COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 60, No. 1, 2 2011

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

Explicit Formulas for the Pair Correlation of Vertical Shifts of Zeros of the Riemann Zeta-Function

by

Andrew LEDOAN and Alexandru ZAHARESCU*

(Received March 1, 2011)

This paper is dedicated to Professor Akio Fujii on the occasion of his retirement.

Abstract. Let $\zeta(s)$ denote the Riemann zeta-function. In this paper we obtain explicit formulas for the pair correlation of zeros of the function $H_\lambda(s) = \zeta(s - i\lambda/2)\zeta(s + i\lambda/2)$, where λ is a fixed positive real number.

1. Introduction and statement of results

Since Riemann's study of the distribution of prime numbers, in particular, his *Memoir* "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" [29], a deep and careful work of striking originality and unexpected insights, first appeared in the *Monatsberichte der Berliner Akademie* in 1859, a vast amount of further work on the distribution of zeros of the Riemann zeta-function $\zeta(s)$ has been done. (See the wonderful expositions of the classical computations by Titchmarsh [32], [33], Ingham [16], Davenport [3], Edwards [4], Ivić [17], Iwaniec and Kowalski [18], Patterson [27], and Karatsuba and Voronin [19].) For example, it was shown successfully by Professor Akio Fujii [10] in 1993, without assuming any unproved hypothesis, that the discrepancy of the set of fractional parts $\{\{\alpha\gamma\}: 0 < \gamma \leq T\}$, where α is a fixed positive real number and γ ranges over the imaginary parts of all nontrivial zeros of $\zeta(s)$, is at most $O(\log\log T/\log T)$. Using further work of Fujii [7], [11] certain measures naturally associated with this set of fractional parts and some connections to the pair correlation of zeros of $\zeta(s)$ were recently investigated by Ford, Soundarajan, and one of the authors [5], [6].

The pair correlation of zeros of $\zeta(s)$ was studied for the first time by Montgomery [24] in the early years of the 1970's. (See the notes by Goldston [12] for a complete discussion of Montgomery's important results and their relations to prime numbers.) Montgomery's work was later generalized to triple correlation by Hejhal [15] and to higher correlations for more general L-functions by Rudnick and Sarnak [30], followed by further developments from Katz and Sarnak [20], [21]. A heuristic derivation of the n-level correlations of zeros of

2010 *Mathematics Subject Classification.* Primary 11M26. Secondary 11M06, 11M36.

Key words and phrases. Explicit formula; Pair correlation; Riemann zeta-function.

*During the preparation of this work, the second author received support from the National Science Foundation Grant DMS-0901621. The authors would like to express their sincere gratitude to Professor Daniel A. Goldston for pointing out an error in an earlier version of this paper.

$\zeta(s)$ without restrictions on the test functions was obtained by Bogomolny and Keating [1], [2]. Also, the pair correlation of zeros of functions in the Selberg class $\mathcal{S}$ was researched by Murty and Perelli [25] and Murty and one of the authors [26]. (See Selberg's lecture [31] at the Amalfi Conference in 1989 for the precise definition of the class $\mathcal{S}$ and important fundamental conjectures surrounding $\mathcal{S}$.)

In the present paper we consider the function

$$H_\lambda(s) = \zeta\left(s - \frac{i\lambda}{2}\right)\zeta\left(s + \frac{i\lambda}{2}\right),$$

where λ is a fixed positive real number. As is well known, explicit formulas in number theory were originally motivated by the counting of prime numbers. It is our purpose to carry through the derivation and prove explicit formulas for the pair correlation of zeros of $H_\lambda(s)$. We will use the order relations $f = O(g)$ and $f \ll g$ synonymously. It is to be understood, here and in all that follows, that $f(x) = O(g(x))$ in a set X means that there exists a nonnegative constant k so that $|f(x)| \le k g(x)$ for all $x \in X$. The implied constant k will sometimes depend on other parameters, which is usually clear from context. We can summarize our first result as follows.

THEOREM 1. *Fix a positive real number λ. For all $2 \le x \le T$, we have*

$$\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\le\Im(\rho),\Im(\rho')\le T}} \frac{x^{\rho+\rho'}}{\rho+\rho'} = \frac{2xT}{\pi}\left\{\left[1 + \Re\left(\frac{x^{i\lambda}}{1+i\lambda}\right)\right]\log x - \Re\left(\frac{x^{i\lambda}}{(1+i\lambda)^2}\right) - 1\right\}$$
$$+ O_\lambda(xT\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}))$$
$$+ O(x^2(\log T)^4) + O(T(\log T)^3),$$

where c is a positive absolute constant.

As an application of Theorem 1 we obtain general formulas for the correlation of zeros of $H_\lambda(s)$ for a class of smooth test functions. We prove in particular

THEOREM 2. *Fix a positive real number λ. For any $T > 2$ and any continuously differentiable complex-valued function g with support contained in the interval $(2, T)$ we have*

$$\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\le\Im(\rho),\Im(\rho')\le T}} f(\rho+\rho') = \frac{2T}{\pi}\int_2^\infty g(x)(1+\cos(\lambda\log x))\log x\,dx$$
$$+ O_\lambda(T\|g'(x)x\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})\|_1)$$
$$+ O((\log T)^4\|g'(x)x^2\|_1) + O(T(\log T)^3\|g'(x)\|_1),$$

where f is the Mellin transform of g and c is a positive absolute constant.

We will consider Montgomery's weight function $w(z)$ given by $w(z) = 4/(4 - z^2)$ and define for any real number α

$$\mathscr{F}_{H_\lambda}(\alpha, T) = \frac{\pi}{T \log T} \sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T \le \Im(\rho), \Im(\rho') \le T}} T^{\alpha(\rho+\rho'-1)} w(\rho + \rho' - 1)\,.$$

Then the following asymptotic formula for $\mathscr{F}_{H_\lambda}(\alpha, T)$ can be established.

THEOREM 3. *Fix a positive real number λ and an $\alpha \in (0, 1)$. We have*

$$\begin{aligned}\mathscr{F}_{H_\lambda}(\alpha, T) = {} & \frac{1}{(4+\lambda^2)^2 \log T}[2\alpha(4+\lambda^2)(4\cos(\alpha\lambda \log T) + 4 + \lambda^2) \log T \\ & - 16\lambda \sin(\alpha\lambda \log T)] \\ & + O_\lambda(\exp(-\alpha c (\log T)^{3/5} (\log\log T)^{-1/5}))\,,\end{aligned}$$

where c is a positive absolute constant.

Some general variants of the above results have been formulated and proved for pairs of functions in the Selberg class $\mathscr{S}$ by Murty and one of the authors [26]. It should be emphasized that although $H_\lambda(s)$ clearly is not a function in $\mathscr{S}$, the general method applies to the present case as well and is carried out in rigorous detail.

2. Proof of Theorem 1

We begin the proof by fixing a positive real number λ and taking an $x \in (1, T]$. We may write

$$\sum_{\substack{H_\lambda(\rho)=0 \\ -T \le \Im(\rho) \le T}} x^\rho = S_1 + S_2 - S_3 + S_4 + S_5 - S_6\,,$$

where

$$S_1 = x^{i\lambda/2} \sum_{\substack{\zeta(\tilde\rho)=0 \\ -T \le \Im(\tilde\rho) \le T}} x^{\tilde\rho}\,, \quad S_2 = x^{i\lambda/2} \sum_{\substack{\zeta(\tilde\rho)=0 \\ -T-\lambda/2 \le \Im(\tilde\rho) < -T}} x^{\tilde\rho}\,,$$

$$S_3 = x^{i\lambda/2} \sum_{\substack{\zeta(\tilde\rho)=0 \\ T-\lambda/2 < \Im(\tilde\rho) \le T}} x^{\tilde\rho}\,, \; S_4 = x^{-i\lambda/2} \sum_{\substack{\zeta(\tilde\rho)=0 \\ -T \le \Im(\tilde\rho) \le T}} x^{\tilde\rho}\,,$$

$$S_5 = x^{-i\lambda/2} \sum_{\substack{\zeta(\tilde\rho)=0 \\ T < \Im(\tilde\rho) \le T+\lambda/2}} x^{\tilde\rho}, \quad S_6 = x^{-i\lambda/2} \sum_{\substack{\zeta(\tilde\rho)=0 \\ -T \le \Im(\tilde\rho) < -T+\lambda/2}} x^{\tilde\rho}\,.$$

Since the number of zeros of $\zeta(s)$ with imaginary parts in the interval $[-T-\lambda/2, -T+\lambda/2] \cup [T - \lambda/2, T + \lambda/2]$ is at most $O_\lambda(\log T)$ where the implied constant depends only on λ (see Chapter 15 of the reference by Davenport [3]) and since $|x^{\tilde\rho}| < x$ for all $\tilde\rho$, it follows that each one of the sums S_2, S_3, S_5, and S_6 is at most $O_\lambda(x \log T)$. To examine the remaining sums S_1 and S_4, we employ the Landau-Gonek asymptotic formula (see the

papers by Landau [23], Gonek [13], [14], Fujii [8], [9], and Ford, Soundarajan, and one of the authors [5], [6]) in the form:

$$\sum_{\substack{\zeta(\rho)=0\\ -T\le\Im(\rho)\le T}} x^{\rho} = -\frac{\Lambda(n_x)}{\pi}\frac{\sin(T\log(x/n_x))}{\log(x/n_x)} + O\left(x(\log(2xT))^2 + \frac{\log(2T)}{\log x}\right),$$

where $x, T > 1$, n_x is the nearest prime power to x, and $\Lambda(n)$ is the von Mangoldt function. This remarkable formula enables us to derive that

$$\sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} x^{\rho} = -\frac{\Lambda(n_x)}{\pi}\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}(x^{i\lambda/2}+x^{-i\lambda/2}) + O_\lambda\left(x(\log(2xT))^2 + \frac{\log(2T)}{\log x}\right).$$

Then squaring both sides of this relation and multiplying the result by x^{-1}, we get

$$\begin{aligned}\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\le\Im(\rho),\Im(\rho')\le T}} x^{\rho+\rho'-1} &= \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right)^2 (x^{-1+i\lambda}+x^{-1-i\lambda}+2x^{-1})\\ &\quad + O_\lambda\left(x(\log(2xT))^4 + x^{-1}\left(\frac{\log(2T)}{\log x}\right)^2\right)\\ &\quad + O_\lambda\left(\log x\left((\log(2xT))^2+\frac{\log(2T)}{x\log x}\right)\left|\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right|\right).\end{aligned}$$

If we integrate this with respect to x from 2 to y for some $y \in [2, T]$, we obtain

$$\begin{aligned}&\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\le\Im(\rho),\Im(\rho')\le T}} \frac{y^{\rho+\rho'}-2^{\rho+\rho'}}{\rho+\rho'}\\ &= G(\lambda,T,y)+G(-\lambda,T,y)+G(0,T,y)+O_\lambda(y^2(\log T)^4)\\ &\quad + O_\lambda\left(\int_2^y \log x\left((\log(2xT))^2+\frac{\log(2T)}{x\log x}\right)\left|\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right|4dx\right),\end{aligned} \tag{1}$$

where

$$G(v,T,y) = \int_2^y \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right)^2 x^{-1+iv}dx\,.$$

To deal with the integral $G(v, T, y)$, we consider the sequence of prime powers $q_1, q_2, q_3, q_4, \ldots$ which satisfies the inequalities $q_1 < q_2 < q_3 < q_4 < \ldots$ and has as midpoints the numbers $x_m = (q_m + q_{m+1})/2$ where $m > 0$. Thus the sequence of prime powers $2, 3, 4, 5, 7, 8, 9, 11, \ldots$ has midpoints $x_1 = 2.5$, $x_2 = 3.5$, $x_3 = 4.5$, $x_4 = 6$, $x_5 = 7.5$, $x_6 = 8.5$, $x_7 = 10$, and so on. Moreover, we note that $n_x = q_2 = 3$ for

$x \in (x_1, x_2)$, that $n_x = q_3 = 4$ for $x \in (x_2, x_3)$, that $n_x = q_4 = 5$ for $x \in (x_3, x_4)$, and so on. Continuing like this we see that $n_x = q_{k+1}$ whenever $x \in (x_k, x_{k+1})$ where $k > 0$. As a result, we have $\Lambda(n_x) = \Lambda(q_{k+1})$.

Now let $x_l < x < y < x_{l+1}$ where $l > 0$. Then we may decompose $G(v, T, y)$ into parts as follows:

$$G(v, T, y) = \left(\int_2^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_{l-1}}^{x_l} + \int_{x_l}^{y} \right) \frac{\Lambda(n_x)^2}{\pi^2} \left(\frac{\sin(T \log(x/n_x))}{\log(x/n_x)} \right)^2 x^{-1+iv} dx \,.$$

Here we make a change of variable $t = T \log(x/q_m)$ and compute

$$\begin{aligned} &\int_{x_{m-1}}^{x_m} \frac{\Lambda(n_x)^2}{\pi^2} \left(\frac{\sin(T \log(x/n_x))}{\log(x/n_x)} \right)^2 x^{-1+iv} dx \\ &\quad = \frac{\Lambda(q_m)^2}{\pi^2} \int_{(q_{m-1}+q_m)/2}^{(q_m+q_{m+1})/2} \left(\frac{\sin(T \log(x/q_m))}{\log(x/q_m)} \right)^2 x^{-1+iv} dx \\ &\quad = \frac{\Lambda(q_m)^2 q_m^{iv} T}{\pi^2} \int_{T \log((q_{m-1}+q_m)/2q_m)}^{T \log((q_m+q_{m+1})/2q_m)} e^{ivt/T} \left(\frac{\sin t}{t} \right)^2 dt \,. \end{aligned}$$

We observe that

$$\int_{T \log((q_{m-1}+q_m)/2q_m)}^{T \log((q_m+q_{m+1})/2q_m)} e^{ivt/T} \left(\frac{\sin t}{t} \right)^2 dt = \int_{-\infty}^{\infty} e^{ivt/T} \left(\frac{\sin t}{t} \right)^2 dt - E_I - E_{II} \,,$$

where

$$E_I = \int_{-\infty}^{T \log(q_{m-1}+q_m)/2q_m} e^{ivt/T} \left(\frac{\sin t}{t} \right)^2 dt \ll_v \int_{-\infty}^{T \log(q_{m-1}+q_m)/2q_m} \frac{dt}{t^2}$$

and

$$E_{II} = \int_{T \log(q_m+q_{m+1})/2q_m}^{\infty} e^{ivt/T} \left(\frac{\sin t}{t} \right)^2 dt \ll_v \int_{T \log(q_m+q_{m+1})/2q_m}^{\infty} \frac{dt}{t^2} \,.$$

If we use the Taylor series expansion for $\exp(ivt/T)$ with $|t| < \sqrt{T}$, we obtain

$$\int_{-\infty}^{\infty} e^{ivt/T} \left(\frac{\sin t}{t} \right)^2 dt = \int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt + O_v\left(\frac{1}{\sqrt{T}} \right) = \pi + O_v\left(\frac{1}{\sqrt{T}} \right).$$

We consider now the integrals E_I and E_{II} and focus our attention on E_{II}. We note that since

$$T \log\left(\frac{q_m + q_{m+1}}{2q_m} \right) \geq T \log\left(\frac{1 + 2q_m}{2q_m} \right) \gg \frac{T}{2q_m} \gg \frac{T}{y} \,,$$

we have

$$E_{II} \ll_v \int_{T/y}^{\infty} \frac{dt}{t^2} \ll_v \frac{y}{T}.$$

It would follow by similar reasoning that

$$E_I \ll_v \frac{y}{T}.$$

Combining our estimates, we get

$$\int_{T\log((q_{m-1}+q_m)/2q_m)}^{T\log((q_m+q_{m+1})/2q_m)} e^{ivt/T}\left(\frac{\sin t}{t}\right)^2 dt = \pi + O_v\left(\frac{1}{\sqrt{T}}\right) + O_v\left(\frac{y}{T}\right),$$

and it follows that

$$\int_{x_{m-1}}^{x_m} \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right)^2 x^{-1+iv}dx$$
$$= \frac{\Lambda(q_m)^2 q_m^{iv} T}{\pi^2}\left[\pi + O_v\left(\frac{1}{\sqrt{T}}\right) + O_v\left(\frac{y}{T}\right)\right].$$

Hence we obtain

$$G(v,T,y) = \frac{T}{\pi}\sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q)^2 q^{iv} + O_v\Bigg(\sqrt{T}\sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q)^2\Bigg)$$
$$+ O_v\Bigg(y\sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q)^2\Bigg).$$

Since

$$\sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q)^2 \le \log y \sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q) \sim y\log y \quad (\text{as } y\to\infty),$$

we see that

$$G(v,T,y) = \frac{T}{\pi}\sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q)^2 q^{iv} + O_v(y^2\log y).$$

However

$$\sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q)^2 q^{iv} = \sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q) q^{iv}\log q$$
$$+ \sum_{\substack{q \text{ prime power}\\ q\le y}} \Lambda(q) q^{iv}(\Lambda(q) - \log q), \tag{2}$$

and we examine the two sums on the right-hand side in turn.

Since $\Lambda(q) = \log q = 0$ for q prime and since the number of prime powers that are not prime up to y is $\pi(\sqrt{y}) + \pi(\sqrt[3]{y}) + \cdots = O(\sqrt{y}/\log y)$, an easy calculation shows that

$$\left| \sum_{\substack{q \text{ prime power} \\ q \le y}} \Lambda(q) q^{iv} (\Lambda(q) - \log q) \right| \ll \sqrt{y} \log y \,. \tag{3}$$

Next, if we write

$$\sum_{\substack{q \text{ prime power} \\ q \le y}} \Lambda(q) q^{iv} \log q = \sum_{n \le y} \Lambda(n) n^{iv} \log n$$

and apply the summation by parts formula (see Section 1.5 of the reference by Iwaniec and Kowalski [18]), we obtain

$$\sum_{2 < n \le y} \Lambda(n) n^{iv} \log n = \psi(y) y^{iv} \log y - \psi(2) 2^{iv} \log 2 - \int_2^y \psi(t) t^{-1+iv} (1 + iv \log t) dt \,, \tag{4}$$

where $\psi(x) = \sum_{n \le x} \Lambda(n)$ with $x > 0$. It follows by the prime number theorem deduced from the sharpest known zero-free region for $\zeta(s)$, essentially due to Korobov [22] and Vinogradov [34] (see the paper by Richert [28], Chapters 2 and 5 of the reference by Walfisz [35], Chapters 13 and 18 of the reference by Davenport [3], Chapter 6 and 12 of the reference by Ivić [17], and Chapter 4 of the reference by Karatsuba and Voronin [19] for alternative expositions),

$$\psi(x) = x + O(x \exp(-c(\log x)^{3/5} (\log \log x)^{-1/5})) \,, \tag{5}$$

where c is a positive absolute constant, that we have

$$\int_2^y \psi(t) t^{-1+iv} (1 + iv \log t) dt = \int_2^y t^{iv} (1 + iv \log t) dt + O_v \left(\int_2^y \exp(-c(\log t)^{3/5} (\log \log t)^{-1/5}) \log t dt \right) .$$

Here we note that

$$\int_2^y \exp(-c(\log t)^{3/5} (\log \log t)^{-1/5}) \log t dt = \int_2^{\sqrt{y}} \exp(-c(\log t)^{3/5} (\log \log t)^{-1/5}) \log t dt$$

$$+\int_{\sqrt{y}}^{y}\exp(-c(\log t)^{3/5}(\log\log t)^{-1/5})\log t dt\,.$$

Since

$$\int_{2}^{\sqrt{y}}\exp(-c(\log t)^{3/5}(\log\log t)^{-1/5})\log t dt \ll \int_{2}^{\sqrt{y}} dt \ll \sqrt{y}$$

and

$$\begin{aligned}&\int_{\sqrt{y}}^{y}\exp(-c(\log t)^{3/5}(\log\log t)^{-1/5})\log t dt\\ &\qquad\le \int_{\sqrt{y}}^{y}\exp(-c(\log\sqrt{y})^{3/5}(\log\log\sqrt{y})^{-1/5})\log\sqrt{y}dt\\ &\qquad\ll y\exp(-c'(\log y)^{3/5}(\log\log y)^{-1/5})\,,\end{aligned}$$

where c' is a positive absolute constant, we see that

$$\begin{aligned}&\int_{2}^{y}\psi(t)t^{-1+iv}(1+iv\log t)dt\\ &\qquad=\int_{2}^{y}t^{iv}(1+iv\log t)dt+O_v(y\exp(-c'(\log y)^{3/5}(\log\log y)^{-1/5}))\,.\end{aligned}$$

Integration by parts shows that

$$\begin{aligned}&\int_{2}^{y}t^{iv}(1+iv\log t)dt\\ &\qquad=\frac{y^{1+iv}}{1+iv}\left(1+iv\log y-\frac{iv}{1+iv}\right)-\frac{2^{1+iv}}{1+iv}\left(1+iv\log 2-\frac{iv}{1+iv}\right),\end{aligned}$$

and so we must have

$$\begin{aligned}\int_{2}^{y}\psi(t)t^{-1+iv}(1+iv\log t)dt=\frac{y^{1+iv}}{1+iv}&\left(1+iv\log y-\frac{iv}{1+iv}\right)\\ &+O_v(y\exp(-c'(\log y)^{3/5}(\log\log y)^{-1/5}))\,.\end{aligned}$$

Inserting this into (4), we apply the prime number theorem (5) and combine the result and (3) in (2) to conclude

$$\sum_{\substack{q \text{ prime power} \\ q \le y}} \Lambda(q)^2 q^{iv} = y^{1+iv}\left(1 - \frac{iv}{1+iv}\right)\left(\log y - \frac{1}{1+iv}\right) + O_v(y \exp(-c'(\log y)^{3/5}(\log\log y)^{-1/5}))\,,$$

and it follows that

$$\begin{aligned} G(v, T, y) = \frac{y^{1+iv}T}{\pi}&\left(1 - \frac{iv}{1+iv}\right)\left(\log y - \frac{1}{1+iv}\right) \\ &+ O_v(yT \exp(-c'(\log y)^{3/5}(\log\log y)^{-1/5})) \\ &+ O(y^2 \log y)\,. \end{aligned}$$

Putting $v = \lambda$, $v = -\lambda$, and $v = 0$ and inserting the results into (1), we get

$$\begin{aligned} &\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T \le \Im(\rho), \Im(\rho') \le T}} \frac{y^{\rho+\rho'} - 2^{\rho+\rho'}}{\rho+\rho'} \\ &\quad = \frac{2yT}{\pi}\left\{\left[1 + \Re\left(\frac{y^{i\lambda}}{1+i\lambda}\right)\right]\log y - \Re\left(\frac{y^{i\lambda}}{(1+i\lambda)^2}\right) - 1\right\} \\ &\quad\quad + O_\lambda(yT \exp(-c'(\log y)^{3/5}(\log\log y)^{-1/5})) \\ &\quad\quad + O(y^2 \log y) + O(y^2 (\log T)^4) \\ &\quad\quad + O_\lambda\left(\int_2^y \log x \left((\log(2xT))^2 + \frac{\log(2T)}{x \log x}\right)\left|\frac{\sin(T \log(x/n_x))}{\log(x/n_x)}\right| dx\right). \end{aligned} \tag{6}$$

Now, let us note that

$$\begin{aligned} &\int_2^y \log x \left((\log(2xT))^2 + \frac{\log(2T)}{x \log x}\right)\left|\frac{\sin(T \log(x/n_x))}{\log(x/n_x)}\right| dx \\ &\qquad \le (\log y)(\log(2yT))^2 \int_2^y \left|\frac{\sin(T \log(x/n_x))}{\log(x/n_x)}\right| dx\,. \end{aligned}$$

The integral on the right-hand side can be estimated as follows. For each $2 \le m \le l$, we have

$$\int_{x_{m-1}}^{x_m} \left|\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right| dx = \int_{x_{m-1}}^{x_m} \left|\frac{\sin(T\log(x/q_m))}{\log(x/q_m)}\right| dx$$
$$= q_m \int_{T\log((q_m+q_{m+1})/2q_m)}^{T\log((q_m+q_{m+1})/2q_m)} e^{t/T} \left|\frac{\sin t}{t}\right| dt\,,$$

after the change of variable $t = T\log(x/q_m)$. Since $q_{m+1} < 2q_m$ and $q_{m-1} > q_m/2$, it follows that $T\log((q_m+q_{m+1})/2q_m) \le T\log(3/2)$ and $T\log((q_{m-1}+q_m)/2q_m) \ge -T\log(4/3)$, respectively. Thus

$$\int_{x_{m-1}}^{x_m} \left|\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right| dx < q_m \int_{-T\log(4/3)}^{T\log(3/2)} e^{t/T} \left|\frac{\sin t}{t}\right| dt$$
$$\le \frac{3q_m}{2} \int_{-T\log(4/3)}^{T\log(3/2)} \left|\frac{\sin t}{t}\right| dt$$
$$\ll \log T\,,$$

when $t \le T\log(3/2)$ or in other words when $e^{t/T} \le 3/2$. Hence

$$\int_2^y \left|\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right| dx \ll \sum_{\substack{q \text{ prime power}\\ q\le y}} q\log T \ll \frac{y^2\log T}{\log y}\,,$$

and we combine all estimates to find that

$$\int_2^y \log x \left((\log(2xT))^2 + \frac{\log(2T)}{x\log x}\right) \left|\frac{\sin(T\log(x/n_x))}{\log(x/n_x)}\right| dx \ll y^2(\log(2yT))^2\log T\,. \tag{7}$$

From (6) and (7) we see that

$$\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\le\Im(\rho),\Im(\rho')\le T}} \frac{y^{\rho+\rho'}-2^{\rho+\rho'}}{\rho+\rho'}$$
$$= \frac{2yT}{\pi}\left\{\left[1+\Re\left(\frac{y^{i\lambda}}{1+i\lambda}\right)\right]\log y - \Re\left(\frac{y^{i\lambda}}{(1+i\lambda)^2}\right) - 1\right\}$$
$$+ O_\lambda(yT\exp(-c'(\log y)^{3/5}(\log\log y)^{-1/5})) + O(y^2(\log T)^4)\,. \tag{8}$$

Here we note that

$$\left|\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\le\Im(\rho),\Im(\rho')\le T}} \frac{2^{\rho+\rho'}}{\rho+\rho'}\right| \le 4 \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} \sum_{0\le k\le 2T} \sum_{\substack{H_\lambda(\rho')=0\\ -T\le\Im(\rho')\le T\\ |\Im(\rho+\rho')|\in[k,k+1]}} \frac{1}{|\rho+\rho'|}$$
$$= S_I + S_{II}\,,$$

where

$$S_I = 4 \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} \sum_{\substack{H_\lambda(\rho')=0\\ -T\le\Im(\rho')\le T\\ |\Im(\rho+\rho')|\in[0,1]}} \frac{1}{|\rho+\rho'|}$$

and

$$S_{II} = 4 \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} \sum_{1\le k\le 2T} \sum_{\substack{H_\lambda(\rho')=0\\ -T\le\Im(\rho')\le T\\ |\Im(\rho+\rho')|\in[k,k+1]}} \frac{1}{|\rho+\rho'|}\,.$$

We leave the sum S_I for the moment and consider the sum S_{II}, which we estimate as follows:

$$S_{II}$$
$$\le 4 \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} \sum_{1\le k\le 2T} \frac{1}{k} |\{\rho' \colon H_\lambda(\rho')=0, -T\le\Im(\rho')\le T, |\Im(\rho+\rho')|\in[k,k+1]\}|$$
$$\ll \log T \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} \sum_{1\le k\le 2T} \frac{1}{k} \ll (\log T)^2 \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} 1 \ll T(\log T)^3\,.$$

We bear in mind that $\Re(\rho)$ and $\Re(\rho')$ are each $\gg 1/\log T$ by the classical zero-free region for $\zeta(s)$ (see Chapters 13 and 15 of the reference by Davenport [3]) and thus estimate the sum S_I as follows:

$$S_I \le 4 \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} \sum_{\substack{H_\lambda(\rho')=0\\ -T\le\Im(\rho')\le T\\ |\Im(\rho+\rho')|\in[0,1]}} \frac{1}{\Re(\rho+\rho')} \ll \log T \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} \sum_{\substack{H_\lambda(\rho')=0\\ -T\le\Im(\rho')\le T\\ |\Im(\rho+\rho')|\in[0,1]}} 1$$
$$\ll \log T \sum_{\substack{H_\lambda(\rho)=0\\ -T\le\Im(\rho)\le T}} |\{\rho' \colon -T\le\Im(\rho')\le T, |\Im(\rho+\rho')\}| \in [0,1]\}|$$
$$\ll (\log T)^2 |\rho \colon H_\lambda(\rho)=0, -T\le\Im(\rho)\le T| \ll T(\log T)^3\,.$$

Therefore

$$\left| \sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le\Im(\rho),\Im(\rho')\le T}} \frac{2^{\rho+\rho'}}{\rho+\rho'} \right| \ll T(\log T)^3\,, \tag{9}$$

and it follows from (8) and (9) that

$$\begin{aligned} &\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le\Im(\rho),\Im(\rho')\le T}} \frac{y^{\rho+\rho'}}{\rho+\rho'} \\ &\quad = \frac{2yT}{\pi}\left\{\left[1+\Re\left(\frac{y^{i\lambda}}{1+i\lambda}\right)\right]\log y - \Re\left(\frac{y^{i\lambda}}{(1+i\lambda)^2}\right) - 1\right\} \\ &\qquad + O_\lambda(yT\exp(-c'(\log y)^{3/5}(\log\log y)^{-1/5})) + O(y^2(\log T)^4) + O(T(\log T)^3)\,, \end{aligned}$$

as asserted. This completes the proof of Theorem 1.

It is worthwhile to remark that the main term in Theorem 1 dominates the three error terms when $(\log T)^4 \le y \le T/(\log T)^5$.

3. Proof of Theorem 2

Let us fix a positive real number λ, take $T > 2$ and consider a continuously differentiable complex-valued function g with support contained in the interval $(2, T)$. We shall use Theorem 1 for each x in the interval $[2, T]$ and have frequent recourse to the function

$$f(s) = \int_2^\infty g(y)y^{s-1}dy = -\frac{1}{s}\int_2^\infty g'(y)y^s dy\,. \tag{10}$$

Thus f denotes the Mellin transform of g where (10) converges. (See Chapter 4 of the reference by Iwaniec and Kowalski [18] and Appendix 2 of the reference by Patterson [27].) Applying (10) with $s = \rho + \rho'$ and summing the result over all ρ and ρ' with $-T \le \Im(\rho), \Im(\rho') \le T$, we get

$$\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le\Im(\rho),\Im(\rho')\le T}} f(\rho+\rho') = -\int_2^\infty g'(y) \sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le\Im(\rho),\Im(\rho')\le T}} \frac{y^{\rho+\rho'}}{\rho+\rho'}dy\,.$$

By Theorem 1, the right-hand side above equals

$$-\int_2^\infty \frac{2yTg'(y)}{\pi}\left\{\left[1+\Re\left(\frac{y^{i\lambda}}{1+i\lambda}\right)\right]\log y - \Re\left(\frac{y^{i\lambda}}{(1+i\lambda)^2}\right) - 1\right\}dy$$

$$+ O_\lambda\left(\int_2^\infty |g'(y)| yT \exp(-c(\log y)^{3/5}(\log\log y)^{-1/5})dy\right)$$
$$+ O\left(\int_2^\infty |g'(y)| y^2(\log T)^4 dy\right) + O\left(\int_2^\infty |g'(y)| T(\log T)^3 dy\right)$$
$$= -\int_2^\infty \frac{2yTg'(y)}{\pi}\left\{\left[1+\Re\left(\frac{y^{i\lambda}}{1+i\lambda}\right)\right]\log y - \Re\left(\frac{y^{i\lambda}}{(1+i\lambda)^2}\right) - 1\right\} dy$$
$$+ O_\lambda(T\|g'(y) y \exp(-c(\log y)^{3/5}(\log\log y)^{-1/5})\|_1) + O((\log T)^4\|g'(y)y^2\|_1)$$
$$+ O(T(\log T)^3\|g'(y)\|_1)\,.$$

We integrate by parts to find that

$$\int_2^\infty g'(y) y^{1+it}\left(\log y - \frac{1}{1+it}\right) dy = -(1+it)\int_2^\infty g(y) y^{it}\log y dy$$

for any real number t. Then putting $t = \lambda$, $t = -\lambda$, and $t = 0$, we obtain

$$\int_2^\infty \frac{2yTg'(y)}{\pi}\left\{\left[1+\Re\left(\frac{y^{i\lambda}}{1+i\lambda}\right)\right]\log y - \Re\left(\frac{y^{i\lambda}}{(1+i\lambda)^2}\right) - 1\right\} dy$$
$$= -\frac{2T}{\pi}\int_2^\infty g(y)(1+\cos(\lambda\log y))\log y dy\,.$$

Altogether

$$\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\le\Im(\rho),\Im(\rho')\le T}} f(\rho+\rho') = \frac{2T}{\pi}\int_2^\infty g(y)(1+\cos(\lambda\log y))\log y dy$$
$$+ O_\lambda(T\|g'(y) y\exp(-c(\log y)^{3/5}(\log\log y)^{-1/5})\|_1)$$
$$+ O((\log T)^4\|g'(y)y^2\|_1) + O(T(\log T)^3\|g'(y)\|_1)\,,$$

which is precisely the statement of Theorem 2. Hence this theorem is now proved.

4. Proof of Theorem 3

We first fix a positive real number λ and an $\alpha \in (0, 1)$. Then we take T to be large and consider the test function

$$g(y) = \begin{cases} \dfrac{y}{T^{\alpha}}, & \text{if } 0 < y \leq T^{\alpha}; \\ \dfrac{T^{3\alpha}}{y^3}, & \text{if } y > T^{\alpha}. \end{cases}$$

The test function g is clearly continuous on the interval $(0, \infty)$. Although g is not differentiable at $y = T^{\alpha}$, it is differentiable on $(0, T^{\alpha}) \cup (T^{\alpha}, \infty)$. Even though g is not compactly supported, it in fact goes to 0 as y approaches 0 and decays in y rapidly enough as y tends to infinity. Thus one easily sees that the proof of Theorem 2 applies here and the statement of the theorem holds true for g. In this case the three error terms appearing on the right-hand side of the statement of Theorem 2 can be dealt with as follows. In the first error term, we see that

$$\begin{aligned} T \int_0^{\infty} |g'(y)| y \exp(-c(\log y)^{3/5}(\log\log y)^{-1/5}) dy \\ &= T^{1-\alpha} \int_0^{T^{\alpha}} y \exp(-c(\log y)^{3/5}(\log\log y)^{-1/5}) dy \\ &\quad + 3T^{1+3\alpha} \int_{T^{\alpha}}^{\infty} \frac{1}{y^3} \exp(-c(\log y)^{3/5}(\log\log y)^{-1/5}) dy \\ &\ll T^{1+\alpha} \exp(-\alpha c(\log T)^{3/5}(\log\log T)^{-1/5}) . \end{aligned}$$

In the second error term, we see that

$$\begin{aligned} (\log T)^4 \int_0^{\infty} |g'(y)| y^2 dy &= (\log T)^4 \int_0^{T^{\alpha}} |g'(y)| y^2 dy + (\log T)^4 \int_{T^{\alpha}}^{\infty} |g'(y)| y^2 dy \\ &= \frac{(\log T)^4}{T^{\alpha}} \int_0^{T^{\alpha}} y^2 dy + 3T^{3\alpha} (\log T)^4 \int_{T^{\alpha}}^{\infty} \frac{dy}{y^2} \\ &\ll T^{2\alpha} (\log T)^4 . \end{aligned}$$

In the third error term, we see that

$$\begin{aligned} T(\log T)^3 \int_0^\infty |g(y)|dy &= T(\log T)^3 \int_0^{T^\alpha} |g'(y)|dy + T(\log T)^3 \int_{T^\alpha}^\infty |g'(y)|dy \\ &= T^{1-\alpha}(\log T)^3 \int_0^{T^\alpha} dy + 3T^{1+3\alpha}(\log T)^3 \int_{T^\alpha}^\infty \frac{dy}{y^4} \\ &\ll T(\log T)^3 . \end{aligned}$$

For a fixed $\alpha \in (0, 1)$ the first of the above three big O terms is the dominant, and so we get

$$\begin{aligned} \sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le\Im(\rho),\Im(\rho')\le T}} f(\rho+\rho') &= \frac{2T}{\pi}\int_0^\infty g(y)(1+\cos(\lambda\log y))\log y dy \\ &\quad + O_\lambda(T^{1+\alpha}\exp(-\alpha c(\log T)^{3/5}(\log\log T)^{-1/5})) . \end{aligned} \tag{11}$$

By (10)

$$f(s) = \int_0^\infty g(y)y^{s-1}dy = \frac{1}{T^\alpha}\int_0^{T^\alpha} y^s dy + T^{3\alpha}\int_{T^\alpha}^\infty y^{s-4}dy = \frac{4T^{\alpha s}}{4-(s-1)^2} .$$

We then conclude that

$$\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le\Im(\rho),\Im(\rho')\le T}} f(\rho+\rho') = \sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le\Im(\rho),\Im(\rho')\le T}} T^{\alpha(\rho+\rho')} w(\rho+\rho'-1) ,$$

from which we obtain

$$\mathscr{F}_{H_\lambda}(\alpha, T) = \frac{\pi}{T^{1+\alpha}\log T} \sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le\Im(\rho),\Im(\rho')\le T}} f(\rho+\rho') . \tag{12}$$

By (11) and (12)

$$\begin{aligned} &\mathscr{F}_{H_\lambda}(\alpha, T) \\ &= \frac{2}{T^\alpha\log T}\int_0^\infty g(y)(1+\cos(\lambda\log y))\log y dy + O_\lambda(\exp(-\alpha c(\log T)^{3/5}(\log\log T)^{-1/5})) \\ &= \frac{2}{T^{2\alpha}\log T}\int_0^{T^\alpha} y(1+\cos(\lambda\log y))\log y dy + \frac{2T^{2\alpha}}{\log T}\int_{T^\alpha}^\infty y^{-3}(1+\cos(\lambda\log y))\log y dy \\ &\quad + O_\lambda(\exp(-\alpha c(\log T)^{3/5}(\log\log T)^{-1/5})) . \end{aligned}$$

After a short calculation, we find that

$$\int_0^{T^\alpha} y(1+\cos(\lambda\log y))\log y\,dy$$
$$= \frac{T^{2\alpha}}{4(4+\lambda^2)^2}[(4+\lambda^2)^2(2\alpha\log T-1)+4(2\alpha(4+\lambda^2)\log T-4+\lambda^2)$$
$$\times\cos(\alpha\lambda\log T)+4\lambda(\alpha(4+\lambda^2)\log T-4)\sin(\alpha\lambda\log T)]$$

and

$$\int_{T^\alpha}^{\infty} y^{-3}(1+\cos(\lambda\log y))\log y\,dy$$
$$= \frac{T^{-2\alpha}}{4(4+\lambda^2)^2}[(4+\lambda^2)^2(2\alpha\log T+1)+4(2\alpha(4+\lambda^2)\log+4-\lambda^2)$$
$$\times\cos(\alpha\lambda\log T)-4\lambda(\alpha(4+\lambda^2)\log T+4)\sin(\alpha\lambda\log T)]\,.$$

Piecing this together, we obtain

$$\mathscr{F}_{H_\lambda}(\alpha,T)$$
$$= \frac{1}{(4+\lambda^2)^2\log T}[2\alpha(4+\lambda^2)(4\cos(\alpha\lambda\log T)+4+\lambda^2)\log T-16\lambda\sin(\alpha\lambda\log T)]$$
$$+ O_\lambda(\exp(-\alpha c(\log T)^{3/5}(\log\log T)^{-1/5}))\,,$$

which finishes the proof of Theorem 3.

References

[1] E. B. Bogomolny and J. P. Keating, *Random matrix theory and the Riemann zeros I: three- and four-point correlations*, Nonlinearity **8** (1995), 1115–1131.

[2] E. B. Bogomolny and J. P. Keating, *Random matrix theory and the Riemann zeros II: n-point correlations*, Nonlinearity **9** (1996), 911–935.

[3] H. Davenport, *Multiplicative Number Theory*, Graduate Studies in Mathematics **74**, Third edition (Edited by H. L. Montgomery), Springer-Verlag, New York, 2000.

[4] H. M. Edwards, *Riemann's Zeta Function*, Pure and Applied Mathematics **58**, A Series of Monographs and Textbooks, Academic Press, New York, London, 1974.

[5] K. Ford and A. Zaharescu, *On the imaginary parts of zeros of the Riemann zeta function*, J. reine angew. Math. **579** (2005), 145–158.

[6] K. Ford, K. Soundararajan, and A. Zaharescu, *On the distribution of imaginary parts of zeros of the Riemann zeta function, II*, Math. Ann. **343** (2009), 487–505.

[7] A. Fujii, *On the uniformity of the distribution of the zeros of the Riemann zeta function*, J. reine angew. Math. **302** (1978), 167–205.

[8] A. Fujii, *On a theorem of Landau*, Proc. Japan Acad., Ser. A **65**, no. 2 (1989), 51–54.

[9] A. Fujii, *On a theorem of Landau. II*, Proc. Japan Acad. **66** (1990), 291–296.

[10] A. Fujii, *Some problems of Diophantine approximation in the theory of the Riemann zeta function* (*III*), Comment. Math. Univ. St. Pauli **42** (1993), 161–187.

[11] A. Fuji, *Some problems of Diophantine approximation in the theory of the Riemann zeta function* (*IV*), Comment. Math. Univ. St. Pauli **43** (1994), 217–244.

[12] D. A. Goldston, *Notes on pair correlation of zeros and prime numbers*, in *Recent Perspectives in Random Matrix Theory and Number Theory*, London Math. Soc. Lecture Notes Ser. **322**, 2005 (Edited by F. Mezzadri and N. C. Snaith), Cambridge University Press, New York, 79–110.

[13] S. M. Gonek, *A formula of Landau and mean values of $\zeta(s)$*, in *Topics in Analytic Number Theory*, University of Texas Press, 1985, 92–97.

[14] S. M. Gonek, *An explicit formula of Landau and its applications to the theory of the zeta-function*, in *A Tribute to Emil Grosswald: Number Theory and Related Analysis*, Contemp. Math. **143**, Amer. Math. Soc., Providence, RI, 1993, 395–413.

[15] D. A. Hejhal, *On the triple correlation of the zeros of the zeta function*, Int. Math. Res. Not. (1994), 293–302.

[16] A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge Tracts in Mathematics and Mathematical Physics **30**, Cambridge University Press, London, 1932.

[17] A. Ivić, *The Riemann Zeta-Function*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1985.

[18] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Colloquium Publications **53**, Amer. Math. Soc., Providence, RI. 2004.

[19] A. A. Karatsuba and S. M. Voronin, *The Riemann Zeta-Function*, de Gruyter Expositions in Mathematics **5** (Translated from the Russian by N. Koblitz), Walter de Gruyter, Berlin, New York, 1992.

[20] N. M. Katz and P. Sarnak, *Zeros of zeta functions and symmetry*, Bull. Amer. Math. Soc. (N. S.) **36** (1999), no. 1, 1–26.

[21] N. M. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues, and Monodromy*, Colloquium Publications **45**, Amer. Math. Soc., Providence, RI. 1999.

[22] N. M. Korobov, *Estimates of trigonometric sums and their applications*, Uspehi Mat. Nauk **13** (1958), 185–192.

[23] E. Landau, *Über die Nullstellen der ζ-Funktion*, Math. Annalen. **71** (1911), 548–564.

[24] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, in *Analytic Number Theory*, Proceedings of Symposium on Pure Mathematics – Vol. XXIV, St. Louis University, 1972 (Edited by H. G. Diamond), Proc. Symp. Pure Math. **24**, Amer. Math. Soc., Providence, RI, 1973, 181–193.

[25] M. R. Murty and A. Perelli, *The pair correlation of zeros of functions in the Selberg class*, Int. Math. Res. Not. **10** (1999), 531–545.

[26] M. R. Murty and A. Zaharescu, *Explicit formulas for the pair correlation of zeros of functions in the Selberg class*, Forum Math. **14** (2002), no. 1, 65–83.

[27] S. J. Patterson, *An Introduction to the Theory of the Riemann Zeta-Function*, Cambridge Studies in Advanced Mathematics **14**, Cambridge University Press, Cambridge, 1988.

[28] H.-E. Richert, *Zur Abschätzung der Riemannschen Zetafunktion in der Nähe der Vertikalen $\sigma = 1$*, Math. Ann. **169** (1967), 97–101.

[29] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. der Berlin. Akad. (1859), 671–680; *Gesammelte Mathematische Werke, Wissenschaftlicher Nachlass und Nachträge. Collected Papers*, Springer-Verlag, Berlin, Heidelberg, 1990, 177–185.

[30] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, in *A celebration of John F. Nash, Jr.*, Duke Math. J. **81**, Issue 2 (1996), 269–322.

[31] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, in *Proceedings of the Amalfi Conference on Analytic Number Theory*, Maiori, 1989 (Edited by E. Bombieri et al.), Università di Salerno, Salerno, 1992, 376–385; *Collected Papers*, Vol. II, Springer-Verlag, Berlin, 1991, 47–63.

[32] E. C. Titchmarsh, *The Zeta-Function of Riemann*, Hafner Publishing Co., New York. 1972.

[33] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Second edition (Revised by D. R. Heath-Brown), The Clarendon Press, Oxford University Press, New York, 1986.

[34] I. M. Vinogradov, *A new estimate for $\zeta(1 + it)$*, Izv. Akad. Nauk SSSR, Ser. Mat. **22** (1958), 161–164.

[35] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, VEB Deutsche Verlag der Wissenschaften, Berlin, 1963.

Andrew LEDOAN
Department of Mathematics
Boston College, 228 Carney Hall
140 Commonwealth Avenue, Chestnut Hill
Massachusetts 02467–3806
United States of America
e-mail: ledoan@bc.edu

Alexandru ZAHARESCU
Department of Mathematics
University of Illinois at Urbana-Champaign
449 Altgeld Hall, 1409 W. Green Street (MC-382)
Urbana, Illinois 61801–2975, United States of America
e-mail: zaharesc@math.uiuc.edu