td>$O_3$</td>
<td>0</td>
<td>0</td>
<td>$(q-1)^2q(q+1)$</td>
</tr>
</tbody>
</table>

By Proposition 2.1, we obtain the following 3 equations.

$$\hat{e}_1 = \frac{1}{q^4}(e_1 + e_2 + e_3),$$

$$\hat{e}_1 + \frac{1}{q+1}\hat{e}_2 = \frac{1}{q^2}e_1 + \frac{1}{q^2(q+1)}e_2,$$

$$\hat{e}_1 + \hat{e}_2 + \hat{e}_3 = e_1.$$

So we obtain

$$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix} = \frac{1}{q^4} \begin{pmatrix} 1 \\ (q-1)(q+1)^2 \\ (q-1)^2q(q+1) \end{pmatrix} \begin{pmatrix} 1 \\ q - 1 \\ -(q-1)q \\ (-q - 1)q \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

**Remark 2.1.** In what follows, when two subspaces $W$ and $W'$ of a prehomogeneous vector space $(G, V)$ satisfy the condition that there exists $g \in G$ such that $gW = W'$, we identify the two and write $W = W'$.

3. Preliminaries

In this section, we review and summarize some basic results which we use in later sections.

3.1. The space of quadratic forms.

Let $\text{Sym}^2(\mathbb{F}_q^n)$ be the vector space of $n$ variable quadratic forms over $\mathbb{F}_q$. We write an element of $\text{Sym}^2(n)$ as $x(u_1, \ldots, u_n)$ where $u_1, \ldots, u_n$ are the variables. The group $\text{GL}_1(\mathbb{F}_q) \times \text{GL}_n(\mathbb{F}_q)$ acts on $\text{Sym}^2(\mathbb{F}_q^n)$ by

$$(\text{GL}_1(\mathbb{F}_q) \times \text{GL}_n(\mathbb{F}_q)) \times \text{Sym}^2(\mathbb{F}_q^n) \rightarrow \text{Sym}^2(\mathbb{F}_q^n)$$

$$(g_1, g_2, x(u_1, \ldots, u_n)) \mapsto g_1 \cdot x((u_1, \ldots, u_n)g_2^T)$$

We recall the orbit decomposition with respect to this action. In this paper we use the cases $n = 2, 3, 4$. The orbit decomposition of $(\text{GL}_1(\mathbb{F}_q) \times \text{GL}_n(\mathbb{F}_q), \text{Sym}^2(\mathbb{F}_q^n))$ is given as follows:
- $n = 2$

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_{\xi_0}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_{\xi_1}$</td>
<td>$u_1^2$</td>
<td>1</td>
</tr>
<tr>
<td>$O_{\xi_2}$</td>
<td>$u_1 u_2$</td>
<td>2</td>
</tr>
<tr>
<td>$O_{\xi_3}$</td>
<td>$u_1^2 + \mu_1 u_1 u_2 + \mu_0 u_2^2$</td>
<td>2</td>
</tr>
</tbody>
</table>

- $n = 3$

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_{\xi_0}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_{\xi_1}$</td>
<td>$u_1^2$</td>
<td>1</td>
</tr>
<tr>
<td>$O_{\xi_2}$</td>
<td>$u_1 u_2$</td>
<td>2</td>
</tr>
<tr>
<td>$O_{\xi_3}$</td>
<td>$u_1^2 + \mu_1 u_1 u_2 + \mu_0 u_2^2$</td>
<td>2</td>
</tr>
<tr>
<td>$O_{\xi_4}$</td>
<td>$u_1^2 + u_2^2 + u_3^2$</td>
<td>3</td>
</tr>
</tbody>
</table>

- $n = 4$

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_{\xi_0}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_{\xi_1}$</td>
<td>$u_1^2$</td>
<td>1</td>
</tr>
<tr>
<td>$O_{\xi_2}$</td>
<td>$u_1 u_2$</td>
<td>2</td>
</tr>
<tr>
<td>$O_{\xi_3}$</td>
<td>$u_1^2 + \mu_1 u_1 u_2 + \mu_0 u_2^2$</td>
<td>2</td>
</tr>
<tr>
<td>$O_{\xi_4}$</td>
<td>$u_1^2 + u_2^2 + u_3^2$</td>
<td>3</td>
</tr>
<tr>
<td>$O_{\xi_5}$</td>
<td>$u_1^2 + u_2^2 + u_3^2 - \lambda u_4^2$</td>
<td>4</td>
</tr>
</tbody>
</table>

Here, the word “rank” means the rank of the symmetric matrix corresponding to the quadratic form and $u_1^2 + \mu_1 u_1 u_2 + \mu_0 u_2^2 \in \text{Sym}^2(\mathbb{F}_q^2)$ is an arbitrary irreducible polynomial and $\lambda$ is an arbitrary quadratic non-residue in $\mathbb{F}_q$.

3.2. The cardinality of certain sets of matrices.

We introduce the following notation.

\[(n_1, n_2), m| := |\{M \in M(n_1, n_2)(\mathbb{F}_q) | \text{rank}(M) = m\}|\]

\[= \frac{\prod_{i=0}^{m-1} (q^{n_2-i} - 1) \prod_{j=0}^{m-1} (q^{n_1} - q^j)}{\prod_{k=1}^{m} (q^k - 1)},\]

\[|n, m| := |(n, n), m| = \prod_{i=0}^{m-1} \frac{(q^{n-i} - 1)(q^n - q^i)}{q^{m-i} - 1},\]

\[\text{gl}_n := |\text{GL}_n(\mathbb{F}_q)| = \prod_{i=0}^{n-1} (q^n - q^i),\]

\[\text{sl}_n := |\text{SL}_n(\mathbb{F}_q)| = \text{gl}_n/(q - 1).\]

Here, $M(n_1, n_2)(\mathbb{F}_q)$ is the set of all $n_1$-by-$n_2$ matrices over $\mathbb{F}_q$. 
3.3. Orbit correspondence and their cardinalities.

We identify $V_n = \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^n$ as the vector space of $n$-tuples of square matrices of order 2. For $x = (X_1, \ldots, X_n) \in V_n$, we define $r_1(x)$ as the dimension of the subspace of $M_2(\mathbb{F}_q)$ generated by $X_1, \ldots, X_n$. Let $G_n = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_n$. $G_n$ acts on $V_n$ by

$$G_n \times V_n \ni ((g_1, g_2, g_3), (X_1, \ldots, X_n)) \mapsto (g_1 X g_2^T, \ldots, g_1 X_n g_2^T) \in V_n.$$ 

For $n < k$, we consider the embeddings $f^k_n : V_n \ni x \mapsto (0, \ldots, 0, x) \in V_k$ and $h^k_n : G_n \ni (g_1, g_2, g_3) \mapsto (g_1, g_2, \begin{pmatrix} I_{k-n} & 0 \\ 0 & g_3 \end{pmatrix}) \in G_k$ where $I_{k-n}$ is the identity matrix of order $k - n$. For all $m < n < k$, we have

$$f_m^k = f_n^k \circ f_m^n$$

and

$$h_m^k = h_n^k \circ h_m^n.$$ 

In addition, for all $x \in V_n$ and $g \in G_n$,

$$f_n^k(gx) = h_n^k(g) f_n^k(x)$$

holds. By these embeddings, we regard $V_n$ as a subspace of $V_k$ and regard $G_n$ as a subgroup of $G_k$. Then we have the following proposition.

**Proposition 3.1.** For $x, y \in V_n \subset V_k$, $x$ and $y$ are $G_n$-equivalent if and only if $x$ and $y$ are $G_k$-equivalent.

[Proof]

If $x$ and $y$ are $G_n$-equivalent, we easily have $x$ and $y$ are $G_k$-equivalent. We consider its conversion.

Assume $x$ and $y$ are $G_k$-equivalent. When $r_1(x) = m \leq n$, we can let $x = (0, \ldots, 0, X_1, \ldots, X_m)$ such that $X_1, \ldots, X_m$ are linearly independent, by the action of $G_n$. Since $r_1(y)$ is also $m$, we also may assume $y = (0, \ldots, 0, Y_1, \ldots, Y_m)$ such that $Y_1, \ldots, Y_m$ are linearly independent. Let $(g_1, g_2, \begin{pmatrix} \vdots & \vdots \\ g_{k,1} & \cdots & g_{k,k} \end{pmatrix})x = y$. Then we have

$$(g_1 \left( \sum_{j=k-m+1}^k g_{1,j} X_{j-k+m} \right) g_2^T, \ldots, g_1 \left( \sum_{j=k-m+1}^k g_{k,j} X_{j-k+m} \right) g_2^T) = (0, \ldots, 0, Y_1, \ldots, Y_m),$$

i.e.,

$$g_1 \left( \sum_{j=1}^m g_{i,j+k-m} X_j \right) g_2^T = 0 \quad \text{where} \quad 1 \leq i \leq k-m.$$ 

(3)
Since $X_1, \ldots, X_m$ are linearly independent, we obtain $g_{i,j} = 0$ where $1 \leq i \leq k - m$ and $k - m + 1 \leq j \leq k$ by (3). It follows that $g'_2 := (g_{i,j})_{k-m+1 \leq i \leq k, k-m+1 \leq j \leq k} \in \text{GL}_m$. By (4), we obtain
\[
\begin{pmatrix}
g_1, g_2, \begin{pmatrix} I_{k-m} & 0 \\ 0 & g'_2 \end{pmatrix} \end{pmatrix} \cdot (0, \ldots, 0, X_1, \ldots, X_m) = (0, \ldots, 0, Y_1, \ldots, Y_m)
\]
\[\Box\]

Next we consider particular subspaces in $V_n$ and $V_k$. Let $U_1$ be an arbitrary subspace of $V_1$, and we let $U_n := U_1 \otimes \mathbb{F}_q^m \subset V_n$. For $n < k$, we regard $U_n$ as a subspace of $U_k$ by the embedding $f^k_n$. We consider a relation between $|\langle G_n x \rangle \cap U_n|$ and $|\langle G_k x \rangle \cap U_k|$.

**Proposition 3.2.** For $x \in V_n$, let $r_1(x) = m \leq n$. Then we have
\[
|\langle G_k x \rangle \cap U_k| = \prod_{i=1}^{m} (q^k - q^i) \prod_{i=1}^{m} (q^n - q^i) \cdot |\langle G_n x \rangle \cap U_n|.
\]

**Proof**
Since the case $m = 0$ is obvious, we assume $m \geq 1$. For $x \in U_n$ where $r_1(x) = m$, we have $x \sim (0, \ldots, 0, X_1, \ldots, X_m) \in U_n$ such that $X_1, \ldots, X_m$ are linearly independent, by the action of $G_n$. Therefore we assume $x = (0, \ldots, 0, X_1, \ldots, X_m)$. Let $\text{Stab}_n(x) := \{ g \in G_n \mid gx = x \}$ and $G_n(x, U_n) := \{ g \in G_n \mid gx \in U_n \}$. Then $\text{Stab}_n(x)$ is a subgroup of $G$ and we have
\[
|\langle G_n x \rangle \cap U_n| = |G_n(x, U_n)|/|\text{Stab}_n(x)|.
\]

By
\[
|G_n(x, U_n)| = \frac{\text{gl}_n}{\text{gl}_m} \cdot |\langle G_m(x, U_m) \rangle|
\]
and
\[
|\text{Stab}_n(x)| = q^{m(n-m)} \text{gl}_{n-m} \cdot |\text{Stab}_m(x)|,
\]
we have
\[
|\langle G_n x \rangle \cap U_n| = \frac{\text{gl}_n}{q^{m(n-m)} \text{gl}_{n-m} \text{gl}_m} \cdot |\langle G_m x \rangle \cap U_m| = \prod_{i=1}^{m} (q^n - q^i) \cdot |\langle G_m x \rangle \cap U_m|.
\]

Therefore we obtain
\[
|\langle G_k x \rangle \cap U_k| = \prod_{i=1}^{m} (q^k - q^i) \text{gl}_m \cdot |\langle G_m x \rangle \cap U_m| = \prod_{i=1}^{m} (q^k - q^i) \prod_{i=1}^{m} (q^n - q^i) \cdot |\langle G_m x \rangle \cap U_m|.
\]
\[
\prod_{i=1}^{m} \frac{(q^k - q^i)}{(q^n - q^i)} \cdot |(G_n x) \cap U_n|.
\]

4. \(2 \otimes 2 \otimes 2\)

Let \(V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2\) and \(G = G_1 \times G_2 \times G_3 = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_2\). We write \(x \in V\) as \(x = (A, B)\) where \(A\) and \(B\) are 2-by-2 matrices, and write \(g \in G\) as \(g = (g_1, g_2, g_3)\) where \(g_1, g_2, g_3 \in \text{GL}_2\). \(G\) acts on \(V\) by

\[
g x = (g_1 A g_2^T, g_1 B g_2^T) g_3^T.
\]

Define a bilinear form \(\beta\) of \(V\) as

\[
\beta((A_1, B_1), (A_2, B_2)) = \text{Tr}(A_1 A_2^T + B_1 B_2^T).
\]

In addition, define an automorphism \(\iota\) of \(G\) as

\[
(g_1, g_2, g_3)\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}, (g_3^T)^{-1})\).
\]

By an easy computation, we see that these \(\beta\) and \(\iota\) satisfy Assumption 2.1.

4.1. Orbit decomposition.

For \(x = (A, B) = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right), \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array}\right) \in V\), we define

\[
\begin{align*}
\ r_1(x) &:= \text{rank}\left(\begin{array}{cccc} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \end{array}\right), \\
\ r_2(x) &:= \text{rank}\left(\begin{array}{cccc} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{array}\right), \\
\ r_3(x) &:= \text{rank}\left(\begin{array}{cccc} a_{11} & a_{21} & b_{11} & b_{21} \\ a_{12} & a_{22} & b_{12} & b_{22} \end{array}\right).
\end{align*}
\]

\(r_1(x), r_2(x), r_3(x)\) are invariants of the orbits. We also define

\[
\det_{x}(u, v) := \det(uA + vB) \in \text{Sym}^2(\mathbb{F}_q^2)\text{ where }u, v \text{ are variables},
\]

\[
T(x) := \langle \alpha \rangle \text{ if and only if } \det_{x}(u, v) \in O_{\langle \alpha \rangle} \text{ in } \text{Sym}^2(\mathbb{F}_q^2).
\]

Note that we introduced the representation \((\text{GL}_1(\mathbb{F}_q) \times \text{GL}_2(\mathbb{F}_q), \text{Sym}^2(\mathbb{F}_q^2))\) in Section 3.1. For \(x \in V\) and \(g = (g_1, g_2, g_3) \in G\), we have

\[
\det_{g x}(u, v) = \det(g_1 g_2) \det_{x}((u, v) g_3).
\]

Therefore \(T(x)\) is also an invariant of the orbits.
**Proposition 4.1.** \( V \) consists of 8 \( G \)-orbits in all.

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>( r_1(x) )</th>
<th>( r_2(x) )</th>
<th>( r_3(x) )</th>
<th>( T(x) )</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_1 )</td>
<td>( \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} \cdot \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \langle 0 \rangle )</td>
<td>1</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>( \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} \cdot \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \langle 0 \rangle )</td>
<td>[1, 0, 3]</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>( \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} \cdot \begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix} )</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>( \langle 1 \rangle )</td>
<td>[2, 1, 2]</td>
</tr>
<tr>
<td>( O_4 )</td>
<td>( \begin{bmatrix} 1 &amp; 0 \ 0 &amp; 0 \end{bmatrix} \cdot \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} )</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>( \langle 0 \rangle )</td>
<td>[2, 1, 2]</td>
</tr>
<tr>
<td>( O_5 )</td>
<td>( \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} \cdot \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} )</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>( \langle 0 \rangle )</td>
<td>[2, 1, 2]</td>
</tr>
<tr>
<td>( O_6 )</td>
<td>( \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} \cdot \begin{bmatrix} 0 &amp; 1 \ 0 &amp; 0 \end{bmatrix} )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( \langle 1 \rangle )</td>
<td>[3, 1, 3]</td>
</tr>
<tr>
<td>( O_7 )</td>
<td>( \begin{bmatrix} 1 &amp; 0 \ 0 &amp; 0 \end{bmatrix} \cdot \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix} )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( \langle 2r \rangle )</td>
<td>( \frac{1}{2} ) [2, 3, 3]</td>
</tr>
<tr>
<td>( O_8 )</td>
<td>( \begin{bmatrix} 1 &amp; 0 \ 1 &amp; 0 \end{bmatrix} \cdot \begin{bmatrix} 0 &amp; -1 \ \mu_0 &amp; \mu_1 \end{bmatrix} )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( \langle 2i \rangle )</td>
<td>( \frac{1}{2} ) [4, 3, 1]</td>
</tr>
</tbody>
</table>

Here \( [a, b, c] = (q - 1)^a q^b (q + 1)^c \) and \( \mu_1, \mu_0 \) are elements of \( \mathbb{F}_q \) such that \( X^2 + \mu_1 X + \mu_0 \in \mathbb{F}_q[X] \) is irreducible.

[Proof]

The invariants \( r_1(x), r_2(x), r_3(x) \) and \( T(x) \) for the 8 elements in the table are easily calculated. Since they do not coincide, these 8 elements belong to different orbits. Let \( O_i \) be the orbit of each element.

First we prove \( V = \bigcup_{i=1}^{8} O_i \). Let \( x \in V \). Let \( (r_1(x), r_2(x), r_3(x)) \neq (2, 2, 2) \). When \( r_1(x) = 0 \), since \( x = 0 \) we have \( x \in O_1 \). When \( r_1(x) \geq 1 \), we have \( r_2(x) \geq 1 \). When \( (r_1(x), r_2(x)) = (1, 1) \), we have \( x \sim (0, B) \sim (0 0) \cdot (0 0) \cdot (0 1) \) by the action of \( G \) and thus \( x \in O_2 \). When \( (r_1(x), r_2(x)) = (1, 2) \), \( x \sim (0 0) \cdot (1 0) \) by the action of \( G \). When \( (r_1(x), r_2(x)) = (2, 1) \), \( x \sim (0 0) \cdot (0 0) \cdot (1 1) \) by the action of \( G \). When \( (r_1(x), r_2(x), r_3(x)) = (2, 2, 1) \), \( x \sim (0 0) \cdot (0 0) \cdot (0 1) \) by the action of \( G \). We have \( \det(u, v) = a_2 u^2 + b_2 v^2 - b_1 u \) if \( \det(u, v) = 0 \), we have \( a_2 = b_2 = b_1 u = 0 \), which contradicts to \( (r_1(x), r_2(x), r_3(x)) = (2, 2, 2) \). It
follows that $T(x) = \langle 1 \rangle, \langle 2r \rangle$ or $\langle 2i \rangle$. When $T(x) = \langle 1 \rangle$, there exists a root of $\det x(u, v)$ which belongs to $\mathbb{P}^1(\mathbb{F}_q)$. Thus we can let $\text{rank}(A) = 1$ by the action of $G_3$. Therefore $x \sim \begin{pmatrix} 1 & 0 \\ 0 & b_{21} \end{pmatrix} \begin{pmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $\det x(u, v) \sim b_{22}uv - b_{12}b_{21}v^2$. Since $T(x) = \langle 1 \rangle$, we have $b_{22} = 0$ and $b_{12}b_{21} \neq 0$. Thus we have $x \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by the action of $G$. When $T(x) = \langle 2r \rangle$, we have $x \sim \begin{pmatrix} 1 & 0 \\ 0 & b_{21} \end{pmatrix} \begin{pmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ in the same way as in the case of $T(x) = \langle 1 \rangle$. Since $b_{22} \neq 0$ in this case, we have $x \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b_{22} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. When $T(x) = \langle 2i \rangle$, $\det x(u, v) \in \text{Sym}^2(\mathbb{F}_q^2)$ is irreducible. By the fact stated in Section 3.1, irreducible polynomials in $\text{Sym}^2(\mathbb{F}_q^2)$ belong to the same $\text{GL}_1 \times \text{GL}_2$-orbit. This fact and the surjectivity of the map $G \ni (g_1, g_2, g_3) \mapsto (\det(g_1g_2), g_3^T) \in \text{GL}_1 \times \text{GL}_2$ means that we can move $\det x(u, v)$ to an arbitrary irreducible polynomial by the action of $G$. Therefore we assume $\det x(u, v) = u^2 + \mu_1uv + \mu_0v^2 = (u - \gamma v)(u - \bar{\gamma}v)$ where $\gamma \in \mathbb{F}_q \setminus \mathbb{F}_q$. We can move $x$ to $y := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ with $a_{11} \neq 0$ by the action of $G_1$ and $G_2$. Thus there exists a pair $(r, s) \in \mathbb{F}_q^2 \setminus \{0\}$ such that

$$a_{11}r + b_{11}s = 0. \tag{5}$$

Besides, for such $r$ and $s$, there exist a pair $(p, q) \in \mathbb{F}_q^2 \setminus \{0\}$ such that

$$-r + s\gamma = p\gamma - q\gamma^2. \tag{6}$$

since $\{\gamma, \gamma^2\}$ is a basis of $\mathbb{F}_q^2$ as a vector space over $\mathbb{F}_q$. The equation (6) is equivalent to

$$\frac{r - s\gamma}{p - q\gamma} = \gamma. \tag{7}$$

If $(p, q) \parallel (r, s)$, we have $\gamma \in \mathbb{F}_q$, which contradicts to the assumption. Therefore $\begin{pmatrix} p \\ r \\ q \\ s \end{pmatrix} \in \text{GL}_2$. Let $g := \begin{pmatrix} 1 & 1 \\ p & q \\ r & s \end{pmatrix}$. By (5), the $(1, 1)$-entry of the first matrix of $gy$ is nonzero and the $(1, 1)$-entry of the second matrix of $gy$ is 0. In addition, by (7), $\det_{gy}(u, v) = ((p - q\gamma)u + (r - s\gamma)v)((p - q\bar{\gamma})u + (r - s\bar{\gamma})v)$ is a nonzero scalar multiple of $u^2 + \mu_1uv + \mu_0v^2$. Since the rank of the matrix $uA + vB$ is 2 for all $(u, v) \in \mathbb{P}^1(\mathbb{F}_q)$, we have $x \sim y' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ b_{21}' & -1 \\ b_{21}' & b_{22}' \end{pmatrix}$ by the action of $G_1$ and $G_2$. Therefore $\det y'(u, v) = g'(u^2 + \mu_1uv + \mu_0v^2)$ for certain $g' \in \text{GL}_1$. On the other hand, $\det y'(u, v) = u^2 + b_{22}'uv + b_{21}'v^2$. Therefore we have $g' = 1$ and $b_{22}' = \mu_1, b_{21}' = \mu_0$. 
Next we count the cardinality of each orbit. \(|O_1|, |O_2|, \text{ and } |O_3|\) can be calculated by means of the cardinality of each orbit in Example 2.1 and Proposition 3.2. \(|O_4|\) can be calculated in the same way as \(|O_3|\) by regarding \(x \in V\) as the pair \(\begin{pmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{pmatrix}, \begin{pmatrix} a_{21} & a_{22} \\ b_{21} & b_{22} \end{pmatrix}\).

For the count of \(|O_5|\), we regard \(x \in V\) as \(\begin{pmatrix} a_{11} & a_{21} \\ b_{11} & b_{21} \end{pmatrix}, \begin{pmatrix} a_{12} & a_{22} \\ b_{12} & b_{22} \end{pmatrix}\) and calculate it in the same way as \(|O_3|\) and \(|O_4|\). Next we count \(|O_6|\). Let \(x_6 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\).

and \(\text{Stab}(x_6) := \{g \in G \mid gx_6 = x_6\}\). Let \(g = (g_1, g_2, g_3) = \begin{pmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \end{pmatrix}\),
\[
\begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}, \begin{pmatrix} p_3 & q_3 \\ r_3 & s_3 \end{pmatrix} \in \text{Stab}_{(x_6)}.
\]
We have \(g_1 \begin{pmatrix} 0 & q_3 \\ q_2 & q_3 \end{pmatrix} q^T_1, g_1 \begin{pmatrix} 0 & s_3 \\ s_2 & s_3 \end{pmatrix} q^T_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \) and therefore \(q_1 = q_2 = 0, \) and \(s_1s_2p_3 \neq 0. \) Thus we have
\[
g_1 \begin{pmatrix} 0 & s_3 \\ s_2 & s_3 \end{pmatrix} q^T_2 = \begin{pmatrix} 0 & p_1s_2s_3 \\ s_1s_2s_3 + s_1r_2s_3 + r_1s_2s_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

and
\[
\text{Stab}_{(x_6)} = \left\{ \begin{pmatrix} (s_2s_3)^{-1} & 0 \\ r_1 & s_1 \end{pmatrix}, \begin{pmatrix} (s_3s_1)^{-1} & 0 \\ r_2 & s_2 \end{pmatrix}, \begin{pmatrix} (s_1s_2)^{-1} & 0 \\ r_3 & s_3 \end{pmatrix} \right\} \subset G q_3 = -s_3 \frac{q_1 + q_2}{s_2} \text{ if } q_3 \neq 0.
\]

Therefore \(|\text{Stab}_{(x_6)}| = (q - 1)^2q^2\) and we obtain \(|O_6| = |G|/|\text{Stab}_{(x_6)}| = (q^2 / (q - 1)^3q^2 = (q - 1)^3q(q + 1)^3. \) Next we count \(|O_7|\). Let \(x_7 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\) and \(\text{Stab}_{(x_7)} := \{g \in G \mid gx_7 = x_7\}\). Let \(g = (g_1, g_2, g_3) = \begin{pmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \end{pmatrix}\),
\[
\begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}, \begin{pmatrix} p_3 & q_3 \\ r_3 & s_3 \end{pmatrix} \in \text{Stab}_{(x_7)}.
\]
We have
\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} q^T_1, g_1 \begin{pmatrix} 0 & 0 \\ 0 & s_3 \end{pmatrix} q^T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

By comparing the rank of each entry, we obtain the following propositions:
\begin{align*}
(8) & \quad \text{If } p_3 \neq 0, \text{ then } q_3 = r_3 = 0 \text{ and } s_3 \neq 0.
(9) & \quad \text{If } p_3 = 0, \text{ then } q_3r_3 \neq 0 \text{ and } s_3 = 0.
\end{align*}
First we assume the case (8). Then we have
\[
\left( p_3 \begin{bmatrix} p_1 p_2 & p_1 r_2 \\ r_1 p_2 & r_1 r_2 \end{bmatrix}, s_3 \begin{bmatrix} q_1 q_2 & q_1 s_2 \\ s_1 q_2 & s_1 s_2 \end{bmatrix} \right) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right),
\]
and therefore
\[
(10) \quad q_1 = q_2 = r_1 = r_2 = 0, \quad p_3 = (p_1 p_2)^{-1}, \quad \text{and} \quad s_3 = (s_1 s_2)^{-1}.
\]
Next we assume the case (9). Then we have
\[
\left( q_3 \begin{bmatrix} q_1 q_2 & q_1 s_2 \\ s_1 q_2 & s_1 s_2 \end{bmatrix}, r_3 \begin{bmatrix} p_1 p_2 & p_1 r_2 \\ r_1 p_2 & r_1 r_2 \end{bmatrix} \right) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right),
\]
and therefore
\[
(11) \quad p_1 = p_2 = s_1 = s_2 = 0, \quad q_3 = (q_1 q_2)^{-1}, \quad \text{and} \quad r_3 = (r_1 r_2)^{-1}.
\]
By (10) and (11), we obtain
\[
\text{Stab}(x_7) = \left\{ \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\} \cong \mathbb{Z}/2\mathbb{Z} \rtimes (GL_2)^4.
\]
It follows that \(|O_7| = |G|/|\text{Stab}(x_7)| = (GL_2)^3/2(q - 1)^4 = (q - 1)^2 q^3 (q + 1)^3/2. In the end, \(|O_8| = q^8 - \sum_{i=1}^{7} |O_i| = (q - 1)^4 q^3 (q + 1)/2. \]

\section{The intersection between the orbits and the subspaces.}
The subspaces we choose to calculate the Fourier transform are as follows:
\[
W_1 = 0, \quad W_2 = \left( \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} \right), \quad W_3 = \left( \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix} \right),
\]
\[
W_4 = \left( \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix} \right), \quad W_5 = \left( \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \right),
\]
\[
W_6 = \left( \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix} \right), \quad W_7 = \left( \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix} \right)
\]
and \(W_8 = V\).

Here, the notations mean, for example,
\[
\left( \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix} \right) = \left\{ \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) \in V \mid b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{F}_q \right\}.
\]

Orthogonal complements of them are as follows (See Remark 2.1 for the convention for some of these equalities):
\[
W_1^\perp = W_8, \quad W_2^\perp = W_7, \quad W_3^\perp = W_5, \quad W_4^\perp = W_4, \\
W_5^\perp = W_5, \quad W_6^\perp = W_6, \quad W_7^\perp = W_2 \quad \text{and} \quad W_8^\perp = W_1.
PROPOSITION 4.2. The cardinalities \(|O_i \cap W_j|\) for the orbit \(O_i\) and the subspace \(W_j\) are given as follows:

<table>
<thead>
<tr>
<th>(O_i)</th>
<th>(W_1)</th>
<th>(W_2)</th>
<th>(W_3)</th>
<th>(W_4)</th>
<th>(W_5)</th>
<th>(W_6)</th>
<th>(W_7)</th>
<th>(W_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(O_1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(O_2)</td>
<td>0</td>
<td>[1, 0, 1]</td>
<td>[1, 0, 2]</td>
<td>[1, 0, 2]</td>
<td>[1, 0, 0](3q + 1)</td>
<td>[1, 0, 1](2q + 1)</td>
<td>[1, 0, 3]</td>
<td></td>
</tr>
<tr>
<td>(O_3)</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 1]</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 0]</td>
<td>[2, 1, 1]</td>
<td>[2, 1, 2]</td>
</tr>
<tr>
<td>(O_4)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 1]</td>
<td>0</td>
<td>[2, 1, 0]</td>
<td>[2, 1, 1]</td>
<td>[2, 1, 2]</td>
</tr>
<tr>
<td>(O_5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 0]</td>
<td>[2, 1, 1]</td>
<td>[2, 1, 2]</td>
</tr>
<tr>
<td>(O_6)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[3, 1, 0]</td>
<td>[3, 1, 1]</td>
<td>[3, 1, 3]</td>
</tr>
<tr>
<td>(O_7)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[2, 3, 1]</td>
<td>(1/2[2, 3, 3])</td>
</tr>
<tr>
<td>(O_8)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(1/2[4, 3, 1])</td>
</tr>
</tbody>
</table>

[Proof]

For \(W_1, W_2, W_3, W_4\) and \(W_5\) we can calculate by means of Example 2.1 and Proposition 3.2. Let \(x \in W_6\) be

\[
\begin{bmatrix}
0 & 0 \\
0 & a_{22}
\end{bmatrix}
\begin{bmatrix}
b_{12} \\
b_{21}
\end{bmatrix}.
\]

When \(a_{22}b_{12}b_{21} \neq 0\), we have \(x \in O_6\). The number of such \(x\) is \((q - 1)^3q\). When \(a_{22} = 0\) and \(b_{12}b_{21} \neq 0\), we have \(x \in O_3\). When \(b_{12} = 0\) and \(a_{22}b_{21} \neq 0\), we have \(x \in O_5\). For each of these cases, there are \((q - 1)^2q\) of such \(x\). The remaining elements all belong to \(O_1\) or \(O_2\). Let \(x \in W_7\) be

\[
\begin{bmatrix}
0 & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
b_{12} \\
b_{21}
\end{bmatrix}.
\]

When \((a_{12}, b_{12}) (a_{21}, b_{21})\), we have \(x \in O_7\).

The number of such \(x\) is \(q^2 gl_2\). When \((a_{12}, b_{12}) \neq 0, (a_{21}, b_{21}) \neq 0, (a_{12}, b_{12}) (a_{21}, b_{21})\) and \((a_{12}, b_{12}) (a_{21}, b_{22})\), we have \(x \in O_6\). The number of such \(x\) is \((q - 1) gl_2\). When \((a_{12}, b_{12}) \neq 0, (a_{21}, b_{21}) \neq 0, (a_{12}, b_{12}) (a_{21}, b_{21})\) and \((a_{12}, b_{12}) (a_{22}, b_{22})\), we have \(x \in O_3\). The number of such \(x\) is \((q^2 - 1)(q - 1)q\). When \((a_{12}, b_{12}) = 0\) and \((a_{21}, b_{21}) \neq 0, (a_{21}, b_{22}) \neq 0, (a_{22}, b_{22}) \neq 0, (a_{12}, b_{12}) (a_{22}, b_{22})\), we have \(x \in O_4\). The number of such \(x\) is \(gl_2\). When \((a_{21}, b_{21}) = 0\) and \((a_{12}, b_{12}) \neq 0, (a_{22}, b_{22}) \neq 0, (a_{22}, b_{22}) \neq 0, (a_{12}, b_{12}) (a_{22}, b_{22})\), we have \(x \in O_5\). The number of such \(x\) is \(gl_2\). The remaining elements all belong to \(O_1\) or \(O_2\).

4.3. Fourier transform.

By applying these results to Proposition 2.1, we obtain an explicit formula for the Fourier transform.
Theorem 4.1. The representation matrix \( M \) of the Fourier transform on \( F^G \) with respect to the basis \( e_1, \ldots, e_8 \) is given as follows:

\[
\begin{bmatrix}
1 & [1, 0, 3] & [2, 1, 2] & [2, 1, 2] & [3, 1, 3] & \frac{1}{2}[2, 3, 3] & \frac{1}{2}[4, 3, 1] \\
1 & c_1 & [1, 1, 0]b_1 & [1, 1, 0]b_1 & [1, 1, 0]b_1 & -[2, 1, 0]a_1 & \frac{1}{2}[1, 3, 0]b_2 & -\frac{1}{2}[3, 3, 0] \\
1 & [0, 0, 1]b_1 & g \sigma_2 & -[1, 1, 1] & -[1, 1, 1] & [1, 1, 1] & \frac{1}{2}[1, 3, 1] & \frac{1}{2}[2, 3, 0] \\
1 & [0, 0, 1]b_1 & -[1, 1, 1] & q \sigma_2 & -[1, 1, 1] & [1, 1, 1] & -\frac{1}{2}[1, 3, 1] & \frac{1}{2}[2, 3, 0] \\
1 & [0, 0, 1]b_1 & -[1, 1, 1] & -[1, 1, 1] & q \sigma_2 & [1, 1, 1] & -\frac{1}{2}[1, 3, 1] & \frac{1}{2}[2, 3, 0] \\
1 & -a_1 & q & q & q \sigma_3 & -\frac{1}{2}[1, 3, 0] & -\frac{1}{2}[1, 3, 0] \\
1 & b_2 & -[1, 1, 0] & -[1, 1, 0] & -[1, 1, 0] & -[2, 1, 0] & q^3 & 0 \\
1 & -[0, 0, 2] & [0, 1, 1] & [0, 1, 1] & [0, 1, 1] & -[0, 1, 2] & 0 & q^3
\end{bmatrix}
\]

Here \([a, b, c] = (q - 1)^u q^b (q + 1)^c\), \(a_1 = 2a + 1, b_1 = q^2 - q - 1, b_2 = q^2 - 2q - 1, c_1 = 2q^3 - 2q - 1, c_2 = q^3 - q^2 + 1\) and \(c_3 = q^3 - q^2 - 1\).

We used PARI/GP [8] to calculate the matrix from Proposition 4.2.

By Theorem 4.1, we can calculate the Fourier transform of the indicator function \( \Psi \) of the singular set \( S = \{ x \in V \mid \text{Disc}(|\text{det}_x(u, v)) = 0\} = \bigcup_{i=1}^6 O_i \), i.e., \( \Psi = \sum_{i=1}^6 e_i \).

Corollary 4.1. The Fourier transform of \( \Psi \) is given as follows:

\[
\hat{\Psi}(x) = \begin{cases} 
q^{-1} + q^{-4} - q^{-5} & x = 0, 
q^{-4} - q^{-5} & x \neq 0, \text{ Disc}(|\text{det}_x(u, v)) = 0, 
-q^{-5} & \text{Disc}(|\text{det}_x(u, v)) \neq 0.
\end{cases}
\]

In particular, we have the following \( L_1 \)-norm bound of \( \hat{\Psi} \):

\[
\sum_{x \in V} |\hat{\Psi}(x)| = O(q^3).
\]

5. \( 2 \otimes 2 \otimes 3 \)

Let \( V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \) and \( G = G_1 \times G_2 \times G_3 = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_3 \). We write \( x \in V \) as \( x = (A, B, C) \) where \( A, B \) and \( C \) are \( 2 \times 2 \) matrices, and write \( g \in G \) as \( g = (g_1, g_2, g_3) \) where \( g_1, g_2 \in \text{GL}_2 \) and \( g_3 \in \text{GL}_3 \). We define the action of \( G \) on \( V \) by

\[
gx = (g_1 A g_2^T, g_1 B g_2^T, g_1 C g_2^T) g_3^T.
\]

Define a bilinear form \( \beta \) of \( V \) as

\[
\beta((A_1, B_1, C_1), (A_2, B_2, C_2)) = \text{Tr}(A_1 A_2^T + B_1 B_2^T + C_1 C_2^T).
\]

In addition, define an automorphism \( \iota \) of \( G \) as

\[
(g_1, g_2, g_3)^\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}, (g_3^T)^{-1}).
\]

These \( \beta \) and \( \iota \) satisfy Assumption 2.1.
Note that we introduced the representation $(\det x^1)$. For $x(u_1)$, we define

$$r_1(x) := \operatorname{rank} \left( \begin{bmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \end{bmatrix} \right).$$

$$r_2(x) := \operatorname{rank} \left( \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \end{bmatrix} \right).$$

$$r_3(x) := \operatorname{rank} \left( \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \end{bmatrix} \right).$$

$\det_x(u_1, u_2, u_3) := \det(u_1A + u_2B + u_3C) \in \operatorname{Sym}^2(\mathbb{F}_q^3)$ where $u_1, u_2, u_3$ are variables.

$T(x) := \langle \alpha \rangle$ if and only if $\det_x(u_1, u_2, u_3) \in O_{\langle \alpha \rangle}$ in $\operatorname{Sym}^2(\mathbb{F}_q^3)$.

Note that we introduced the representation $(\operatorname{GL}_1(\mathbb{F}_q) \times \operatorname{GL}_3(\mathbb{F}_q), \operatorname{Sym}^2(\mathbb{F}_q^3))$ in Section 3.1. For $x \in V$ and $g = (g_1, g_2, g_3) \in G$, we have

$$\det_{gx}(u_1, u_2, u_3) = \det(g_1 g_2) \det_x((u_1, u_2, u_3)g_3).$$

**Proposition 5.1.** $V$ consists of 10 $G$-orbits in all.

<table>
<thead>
<tr>
<th>Orbit name $\mathcal{O}$</th>
<th>Representative</th>
<th>$r_1(x)$</th>
<th>$r_2(x)$</th>
<th>$r_3(x)$</th>
<th>$T(x)$</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{O}_1$</td>
<td>$\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\langle 0 \rangle</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{O}_2$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>\langle 0 \rangle</td>
<td>[1, 0, 2, 1]</td>
</tr>
<tr>
<td>$\mathcal{O}_3$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>\langle 1 \rangle</td>
<td>[2, 1, 1, 1]</td>
</tr>
<tr>
<td>$\mathcal{O}_4$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>\langle 0 \rangle</td>
<td>[2, 1, 2, 1]</td>
</tr>
<tr>
<td>$\mathcal{O}_5$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>\langle 0 \rangle</td>
<td>[2, 1, 2, 1]</td>
</tr>
<tr>
<td>$\mathcal{O}_6$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>\langle 1 \rangle</td>
<td>[3, 1, 3, 1]</td>
</tr>
<tr>
<td>$\mathcal{O}_7$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>\langle 2r \rangle</td>
<td>\frac{1}{2}[2, 3, 3, 1]</td>
</tr>
<tr>
<td>$\mathcal{O}_8$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>\langle 2r \rangle</td>
<td>\frac{1}{2}[4, 3, 1, 1]</td>
</tr>
<tr>
<td>$\mathcal{O}_9$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>\langle 2r \rangle</td>
<td>[3, 3, 3, 1]</td>
</tr>
<tr>
<td>$\mathcal{O}_{10}$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>\langle 3 \rangle</td>
<td>[4, 4, 2, 1]</td>
</tr>
</tbody>
</table>
Here \([a, b, c, d] = (q - 1)^a q^b (q + 1)^c (q^2 + q + 1)^d\), and \(\mu_1, \mu_0\) are elements of \(\mathbb{F}_q\) such that \(X^2 + \mu_1 X + \mu_0 \in \mathbb{F}_q[X]\) is irreducible.

[Proof]
The invariants \(r_1(x), r_2(x), r_3(x)\) and \(T(x)\) for the 10 elements in the table are easily calculated. Since they do not coincide, these 10 elements belong to different orbits. Let \(O_i\) be the orbit of each element.

First we prove that \(\bigcup_{i=1}^{10} O_i = V\). Let \(x \in V\). When \(r_1(x) \leq 2\), we have \(x \sim (0, B, C)(B, C \in M_2(\mathbb{F}_q))\). Therefore by Propositions 3.1 and 4.1 we see that

\[
\{x \in V \mid r_1(x) \leq 2\} = \bigcup_{i=1}^{8} O_i.
\]

When \(r_1(x) = 3\) we have \(x \sim \left(\begin{array}{cc} 1 & 0 \\ 0 & a_{22} \end{array}\right) \cdot \left(\begin{array}{cc} b_{12} & 0 \\ b_{21} & b_{22} \end{array}\right) \cdot \left(\begin{array}{cc} 0 & c_{12} \\ c_{21} & c_{22} \end{array}\right)\). If \(b_{12} = c_{12} = 0\), we have \(x \sim \left(\begin{array}{cc} 1 & 0 \\ 0 & a_{22} \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right)\).

If \(b_{12} \neq 0\), we have \(x \sim \left(\begin{array}{cc} 1 & 0 \\ 0 & a_{22} \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 1 \\ b_{21} & b_{22} \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 1 \\ c_{21} & c_{22} \end{array}\right)\).

In any case, the orbit of \(x\) contains an element \((A, B, C) \in V\) where at least one of \(A, B, C\) is of rank 1. Therefore we have \(x \sim \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \cdot \left(\begin{array}{cc} 0 & b_{12} \\ b_{21} & b_{22} \end{array}\right) \cdot \left(\begin{array}{cc} 0 & c_{12} \\ c_{21} & c_{22} \end{array}\right)\).

Thus \(\det_x (u_1, u_2, u_3) \sim u_1(b_{22}u_2 + c_{22}u_3) - (b_{12}u_2 + c_{12}u_3)(b_{21}u_2 + c_{21}u_3)\).

If \(\det_x (u_1, u_2, u_3)\) is reducible, we see that \((b_{22}, c_{22})\parallel (b_{21}, c_{21})\) or \((b_{22}, c_{22})\parallel (b_{12}, c_{12})\).

- When \((b_{22}, c_{22}) = 0\), by \((b_{22}, c_{22})\parallel (b_{21}, c_{21})\) we have \(x \sim \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) \cdot \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \in O_9\).

- When \((b_{22}, c_{22}) \neq 0\), we can assume \((b_{21}, c_{21}) = t(b_{22}, c_{22})\) where \(t \in \mathbb{F}_q\) without loss of generality.

It follows that \(x \sim \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right) \in O_9\).

If \(\det_x (u_1, u_2, u_3)\) is irreducible, since \((b_{22}, c_{22})\parallel (b_{21}, c_{21})\) and \((b_{22}, c_{22})\parallel (b_{12}, c_{12})\), we have \(x \sim \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \in O_{10}\).

Next we count the number of the orbits. \(|O_1|, \ldots, |O_8|\) can be calculated by Propositions 3.2 and 4.1. For the count of \(|O_9|\), we calculate the order of the stabilizer subgroup of \(x_9 := \left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right)\) and \(\text{Stab}(x_9) := \{g \in G \mid gx_9 = x_9\}\). Let \(g = (g_1, g_2, g_3) = \left(\begin{array}{ccc} p_1 & q_1 & g_{11} \\ r_1 & s_1 & g_{12} \\ g_{31} & g_{32} & g_{33} \end{array}\right), \left(\begin{array}{ccc} p_2 & q_2 & g_{12} \\ r_2 & s_2 & g_{22} \\ g_{32} & g_{33} & g_{33} \end{array}\right) \in \text{Stab}(x_9)\). We
have
\[(1, 1, g_3) \cdot \left( \begin{bmatrix} q_1 p_2 & q_1 r_2 \\ s_1 p_2 & s_1 r_2 \end{bmatrix}, \begin{bmatrix} p_1 q_2 & p_1 s_2 \\ r_1 q_2 & r_1 s_2 \end{bmatrix}, \begin{bmatrix} q_1 q_2 & q_1 s_2 \\ s_1 q_2 & s_1 s_2 \end{bmatrix} \right) = \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right). \]

By comparing the (1, 1)-entry of each matrix, we obtain \([q_1 p_2, p_1 q_2, q_1 q_2] g_3 = [0, 0, 0]\), and therefore \(q_1 = q_2 = 0\). Thus
\[(1, 1, g_3) \cdot \left( \begin{bmatrix} 0 & 0 \\ s_1 p_2 & s_1 r_2 \\ 0 & 0 \\ 0 & r_1 s_2 \\ 0 & 0 & s_1 s_2 \end{bmatrix} \right) = \left( \begin{bmatrix} 0 & 0 & p_1 s_2 g_{12} \\ s_1 p_2 g_{11} & s_1 r_2 g_{11} + r_1 s_2 g_{12} + s_1 s_2 g_{13} \\ 0 & s_1 p_2 g_{21} & s_1 r_2 g_{21} + r_1 s_2 g_{22} + s_1 s_2 g_{23} \\ 0 & s_1 p_2 g_{31} & s_1 r_2 g_{31} + r_1 s_2 g_{32} + s_1 s_2 g_{33} \end{bmatrix} \right) = \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right). \]

Therefore we obtain \(s_1 p_2 g_{11} = p_1 s_2 g_{22} = 1\) and \(g_{12} = g_{21} = g_{31} = g_{32} = 0\). It follows that
\[
\begin{bmatrix} 0 & 0 & 0 \\ s_1 p_2 g_{11} & s_1 r_2 g_{11} + s_1 s_2 g_{13} \\ 0 & r_1 s_2 g_{22} + s_1 s_2 g_{23} \\ 0 & s_1 s_2 g_{33} \end{bmatrix} = \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right). \]

Thus we obtain \(g_{11} = (s_1 p_2)^{-1}, g_{22} = (p_1 s_2)^{-1}, g_{33} = (s_1 s_2)^{-1}, g_{13} = -\frac{p_2 g_{11}}{s_2}, \) and \(g_{23} = -\frac{r_2 g_{31}}{s_2}\). Therefore
\[
\text{Stab}(x_0) = \left\{ \begin{pmatrix} p_1 & 0 \\ r_1 & s_1 \end{pmatrix}, \begin{pmatrix} p_2 & 0 \\ r_2 & s_2 \end{pmatrix}, \begin{pmatrix} (s_1 p_2)^{-1} & 0 \\ 0 & (p_1 s_2)^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\frac{r_2}{s_2} \\ 0 & \frac{s_1 p_2 r_2}{s_2 s_1} \end{pmatrix} \right\} \subset G \]
\[
\cong ((\text{GL}_1)^2 \ltimes \mathbb{F}_q) \times ((\text{GL}_1)^2 \ltimes \mathbb{F}_q). \]

Thus we obtain \(|\text{Stab}(x)| = (q - 1)^4 q^2\), and \(|O_0| = |G|/|\text{Stab}(x)| = (q - 1)^3 q^3(q + 1)^3(q^2 + q + 1)\). Lastly, we obtain \(|O_{10}| = q^{12} - \sum_{i=1}^{9} |O_i| = (q - 1)^4 q^4(q + 1)(q^2 + q + 1)\).

5.2. The intersection between the orbits and the subspaces.

The subspaces we choose to calculate the Fourier transform are as follows:

\[W_1 = \emptyset, \quad W_2 = \left( \begin{bmatrix} 0 & 0 \\ 0 & * \\ 0 & * \\ 0 & * \end{bmatrix} \right), \quad W_3 = \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right), \quad W_4 = \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right), \quad W_5 = \left( \begin{bmatrix} 0 & * \\ 0 & * \\ 0 & * \end{bmatrix} \right).\]
Here \([a, b, c, d] = (q - 1)^d q^b (q + 1)^c (q^2 + q + 1)^d\), \(a_1 = 2q + 1\), \(b_1 = q^2 + 3q + 1\) and \(b_2 = 3q^2 + 3q + 1\).
5.3. Fourier transform.

[Proof]

We can calculate for all subspaces but $W_6$ and $W_6^\perp$ by means of Propositions 3.2, 4.1 and 4.2. Let $x \in W_6$ be
\[
\begin{bmatrix}
0 & 0 \\
0 & a_{22} \\
0 & b_{22}
\end{bmatrix}
\begin{bmatrix}
0 & c_{12} \\
c_{21} & c_{22}
\end{bmatrix}.
\]
For the case $a_{22} = 0$, we calculated in the proof of Proposition 4.2. We only count the case $a_{22} \neq 0$ and add up the two result. When $c_{12}c_{21} \neq 0$, we have $x \in \mathcal{O}_6$. The number of such elements is $(q - 1)^3q^2$. When $c_{12} = 0$ and $c_{21} \neq 0$, we have $x \in \mathcal{O}_4$. When $c_{21} = 0$ and $c_{12} \neq 0$, we have $x \in \mathcal{O}_5$. For both cases, there are $(q - 1)^2q^2$ of such $x$. The remaining elements all belong to $\mathcal{O}_2$. Next, let $x \in W_6^\perp$ be
\[
\begin{bmatrix}
0 & 0 \\
0 & a_{22} \\
b_{21} & b_{22}
\end{bmatrix}
\begin{bmatrix}
0 & c_{12} \\
c_{21} & c_{22}
\end{bmatrix}.
\]
Again, we only count the case $a_{22} \neq 0$. When $(b_{12}, c_{12}) \neq (b_{21}, c_{21})$, we have $x \in \mathcal{O}_9$. The number of such elements is $(q - 1)q^2g_1$. When $(b_{12}, c_{12}) \neq 0$ and $(b_{21}, c_{21}) \neq 0$ and $(b_{12}, c_{12}) = (b_{21}, c_{21})$, we have $x \in \mathcal{O}_6$. The number of such elements is $(q - 1)^2q^2(q^2 - 1)$. When $(b_{12}, c_{12}) = 0$ and $(b_{21}, c_{21}) \neq 0$, we have $x \in \mathcal{O}_4$. When $(b_{12}, c_{12}) \neq 0$ and $(b_{21}, c_{21}) = 0$, we have $x \in \mathcal{O}_5$. For both cases, there are $(q^2 - 1)(q - 1)^2q^2$ of such $x$. The remaining elements all belong to $\mathcal{O}_2$.

\[\square\]

5.3. Fourier transform.

THEOREM 5.1. The representation matrix $M$ of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, ..., e_{10}$ is given as follows:

\[
\begin{array}{cccccc}
1 & [1, 0, 2, 1] & [2, 1, 1, 1] & [2, 1, 2, 1] & [2, 1, 2, 1] & [3, 1, 3, 1] \\
1 & d_1 & [1, 1, 0, 0]c_1 & [1, 1, 1, 0]c_1 & [1, 1, 1, 0]c_1 & [2, 1, 1, 0]c_2 \\
1 & [0, 0, 1, 0]c_1 & qd_2 & [1, 1, 2, 0] & [1, 1, 2, 0] & [1, 1, 2, 0]c_4 \\
1 & [0, 0, 1, 0]c_1 & [1, 1, 1, 0] & qe_1 & [1, 1, 2, 0] & [1, 1, 1, 0]b_1 \\
1 & [0, 0, 1, 0]c_1 & [1, 1, 1, 0] & [1, 1, 2, 0] & qe_1 & [1, 1, 1, 0]b_1 \\
1 & q^{12} & c_2 & qc_4 & -qb_1 & -qb_1 & qe_2 \\
1 & c_3 & [1, 1, 0, 0]b_1 & [-1, 1, 0, 0]a_1 & [-1, 1, 0, 0]a_1 & [-2, 1, 0, 0]b_3 \\
1 & [-2, 1, 1, 0]b_2 & [0, 1, 1, 0] & [0, 1, 1, 0] & [0, 1, 1, 0] & [-2, 1, 2, 0]b_2 \\
1 & [-2, 1, 1, 0]b_2 & [-1, 1, 0, 0] & [-1, 1, 0, 0] & [-1, 1, 0, 0] & [-2, 1, 0, 0]a_1 \\
1 & [-2, 1, 0, 0] & q & [0, 1, 1, 0] & [0, 1, 1, 0] & [0, 1, 1, 0]
\end{array}
\]
We used PARI/GP [8] to calculate the matrix from Proposition 5.

\[\hat{x} \in \Psi(\hat{x})\]

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Here \([a, b, c, d] = (q - 1)^d q^h(q + 1)^c(q^2 + q + 1)^d\) and

\[
\begin{align*}
a_1 &= 2q + 1, & c_1 &= q^3 - q - 1, & d_1 &= 2q^4 + q^3 - q^2 - 2q - 1, \\
b_1 &= q^2 - q - 1, & c_2 &= q^3 - q^2 - 2q - 1, & d_2 &= q^5 - q^2 + 1, \\
b_2 &= q^2 - q + 1, & c_3 &= 2q^3 - q^2 - 2q - 1, & e_1 &= q^5 - q^3 - q^2 + q + 1, \\
b_3 &= q^2 + 3q + 1, & c_4 &= q^3 - q^2 + 1, & e_2 &= q^5 - q^4 - q^3 + 2q^2 - q - 1, \\
b_4 &= q^2 - 2q - 1, & c_5 &= q^3 - 3q^2 + 3q + 1, & c_6 &= q^3 + q^2 - q + 1.
\end{align*}
\]

We used PARI/GP [8] to calculate the matrix from Proposition 5.2.

**Corollary 5.1.** The indicator function of singular set of \(V\) is \(\Psi = \sum_{i=1}^{9} e_i\). Its Fourier transform \(\hat{\Psi}\) is given as follows:

\[
\hat{\Psi}(x) = \begin{cases} 
q^{-1} + 2q^{-2} - q^{-3} - 2q^{-4} - q^{-5} + 2q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_1, \\
q^{-3} - q^{-4} - 2q^{-5} + 2q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_2, \\
-q^{-5} + q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \\
-q^{-6} + 2q^{-7} - q^{-8} & x \in \mathcal{O}_7, \\
q^{-6} - q^{-8} & x \in \mathcal{O}_8, \\
q^{-7} - q^{-8} & x \in \mathcal{O}_9, \\
-q^{-8} & x \in \mathcal{O}_{10}.
\end{cases}
\]

In particular, we have the following \(L_1\)-norm bound of \(\hat{\Psi}\):

\[
\sum_{x \in V} |\hat{\Psi}(x)| = O(q^4).
\]

6. **2 \otimes 2 \otimes 4**

Let \(V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^4\) and \(G = G_1 \times G_2 \times G_3 = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_4\). We write \(x \in V\) as \(x = (A, B, C, D)\) where \(A, B, C\) and \(D\) are 2-by-2 matrices, and write \(g \in G\) as...
where \( g_1, g_2 \in \text{GL}_2 \) and \( g_3 \in \text{GL}_4 \). The action of \( G \) on \( V \) is defined by
\[
gx = (g_1 A g_2^T, g_1 B g_2^T, g_1 C g_2^T, g_1 D g_2^T) g_3^T.
\]
Define a bilinear form \( \beta \) of \( V \) as
\[
\beta((A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2)) = \text{Tr}(A_1 A_2^T + B_1 B_2^T + C_1 C_2^T + D_1 D_2^T).
\]
In addition, define an automorphism \( \iota \) of \( G \) as
\[
(g_1, g_2, g_3) \iota = ((g_1^T)^{-1}, (g_2^T)^{-1}, (g_3^T)^{-1})
\]
These \( \beta \) and \( \iota \) satisfy Assumption 2.1.

### 6.1. Orbit decomposition.

For \( x = (A, B, C, D) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\right) \in V \), we define
\[
r_1(x) := \text{rank} \left(\begin{bmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ d_{11} & d_{12} & d_{21} & d_{22} \end{bmatrix}\right),
\]
\[
r_2(x) := \text{rank} \left(\begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} & d_{11} & d_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix}\right),
\]
\[
r_3(x) := \text{rank} \left(\begin{bmatrix} a_{11} & a_{21} & b_{11} & b_{12} & c_{11} & c_{21} & d_{11} & d_{12} \\ a_{12} & a_{22} & b_{21} & b_{22} & c_{12} & c_{22} & d_{21} & d_{22} \end{bmatrix}\right),
\]
\[
det_x(u_1, u_2, u_3, u_4) := \det(u_1 A + u_2 B + u_3 C + u_4 D) \in \text{Sym}^2(\mathbb{F}_q^4)
\]
where \( u_1, u_2, u_3, u_4 \) are variables, \( T(x) := \langle \alpha \rangle \) if and only if \( \det_x(u, v) \in O_{\langle \alpha \rangle} \) in \( \text{Sym}^2(\mathbb{F}_q^4) \).

Note that we introduced the representation \((\text{GL}_1(\mathbb{F}_q) \times \text{GL}_4(\mathbb{F}_q), \text{Sym}^2(\mathbb{F}_q^4))\) in Section 3.1. For \( x \in V \) and \( g = (g_1, g_2, g_3) \in G \), we have
\[
det_{g_3}(u_1, u_2, u_3, u_4) = \det(g_1 g_2) \det_x((u_1, u_2, u_3, u_4) g_3).
\]
**PROPOSITION 6.1.** \( V \) consists of 11 \( G \)-orbits in all.

<table>
<thead>
<tr>
<th>Orbit name ( O )</th>
<th>Representative</th>
<th>( r_1(x) )</th>
<th>( r_2(x) )</th>
<th>( r_3(x) )</th>
<th>( T(x) )</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_1 )</td>
<td>((0, 0))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \emptyset )</td>
<td>1</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>((0, 0))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \emptyset )</td>
<td>(1, 0, 3, 0, 1)</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>((0, 0))</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>( {1} )</td>
<td>(2, 1, 2, 0, 1)</td>
</tr>
<tr>
<td>( O_4 )</td>
<td>((0, 0))</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>( \emptyset )</td>
<td>(2, 1, 2, 1, 1)</td>
</tr>
<tr>
<td>( O_5 )</td>
<td>((0, 0))</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>( \emptyset )</td>
<td>(2, 1, 2, 1, 1)</td>
</tr>
<tr>
<td>( O_6 )</td>
<td>((0, 0))</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( {1} )</td>
<td>(3, 1, 3, 1, 1)</td>
</tr>
<tr>
<td>( O_7 )</td>
<td>((0, 0))</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( {2} )</td>
<td>(2, 3, 3, 1, 1)</td>
</tr>
<tr>
<td>( O_8 )</td>
<td>((0, 0))</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>( {2} )</td>
<td>(4, 3, 3, 1, 1)</td>
</tr>
<tr>
<td>( O_9 )</td>
<td>((0, 0))</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>( {2} )</td>
<td>(3, 3, 4, 1, 1)</td>
</tr>
<tr>
<td>( O_{10} )</td>
<td>((0, 0))</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>( {3} )</td>
<td>(4, 4, 3, 1, 1)</td>
</tr>
<tr>
<td>( O_{11} )</td>
<td>((0, 0))</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>( {4} )</td>
<td>(4, 6, 2, 1, 1)</td>
</tr>
</tbody>
</table>

Here \( [a, b, c, d, e] = (q - 1)^a q^b (q + 1)^c (q^2 + q + 1)^d (q^2 + 1)^e \), and \( \mu_1, \mu_0 \) are elements of \( \mathbb{F}_q \) such that \( X^2 + \mu_1 X + \mu_0 \in \mathbb{F}_q[X] \) is irreducible.

[Proof]

First we consider the orbit decomposition. When \( r_1(x) \leq 3 \), we have \( x \sim (0, B, C, D) \) where \( B, C, D \in M_2(\mathbb{F}_q) \). Therefore by Propositions 3.1 and 5.1, we have

\[
\{ x \in V | r_1(x) \leq 3 \} = \bigcup_{i=1}^{10} O_i.
\]

When \( r_1(x) = 4 \), it is easy to show that \( x \sim \left( \begin{array}{cccc} 1 & 0 \\ 0 & 0 \end{array} \right) \cdot \left( \begin{array}{cccc} 0 & 0 \\ 1 & 0 \end{array} \right) \cdot \left( \begin{array}{cccc} 0 & 0 \\ 0 & 0 \end{array} \right) \cdot \left( \begin{array}{cccc} 0 & 0 \\ 1 & 0 \end{array} \right) \) by the action of \( \text{GL}_4 \). Therefore the number of orbits is 11.

\(|O_1|, \ldots, |O_{10}| \) can be calculated by means of Propositions 3.2 and 5.1. By subtracting these numbers from \(|V| = q^{16}\), we obtain \(|O_{11}| \). Alternatively, we may say that \(|O_{11}| \) coincides with \( \text{gl}_4 \).

\( \square \)

6.2. The intersection between the orbits and the subspaces.

The subspaces we choose to calculate the Fourier transform are as follows:
Orthogonal complements of them are as follows:

\[ W_1 = 0, \quad W_2 = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right), \]

\[ W_3 = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right), \]

\[ W_4 = \left( \begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\end{array} \right), \]

\[ W_5 = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right), \]

\[ W_6 = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right), \]

\[ W_7 = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right), \]

\[ W_8 = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right), \]

\[ W_9 = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right), \]

\[ W_{10} = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right) \text{ and } W_{11} = V. \]

The cardinalities \(|\mathcal{O}_i \cap W_j|\) for the orbit \(\mathcal{O}_i\) and the subspace \(W_j\) are given as follows:

<table>
<thead>
<tr>
<th>(\mathcal{O}_i)</th>
<th>(W_1)</th>
<th>(W_2)</th>
<th>(W_3)</th>
<th>(W_4)</th>
<th>(W_5)</th>
<th>(W_6)</th>
<th>(W_7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{O}_1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\mathcal{O}_2)</td>
<td>0</td>
<td>[1, 0, 1, 0, 1]</td>
<td>[1, 0, 2, 0, 0]</td>
<td>[1, 0, 2, 0, 1]</td>
<td>[1, 0, 2, 0, 1]</td>
<td>(q - 1)c_1</td>
<td>2[1, 0, 1, 0, 1]</td>
</tr>
<tr>
<td>(\mathcal{O}_3)</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 1, 0, 0]</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 0, 0, 0]</td>
<td>[2, 0, 1, 0, 1]</td>
</tr>
<tr>
<td>(\mathcal{O}_4)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 1, 1, 1]</td>
<td>0</td>
<td>[2, 1, 0, 1, 1]</td>
<td>0</td>
</tr>
<tr>
<td>(\mathcal{O}_5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 1, 1, 1]</td>
<td>[2, 1, 0, 1, 0]</td>
<td>0</td>
</tr>
<tr>
<td>(\mathcal{O}_6)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[3, 1, 0, 1, 0]</td>
<td>0</td>
</tr>
<tr>
<td>(\mathcal{O}_7)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 1, 1, 1]</td>
</tr>
<tr>
<td>(\mathcal{O}_8)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\mathcal{O}_9)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\mathcal{O}_{10})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\mathcal{O}_{11})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
When \(b_{12}, c_{12}, d_{12} \neq 0\), we have \(x \in O_9\). The number of such elements is \((q - 1)^3 - 1\). When \(b_{12}, c_{12}, d_{12} = 0\), we have \(x \in O_8\). The number of elements is \((q^2 - 1)(q - 1)^2\). The remaining elements all belong to \(O_2\). \(\square\)

6.3. Fourier transform.

**Theorem 6.1.** The representation matrix \(M\) of the Fourier transform on \(F^G_v\) with respect to the basis \(e_1, \ldots, e_{11}\) is given as follows:
We used PARI/GP [8] to calculate the matrix from Proposition 6.2.
In addition, define an automorphism \( \iota \) of \( G \) as
\[
\iota(F) = (g_1^T)^{-1}, (g_2^T)^{-1}.
\]
These \( \beta \) and \( \iota \) satisfy Assumption 2.1.

### 7.1. Orbit decomposition.

For \( x = (A, B) \in \mathbb{F}_q^2 \), we define
\[
r_1(x) := \text{rank} \left( \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \right),
\]
\[
r_2(x) := \text{rank} \left( \begin{bmatrix} a_{11} & a_{12} & a_{12} & a_{22} \\ b_{11} & b_{12} & b_{12} & b_{22} \end{bmatrix} \right),
\]
\[
det_x(u, v) := \det(uA + vB) \in \text{Sym}^2(\mathbb{F}_q^2) \text{ where } u, v \text{ are variables},
\]
\[
T(x) := \langle \alpha \rangle \text{ if and only if } \det_x(u, v) \in O_{\langle \alpha \rangle} \text{ in } \text{Sym}^2(\mathbb{F}_q^2).
\]
For \( x \in V \) and \( g = (g_1, g_2) \in G \), we have
\[
\det_{q, x}(u, v) = N_2(\det(g_2))\det_x((u, v)g_1),
\]
where \( N_2 : \mathbb{F}_q \to \mathbb{Z} \to \mathbb{F}_q \) is the norm map.

**Proposition 7.1.** \( V \) consists of 6 \( G \)-orbits in all.

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>( r_1(x) )</th>
<th>( r_2(x) )</th>
<th>( T(x) )</th>
<th>Cardinality</th>
</tr>
</thead>
</table>
| \( O_1 \)  | \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\] | 0 | 0 | \( \langle 0 \rangle \) | 1 |
| \( O_2 \)  | \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\] | 1 | 1 | \( \langle 0 \rangle \) | \( [1, 0, 1, 1] \) |
| \( O_3 \)  | \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] | 1 | 2 | \( \langle 1 \rangle \) | \( [1, 1, 1, 1] \) |
| \( O_4 \)  | \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\] | 2 | 2 | \( \langle 1 \rangle \) | \( [2, 1, 2, 1] \) |
| \( O_5 \)  | \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\] | 2 | 2 | \( \langle 2r \rangle \) | \( \frac{1}{2}[2, 3, 1, 1] \) |
| \( O_6 \)  | \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & \mu_0 \\
\mu_0 & \mu_1
\end{bmatrix}
\] | 2 | 2 | \( \langle 2i \rangle \) | \( \frac{1}{2}[2, 3, 1, 1] \) |

Here \( [a, b, c, d] = (q - 1)^{a-1}a^{b}(q + 1)^{c+1}d, \) and \( \mu_1, \mu_0 \in \mathbb{F}_q^* \) are elements such that \( X^2 + \mu_1X - N_2(\mu_0) \in \mathbb{F}_q[X] \) is irreducible.

Note that there exist such \( \mu_1 \) and \( \mu_0 \) of the lowest row of the table because of the surjectivity of the norm map \( N_2 \).

[Proof]

The invariants \( r_1(x), r_2(x) \) and \( T(x) \) for the 6 elements in the table are easily calculated. Since they do not coincide, these 6 elements belong to different orbits. Let \( O_i \) be the orbit of each element.

First we consider the orbit decomposition. When \( r_1(x) = 0 \), we easily have \( x \in O_1 \). When \( r_1(x) = 1 \), we have \( x \sim (0, B) \). If \( \text{rank}(B) = 1 \) we have \( x \in O_2 \), and if \( \text{rank}(B) = 2 \) we have \( x \in O_3 \). When \( r_1(x) = 2 \), we have \( T(x) = \langle 1 \rangle, \langle 2r \rangle \) or \( \langle 2i \rangle \). If \( T(x) = \langle 1 \rangle \) or \( \langle 2r \rangle \), we have \( x \sim \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & b_1 \\ b_1 & b_2 \end{bmatrix} \right) \). Since \( \det_x(u, v) = v(b_{22}u - N_2(b_{12})v) \), we have \( x \in O_4 \) if \( b_{22} = 0 \) and \( x \in O_5 \) if \( b_{22} \neq 0 \). If \( T(x) = \langle 2i \rangle \), we have \( \det_x(u, v) \sim u^2 + \mu_1uv - N_2(\mu_0)v^2 = (u - v)(u - \overline{v}) \) where \( v \in \mathbb{F}_q^* \) by Section 3.1 and the surjectivity of the map \( G \ni (g_1, g_2) \mapsto (N_2(\det(g_2)), g_1^T) \in GL_1(\mathbb{F}_q) \times GL_2(\mathbb{F}_q) \). Therefore we assume \( \det_x(u, v) = u^2 + \mu_1uv - N_2(\mu_0)v^2 \). We can move \( x \) to \( y := \left( \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \right) \) with \( a_{11} \neq 0 \) by the action of \( G_2 \). As we saw in the proof of Proposition 4.1, there exists \( \left( \begin{bmatrix} p \\ r \end{bmatrix}, q \right) \in GL_2(\mathbb{F}_q) \) such that \( p, q, r, s \) satisfy the equation (5) and (7). Let \( g := \left( \begin{bmatrix} p \\ r \end{bmatrix}, q \right) \in G \). Then the (1, 1)-entry of the first matrix of
gy is nonzero and the $(1, 1)$-entry of the second matrix of $gy$ is zero. Thus we can move $x$ to $y' := \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & b'_{12} \\ b'_{12} & b'_{22} \end{bmatrix} \right)$ such that $\det_{y'}(u, v) = g'(u^2 + \mu_1 uv - N_2(\mu_0)v^2)$ for certain $g' \in \text{GL}_1(F_q)$. On the other hand, $\det_{y'}(u, v) = u^2 + b'_{22} uv - N_2(b'_{12})v^2$. Therefore we have $g' = 1$ and $b'_{22} = \mu_1, N_2(b'_{12}) = N_2(\mu_0)$. Since $N_2(b'_{12}) = 1$ we obtain

$$
\left( 1, \left( \begin{bmatrix} 0 & \mu_0 \\ \mu_0 & \mu_1 \end{bmatrix} \right) \right)
$$

Next we consider the cardinality of the each orbit. We use the following facts for the calculation:

$$
|M \in \text{H}_2(F_q) \mid \text{rank}(M) = 1| = (q - 1)(q^2 + 1),
$$

$$
|M \in \text{H}_2(F_q) \mid \text{rank}(M) = 2| = (q - 1)q(q^2 + 1).
$$

It is clear that $|O_1| = 1$. In addition, we easily have $|O_2| = (q + 1) \cdot |\{M \in \text{H}_2(F_q) \mid \text{rank}(M) = 1\}|$ and $|O_3| = (q + 1) \cdot |\{M \in \text{H}_2(F_q) \mid \text{rank}(M) = 2\}|$. Next we count $|O_4|$. Let $x_4 = \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$, and $\text{Stab}(x_4) := \{g \in G \mid gx_4 = x_4\}$. Let $g = (g_1, g_2) = \left( \begin{bmatrix} p_1 \\ q_1 \\ s_1 \\ r_1 \end{bmatrix}, \begin{bmatrix} p_2 \\ q_2 \\ s_2 \\ r_2 \end{bmatrix} \right) \in \text{Stab}(x_4)$. We have $g_2 \left[ \begin{bmatrix} 0 & q_1 \\ p_1 & q_2 \end{bmatrix} g_1^{-1}, g_2 \left[ \begin{bmatrix} 0 & s_1 \\ s_1 & r_1 \end{bmatrix} g_1^{-1} \right] g_2^{-1} \right] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. By comparing the rank of the first entry, we have $q_1 = 0$, and $p_1 s_1 \neq 0$. It follows that $p_1 \begin{bmatrix} N_2(q_2) & q_2 s_2 \\ s_2 q_2 & N_2(s_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and therefore $q_2 = 0$ and $p_1 N_2(s_2) = 1$. Thus we have $g_2 \left[ \begin{bmatrix} 0 & s_1 \\ s_1 & r_1 \end{bmatrix} g_1^{-1} \right] g_2^{-1} = \begin{bmatrix} 0 & s_1 p_2 s_2 \\ s_1 s_2 r_2 & r_1 N_2(s_2) + s_1 \text{Tr}_2(s_2 r_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and therefore

$$
\text{Stab}(x_4) = \left\{ \left. \begin{bmatrix} (N_2(s_2))^{-1} & 0 \\ s_1 & r_1 \end{bmatrix}, \begin{bmatrix} (s_1 s_2)^{-1} & 0 \\ r_2 & s_2 \end{bmatrix} \right) \in G \mid r_1 = -s_1 \text{Tr}_2(s_2 r_2) / N_2(s_2) \right\}
$$

$$
\cong (\text{GL}_1(F_q) \times \text{GL}_1(F_q^2)) \rtimes F_q^2.
$$

Thus we obtain $|\text{Stab}(x_4)| = (q - 1)^2 q^2 (q + 1)$, and $|O_4| = |G| / (q - 1)^2 q^2 (q + 1) = (q - 1)^2 q (q + 1)^2 (q^2 + 1)$. Next we count $|O_5|$. Let $x_5 := \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$ and $\text{Stab}(x_5) := \{g \in G \mid gx_5 = x_5\}$. Let $g = (g_1, g_2, g_3) = \left( \begin{bmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \end{bmatrix}, \begin{bmatrix} p_2 \\ q_2 \\ r_2 \\ s_2 \end{bmatrix} \right) \in \text{Stab}(x_5)$. We have $g_2 \left[ \begin{bmatrix} p_1 \\ 0 \\ q_1 \\ r_1 \\ s_1 \end{bmatrix} g_1^{-1}, g_2 \left[ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} g_1^{-1} \right] g_2^{-1} \right] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. By comparing the rank of each entry, we obtain the following propositions:

\[(12) \quad \text{If } p_1 \neq 0, \text{ then } q_1 = r_1 = 0 \quad \text{and} \quad s_1 \neq 0.\]
If $p_1 = 0$, then $q_1 r_1 \neq 0$ and $s_1 = 0$.

First we assume the case (12). Then we have
\[
(p_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, s_1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]
and therefore
(14) \[q_2 = r_2 = 0, \quad p_1 = N_2(p_2)^{-1}, \quad s_1 = N_2(s_2)^{-1}.
\]

Next we assume the case (13). Then we have
\[
(q_1 \begin{bmatrix} 0 & 0 \\ s_2 & 0 \end{bmatrix}, r_1 \begin{bmatrix} 0 & 0 \\ r_2 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]
and therefore
(15) \[p_2 = s_2 = 0, \quad q_1 = N_2(q_2)^{-1}, \quad r_1 = N_2(r_2)^{-1}.
\]

By (14) and (15), we obtain
\[
\text{Stab}(x_5) = \left\{ \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \right\} \ltimes \left\{ \left( \begin{bmatrix} N_2(p_2)^{-1} & 0 \\ 0 & N_2(s_2)^{-1} \end{bmatrix}, \begin{bmatrix} p_2 & 0 \\ 0 & s_2 \end{bmatrix} \right) \in G \right\}
\]
\[\cong \mathbb{Z}/2\mathbb{Z} \rtimes \text{GL}_1(F_{q^2}).\]

Thus we obtain $|\text{Stab}(x_5)| = 2(q - 1)^2(q + 1)^2$, and $|O_5| = |G|/2(q - 1)^2(q + 1)^2 = (q - 1)^2q^3(q + 1)(q^2 + 1)/2$. $|O_6|$ can be calculated by subtracting $|O_1|, \ldots, |O_5|$ from $|V| = q^8$. \hfill \Box

### 7.2. The intersection between the orbits and the subspaces.

The subspaces we choose to calculate the Fourier transform are as follows:

$W_1 = 0$, $W_2 = \left( \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} \right)$, $W_3 = \left( \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} \right)$,

$W_4 = \left( \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \right)$,

$W_5 = \left( \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix}, \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \right)$ and $W_6 = V$.

Orthogonal complements of them are as follows:

$W_1^\perp = W_6$, $W_2^\perp = \left( \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \right)$, $W_3^\perp = W_3$, $W_4^\perp = W_4$, $W_5^\perp = \left( \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \right)$ and $W_6^\perp = W_1$.\hfill \Box
PROPOSITION 7.2. The cardinalities $|O_i \cap W_j|$ for the orbit $O_i$ and the subspace $W_j$ are given as follows:

<table>
<thead>
<tr>
<th></th>
<th>$W_1$</th>
<th>$W_2$</th>
<th>$W_3$</th>
<th>$W_4$</th>
<th>$W_5$</th>
<th>$W_6$</th>
<th>$W_7^\perp$</th>
<th>$W_8^\perp$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$O_2$</td>
<td>0</td>
<td>[1, 0, 1, 0]</td>
<td>[1, 0, 1, 0]</td>
<td>2[1, 0, 1, 0]</td>
<td>[1, 0, 1, 0]</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_3$</td>
<td>0</td>
<td>0</td>
<td>[1, 1, 0, 1]</td>
<td>[1, 1, 1, 0]</td>
<td>[2, 0, 1, 0]</td>
<td>[1, 1, 1, 0]</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 1, 0]</td>
<td>0</td>
<td>[2, 1, 2, 1]</td>
<td>[2, 1, 2, 0]</td>
<td>0</td>
</tr>
<tr>
<td>$O_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[2, 1, 1, 0]</td>
</tr>
</tbody>
</table>

Here $[a, b, c, d] = (q - 1)^a q^b (q + 1)^c (q^2 + 1)^d$.

[Proof]

We only consider the cases of $W_2^\perp$ and $W_5^\perp$, since the rest cases are easy. We write $x \in W_2^\perp$ as $x = (A, B) = \begin{bmatrix} 0 & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \begin{bmatrix} 0 \\ b_{12} \end{bmatrix}$. We consider the case $(a_{12}, b_{12}) \neq 0$.

We have $\det_x(u, v) = N_2(a_{12})u^2 + (a_{12}b_{12} + a_{12}b_{12})uv + N_2(b_{12})v^2$ with the discriminant $\text{Disc}(\det_x(u, v)) = (a_{12}b_{12} - \overline{a_{12}b_{12}})^2$. Thus we have

\begin{align*}
(16) & \quad a_{12} \text{ and } b_{12} \text{ are parallel over } \mathbb{F}_q \text{ if and only if } a_{12}b_{12} = \overline{a_{12}b_{12}}, \\
(17) & \quad a_{12} \text{ and } b_{12} \text{ are not parallel over } \mathbb{F}_q \text{ if and only if } a_{12}b_{12} - \overline{a_{12}b_{12}} \notin \mathbb{F}_q.
\end{align*}

When $a_{12}$ and $b_{12}$ are not parallel over $\mathbb{F}_q$, we have $x \in O_6$. The number of such elements is $q^2 \# \mathbb{G}$. When $a_{12}$ and $b_{12}$ are parallel over $\mathbb{F}_q$ and $(a_{12}, b_{12}) \not\parallel (a_{22}, b_{22})$, we have $x \in O_4$. The number of such elements is $q(q^2 - 1)(q^2 - q)$. When $(a_{22}, b_{22}) \neq 0$ and $\forall B$ over $\mathbb{F}_q$, we have $x \in O_3$. The number of such elements is $(q^2 - 1)(q + 1)$. Remaining elements belong to $O_1$ or $O_2$. We write $x \in W_5^\perp$ as $x = \begin{bmatrix} 0 & a_{12} \\ a_{12} & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b_{12} \end{bmatrix}$. We only consider the case $(a_{12}, b_{12}) \neq 0$. By (17), we find that when $a_{12}$ and $b_{12}$ are not parallel over $\mathbb{F}_q$, we have $x \in O_6$. The number of such elements is $\# \mathbb{G}$. By (16), when $a_{12}$ and $b_{22}$ are parallel over $\mathbb{F}_q$, we have $x \in O_3$. The number of such elements is $(q^2 - 1)(q + 1)$. \hfill \Box

7.3. Fourier transform.

THEOREM 7.1. The representation matrix $M$ of the Fourier transform on $X_V^G$ with respect to the basis $e_1, \ldots, e_6$ is given as follows:

$$
\begin{bmatrix}
1 & 1, 0, 1, 1 & 1, 1, 1 & 1, 1, 1 & 1, 1, 1 & 1, 1, 1 & 1, 1, 1 \\
0 & -1 & 1, 1, 0 & 0 & 1, 1, 0 & 0 & 1, 1, 0 \\
0 & 1, 0, 0 & 0 & b_1 & q c_1 & -1, 1, 0, 0 & -1, 1, 0, 0 \\
0 & -1 & 0 & q & c_2 & -1, 1, 0, 0 & -1, 1, 0, 0 \\
1 & 0, 0, 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1, 0, 1, 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Here $[a, b, c, d] = (q - 1)^a q^b (q + 1)^c (q^2 + 1)^d$, $b_1 = q^2 + q + 1$, $c_1 = q^3 - q^2 - 1$ and $c_2 = q^3 - q^2 + 1$.

We used PARI/GP [8] to calculate the matrix from Proposition 7.2.
COROLLARY 7.1. The indicator function of singular set of $V$ is $Ψ = \sum_{i=1}^{4} e_i$. Its Fourier transform $\hat{Ψ}$ is given as follows:

$$
\hat{Ψ}(x) = \begin{cases} 
q^{-1} + q^{-4} - q^{-5} & x = 0, \\
q^{-4} - q^{-5} & x \neq 0, \text{Disc}(\det_x(u, v)) = 0, \\
- q^{-5} & \text{Disc}(\det_x(u, v)) \neq 0.
\end{cases}
$$

In particular, we have the following $L_1$-norm bound of $\hat{Ψ}$:

$$
\sum_{x \in V} |\hat{Ψ}(x)| = O(q^3).
$$

8. $2 \otimes \wedge^2(4)$

Let $\wedge^2(\mathbb{F}_q^4)$ be the set of all alternating matrices of order 4 over $\mathbb{F}_q$. We write $A \in \wedge^2(4)$ as

$$
A = \begin{bmatrix}
0 & a_{12} & a_{13} & a_{14} \\
-a_{12} & 0 & a_{23} & a_{24} \\
-a_{13} & -a_{23} & 0 & a_{34} \\
-a_{14} & -a_{24} & -a_{34} & 0
\end{bmatrix}
$$

where $a_{ij} \in \mathbb{F}_q$.

The Pfaffian of $A$ is defined by

$$
Pfaff(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.
$$

Let $V = \mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^4)$ and $G = G_1 \times G_2 = GL_2 \times GL_4$. We write $x \in V$ as $x = (A, B)$ where $A, B \in \wedge^2(4)$, and write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in GL_2$ and $g_2 \in GL_4$. The action of $G$ on $V$ is defined by

$$
gx = (g_2Ag_2^T, g_2Bg_2^T)g_1^T.
$$

Define a bilinear form $β$ of $V$ as

$$
β((A_1, B_1), (A_2, B_2)) = \text{Tr}(A_1A_2^T + B_1B_2^T).
$$

In addition, define an automorphism $ι$ of $G$ as

$$
(g_1, g_2)^ι = ((g_1^T)^{-1}, (g_2^T)^{-1}).
$$

These $β$ and $ι$ satisfy Assumption 2.1.

8.1. Orbit decomposition.

For $x = (A, B) = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\
a_{12} & 0 & a_{23} & a_{24} \\
a_{13} & -a_{23} & 0 & a_{34} \\
a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}$, we define

$$
r_1(x) := \dim((A, B)_{\mathbb{F}_q^2}), \text{ i.e., the dimension of the subspace of } \wedge^2(4) \text{ which is}
$$
generated by $A$ and $B$, 

$$r_2(x) := \text{rank} \begin{bmatrix}
0 & a_{12} & a_{13} & a_{14} & 0 & b_{12} & b_{13} & b_{14} \\
-a_{12} & 0 & a_{23} & a_{24} & -b_{12} & 0 & b_{23} & b_{24} \\
-a_{13} & -a_{23} & 0 & a_{34} & -b_{13} & -b_{23} & 0 & b_{34} \\
-a_{14} & -a_{24} & -a_{34} & 0 & -b_{14} & -b_{24} & -b_{34} & 0
\end{bmatrix},$$

$\text{Pf}_x(u, v) := \text{Pfaff}(uA + vB) \in \text{Sym}^2(\mathbb{F}_q^2)$ where $u, v$ are variables,

$$T(x) := \langle \alpha \rangle \text{ if and only if } \text{Pf}_x(u, v) \in O_{\beta \alpha} \text{ in } \text{Sym}^2(\mathbb{F}_q^2).$$

For $x \in V$ and $g = (g_1, g_2) \in G$, we have

$$\text{Pf}_{gX}(u, v) = \det(g_2)\text{Pf}_x((u, v)g_1).$$

Since alternating matrix is determined by its upper triangular part, we write

$$\begin{bmatrix}
0 & a_{12} & a_{13} & a_{14} \\
-a_{12} & 0 & a_{23} & a_{24} \\
-a_{13} & -a_{23} & 0 & a_{34} \\
-a_{14} & -a_{24} & -a_{34} & 0
\end{bmatrix}, \begin{bmatrix}
0 & b_{12} & b_{13} & b_{14} \\
-b_{12} & 0 & b_{23} & b_{24} \\
-b_{13} & -b_{23} & 0 & b_{34} \\
-b_{14} & -b_{24} & -b_{34} & 0
\end{bmatrix}$$

as

$$\begin{bmatrix}
a_{12} & a_{13} & a_{14} & a_{23} & a_{24} & a_{34} \\
b_{12} & b_{13} & b_{14} & b_{23} & b_{24} & b_{34}
\end{bmatrix}.$$ \hfill (PROPOSITION 8.1.)

$V$ consists of 7 $G$-orbits in all.

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>$r_1(x)$</th>
<th>$r_2(x)$</th>
<th>T(x)</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>0</td>
<td>0</td>
<td>$\langle 0 \rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$O_2$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>1</td>
<td>2</td>
<td>$[1, 0, 1, 1, 1]$</td>
<td></td>
</tr>
<tr>
<td>$O_3$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>1</td>
<td>4</td>
<td>$[2, 2, 1, 1, 0]$</td>
<td></td>
</tr>
<tr>
<td>$O_4$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>2</td>
<td>3</td>
<td>$[0, 2, 1, 2, 1]$</td>
<td></td>
</tr>
<tr>
<td>$O_5$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>2</td>
<td>4</td>
<td>$[1, 2, 2, 1, 1]$</td>
<td></td>
</tr>
<tr>
<td>$O_6$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>2</td>
<td>4</td>
<td>$[2r, 2, 1, 1, 1]$</td>
<td></td>
</tr>
<tr>
<td>$O_7$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>2</td>
<td>4</td>
<td>$[2, 5, 1, 1, 1]$</td>
<td></td>
</tr>
</tbody>
</table>

Here $[a, b, c, d, e] = (q - 1)^a q^b (q + 1)^c (q^2 + q + 1)^d (q^2 + 1)^e$, and $\mu_1, \mu_0$ are elements of $\mathbb{F}_q$ such that $X^2 + \mu_1 X + \mu_0 \in \mathbb{F}_q[X]$ is irreducible.

[Proof]

The invariants $r_1(x), r_2(x)$ and $T(x)$ for the 7 elements in the table are easily calculated. Since they do not coincide, these 7 elements belong to different orbits. Let $O_i$ be the orbit of each element.
First we prove that \( V = \bigcup_{i=1}^{r} O_i \). Let \( x \in V \). When \( r_1(x) = 0 \) we easily have \( x \in O_1 \). When \((r_1(x), r_2(x)) = (1, 2)\), we have \( x \sim (0, B) \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \) by the action of \( G \). It follows that \( x \in O_2 \). When \((r_1(x), r_2(x)) = (1, 4)\), we have \( x \sim (0, B) \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \). It follows that \( x \in O_3 \). When \( r_1(x) = 2 \), we have \( r_2(x) \geq 2 \). If \((r_1(x), r_2(x)) = (2, 3)\), we have \( x \sim \begin{bmatrix} 0 & 0 & a_{23} & a_{24} & a_{34} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \). It follows that \( x \in O_4 \). When \((r_1(x), r_2(x)) = (2, 4)\), we have \( x \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix} \). Thus we have \( P f_\times(u, v) \sim a_{34}u^2 + b_{34}uv - \begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{vmatrix} v^2 \).

If \( T(x) = \{0\} \), we have \( a_{34} = b_{34} = \begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{vmatrix} = 0 \), which contradicts to the assumption \( r_2(x) = 4 \). It follows that \( T(x) = \{1\} \), \( \{2r\} \) or \( \{2i\} \). If \( T(x) = \{1\} \), we can let \( \text{rank}(A) = 2 \) and therefore \( x \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix} \). Thus we have \( P f_\times(u, v) \sim b_{34}uv - \begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{vmatrix} v^2 \). Since \( T(x) = \{1\} \), we have \( b_{34} = 0 \) and \( \begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{vmatrix} \neq 0 \). It follows that \( x \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix} \) as in the case of \( T(x) = \{1\} \). Since in this case we have \( b_{34} \neq 0 \), we have \( x \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix} \).

If \( T(x) = \{2i\} \), we have \( P f_\times(u, v) \sim u^2 + \mu_1uv + \mu_0v^2 = (u - \gamma v)(u - \overline{\gamma}v) \) where \( \gamma \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) by Section 3.1 and the surjectivity of the map \( G \ni (g_1, g_2) \mapsto (\det(g_2), g_1^T) \in \text{GL}_1(\mathbb{F}_q) \times \text{GL}_2(\mathbb{F}_q) \). Therefore we assume \( P f_\times(u, v) = u^2 + \mu_1uv + \mu_0v^2 \). We can move \( x \) to \( y := \begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{23} & a_{24} & a_{34} \\ b_{12} & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix} \) with \( a_{12} \neq 0 \) by the action of \( G_2 \). As we saw in the proof of Proposition 4.1, there exists \( \begin{pmatrix} p \\ r \\ q \\ s \end{pmatrix} \in \text{GL}_2 \) such that \( p, q, r, s \) satisfy the equation (7) and
\[
(18) \quad a_{12}r + b_{12}s = 0. 
\]
Let \( g := \begin{pmatrix} p & q \\ r & s \end{pmatrix}, 1 \in G \). Then the (1, 2)-entry of the first matrix of \( gy \) is nonzero and the (1, 2)-entry of the second matrix of \( gy \) is zero. Thus we can move \( x \) to \( y' := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{13}' & b_{14}' & b_{23}' & b_{24}' & b_{34}' \end{bmatrix} \) such that \( P f_{y'}(u, v) = g'(u^2 + \mu_1uv + \mu_0v^2) \) for certain \( g' \in \text{GL}_1 \). On the other hand, \( P f_{y'}(u, v) = u^2 + b_{34}'uv - \begin{vmatrix} b_{13}' & b_{14}' \\ b_{23}' & b_{24}' \end{vmatrix} v^2 \). Therefore we have \( g' = 1 \) and \( b_{34}' = \mu_1, \begin{vmatrix} b_{13}' & b_{14}' \\ b_{23}' & b_{24}' \end{vmatrix} = -\mu_0 \). Thus there exists \( h \in \text{SL}_2 \) such that \( h \begin{bmatrix} b_{13}' & b_{14}' \\ b_{23}' & b_{24}' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mu_0 & 0 \end{bmatrix} \). Thus we obtain \( (1, \begin{pmatrix} h & 0 \\ 0 & I_2 \end{pmatrix} )y' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mu_0 & 0 & \mu_1 \end{bmatrix} \).
Next we calculate the cardinality of each orbit. We use the following facts:
\[ |\text{Sp}_2(\mathbb{F}_q)| = (q - 1)q(q + 1), \]
\[ |\text{Sp}_4(\mathbb{F}_q)| = (q - 1)^2q^4(q + 1)^2(q^2 + 1), \]
\[ |\{M \in \wedge^2(\mathbb{F}_q^4) \mid \text{rank}M = 2\}| = (q - 1)(q^2 + q + 1)(q^2 + 1), \]
\[ |\{M \in \wedge^2(\mathbb{F}_q^4) \mid \text{rank}M = 4\}| = (q - 1)^2q^2(q^2 + q + 1). \]

It is obvious that \( |O_1| = 1. \) We easily have \( |O_2| = (q + 1)|\{M \in \wedge^2(\mathbb{F}_q^4) \mid \text{rank}M = 2\}| \) and \( |O_3| = (q + 1)|\{M \in \wedge^2(\mathbb{F}_q^4) \mid \text{rank}M = 4\}|. \) To count \( |O_4|, \) we calculate the order of the stabilizer subgroup \( \text{Stab}(x_4) \) of \( x_4 := (A, B) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \) In other words, we count the number of \( g = (g_1, g_2^{-1}) = \begin{pmatrix} p & r \\ q & s \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}^{-1} \in G \)
such that
\[ (19) \quad (pA + qB, rA + sB) = (g_2Ag_2^T, g_2Bg_2^T). \]

By (19), we have
\[ \begin{vmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{vmatrix} = 0, \begin{vmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{vmatrix} = 0, \begin{vmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{vmatrix} = 0, \begin{vmatrix} g_{23} & g_{24} \\ g_{33} & g_{34} \end{vmatrix} = 0, \begin{vmatrix} g_{13} & g_{14} \\ g_{33} & g_{34} \end{vmatrix} = 0, \begin{vmatrix} g_{23} & g_{24} \\ g_{32} & g_{34} \end{vmatrix} = 0, \begin{vmatrix} g_{23} & g_{24} \\ g_{43} & g_{44} \end{vmatrix} = 0, \begin{vmatrix} g_{23} & g_{24} \\ g_{42} & g_{44} \end{vmatrix} = 0, \begin{vmatrix} g_{23} & g_{24} \\ g_{42} & g_{44} \end{vmatrix} = 0. \]

If \( q \neq 0, \) we have \( \begin{vmatrix} g_{23} & g_{24} \\ g_{43} & g_{44} \end{vmatrix} \neq 0. \) Since \( \begin{vmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{vmatrix} = 0 \) and \( \begin{vmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{vmatrix} = 0, \)
we obtain \( g_{13} = g_{14} = 0. \) If \( q = 0, \) we have \( \begin{vmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{vmatrix} \neq 0. \) Since \( \begin{vmatrix} g_{13} & g_{14} \\ g_{33} & g_{34} \end{vmatrix} = 0 \) and \( \begin{vmatrix} g_{13} & g_{14} \\ g_{43} & g_{44} \end{vmatrix} = 0, \) we also obtain \( g_{13} = g_{14} = 0. \) Furthermore, we similarly obtain \( g_{12} = g_{14} = 0 \) by \( \begin{vmatrix} g_{12} & g_{14} \\ g_{22} & g_{24} \end{vmatrix} = 0, \begin{vmatrix} g_{12} & g_{14} \\ g_{32} & g_{34} \end{vmatrix} = 0, \begin{vmatrix} g_{12} & g_{14} \\ g_{42} & g_{44} \end{vmatrix} = 0, \begin{vmatrix} g_{22} & g_{24} \\ g_{42} & g_{44} \end{vmatrix} = 0, \begin{vmatrix} g_{22} & g_{24} \\ g_{32} & g_{34} \end{vmatrix} = 0. \]

Thus we have \( g_{12} = g_{13} = g_{14} = 0. \) If \( (g_{24}, g_{34}) \neq 0, \) then we have \( (g_{22}, g_{23}, g_{24}) \) by \( \begin{vmatrix} g_{23} & g_{24} \\ g_{33} & g_{34} \end{vmatrix} = \begin{vmatrix} g_{22} & g_{24} \\ g_{32} & g_{34} \end{vmatrix} = 0, \) which contradicts to \( g_2 \in \text{GL}_4. \) It also follows that \( g_{24} = g_{34} = 0. \) Hence we have \( \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} s & q \\ r & p \end{vmatrix}. \)
Therefore we obtain

\[
\text{Stab}(x_4) = \begin{bracket}
\begin{pmatrix}
g_{44} \begin{pmatrix} g_{23} & g_{22} \\
g_{33} & g_{33} \end{pmatrix},
\begin{pmatrix} g_{11} & 0 & 0 \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33} \end{pmatrix}
\end{pmatrix}
\end{bracket}
\begin{pmatrix}
g_{11} & 0 & 0 \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33} \\
g_{41} & g_{42} & g_{43} \end{pmatrix}
\end{bracket}^{-1} \in G
\]
\]

\[
= \begin{bracket}
\begin{pmatrix}
g_{44}^{-1} \begin{pmatrix} g_{23} & g_{22} \\
g_{33} & g_{33} \end{pmatrix},
\begin{pmatrix} g_{11} & 0 & 0 \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33} \\
g_{41} & g_{42} & g_{43} \end{pmatrix}
\end{pmatrix}
\end{bracket}
\begin{pmatrix}
g_{11} & 0 & 0 \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33} \\
g_{41} & g_{42} & g_{43} \end{pmatrix}
\end{bracket}^{-1} \in G
\]
\]

\[
\cong ((GL_1)^2 \times GL_2) \ltimes \mathbb{F}_q^5,
\]

and \(|\text{Stab}(x_4)| = (q - 1)^2q^5g_2\). By this result, we can calculate \(|O_4|\). Next we count \(|O_5|\).

Let \(J = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}\) and \(O = \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix}\). Let \(x_5 := \left( \begin{pmatrix} O & O \\
O & J \end{pmatrix}, \begin{pmatrix} G_{11} & G_{12} \\
G_{21} & G_{22} \end{pmatrix} \right) \in \text{Stab}(x_5)\), where \(G_{ij} \in M_2(\mathbb{F}_q)\). We have \(g_2 \begin{pmatrix} O & qJ \\
qJ & pJ \end{pmatrix} g_2^T, g_2 \begin{pmatrix} O & sJ \\
sJ & rJ \end{pmatrix} g_2^T \Rightarrow \left( \begin{pmatrix} O & O \\
O & J \end{pmatrix}, \begin{pmatrix} O & J \end{pmatrix} \right)\).

By comparing the rank of the first entry, we have \(q = 0\), and \(ps \neq 0\). It follows that

\[
p \begin{pmatrix} |G_{12}|J & G_{12}JG_{22}^T \\
G_{22}JG_{12}^T & |G_{22}|J \end{pmatrix} = \begin{pmatrix} O & O \\
O & J \end{pmatrix},
\]

and therefore \(G_{12} = O\) and \(|G_{22}| = p^{-1}\). Thus we have

\[
g_2 \begin{pmatrix} O & qJ \\
qJ & pJ \end{pmatrix} g_2^T = \begin{pmatrix} 0 & 0 \\
0 & G_{12}JG_{22}^T + sG_{22}JG_{21}^T + sG_{21}JG_{22}^T + rG_{22}|J| \end{pmatrix} = \begin{pmatrix} O & O \\
O & J \end{pmatrix},
\]

and therefore

\[
\text{Stab}(x_5) = \begin{bracket}
\begin{pmatrix}
\left( \begin{pmatrix} G_{22} & 0 \\
0 & 1 \end{pmatrix}, \begin{pmatrix} G_{22}^{-1} & 0 \\
0 & G_{22}^{-1} \end{pmatrix} \right) \end{pmatrix}
\end{bracket}
\begin{pmatrix}
G_{22}JG_{21}^T + sG_{22}JG_{21}^T + sG_{21}JG_{22}^T + rG_{22}|J| = O
\end{pmatrix}
\end{bracket} \cong (GL_1 \times GL_2) \ltimes M_2(\mathbb{F}_q)\).
\]

Thus we obtain \(|\text{Stab}(x_5)| = (q - 1)q^4g_2\), and we can calculate \(|O_5|\). Next we count \(|O_6|\). Let \(x_6 := \left( \begin{pmatrix} J & O \\
O & O \end{pmatrix}, \begin{pmatrix} O & O \\
O & J \end{pmatrix} \right) \in \text{Stab}(x_6)\), where \(G_{ij} \in M_2(\mathbb{F}_q)\). We have

\[
g_2 \begin{pmatrix} O & qJ \\
O & pJ \end{pmatrix} g_2^T, g_2 \begin{pmatrix} O & sJ \\
O & rJ \end{pmatrix} g_2^T = \begin{pmatrix} J & O \\
O & J \end{pmatrix}\).
\]

By comparing the rank of each entry, we obtain the following propositions:

(20) If \(p \neq 0\), then \(q = r = 0\) and \(s \neq 0\).
(21) If \(p = 0\), then \(qr \neq 0\) and \(s = 0\).
First we assume the case (20). Then we have

\[
\begin{pmatrix}
  G_{11} & \xi \\
  G_{21} & \xi
\end{pmatrix}
\begin{pmatrix}
  J \\
  J
\end{pmatrix}
\]

and therefore

\[
G_{12} = G_{21} = 0, \quad p = |G_{11}|^{-1}, \quad s = |G_{22}|^{-1}.
\]

Next we assume the case (21). Then we have

\[
(q \begin{pmatrix}
  |G_{12}| & \xi \\
  G_{22} & \xi
\end{pmatrix}, r \begin{pmatrix}
  |G_{11}| & \xi \\
  G_{21} & \xi
\end{pmatrix}) = \begin{pmatrix}
  J & O \\
  O & J
\end{pmatrix},
\]

and therefore

\[
G_{11} = G_{22} = 0, \quad q = |G_{12}|^{-1}, \quad r = |G_{21}|^{-1}.
\]

By (22) and (23), we obtain

\[
\text{Stab}(x_6) = \left\{ \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix} \cdot \begin{pmatrix}
  0 & I_2 \\
  I_2 & 0
\end{pmatrix} \right\} \ltimes \left\{ \begin{pmatrix}
  |G_{11}|^{-1} & 0 \\
  0 & |G_{22}|
\end{pmatrix} \cdot \begin{pmatrix}
  0 & G_{11} \\
  0 & 0
\end{pmatrix} \right\} \in G
\]

\[
\cong \mathbb{Z}/2\mathbb{Z} \ltimes (GL_2)^2.
\]

Thus we obtain \(|\text{Stab}(x_6)| = 2g_l^2| and we can calculate \(|\mathcal{O}_6|\). Lastly we obtain \(|\mathcal{O}_7| by subtracting the sum of \(|\mathcal{O}_1|, \ldots, |\mathcal{O}_6|\) from \(|V| = q^{12}.

8.2. The intersection between the orbits and the subspaces.

The subspaces we choose to calculate the Fourier transform are as follows:

\[
W_1 = 0, \quad W_2 = \begin{pmatrix}
  * & 0 & 0 & 0 & 0 & * \\
  * & 0 & 0 & 0 & 0 & *
\end{pmatrix}, \quad W_3 = \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & 0 \\
  * & * & * & * & * & *
\end{pmatrix},
\]

\[
W_4 = \begin{pmatrix}
  * & * & 0 & * & 0 \\
  * & 0 & * & 0 & * \\
  * & * & 0 & 0 & 0
\end{pmatrix}, \quad W_5 = \begin{pmatrix}
  0 & 0 & 0 & 0 & * \\
  0 & * & * & * & *
\end{pmatrix},
\]

\[
W_6 = \begin{pmatrix}
  0 & * & * & * & 0 \\
  0 & * & * & * & 0
\end{pmatrix} \quad \text{and} \quad W_7 = V.
\]

Orthogonal complements of them are as follows:

\[
W_1^\perp = W_7, \quad W_2^\perp = W_6, \quad W_3^\perp = W_3, \quad W_4^\perp = \begin{pmatrix}
  0 & * & 0 & 0 & * \\
  0 & 0 & * & 0 & *
\end{pmatrix},
\]

\[
W_5^\perp = W_5, \quad W_6^\perp = W_2 \quad \text{and} \quad W_7^\perp = W_1.
\]

**Proposition 8.2.** The cardinalities \(|\mathcal{O}_i \cap W_j| for the orbit \mathcal{O}_i and the subspace \ W_j are given as follows:

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_4 & \mathcal{O}_5 & \mathcal{O}_6 & \mathcal{O}_7 & \mathcal{O}_8 \\
\hline
W_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
W_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
W_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
W_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
W_5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
W_6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
W_7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
W_8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]
[Proof]

We only consider the cases of 

\[ W_4, W_5 \] and \( W_6 \), since the rest cases are easy. First we calculate for \( W_4 \). Restrict the representation of \( G \) on \( V \) to the subgroup \( H := \left\{ \begin{pmatrix} g_1 & g_2 \\ 0 & 1 \end{pmatrix} \bigg| g_2 \in \text{GL}_3 \right\} \). Then \( H \) acts on \( W_4 \). We can choose three elements \( 0, x = (A, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) and \( y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) as complete representatives of \( W \) with this action of \( H \). We count the cardinalities of these orbits. For \( x \), we calculate the order of the stabilizer subgroup \( \text{Stab}(x) \) of \( x \) in \( H \). We have

\[
\text{Stab}(x) = \left\{ (g_1, g_2^{-1}) = \left( \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}^{-1} \right) \bigg| G_{ij} \in \text{M}_2(\mathbb{F}_q), \begin{pmatrix} g_2 & 0 \\ g_2^T & 0 \end{pmatrix} \right\}. 
\]

We see \( r = 0, G_{21} = O \) and \( |G_{11}| = p \) by calculation. Therefore \( \text{Stab}(x) \cong ((\text{GL}_1)^2 \times \text{GL}_2) \rtimes \mathbb{F}_q^3 \), and its order is \((q - 1)^2 q^3 \text{gl}_2 \) and the cardinality is \(|H|/(q - 1)^2 q^3 \text{gl}_2 = (q - 1)(q + 1)(q^2 + q + 1)\). It follows that \(|H_y| = q^6 - |Hx| - 1 = (q - 1)^2 q(q + 1)(q^2 + q + 1)\). In view of \( 0 \in O_1, x \in O_2 \) and \( y \in O_4 \), we obtain \(|W_4 \cap O_2| = |Hx| \) and \(|W_4 \cap O_4| = |H| \).

Next we consider \( W_5 \). We write \( x \in W_5 \) as \( x = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{34} \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} \end{bmatrix} \). When \( a_{34} = 0 \) and \( b_{13} b_{14} = 0 \) and \( b_{34} \neq 0 \), we have \( x \in O_2 \). The number of such elements is \( q - 1 \). When \( a_{34} = 0 \) and \( \text{rank} \left( \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \right) = 1 \), we have \( x \in O_2 \). The number of such elements is \( q|2, 1| \). When \( a_{34} = 0 \) and \( \text{rank} \left( \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \right) = 2 \), we have \( x \in O_3 \). The number of such elements is \( q\text{gl}_2 \). When \( a_{34} \neq 0 \) and \( \text{rank} \left( \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \right) = 0 \), we have \( x \in O_2 \). The number of such elements is \( (q - 1)q \). When \( a_{34} \neq 0 \) and \( \text{rank} \left( \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \right) = 1 \), we have \( x \in O_4 \). The number of such elements is \( (q - 1)q|2, 1| \).

When \( a_{34} \neq 0 \) and \( \text{rank} \left( \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \right) = 2 \), we have \( x \in O_5 \). The number of such elements is \( (q - 1)q\text{gl}_2 \). Lastly we consider \( W_6 \). Restrict the representation of \( G \) on \( V \) to the subgroup \( H = \left\{ \begin{pmatrix} g_1 & g_2 \\ 0 & h_2 \end{pmatrix} \bigg| g_2, h_2 \in \text{GL}_2 \right\} \). Then \( H \) acts on \( W_6 \). Identify

\[
\left( \begin{pmatrix} 0 & a_{13} & a_{14} & a_{23} & a_{24} \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} \end{pmatrix} \right) \in W_6 \text{ as the pair of the 2-by-2 matrices}
\left( \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}, \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} \right). 
\]

The action of \( H \) on \( W_6 \) is identical to the action of \( \text{GL}_2 \times \text{GL}_2 \) on \( 2 \otimes 2 \otimes 2 \) which we considered in Section 4. By this identification, \( 2 \otimes 2 \otimes 2 \) can be embedded in \( V \). Let \( \sigma \) be this embedding. We easily see \( \sigma(O_1) \subset O_1, \sigma(O_2) \subset O_2, \sigma(O_3) \subset O_3, \sigma(O_4) \subset O_4, \sigma(O_5) \subset O_5, \sigma(O_6) \subset O_5, \sigma(O_7) \subset O_6 \) and \( \sigma(O_8) \subset O_7 \). The cardinalities can be calculated by these results. 

□
8.3. Fourier transform.

**Theorem 8.1.** The representation matrix \( M \) of the Fourier transform on \( \mathcal{F}_V^G \) with respect to the basis \( e_1, \ldots, e_7 \) is given as follows:

\[
\begin{bmatrix}
1 & [1, 0, 1, 1] & [2, 2, 1, 1] & [2, 1, 2, 1, 1] & \frac{1}{2}[2, 5, 1, 1, 1] & \frac{1}{2}[4, 5, 1, 1, 0] \\
1 & c_1 & [1, 2, 0, 0, 0] & [1, 2, 1, 0, 0] & \frac{1}{2}[2, 5, 1, 1, 0] & \frac{1}{2}[3, 5, 1, 1, 0] \\
1 & [0, 0, 0, 0, 0] & q^2d_1 & \frac{1}{2}[2, 1, 1, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] \\
1 & [0, 0, 0, 0, 0] & q^2 & [1, 2, 0, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] \\
1 & [0, 0, 0, 0, 0] & -q^2 & [0, 1, 0, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] \\
1 & c_3 & [1, 2, 0, 0, 0] & [0, 1, 0, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] \\
1 & [0, 0, 0, 0, 0] & [0, 2, 1, 0, 0] & [0, 1, 1, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] & \frac{1}{2}[2, 1, 1, 0, 0] \\
\end{bmatrix}
\]

Here \( [a, b, c, d, e] = (q - 1)^a q^b (q + 1)^c (q^2 + q + 1)^d (q^2 + 1)^e \), \( c_1 = q^3 - q - 1 \), \( c_2 = q^3 - q^2 - 1 \), \( c_3 = q^3 - q^2 - q - 1 \), \( d_1 = q^4 - q^3 + 1 \), \( e_1 = q^3 + q^4 - q^2 - q - 1 \) and \( e_2 = q^5 - q^4 - q^3 + q + 1 \).

We used PARI/GP [8] to calculate the matrix from Proposition 8.2.

**Corollary 8.1.** The indicator function of singular set of \( V \) is \( \Psi = \sum_{i=1}^{5} e_i \). Its Fourier transform \( \hat{\Psi} \) is given as follows:

\[
\hat{\Psi}(x) = \begin{cases} 
q^{-1} + q^{-6} - q^{-7} & x = 0 , \\
q^{-6} - q^{-7} & x \neq 0 , \text{Disc}(P_{x}(u, v)) = 0 , \\
- q^{-7} & \text{Disc}(P_{x}(u, v)) \neq 0 . 
\end{cases}
\]

In particular, we have the following \( L_1 \)-norm bound of \( \hat{\Psi} \):

\[
\sum_{x \in V} |\hat{\Psi}(x)| = O(q^5) .
\]

9. The space of binary tri-Hermitian forms

Fix a non-identity element \( \sigma \) in \( \text{Gal}(\mathbb{F}_{q^3}/\mathbb{F}_q) \). For \( z \in \mathbb{F}_{q^3} \), we write \( \sigma(z) \) and \( \sigma^2(z) \) as \( z' \) and \( z'' \), respectively. Define the trace map and the norm map as follows:

\[
\text{Tr}_3 : \mathbb{F}_{q^3} \ni z \mapsto z + z' + z'' \in \mathbb{F}_q ,
\]

\[
\text{N}_3 : \mathbb{F}_{q^3} \ni z \mapsto zz'z'' \in \mathbb{F}_q .
\]

Both maps are surjective. \( \text{Tr}_3 \) is a \( \mathbb{F}_q \)-linear map. \( \text{N}_3|_{\mathbb{F}_{q^3}^*} : \mathbb{F}_{q^3}^* \to \mathbb{F}_q^* \) is a surjective group homomorphism.

Let

\[
V := \left\{ x = (A, B) = \left( \begin{bmatrix} a_1 & a_2'' \\ a_2' & a_3'' \end{bmatrix}, \begin{bmatrix} a_2 & a_3'' \\ a_3' & a_4'' \end{bmatrix} \right) \mid a_1, a_4 \in \mathbb{F}_q \text{ and } a_2, a_3 \in \mathbb{F}_{q^3} \right\} .
\]

\( V \) is an 8 dimensional vector space over \( \mathbb{F}_q \). Let \( G = G_1 \times G_2 = \text{GL}_1(\mathbb{F}_q) \times \text{GL}_2(\mathbb{F}_{q^3}) \). We write \( g \in G \) as \( g = (g_1, g_2) \) where \( g_1 \in \text{GL}_1(\mathbb{F}_q) \) and \( g_2 \in \text{GL}_2(\mathbb{F}_{q^3}) \). For a matrix \( h = (h_{ij}) \in \text{GL}_2(\mathbb{F}_{q^3}) \), we define \( h' = (h'_{ij}) \) and \( h'' = (h''_{ij}) \). Then the action of \( G \) on \( V \) is defined by

\[
gx = g_1(g_2 A(g_2^T)'', g_2 B(g_2^T)'')(g_2^T)'' .
\]
Define a bilinear form $\beta$ of $V$ as
$$\beta((A_1, B_1), (A_2, B_2)) = \text{Tr}(A_1 A_2^T + B_1 B_2^T).$$

In addition, define an automorphism $\iota$ of $G$ as
$$(g_1, g_2)^i = ((g_1)^{-1}, (g_2^T)^{-1}).$$

These $\beta$ and $\iota$ satisfy Assumption 2.1.

**9.1. Orbit decomposition.**

By substituting $\mathbb{F}_{q^3}$ for $\mathbb{F}_q$ in Section 3.1, we obtain the following orbit decomposition of $(\text{GL}_1(\mathbb{F}_{q^3}) \times \text{GL}_2(\mathbb{F}_{q^3}), \text{Sym}^2(\mathbb{F}_q^2))$:

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_{{0}}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$O_{{1}}$</td>
<td>$u^2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$O_{{2}}$</td>
<td>$uv$</td>
<td>$2$</td>
</tr>
<tr>
<td>$O_{{3}}$</td>
<td>$u^2 + \mu_1uv + \mu_0v^2$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

Here, $u^2 + \mu_1uv + \mu_0v^2 \in \text{Sym}^2(\mathbb{F}_q^2)$ is an arbitrary irreducible polynomial. For $x = (A, B) \in V$, we define
$$r_1(x) := \dim((A, B)_{\mathbb{F}_{q^3}}),$$
i.e., the dimension of the subspace of $\text{M}_2(\mathbb{F}_{q^3})$ generated by $A$ and $B$,

$$\det_x(u, v) := \det(uA + vB) \in \text{Sym}^2(\mathbb{F}_q^2)$$
where $u, v$ are variables,

$$T(x) := \langle x \rangle$$
if and only if $\det_x(u, v) \in O_{\{0\}}$ in $\text{Sym}^2(\mathbb{F}_q^2)$.

For $x \in V$ and $g = (g_1, g_2) \in G$, we have
$$\det_{gx}(u, v) = g_1^2 \det((g_2g_1') \det_x((u, v)g_2')).$$

**PROPOSITION 9.1.** $V$ consists of 5 $G$-orbits in all.

<table>
<thead>
<tr>
<th>Orbit name</th>
<th>Representative</th>
<th>$r_1(x)$</th>
<th>$T(x)$</th>
<th>Cardinality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>$\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$0$</td>
<td>$\langle 0 \rangle$</td>
<td>$1$</td>
</tr>
<tr>
<td>$O_2$</td>
<td>$\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$1$</td>
<td>$\langle 0 \rangle$</td>
<td>$[1, 0, 1, 0, 1]$</td>
</tr>
<tr>
<td>$O_3$</td>
<td>$\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$2$</td>
<td>$\langle 1 \rangle$</td>
<td>$[1, 1, 1, 1]$</td>
</tr>
<tr>
<td>$O_4$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$2$</td>
<td>$\langle 1 \rangle$</td>
<td>$[2, 3, 1, 0, 1]$</td>
</tr>
<tr>
<td>$O_5$</td>
<td>$\begin{bmatrix} 2 &amp; \mu_1 \ \mu_1 &amp; \mu_1^2 - 2\mu_0 \end{bmatrix}, \begin{bmatrix} \mu_1^2 - 2\mu_0 \ \mu_1^2 - 3\mu_1\mu_0 \end{bmatrix}$</td>
<td>$2$</td>
<td>$\langle 2 \rangle$</td>
<td>$[2, 3, 1, 0, 1]$</td>
</tr>
</tbody>
</table>

Here $[a, b, c, d, e] = (q - 1)^a q^b (q + 1)^c (q^2 + q + 1)^d (q^2 - q + 1)^e$, and $\mu_1, \mu_0 \in \mathbb{F}_q$ are elements such that $X^2 + \mu_1X + \mu_0 \in \mathbb{F}_q[X]$ is irreducible.

[Proof]
The invariants \( r_1(x) \) and \( T(x) \) for the 5 elements in the table are easily calculated. Since they do not coincide, these 5 elements belong to different orbits. Let

\[
x_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
x_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_5 = \begin{pmatrix} 2 & \mu_1 \\ \mu_1 & -2\mu_0 \end{pmatrix}, \quad \begin{pmatrix} \mu_1^2 -2\mu_0 & \mu_1^3 - 3\mu_1\mu_0 \end{pmatrix}, \quad \text{and} \quad O_i = G x_i.
\]

For \( x \in V \), let \( \text{Stab}(x) = \{ g \in G \mid gx = x \} \).

In Section 9, we substitute \( F_q^3 \) for \( F_q \) in the notation of Section 3.2, i.e.,

\[
|n, m| := |\{ M \in M(n_1, n_2)(F_q)|\text{rank}(M) = m \}|
\]

\[
= \frac{\prod_{i=0}^{m-1} (q^{3(n_2-i)} - 1) \prod_{i=0}^{n-1} (q^{3n_1} - q^{3i})}{\prod_{k=1}^{m} (q^{3k} - 1)},
\]

\[
|n, m| := |\{ n, m \} = \prod_{i=0}^{m-1} (q^{3(n-i)} - 1)(q^{3n} - q^{3i})
\]

\[
q^{3(m-i)} - 1,
\]

\[
\text{gl}_n := |\text{GL}_n(F_q^3)| = \prod_{i=0}^{n-1} (q^{3n} - q^{3i}),
\]

\[
\text{sl}_n := |\text{SL}_n(F_q^3)| = \text{gl}_n/(q^3 - 1).
\]

We count \(|O_1|\). Clearly \(|O_1| = 1\). To calculate \(|O_2|\), we count the order of \( \text{Stab}(x_2) \). Let

\[
g = (g_1, g_2) = (g_1, (p_2 \begin{pmatrix} q_2 \\ s_2 \end{pmatrix})) \in \text{Stab}(x_2).\]

Then

\[
\begin{pmatrix} N_3(q_2) & q_2 q_2'' s_2' \\ q_2' q_2'' s_2' & N_3(s_2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

holds. Thus we have \( q_2 = 0 \) and \( N_3(s_2) = \mu_1^{-1}. \) By the surjectivity of \( N_3 \), we obtain \( \text{Stab}(x_2) = \text{GL}_1(F_q^3) \times \text{GL}_1(F_q^3) \), and \(|\text{Stab}(x_2)| = q^3(q^3 - 1)^2. \) Therefore \(|O_2| = |G|/|\text{Stab}(x_2)| = (q - 1)g_{12}/q^3(q^3 - 1)^2 = (q - 1)(q + 1)(q^2 - q + 1).\) Next we count \(|O_3|\). If \( g \in \text{Stab}(x_3),\)

\[
\begin{pmatrix} g_2 & 0 \\ q_2'' & g_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & g_1^{-1} \end{pmatrix}
\]

holds. By comparing the rank of the first entry, we have \( q_2 = 0 \). Therefore we have \( p_2'' s_2 g_2' = \mu_1^{-1} \) and \( \text{Tr}_3(r_2 s_2' g_2'') = 0. \) It follows that

\[
\text{Stab}(x_3) = \left\{ (g_1, \begin{pmatrix} 0 \\ \frac{(g_1 h_2')^2 s_2'}{r_2} \end{pmatrix}) \in G \mid \text{Tr}_3(r_2 s_2' g_2'') = 0 \right\}
\]

\[
\cong (\text{GL}_1(F_q) \times \text{GL}_1(F_q)) \times \text{Ker}(\text{Tr}_3),
\]

and \(|\text{Stab}(x_3)| = q^2(q - 1)(q^3 - 1).\) Therefore we obtain \(|O_3| = q(q^6 - 1).\) Next we calculate \(|O_4| \) and \(|O_5| \). Kable and Yukie [2, Proposition (3.9), (3.12), Theorem (3.13)]
proved the following facts:

\begin{align}
(24) \quad \text{Stab}(x_4) & \cong \mathbb{Z}/2\mathbb{Z} \times \{(g_1, g_2) \in \text{GL}_1(\mathbb{F}_q^*) \times \text{GL}_1(\mathbb{F}_q^*) | N_3(g_1) = N_3(g_2)\}, \\
(25) \quad \text{Stab}(x_5) & \cong \mathbb{Z}/2\mathbb{Z} \times \{g \in \text{GL}_1(\mathbb{F}_q^*) | g \delta^3(g) \delta^4(g) \in \mathbb{F}_q^\times\}.
\end{align}

Here $\delta$ is an element of Gal($\mathbb{F}_{q^6}/\mathbb{F}_q$) such that $\langle \delta \rangle = \text{Gal}(\mathbb{F}_{q^6}/\mathbb{F}_q)$. In [2], Kable and Yukie assume that $V$ is defined over infinite field, but the method to determine the structures for Stab($x_4$) and Stab($x_5$) holds for the $\mathbb{F}_q$. By applying (24) and (25), we obtain $|\text{Stab}(x_4)| = 2(q - 1)(q^2 + q + 1)^2$ and $|\text{Stab}(x_5)| = 2(q^3 - 1)(q^2 - q + 1)$. Thus we have $|O_4| = 2q^3(q - 1)^2(q + 1)(q^2 - q + 1)$ and $|O_5| = 4q^3(q - 1)^2(q + 1)(q^2 + q + 1)$. Lastly, since $\sum_{i=1}^5 |O_i| = q^8 = |V|$, we have $\bigcup_{i=1}^5 O_i = V$. \hfill \square

9.2. The intersection between the orbits and the subspaces.

The subspaces we choose to calculate the Fourier transform are as follows:

\begin{align*}
W_1 &= 0, W_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, W_3 = \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \\
W_4 &= \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \text{ and } W_5 = V.
\end{align*}

Orthogonal complements of them are as follows:

\begin{align*}
W_1^\perp &= W_5, \quad W_2^\perp = \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \quad W_3^\perp = W_3, \quad W_4^\perp = W_4 \quad \text{and} \quad W_5^\perp = W_1.
\end{align*}

**Proposition 9.2.** The cardinalities $|O_i \cap W_j|$ for the orbit $O_i$ and the subspace $W_j$ are given as follows:

| $O_i$ | $|W_1|$ | $|W_2|$ | $|W_3|$ | $|W_4|$ | $|W_5|$ | $|W_5^\perp|$ |
|-------|-------|-------|-------|-------|-------|-------|
| $O_1$ | 1     | 1     | 1     | 1     | 1     | 1     |
| $O_2$ | 0     | $[1, 0, 0, 0, 0]$ | $[1, 0, 0, 0, 0]$ | $[1, 0, 1, 0, 1]$ | $[1, 0, 0, 0, 0]$ | 0     |
| $O_3$ | 0     | 0     | $[1, 1, 0, 1, 0]$ | $[1, 0, 1, 1, 1]$ | $[1, 1, 0, 1, 0]$ | $b_2$ |
| $O_4$ | 0     | 0     | 0     | $\frac{1}{2} [2, 0, 0, 1, 0]$ | $\frac{1}{2} [2, 3, 1, 0, 1]$ | $\frac{1}{2} [2, 3, 0, 1, 0]$ |
| $O_5$ | 0     | 0     | 0     | $\frac{1}{2} [2, 0, 0, 1, 0]$ | $\frac{1}{2} [2, 3, 1, 1, 0]$ | $\frac{1}{2} [2, 3, 0, 1, 0]$ |

Here $[a, b, c, d, e] = (q - 1)^a q^b (q + 1)^c (q^2 + q + 1)^d (q^2 - q + 1)^e$ and $b_2 = q^2 + 1$.

**[Proof]**

We only consider the case of $W_2^\perp$, since the rest cases are easy. We easily have $|O_1 \cap W_2^\perp| = 1$ and $|O_2 \cap W_2^\perp| = (q - 1)$. For $1 \leq i \leq 5$ and $W \subset V$, let $G(i, W) = \{g \in G \mid gx_i \in W\}$. Then we have $|O_i \cap W| = |G(i, W)|/|\text{Stab}(x_i)|$. Let $g = \left( \begin{array}{cc} p_2 & q_2 \\ r_2 & s_2 \end{array} \right)$ and assume $g \in G(3, W_2^\perp)$. Then we have

$$
\begin{bmatrix} p_2 & q_2 \\ q_2' & p_2' \end{bmatrix} = \text{Tr}_3(p_2 q_2' q_2'') = 0.
$$

Therefore

\begin{align*}
|G(3, W_2^\perp)| &= |\{g \in G(3, W_2^\perp) \mid q_2 = 0\}| + |\{g \in G(3, W_2^\perp) \mid q_2 \neq 0\}| \\
&= q^3(q - 1)q_1^2 + q^2(q - 1)(q^6 - q^3)q_1
\end{align*}
= q^3(q - 1)^3(q^2 + q + 1)^2(q^2 + 1),
and we obtain |O_3 \cap W_2^+| = q(q - 1)(q^2 + q + 1)(q^2 + 1). Next, assume q \in G(4, W_2^+) . Then we have N_3(p_2) + N_3(q_2) = 0. If p_2 = 0, we have q_2 = 0, which contradicts to \( \begin{pmatrix} p_2 \\ q_2 \\ r_2 \end{pmatrix} \) \in GL_2(\mathbb{F}_q^3) . Thus we have p_2 \neq 0, and G(4, W_2^+) = (q - 1)gl_1^1(q^6 - q^3)/(q - 1). Therefore we obtain |O_4 \cap W_2^+| = \frac{1}{2}q^3(q - 1)^2(q^2 + q + 1) . Lastly, |O_5 \cap W_2^+| = q^3 - \sum_{i=1}^4 |O_i \cap W_2^+| = \frac{1}{2}q^3(q - 1)^2(q^2 + q + 1). \Box

9.3. Fourier transform.

**Theorem 9.1.** The representation matrix M of the Fourier transform on \( \mathcal{F}^G \) with respect to the basis \( e_1, \ldots, e_5 \) is given as follows:

\[
\begin{pmatrix}
1 & [1, 0, 1, 0, 1] & [1, 1, 1, 1, 1] & \frac{1}{2}[2, 3, 1, 0, 1] & \frac{1}{2}[2, 3, 1, 1, 0] \\
1 & -c_1 & [1, 1, 0, 1, 0] & \frac{1}{2}[1, 3, 0, 0, 0] & b_1 \\
\frac{1}{q^n} & 1 & [1, 0, 0, 0, 0] & qc_2 & -\frac{1}{2}[1, 3, 0, 0, 0] \\
1 & b_1 & -[0, 1, 0, 1, 0] & q^3 & 0 \\
1 & -[0, 0, 0, 0, 1] & -[0, 1, 0, 0, 1] & 0 & q^3 \\
\end{pmatrix}
\]

Here \([a, b, c, d, e] = (q - 1)^b(q + 1)^c(q^2 + q + 1)^d(q^2 - q + 1)^e\), \( b_1 = q^2 + q - 1\), \( c_1 = q^3 - q + 1 \) and \( c_2 = q^3 - q^2 - 1 \). We used PARI/GP [8] to calculate the matrix from Proposition 9.2.

**Corollary 9.1.** The indicator function of singular set of \( V \) is \( \Psi = \sum_{i=1}^3 e_i \). Its Fourier transform \( \hat{\Psi} \) is given as follows:

\[
\hat{\Psi}(x) = \begin{cases}
q^{-1} + q^{-4} - q^{-5} & \text{if } x = 0, \\
q^{-4} - q^{-5} & \text{if } x \neq 0, \text{Disc(det}_x(u, v)) = 0, \\
- q^{-5} & \text{if } \text{Disc(det}_x(u, v)) \neq 0.
\end{cases}
\]

In particular, we have the following L_1-norm bound of \( \hat{\Psi} \):

\[
\sum_{x \in V} |\hat{\Psi}(x)| = O(q^3).
\]

10. Concluding Remarks

Here, we state some notices of the paper and what are observed from the results of the calculations.

10.1. Verification for the calculation.

For a general linear representation \( (G, V) \) over \( \mathbb{F}_q \), the matrix \( M \) stated after Proposition 2.1 satisfies the following properties:

**Lemma 10.1.** [5, Lemma 7] 1. Let \( S = \text{diag}(\mid O_i \mid) \). Then SM is symmetric.

2. Suppose that \( x \) and \( -x \) lie in the same \( G \)-orbit for each \( x \in V \). Then \( M^2 = |V|^{-1} I_r \).
These properties are not needed to calculate the explicit formula, but it is effective to verify our calculations for the explicit formulas. We confirmed that for the prehomogeneous vector spaces in this paper, the matrices $M$ satisfy this lemma.

**10.2. Eigenvalue of $M$.**

Let $\dim V$ be the dimension over $\mathbb{F}_q$ of $V$. By 2 of Lemma 10.1, the possible eigenvalues of $M$ are either $q^{-\frac{\dim V}{2}}$ or $-q^{-\frac{\dim V}{2}}$. Let $m_+$ and $m_-$ be the multiplicity of the eigenvalues $q^{-\frac{\dim V}{2}}$ and $-q^{-\frac{\dim V}{2}}$, respectively. We easily have

$$m_+ + m_- = r$$

and

$$m_+ - m_- = q^{\frac{\dim V}{2}} \cdot \text{Tr}(M).$$

Therefore we have $m_+$ and $m_-$ for each prehomogeneous vector space $V$ in this paper.

**COROLLARY 10.1.** The multiplicities $m_+$ and $m_-$ for each $V$ are given as follows:

<table>
<thead>
<tr>
<th>$V$</th>
<th>$m_+$</th>
<th>$m_-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \otimes 2 \otimes 2$</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>$2 \otimes 2 \otimes 3$</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>$2 \otimes 2 \otimes 4$</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>$2 \otimes H_2(\mathbb{F}_q^2)$</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$2 \otimes H_2(4)$</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

**binary tri-Hermitian forms over $\mathbb{F}_q^3$**

It may be interesting if a way to calculate the $m_+$ and $m_-$ systematically is found.

**10.3. Similarity of $2 \otimes 2 \otimes 2$, $2 \otimes H_2(\mathbb{F}_q^2)$ and the space of binary tri-Hermitian forms.**

$2 \otimes H_2(\mathbb{F}_q^2)$ and the space of binary tri-Hermitian forms over $\mathbb{F}_q^3$ are the non-split $\mathbb{F}_q$-forms of $2 \otimes 2 \otimes 2$. In all of the three cases, we have

$$|\{x \in V \mid \text{Disc}(\det_x(u, v)) \neq 0\}| = q^3(q - 1)^2(q + 1)(q^2 + 1)$$

and

$$\hat{\Psi}(x) = \begin{cases} q^{-1} + q^{-4} - q^{-5} & x = 0, \\ q^{-4} - q^{-5} & x \neq 0, \text{Disc}(\det_x(u, v)) = 0, \\ -q^{-5} & \text{Disc}(\det_x(u, v)) \neq 0. \end{cases}$$

By these coincidences, we find that the $L_1$-norms of $\hat{\Psi}$ for these three spaces also coincide:

$$\sum_{x \in V} |\hat{\Psi}(x)| = 2q^3 - 2q^2 + 1 - 2q^{-1} + 2q^{-2}.$$ 

The reason for these coincidences are not yet found.

**Acknowledgement**

I am very thankful to my adviser Takashi Taniguchi for his guidance.
References


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