Overdetermined Elliptic Systems of Nonlinear Differential Equations in Two Independent Variables

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Introduction

This paper is concerned with the problem of proving the existence of solutions to an overdetermined system of nonlinear partial differential equations. To solve this problem, É. Cartan [4] first introduced the notion of involutive systems and established an existence theorem for analytic systems. The existence theorem called today as the Cartan-Kähler theorem demonstrates that an involutive analytic system admits local analytic solutions (cf. Cartan [5], Kähler [12], Kuranishi [19], Goldschmidt [7–8]). Let us consider a smooth system. Here by smooth, we mean differentiable of class $C^\infty$. One cannot deduce such a general existence result for a smooth system, as one knows from the famous fact found by Lewy [21] that there exist smooth linear partial differential equations without solution (cf. Hörmander [11], Chapter VI). To prove the existence of smooth solutions, one needs to make some additional assumption such as ellipticity or hyperbolicity. There has been obtained some existence theorems asserting that if a nonlinear system is involutive and hyperbolic in some sense, it admits local smooth solutions (see Yang [31], Kakié [13,15,16]). However, such an existence theorem for an overdetermined elliptic nonlinear system has not been obtained as yet.

The purpose of this paper is to prove the following existence theorem for systems with two independent variables (Theorem 8.2).

Theorem. If a smooth overdetermined nonlinear system with two independent variables is involutive and elliptic, then it admits local smooth solutions.

We emphasize that the assumption of involutiveness is a reasonable one, since any elliptic system can be prolonged, at least in theory, to an involutive elliptic system.

As is expected, the proof of the theorem requires some theory of overdetermined linear elliptic differential operators which relates closely to existence of local smooth solutions for linear equations. It should be noted that such a theory has not been obtained as yet at least in the general case. In fact, for overdetermined linear elliptic differential equations with more than two independent variables, there has not been established a general existence theorem, although some results were obtained under strong additional assumptions (see MacKichan [22], Sweeney [28, 29]. cf. Bryant et al. [3]). In the case of linear elliptic equations with two independent variables, we have proved a general existence theorem in [17]. Inspired
by the investigation in it and the work of Sweeney [29], we succeed in obtaining a required theory for operators with two independent variables.

Let us state roughly the way we prove the existence theorem. Let \( \mathcal{R}_l \) be an overdetermined smooth system of nonlinear partial differential equations of order \( l \), and let \( X \) denote the space of independent variables. Assume that \( \mathcal{R}_l \) is involutive and elliptic. By a well-known procedure, we reduce solution of \( \mathcal{R}_l \) to solution of an involutive elliptic system \( \mathcal{R}_1 \) of the first order. Let \( x_0 \in X \). We construct an involutive elliptic nonlinear differential operator \( \Phi \) of the first order from a product vector bundle \( E^0 \) to another one \( E^1 \) over a neighborhood of \( x_0 \) in \( X \) such that local solutions of \( \mathcal{R}_1 \) correspond to local sections \( u \) of \( E^0 \) satisfying \( \Phi(u) = 0 \). Choosing a sufficiently large integer \( k \), we consider the nonlinear operator \( \Phi : H^{k+1}(\Omega_\epsilon, E^0) \to H^k(\Omega_\epsilon, E^1) \), where \( H^l(\Omega_\epsilon, E^i) \) denotes the Sobolev space of sections of \( E^i \) over an open ball \( \Omega_\epsilon \subset X \) with center \( x_0 \) and of radius \( \epsilon > 0 \). We can take a smooth section \( u_0 \) of \( E^0 \) such that the \( q \)-jet of \( \Phi(u_0) \) at \( x_0 \) vanishes for any integer \( q > 0 \); \( u_0 \) defines an element of \( H^{k+1}(\Omega_\epsilon, E^0) \) by restriction, which we call also \( u_0 \). The main part of the proof is to show that there exists an element \( u \in H^{k+1}(\Omega_\epsilon, E^0) \) near \( u_0 \) such that \( \Phi(u) = 0 \), provided \( \epsilon \) is small enough. If \( \Phi \) is a determined operator, then one can readily prove it, even when \( \dim X > 2 \), by applying the implicit function theorem in functional analysis (cf. Taylor [30], Chapter 14). However, when \( \Phi \) is overdetermined, the circumstances become quite different. We proceed as follows. Let \( \tau_0 \) be the translation operator in \( H^{k+1}(\Omega_\epsilon, E^0) \) defined by \( \tau_0(u) = u_0 + u \). We choose a closed subspace \( Y \) of \( H^{k+1}(\Omega_\epsilon, E^0) \) and a closed subspace \( Z \) of \( H^k(\Omega_\epsilon, E^1) \) together with a projection \( \rho : H^k(\Omega_\epsilon, E^1) \to Z \), and consider the mapping \( T : Y \to Z \) defined by \( T = \rho \circ \Phi \circ \tau_0 \). We must find \( Y, Z, \rho \) in such a way that we can apply the implicit function theorem to the operator \( T \), and that \( \rho \) has the property that \( \rho \circ \Phi \circ \tau_0(u) = 0 \) with \( u \in Y \) implies that \( \Phi \circ \tau_0(u) = 0 \). Restricting to the case when \( \dim X = 2 \), we show that such spaces \( Y, Z \) can be actually found; In the discussion the Fréchet derivative \( D_{u_0} \Phi \) of \( \Phi \) at \( u_0 \) plays an important role. \( D_{u_0} \Phi \) is indeed an operator induced from a quasi-involutive elliptic linear differential operator \( D^0 \). By virtue of the simple structure of such an operator with two independent variables, we can give a theory for such an operator which enables us to obtain the required results concerning \( D_{u_0} \Phi \). In this way we show that there is an element \( u \in H^{k+1}(\Omega_\epsilon, E^0) \) such that \( \Phi(u) = 0 \), provided \( \epsilon \) is small enough. The elliptic regularity theorem for overdetermined nonlinear equations indicates that \( u \) is smooth. Thus we obtain a local smooth solution of the original system \( \mathcal{R}_l \).

This paper consists of eight sections. In sections 1–5, we recall various notions concerning involutive systems of nonlinear differential equations and involutive differential operators; We also introduce some modified notions, and derive some results needed in this paper. In section 6, we discuss differential operators between Sobolev spaces. In these sections we do not restrict the notions to the ones with two independent variables, for doing so leads to little advantage in the discussions. In section 7, we give a theory of involutive elliptic linear differential operators in two independent variables, and deduce a result, which is not exactly a standard one, but is indispensable in our discussion. In the final section 8,
we first prove an existence theorem for an involutive elliptic nonlinear differential operator with two independent variables (Theorem 8.1) by using various results obtained in the previous sections. By applying this theorem, we finally prove an existence theorem for an involutive elliptic nonlinear system with two independent variables (Theorem 8.2).

Throughout the paper, the terminology “smooth” means “differentiable of class $C^\infty$”, and the notions such as manifolds, vector bundles are assumed to be smooth unless otherwise is expressly stated.

1. Involution symbols

Let $E, T$ be real vector spaces of finite dimensions. The dual space to $T$ will be denoted by $T^*$. Let $\otimes^i T^*$ and $S^i T^*$ denote the tensor product and the symmetric product of $k$ copies of $T^*$, respectively.

By a symbol of order $l \geq 1$, we mean a vector subspace $G_l$ of $S^l T^* \otimes E$. The $k$-th prolongation $G_{k+l}$ of a symbol $G_l$ is defined by

$$G_{l+k} = (\otimes^k T^* \otimes G_l) \cap (S^{l+k} T^* \otimes E).$$

We may consider $G_l$ as a subspace of $T^* \otimes F = \text{Hom}(T, F)$, where $F = S^{l-1} T^* \otimes E$. For a subset $A$ of $T$, let $G_l(A)$ denote the subspace of $G_l$ consisting of all elements which annihilate $A$. Let $r_i = r_i(G_l)$ be the minimum of $\dim G_l(A)$ where $A$ runs through the set of $i$-dimensional subspaces of $T$ ($0 \leq i \leq n = \dim T$). Clearly $r_0 \geq r_1 \geq \ldots \geq r_n = 0$. It can be shown that the inequality $\dim G_l \leq \sum_{i=0}^n r_i$ holds. A symbol $G_l$ is said to be involutive if the equality $\dim G_l = \sum_{i=0}^n r_i$ holds (cf. Kuranishi [19], Guillemin and Sternberg [10], Appendix). Given a basis $B = \{t_1, \ldots, t_n\}$ of $T$, we set $B_i = \{t_1, \ldots, t_i\}$ ($1 \leq i \leq n$), $B_0 = \{0\}$. A symbol $G_l$ is involutive if and only if there exists a basis $B$ of $T$ such that $\dim G_{l+1}$ is equal to $\sum_{i=0}^n \dim G_l(B_i)$. Such a basis $B$ is said to be regular for $G_l$. If $B$ is regular for $G_l$, then $r_i$ is equal to $\dim G_l(B_i)$ ($0 \leq i \leq n$).

Let $G_l$ be an involutive symbol of order $l$. The non-negative integers $s_i = r_i - r_{i-1}$ ($1 \leq i \leq n$) are called the Cartan characters of $G_l$. We denote by $\text{Vc}$ the complexification of a real vector space $V$. A non-zero cotangent vector $\xi \in T^*_C$ is said to be characteristic for $G_l$ if $G_l \cap \{t^i \otimes E_C\} \neq \{0\}$. The characteristic variety $\mathcal{Z}(G_l)$ of $G_l$ is defined to be the set of all characteristic cotangent vectors in $T^*_C \setminus \{0\}$; $\mathcal{Z}(G_l)$ is a projective algebraic variety.

**Lemma 1.1.** Let $G_l \subset S^l T^* \otimes E$ be a symbol of order $l$.

(I) Let $F'$ be a real vector space containing $F = S^{l-1} T^* \otimes E$ as a subspace. Then $G_l$ is involutive if and only if the symbol $G_l \subset T^* \otimes F'$ of order $l$ is involutive. Moreover the characteristic varieties of both symbols coincide with each other.

(II) If $G_l$ is involutive, then the $k$-th prolongation $G_{l+k}$ is involutive for each $k \geq 1$.

(III) If $G_l$ is involutive, $\mathcal{Z}(G_{l+k}) = \mathcal{Z}(G_l)$ ($k \geq 0$).

(IV) Let $p$ be the largest integer for which $s_p \neq 0$. Then $\mathcal{Z}(G_l)$ is of projective dimension $p - 1$. 

Q.E.D.

From now on, let $E, T$ be real vector bundles over a manifold $M$. The fiber of $E$ over a point $a \in M$ will be denoted by $E_a$. By a symbol $G_{i}$ of order $l$, we shall mean the kernel of a vector bundle morphism

$$
\sigma : S^l T^* \otimes E \longrightarrow E^1
$$

$E^1$ being a vector bundle over $M$.

$G_{i} = \ker \sigma$ is a family $\{G_{i,a} \mid a \in M\}$ of vector spaces $G_{i,a} \subset S^l T^*_a \otimes E_a$. A symbol $G_{i}$ is said to be involutive if the fiber $G_{i,a}$ is involutive for each $a \in M$. The $k$-th prolongation $G_{i+k}$ of $G_{i}$ is defined to be the family $\{G_{i+k,a} \mid a \in M\}$ in $S^{l+k} T^* \otimes E$ where each $G_{i+k,a}$ is the $k$-th prolongation of $G_{i,a}$. The prolongation $G_{i+k}$ may be defined also in the following way. The $k$-th prolongation $\sigma_k$ of the morphism $\sigma$ is, by definition, the composition

$$
S^{l+k} T^* \otimes E \xrightarrow{\delta_{k,l}} S^k T^* \otimes (S^l T^* \otimes E) \xrightarrow{1 \otimes \sigma} S^k T^* \otimes E^1,
$$

where $\delta_{k,l} : S^{l+k} T^* \longrightarrow S^k T^* \otimes S^l T^*$ is the canonical injection. Then it is readily seen that the $k$-th prolongation $G_{i+k}$ coincides with $\ker \sigma_k$ for each $k \geq 1$.

Lemma 1.2. Let $G_{1}$ be a symbol of order $l$, and let $a_0 \in M$. Assume that $G_{1,a_0}$ is involutive, and that the first prolongation $G_{1+1}$ is trivial, that is, $\dim G_{1+1,a} \equiv 0$. Let $\{t_1, \ldots, t_n\}$ be a set of continuous sections of $\pi$ over a neighborhood of $a_0$ such that $\{t_1(a_0), \ldots, t_n(a_0)\}$ is a regular basis for $G_{1,a_0}$. Then, there exists a neighborhood $U$ of $a_0$ in $M$ such that $B(a) = \{t_1(a), \ldots, t_n(a)\}$ is a regular basis for $G_{1,a}$ for any $a \in U$. Moreover the integers $r_i(G_{1,a})$ ($0 \leq i \leq n$) are constant on $a \in U$, and so are the Cartan characters $s_i(G_{1,a})$ ($1 \leq i \leq n$).

Proof. Denote $\rho_i(a) = \dim G_{1,a} (B_i(a))$. The inequality $\dim G_{1+1,a} \leq \sum_{i=0}^n \rho_i(a)$ holds. By assumption, when $a = a_0$ the equality holds, and $\dim G_{1+1,a}$ is constant around $a_0$. Since $\rho_i(a)$’s are upper semi-continuous, there exists a neighborhood $U$ of $x_0$ such that $\rho_i(a)$ are constant on $U$, and hence that the equality holds for any $a \in U$. Hence we have all the required assertions.

Q.E.D.

2. Nonlinear systems of partial differential equations

Let $X$ be a manifold. We shall denote $n = \dim X$. The tangent bundle of $X$ will be denoted by $TX$, and the cotangent bundle of $X$ by $T^*X$. Let $f$ be a smooth mapping from a manifold $Y$ to $X$, and $f_* : TY \rightarrow TX$ denote the differential of $f$. we shall call $f$ a submersion if, for each $y \in Y$, the linear mapping $f_* : T_Y Y \rightarrow T_{f(y)} X$ is surjective. The bundle induced by $f$ from a vector bundle $W$ over $X$ will be denoted by $f^{-1} W$.

A fibered manifold over $X$ is, by definition, a manifold $E$ together with a surjective submersion $\pi : E \rightarrow X$ (\pi is called the projection). A submanifold $F$ of $E$ is called a fibered submanifold of $E$ if $F$ is a fibered manifold over $X$ with projection $\pi|_F$. The vertical bundle $V(E)$ of a fibered manifold $E$ is defined to be the kernel of the differential
\[ \pi : T \mathcal{E} \to TX. \] A smooth mapping \( \varphi \) from a fibered manifold \( \mathcal{E} \) to another one \( \mathcal{E}' \) over the same base manifold \( X \) with projection \( \pi' \) is called a fibered morphism if \( \pi = \pi' \circ \varphi \) holds.

Let \( \mathcal{E} \) be a fibered manifold over \( X \) with projection \( \pi \). The bundle of \( k \)-jets of sections of \( \mathcal{E} \) will be denoted by \( J_k(\mathcal{E}) \), and the natural projection from \( J_k(\mathcal{E}) \) to \( J_l(\mathcal{E}) \) by \( \pi^k_l \) (\( k > l \geq 0 \)), \( J_0(\mathcal{E}) \) being identified with \( \mathcal{E} \). The space \( J_k(\mathcal{E}) \) is a fibered manifold over \( X \) with projection \( \pi^k_{k-l} = \pi \circ \pi^k_0 \). If \( u \) is a \( k \) times differentiable section of \( \mathcal{E} \), we have the section \( j_k(u) \) of \( J_k(\mathcal{E}) \) which associates with each point \( x \in X \) the \( k \)-jet of \( u \) at \( x \). One has the following exact sequence of vector bundles over \( J_l(\mathcal{E}) \):

\[
0 \to S^lT^* \otimes_{J_l} V(\mathcal{E}) \overset{\epsilon_l}{\longrightarrow} V(J_l) \overset{(\pi^l_{l-1})^*}{\longrightarrow} (\pi^l_{l-1})^{-1}V(J_{l-1}) \to 0,
\]

where we use the simplified notations \( T^* = T^*X \), \( J_k = J_k(\mathcal{E}) \) (cf. Goldschmidt [8], Proposition 5.2, Pommaret [24], Chapter 1, Proposition 9.11).

A system of nonlinear partial differential equations of order \( l \) on \( \mathcal{E} \) is, by definition, a fibered submanifold \( \mathcal{R}_l \) of \( J_l(\mathcal{E}) \). We shall also call it simply a nonlinear system of order \( l \) on \( \mathcal{E} \). The \( k \)-th prolongation \( \mathcal{R}_{l+k} \) of \( \mathcal{R}_l \) is defined to be the subset \( \mathcal{R}_{l+k} = J_k(\mathcal{R}_l) \cap J_{l+k}(\mathcal{E}) \), where \( J_{l+k}(\mathcal{E}) \) is regarded canonically as a submanifold of \( J_k(J_l) \).

Let \( \mathcal{R}_l \) be a nonlinear system of order \( l \) on \( \mathcal{E} \). By a solution of \( \mathcal{R}_l \), we shall mean a not necessarily smooth section \( u \) of \( \mathcal{E} \) over an open set \( U \subset X \) such that \( u \) is at least \( l \) times differentiable, and that its \( l \)-jet \( j_l(u)(x) \) at \( x \) belongs to \( \mathcal{R}_l \) for each \( x \in U \). Observing that one can define the Sobolev space \( H^k(U, \mathcal{E}) \) of sections of \( \mathcal{E} \) over \( U \) for each \( k > n/2 \), one may define a solution \( u \) of Sobolev class \( H^k \) (that is, a solution \( u \) belonging to \( H^k(U, \mathcal{E}) \)) for each integer \( k > l + (n/2) \) (cf. the discussion at the beginning of section 6).

Let \( \iota : \mathcal{R}_l \to J_l = J_l(\mathcal{E}) \) be the embedding mapping. The vertical bundle \( V(\mathcal{R}_l) \) is regarded as a subbundle of the induced bundle \( \iota^{-1}V(J_l) \). Let \( Q_0 \) denote the quotient bundle \( \iota^{-1}V(J_l)/V(\mathcal{R}_l) \) over \( \mathcal{R}_l \). Let \( \sigma : S^lT^* \otimes_{\mathcal{R}_l} V(\mathcal{E}) \to Q_0 \) be the composition of the monomorphism \( \epsilon_l \) in (2.1) over \( \mathcal{R}_l \) and the canonical projection \( \iota^{-1}V(J_l) \to Q_0 \). The symbol \( G_l \) of \( \mathcal{R}_l \) is defined to be the kernel of \( \sigma \). \( G_l \) is a family of vector spaces over \( \mathcal{R}_l \). The \( k \)-th prolongation of \( G_l \) will be denoted by \( G_{l+k} \). Note that \( G_{l+k} \) coincides with the kernel of the \( k \)-th prolongation \( \sigma_k \) of the morphism \( \sigma \) (cf. section 1).

**Definition** (Kuranishi [19], cf. Pommaret [24]). A nonlinear system \( \mathcal{R}_l \) of order \( l \) on \( \mathcal{E} \) is involutive if the following conditions (i)--(iii) are satisfied:

(i) The family \( G_{l+1} \) is a vector bundle over \( \mathcal{R}_l \);

(ii) The symbol \( G_l \) is involutive (cf. section 1);

(iii) The mapping \( \pi_l^{l+1} : \mathcal{R}_{l+1} \to \mathcal{R}_l \) is surjective.

We note that the conditions (i),(ii) imply that \( G_l \) is a vector bundle over \( \mathcal{R}_l \). Moreover, we have the following prolongation theorem due to Cartan and Matsushima [23] (see Kuranishi [19], Goldschmidt [8]).

**Theorem 2.1.** If \( \mathcal{R}_l \) is involutive, then its \( k \)-th prolongation \( \mathcal{R}_{l+k} \) is an involutive system of order \( l + k \); In particular,
(i) \( R_{l+k} \) is a fibered submanifold of \( J_{l+k}(E) \), and the mapping
\[
\pi_{l+k}^{l+k+1} : R_{l+k+1} \to R_{l+k}
\]
is a surjective submersion \((k = 0, 1, 2, \ldots)\),
(ii) \( G_{l+k} \) is a vector bundle over \( R_l \) \((k = 0, 1, 2, \ldots)\), and
(iii) the symbol of \( R_{l+k} \) is the family induced from \( G_{l+k} \) by the mapping \( \pi_{l+k}^{l+k} : R_{l+k+1} \to R_l \) \((k = 1, 2, 3, \ldots)\).

Let \( R_l \) be an involutive nonlinear system. The Cartan characters \( s_i(P) \) \((1 \leq i \leq n)\) of \( R_l \) at \( P \in R_l \) is defined to be the Cartan characters \( s_i \) \((1 \leq i \leq n)\) of the symbol \( G_{l,P} \), respectively (cf. section 1). The \( s_i(P) \) are (locally) constant on \( P \), and hence we may write \( s_i = s_i(P) \). The characteristic variety \( \mathcal{Z}_P \) of \( R_l \) at \( P \in R_l \) is, by definition, the characteristic variety \( \mathcal{Z}(G_{l,P}) \) of the involutive symbol \( G_{l,P} \) (cf. section 1), and the characteristic variety \( \mathcal{Z} \) of \( R_l \) is the family \( \{ \mathcal{Z}_P; P \in R_l \} \). A system \( R_l \) is said to be elliptic if the characteristic variety \( \mathcal{Z} \) has no real point, that is, \( \mathcal{Z} \cap (\pi_{l-1}^{-1}(T^* \setminus \{ 0 \})) = \emptyset \).

In showing the existence of smooth solutions, we shall use the extension to overdetermined nonlinear equations of the elliptic regularity theorem for determined nonlinear equations due to Bony [2] and Beals and Reed [1].

**Theorem 2.2 (Elliptic regularity theorem).** Assume that \( R_l \) is involutive and elliptic. Set \( N = \max \{ ml - 1, l + 1 \} + (n/2) \), where \( n = \dim X \), \( n + m = \dim E \). If a solution of \( R_l \) is of Sobolev class \( H^s \) with \( s > N \), then it is smooth (differentiable of class \( C^\infty \)).

(For the detailed proof, see Kakié [18].)

We shall give a process of reducing a nonlinear system \( R_l \) of higher order to a system of the first order with some good properties. In discussing it, we find it convenient to use the notion of affine bundles (cf. Goldschmidt [8]). By an affine bundle \( A \) over a manifold \( X \) modeled on a vector bundle \( \xi : W \to X \), we mean a fibered manifold \( A \) over \( X \) with projection \( \pi : A \to X \) together with a morphism of fibered manifolds \( W \times_X A \to A \) sending \((w, a) \in W \times X A \) to \( w + a \in A \) such that, for each \( x \in X \), the fiber \( A_x \) is an affine space modeled on the vector space \( W_x \) under the action of \( W_x \) on \( A_x \) sending \((w, a) \in W_x \times A_x \) to \( w + a \in A_x \). An affine bundle \( A' \) over \( X \) modeled on a vector bundle \( \xi' : W' \to X \) is called an affine subbundle of \( A \) if \( A' \) is a fibered submanifold of \( A \), \( \pi' : A' \to X \) is locally trivial, and \( A_x' \) is an affine subspace of \( A_x \) for each \( x \in X \); The vector bundle \( W' \) is canonically regarded as a subbundle of \( W \).

The bundle \( J_k(E) \) of \( k \)-jets of sections of \( E \) is an affine bundle over \( J_{k-1}(E) \) modeled on the vector bundle \( S^k T^* \otimes V(E) \) over \( J_{k-1}(E) \) \((k \geq 1) \) (see Goldschmidt [8], section 5). Let us describe its structure in terms of local coordinates. Let \((x^i, y^{a}; 1 \leq i \leq n, 1 \leq a \leq m)\) be a local coordinate system in a fibered chart of \( E \) with \((x_i; 1 \leq i \leq n)\) being the (pull-back of ) a local coordinate system of \( X \). For a multi-index \( v = (v_1, \ldots, v_n) \), we shall use the usual notations: \(|v| = v_1 + \cdots + v_n\),
\[
\left( \frac{\partial}{\partial x^i} \right)^v = \left( \frac{\partial}{\partial x^1} \right)^{v_1} \cdots \left( \frac{\partial}{\partial x^n} \right)^{v_n}, \quad (dx)^v = (dx^1)^{v_1} \cdots (dx^n)^{v_n} \in S^{|v|} T^*.
\]
Let \( p_\alpha^u \) be the function on \( J_k(\mathcal{E}) \) defined by
\[
p_\alpha^u(j_k(u)(x)) = (\frac{\partial}{\partial x})^u u^\alpha(x),
\]
where \( u \) is a section of \( \mathcal{E} \) described by \( y^\alpha = u^\alpha(x) \) (\( 1 \leq \alpha \leq m \)). Then
\[
(x^i, y^\alpha, p_\alpha^u; 1 \leq i \leq n, 1 \leq \alpha \leq m, 1 \leq |v| \leq k)
\]
gives a local coordinate system of \( J_k(\mathcal{E}) \). For convenience we make the convention that \( p_\alpha^u \) with \( |v| = 0 \) denotes \( y^\alpha \).

An element \( \zeta \) of \( S^k T^* \otimes V(\mathcal{E}) \) is described by
\[
\zeta = \sum_{\alpha=1}^m \sum_{|v|=k} \xi^\alpha_v (dx)^v \otimes \frac{\partial}{\partial y^\alpha}, \quad \xi^\alpha_v \in \mathbb{R}.
\]

Then the action of affine bundle structure of \( J_k(\mathcal{E}) \) is described as follows: If a point \( P \in J_k(\mathcal{E}) \) has the coordinates (2.2), then \( \zeta + P \) is the point having the coordinates
\[
(x^i, y^\alpha, p_\alpha^u, \xi^\alpha_v + p_\alpha^u; 1 \leq i \leq n, 1 \leq \alpha \leq m, 1 \leq |v| < k, |v| = k).
\]

Let \( R_l \) be an involutive nonlinear system of order \( l \) on \( \mathcal{E} \). The first prolongation \( R_{l+1} \) is a subset of \( J_1(\mathcal{R}_l) \). Set \( \mathcal{E}' = \mathcal{R}_l \) and \( \mathcal{R}'_l = \mathcal{R}_{l+1} \). \( \mathcal{E}' \) is a fibered manifold over \( X \) with projection \( \pi' = \pi_{l+1} : \mathcal{R}_l \to X \), and by Theorem 2.1, \( \mathcal{R}_{l+1} \) is a fibered submanifold of \( J_1(\mathcal{E}') \). Thus \( \mathcal{R}'_l \) is a nonlinear system of order 1 on \( \mathcal{E}' \). We shall denote by \( G'_1 \) the symbol of \( \mathcal{R}'_l \); \( G'_1 \subset T^* \otimes \mathcal{R}'_l V(\mathcal{E}') \).

**Theorem 2.3.** If \( R_l \) is an involutive nonlinear system of order \( l \) on \( \mathcal{E} \), then \( \mathcal{R}'_l \) possesses the following properties:

(i) \( \mathcal{R}'_l \) is an involutive nonlinear system of order \( l \) on \( \mathcal{E}' \).

(ii) The mapping \( j_l \) sending each section \( u \) of \( \mathcal{E} \) to the section \( j_l(u) \) of \( J_1(\mathcal{E}) \) establishes a bijective correspondence between the set of smooth solutions of \( R_l \) and that of \( \mathcal{R}'_l \).

(iii) The characteristic variety \( \mathcal{Z}' \) of \( \mathcal{R}'_l \) coincides with the family induced from that of \( R_l \) by \( \pi'^{-1}_l : \mathcal{R}'_l \to \mathcal{R}_l \).

(iv) The projection \( \pi'^{-1}_l : \mathcal{R}'_l \to \mathcal{E}' \) is a surjective submersion. Moreover the space \( \mathcal{R}'_l \) is an affine subbundle over \( \mathcal{E}' \) modeled on the vector bundle \( W' \), and is an affine subbundle of the affine bundle \( J_1(\mathcal{E}') \) over \( \mathcal{E}' \) modeled on the vector bundle \( T^* \otimes \mathcal{E}' V(\mathcal{E}') \). The symbol \( G'_1 \) is equal to the induced bundle \( (\pi'^{-1}_l)^{-1} W' \).

**Note.** The last property of (iv) may be said that the system \( \mathcal{R}'_l \) is a *quasi-linear* differential equation.

**Proof.** By Theorem 2.1, \( \mathcal{R}_{l+1} \) is an involutive system of order \( l + 1 \) on \( \mathcal{E} \) and its symbol coincides with the family induced from the vector bundle \( G_{l+1} \) by \( \pi'^{-1}_l : \mathcal{R}_{l+1} \to \mathcal{R}_l \). Clearly the set of smooth solutions of \( \mathcal{R}_{l+1} \) is equal to that of \( \mathcal{R}_l \). Moreover \( \mathcal{R}_{l+1} \) is an affine subbundle of \( J_1(\mathcal{E}) | R_l \) over \( \mathcal{R}_l \) modeled on the vector bundle \( G_{l+1} \) (cf. Goldschmidt [8], Proposition 7.1). This fact is described, using local coordinates \((x, y, p)\) in (2.2) with \( k = l \), as follows. If
\[
\psi^\beta(x, y, p) = 0 \quad (\beta = 1, 2, \ldots, N)
\]
is a regular local equation of $\mathcal{R}_l$ in $J_l(\mathcal{E})$, then the prolongation $\mathcal{R}_{l+1}$ is described by the system of equations consisting of (2.3) and

\[
(2.4) \quad \sum_{\alpha=1}^{m} \sum_{|\nu|=d} \frac{\partial \psi_\beta}{\partial p_\nu}(x, y, p)p_\nu^\alpha + \psi_i^\beta (x, y, p) = 0 \quad (i = 1, \ldots, n; \beta = 1, \ldots, N)
\]

with

\[
\psi_i^\beta = \frac{\partial \phi^\beta}{\partial x_i} + \sum_{\alpha=1}^{m} \sum_{0 \leq |\nu| < l} \frac{\partial \phi^\beta}{\partial p_\nu}(x, y, p)p_\nu^\alpha + 1_i + \psi_\beta (x, y, p) = 0 \quad (i = 1, \ldots, n; \beta = 1, \ldots, N)
\]

where 1_i is the multi-index having only one non-zero component 1 in the i-th place.

According to the well-known procedure (cf. Pommaret [24], Chapter 3, section 3), we set $\hat{\mathcal{E}} = J_l(\mathcal{E})$, $\hat{\mathcal{R}}_1 = \mathcal{R}_{l+1} \subset J_1(\hat{\mathcal{E}})$. $\hat{\mathcal{R}}_1$ is a nonlinear system of order 1 on $\hat{\mathcal{E}}$. It is obvious that the mapping $j_l$ gives a bijective correspondence between the set of smooth solutions of $\mathcal{R}_{l+1}$ and that of $\hat{\mathcal{R}}_1$.

Let $\hat{\mathcal{R}}_{1+k}$ denote the k-th prolongations of $\hat{\mathcal{R}}_1$ and the symbol $\hat{G}_1$ of $\hat{\mathcal{R}}_1$, respectively. Using the canonical monomorphisms $J_{l+k}(\mathcal{E}) \to J_k(J_l(\mathcal{E}))$, one has the identifications

\[
\hat{\mathcal{R}}_{1+k} = \mathcal{R}_{l+1+k}, \quad \hat{G}_{1+k} = (\pi_l^{l+1}|_{\hat{\mathcal{R}}_1})^{-1} G_{l+1+k} \quad (k = 0, 1, 2, \ldots)
\]

Since $G_l$ is involutive, (I), (II) in Lemma 1.1 indicate that $\hat{G}_1$ is involutive. From these facts, it follows that $\hat{\mathcal{R}}_1$ is involutive.

We now observe that in terms of the canonical monomorphisms $J_k(\mathcal{E}') \to J_k(\hat{\mathcal{E}})$ and $S^k T^* \otimes \mathcal{V}(\mathcal{E}') \to S^k T^* \otimes \mathcal{V}(\hat{\mathcal{E}})$, we have the identifications

\[
\mathcal{R}'_{l+k} = \hat{\mathcal{R}}_{1+k}, \quad G'_1 = \hat{G}_{1+k} \quad (k = 0, 1, 2, \ldots)
\]

From these we know that $\mathcal{R}'_l$ is involutive, and that it has the property (ii). By virtue of (I), (II) in Lemma 1.1, the identification $G'_1 = (\pi_l^{l+1}|_{\mathcal{R}'_l})^{-1} G_{l+1}$ implies the property (iii). The first property of (iv) is a consequence of Theorem 2.1. Let $W'$ denote the family $G_{l+1}$ over $\mathcal{E}'$ considered as a family in $T^* \otimes \mathcal{V}(\mathcal{E}')$ in terms of the monomorphism $S^{l+1} T^* \otimes \mathcal{V}(\mathcal{E}) \to T^* \otimes \mathcal{V}(\mathcal{E}')$. The fact that $\mathcal{R}_{l+1}$ is an affine bundle over $\mathcal{R}_l$ modeled on $G_{l+1}$ together with the above identifications implies that $\mathcal{R}'_l$ is an affine bundle over $\mathcal{E}'$ modeled on $W'$. Thus $\mathcal{R}'_l$ has the property (iv). Q.E.D.

3. Linear differential operators

Let $E$ be a real vector bundle over a manifold $X$. The space $J_k(E)$ of k-jets of sections of $E$ is a vector bundle, and the canonical projection $\pi^l_k : J_k(E) \to J_l(E)$ is a vector bundle morphism ($k > l \geq 0$). Corresponding to the sequence (2.1), we have the exact sequence

\[
0 \to S^l T^* \otimes E \xrightarrow{\pi^l_{l-1}} J_l(E) \xrightarrow{\pi^l_{l-1}} J_{l-1}(E) \to 0
\]
of vector bundles over $X$ (cf. Goldschmidt [7]). We denote by $E$ the sheaf of germs of smooth sections of $E$.

Let $E^0$, $E^1$ be real vector bundles over $X$. A sheaf morphism $D : E^0 \to E^1$ is called a linear differential operator of order $l$ from $E^0$ to $E^1$ if there exists a vector bundle morphism $\phi = \phi(D) : J_l(E^0) \to E^1$ such that $D$ is the composition of $j_l : E^0 \to J_l(E^0)$ and $\phi : J_l(E^0) \to E^1$; $D = \phi \circ j_l$. The $k$-th prolongation $p_k(\phi)$ of $\phi$ is defined to be the vector bundle morphism from $J_{l+k}(E^0)$ to $J_k(E^1)$ sending $j_{l+k}(u)(x)$ to $j_k(\phi \circ j_l(u))(x)$, $u$ being a section of $E^0$ defined around a point $x \in X$ ($k \geq 0$). The symbol morphism of $D$ is the vector bundle morphism $\sigma(D) : \bigwedge^l T^* \otimes E^0 \to E^1$ defined to be the composition of $\epsilon_l$ and $\phi = \phi(D)$. The $k$-th prolongation of $\sigma(D)$ will be denoted by $\sigma_k(D)$ ($k \geq 0$) (cf. section 1). We shall also denote $\sigma(D) = \sigma(\phi)$, $\sigma_k(D) = \sigma_k(\phi)$. Given a linear differential operator $D$ of order $l$, we set

$$R_l = \ker \phi(D), \quad R_{l+k} = \ker p_k(\phi(D)) \quad (k \geq 1);$$
$$g_l = \ker \sigma(D), \quad g_{l+k} = \ker \sigma_k(D) \quad (k \geq 1).$$

These are families of vector spaces over $X$.

A linear differential operator $D$ of order $l$ is said to be involutive if $R_l$ and $R_{l+1}$ are vector bundles, if the mapping $\pi_l^{l+1} : R_{l+1} \to R_l$ is surjective, and if $g_l \subset \bigwedge^l T^* \otimes E^0$ is involutive (cf. Goldschmidt [9], Spencer [27]). If $D$ is involutive, $g_{l+k}$ is a vector bundle for each $k \geq 0$. In this paper we need to introduce the following notion.

DEFINITION. A linear differential operator $D$ of order $l$ is quasi-involutive if the following conditions (i)–(iii) are satisfied:

(i) $R_l$ is a vector bundle over $X$;
(ii) $g_{l+1}$ is a vector bundle over $X$;
(iii) $g_l \subset \bigwedge^l T^* \otimes E^0$ is involutive.

For a cotangent vector $\xi \in T^*_x$, let $\sigma_\xi(D)$ denote the linear mapping from the fiber $E^0_x$ to $E^1_x$ defined by $\sigma_\xi(D)(e) = \sigma(D)(\xi^l \otimes e/l!)$. A cotangent vector $\xi$ is said to be non-characteristic with respect to $D$ if $\sigma_\xi(D)$ is injective, and characteristic if $\sigma_\xi(D)$ is not injective. Observe that a cotangent vector $\xi$ is characteristic if and only if it is characteristic for the symbol $g_l$ (cf. section 1). A linear differential operator $D$ is said to be elliptic if $D$ admits no non-zero (real) characteristic cotangent vector.

Let $D^0$ be a linear differential operator from $E^0$ to $E^1$. Goldschmidt [9] showed that, if $D^0$ is involutive, one can construct a linear differential operator $D^1 : E^1 \to E^2$ such that $D^0$ and $D^1$ form a differential complex which is formally exact. In the case when $D^0$ is quasi-involutive, we cannot apply this theory. In the following lemma, we say that a differential operator $D : E \to E'$ of order $l$ is of order $\leq k (k < l)$ if there exists a differential operator $D' : E \to E'$ of order $k$ such that $Du = D'u$ for all sections $u$ of $E$.

LEMMA 3.1. Let $D^0$ be a linear differential operator of order $l$ from $E^0$ to $E^1$, and let the base manifold $X$ be paracompact. Assume that $D^0$ is quasi-involutive, and that $\sigma(D^0) : T^* \otimes E^0 \to E^1$ is surjective. Set $E^2 = \coker \sigma_1(D^0)$, where $\sigma_1(D^0)$ is the first prolongation of $\sigma(D^0)$.
Then there exists a linear differential operator $D^1$ of order 1 from $E^1$ to $E^2$ such that the following (i)–(iii) are valid:

(i) The differential operator $D^1 \circ D^0 : E^0 \to E^2$ is of order $\leq 1$;

(ii) The symbol morphism $\sigma(D^1) : T^* \otimes E^1 \to E^2$ is the canonical projection, and we have an exact sequence

$$0 \to S^2T^* \otimes E^0 \sigma(D^0) T^* \otimes E^1 \sigma(D^1) E^2 \to 0;$$

(iii) If $\xi \in T^*_x \{0\}$ is non-characteristic with respect to $D^0$, then the sequence

$$0 \to E_x^0 \sigma_1(D^0) E_x^1 \sigma_1(D^1) E_x^2$$

is exact.

Proof. Since $S^2$ is a vector bundle, $\sigma_1(D^0)$ is of constant rank, and hence $E^2$ is a vector bundle. Let $\mu : T^* \otimes E^1 \to E^2$ be the canonical projection. Construct a vector bundle morphism $s : E^1 \to J_1(E^1)$ such that $\pi_0^1 \circ s = id$. We define a vector bundle morphism $\varphi^1 : J_1(E^1) \to E^2$ by

$$\varphi^1(Q) = \mu \circ (1 - s \circ \pi_0^1)(Q), \quad Q \in J_1(E^1).$$

Let $D^1 : E^1 \to E^2$ be the differential operator of order 1 defined by $\varphi(D^1) = \varphi^1$. Clearly $\sigma(D^1) = \mu$, and hence (ii) holds true.

Let us show (i). Set $\tilde{D} = D^1 \circ D^0$. $\tilde{D}$ is a linear differential operator with $\varphi(\tilde{D}) = \varphi(D^1) \circ p_1(\varphi(D^0))$. The symbol morphism $\sigma(\tilde{D}) : S^2T^* \otimes E^0 \to E^2$ is the zero operator. In fact, $\sigma(\tilde{D}) = \sigma(D^1) \circ \sigma_1(D^0)$, and the latter is the zero operator by (ii). From this fact we readily see that there is a vector bundle morphism $\tilde{\varphi} : J_1(E^0) \to E^2$ with $\varphi(D) = \tilde{\varphi} \circ \pi_0^1$. This implies that $\tilde{D}$ is of order $\leq 1$.

The proof of (iii) in the general case is not easy (cf. Goldschmidt [7], Spencer [27]). When dim $X = 2$, (iii) follows immediately from Lemma 7.1 proved later. We only use this particular case in this paper. Q.E.D.

4. Nonlinear differential operators

Let $X$ be a manifold. Let $E, E^1$ be fibered manifolds over $X$ with projections $\pi : E^0 \to X$, $\pi' : E^1 \to X$, respectively. If $f : E^0 \to E^1$ is a fibered morphism, we shall use the notations:

$$\ker f = \{ e \in E^0 ; f(e) = v(\pi(e)) \},$$

where $v$ is a given section of $E^1$;

$$\text{im } f = \{ f(e) \in E^1; e \in E^0 \}.$$

The sheaf of germs of smooth sections of $E^1$ will be denoted by $\mathcal{E}^1$.

A sheaf morphism $\Phi : \mathcal{E} \to \mathcal{E}^1$ is called a nonlinear differential operator of order 1 from $\mathcal{E}^0$ to $\mathcal{E}^1$ if there exists a fibered morphism $\varphi : J_1(\mathcal{E}^0) \to \mathcal{E}^1$ such that $\Phi$ is the composition of $j_1 : \mathcal{E}^0 \to J_1(\mathcal{E}^0)$ and $\varphi : J_1(\mathcal{E}^0) \to \mathcal{E}^1$. We shall call $\varphi$ the fibered
morphism associated with \( \Phi \), and describe the circumstances by \( \Phi = \varphi \circ j_l \). The \( k \)-th prolongation \( p_k(\varphi) \) of \( \varphi \) is defined to be the fibered morphism from \( J_{l+k}(E^0) \) to \( J_k(E^1) \) over \( X \) which sends \( j_{l+k}(u)(x) \) to \( j_k(\varphi \circ j_l(\cdot))(x) \), \( u \) being a section of \( E^0 \) \((k \geq 0)\).

Let \( \sigma(\varphi) : S^lT^* \otimes J_l(E^0) V(E^0) \rightarrow V(E^1) \) be the vector bundle morphism over \( \varphi : J_l(E^0) \rightarrow E^1 \) defined by \( \sigma(\varphi) = \varphi_* \circ \epsilon_l \), where \( \varphi_* : V(J_l(E^0)) \rightarrow V(E^1) \) is the differential of \( \varphi \) and \( \epsilon_l \) is the monomorphism in (2.1). We shall call \( \sigma(\varphi) \) the symbol morphism of \( \Phi \), and denote it also by \( \sigma(\Phi) \). The \( k \)-th prolongation of \( \sigma(\varphi) \) will be denoted by \( \sigma_k(\varphi) \) or \( \sigma_k(\Phi) \), where \( k \geq 1 \) (cf. section 1). We set

\[
G_l = \ker \sigma(\varphi), \quad G_{l+k} = \ker \sigma_k(\varphi) \quad (k \geq 1).
\]

These are families of vector spaces over \( J_l(E^0) \). We call \( G_l \) the symbol of \( \Phi \).

**DEFINITION.** A nonlinear differential operator \( \Phi = \varphi \circ j_l \) is quasi-involutive if the following conditions (i)–(iii) are satisfied:

(i) \( \varphi : J_l(E^0) \rightarrow E^1 \) is of constant rank;
(ii) \( G_{l+1} \) is a vector bundle over \( J_l(E^0) \);
(iii) The symbol \( G_l \subset S^lT^* \otimes J_l(E^0) V(E^0) \) is involutive.

We shall also need to introduce one more related notion.

**DEFINITION.** Let \( v \) be a smooth section of \( E^1 \). \( \Phi = \varphi \circ j_l \) is involutive over \( v \) if \( \Phi \) is quasi-involutive and if the following condition (iv) is satisfied:

(iv) \( v(x) \in \ker \varphi \) \((x \in X)\), and the mapping \( \pi^{l+1}_l : \ker_{j_l(x)} p_l(\varphi) \rightarrow \ker_v \varphi \) is surjective.

**REMARK.** (1) These notions can be localized to an open set in \( J_l(E^0) \).

(2) It is natural to introduce the notion of involutive nonlinear differential operators as was done in Yang [31]: It is defined in such a way that if \( \Phi \) is involutive, then \( \Phi \) is involutive over any section \( v \) of \( E^1 \) with \( v(x) \in \ker \varphi \) \((x \in X)\). In our investigation, however, it is the notion introduced above that is useful.

Let \( P \in J_l(E^0) \) and \( x = \pi^{l-1}_l(P) \). For each cotangent vector \( \xi \in T^*_x \), let \( \sigma_\xi(\varphi; P) \) denote the linear mapping from \( V_{\pi^{l-1}_l(P)}(E^0) \) to \( V_{\pi^{l-1}_l(P)}(E^1) \) defined by

\[
\sigma_\xi(\varphi; P)(w) = \sigma(\varphi)(\xi^l \otimes w/l!).
\]

It is said that a non-zero cotangent vector \( \xi \) is non-characteristic or characteristic for \( \Phi = \varphi \circ j_l \) at \( P \) according as the mapping \( \sigma_\xi(\varphi; P) \) is injective or not injective. We note that \( \sigma_\xi(\varphi; P) \) is injective if and only if \( G_{l+1} \cap (\xi^l \otimes V_{\pi^{l+1}_l(P)}(E^0)) = \{0\} \), equivalently, \( \xi \notin \mathcal{E}(G_{l, P}) \) (cf. section 1). A nonlinear operator \( \Phi = \varphi \circ j_l \) is said to be elliptic if \( \Phi \) admits no non-zero characteristic (real) cotangent vector.

**LEMMA 4.1.** If \( \Phi = \varphi \circ j_l \) is involutive over \( v \), then, for each integer \( k \geq 1 \), the mapping

\[
\pi^{l+k}_{l+k-1} : \ker_{j_k(v)} p_k(\varphi) \rightarrow \ker_{j_{l-k}(v)} p_{l-k}(\varphi)
\]

is surjective, and in particular \( j_k(v)(x) \in \ker_{j_k(v)} p_k(\varphi) \) \((x \in X)\).
Proof. Since $\phi$ is a fibered morphism of constant rank, by the implicit function theorem, $R_l = \ker_{\phi}$ is a fibered submanifold of $J_1(\mathcal{E}^0)$, we readily see that $R_l$ is an involutive system and that its $k$-th prolongation is equal to $\ker_{\Phi} p_k^*(\phi)$ for each $k \geq 0$ (see Goldschmidt [8], section 7). Hence, by Theorem 2.1, we have the desired assertion. Q.E.D.

Proposition 4.2. Assume that $\Phi = \phi \circ j_l$ is involutive over a smooth section $\nu$, and that $\Phi$ is elliptic. Let $N = \max\{|m_0| - 1, l + 1\} + (n/2)$, where $n = \dim X$ and $n + m_0 = \dim \mathcal{E}^0$. Then, if $u$ is a section of $\mathcal{E}^0$ of Sobolev class $H^s$ with $s > N$, and if $\Phi(u) = v$, then $u$ is smooth.

Proof. Set $R_l = \ker_{\phi} \subset J_l(\mathcal{E}^0)$. Then $R_l$ is an involutive system of order $l$ (cf. Lemma 4.1 and its proof). Moreover the symbol of $R_l$ is the family $G_l = \ker_{\sigma}(\phi)$ restricted to $R_l$. Hence $R_l$ is elliptic. The assumption on $u$ means that $u$ is a solution of $R_l$ of Sobolev class $H^s$. Therefore we can apply Theorem 2.2 to conclude that $u$ is smooth.

Lemma 4.3. Assume that $\Phi = \phi \circ j_l$ is involutive over $\nu_0$, where $\nu_0$ is a smooth section of $\mathcal{E}^1$. Let $P_0 \in J_1(\mathcal{E}^0)$ be a point with $\phi(P_0) = \nu_0(x_0)$. Then there exists a smooth section $u_0$ of $\mathcal{E}^0$ defined over a neighborhood of $x_0$ such that $j_l(u_0)(x_0) = P_0$ and $j_k(\Phi(u_0))(x_0) = j_k(\nu_0)(x_0)$ for all integer $k > 0$.

Proof. By virtue of Lemma 4.1, one can choose a sequence $\{P_k\}_{k=0,1,2,\ldots}$ of points $P_k \in J_{l+k}(\mathcal{E})$ such that $p_k^*(\phi)(P_k) = j_l(\nu_0)(x_0)$ and $\sigma_{l+k}^1(P_{k+1}) = P_k$ for all integer $k \geq 0$. Let $(x^i, y^\alpha; 1 \leq i \leq n, 1 \leq \alpha \leq m_0)$ be coordinates in a fibered chart of $\mathcal{E}^0$ with $x^i(x_0) = 0 (1 \leq i \leq n)$, and let $(x^i, y^\alpha; 1 \leq i \leq n, 1 \leq \alpha \leq m_0, 0 \leq |\nu| \leq k)$ be the coordinate system of $J_k(\mathcal{E}^0)$ (cf. section 2). From the way we choose $\{P_k\}_{k=0,1,2,\ldots}$, it follows that there exists a system $(\epsilon_\alpha^i; 1 \leq \alpha \leq m_0, 0 \leq |\nu| < \infty)$ of scalars such that each point $P_k$ has the coordinates $x_i(P_k) = 0 (1 \leq i \leq n)$, $y^\alpha(P_k) = \epsilon_\alpha^i (1 \leq \alpha \leq m_0, 0 \leq |\nu| \leq k + l)$. Let $\chi(x)$ be a smooth function of $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ such that $\chi(x) = 1$ if $|x| \leq 1/2$, $\chi(x) = 0$ if $|x| \geq 1$, and let $\{\delta_k\}_{k=0,1,2,\ldots}$ be a sequence of positive numbers such that $(\sum_{|\alpha| = 0}^{m_0} \sum_{|\nu| = k} \epsilon_\alpha^i \delta_k < 1/2 (k \geq 0))$. We set

$$u^\alpha(x) = \sum_{0 \leq |\nu| < \infty} \epsilon_\alpha^i \frac{x^i}{\delta_k} \chi(x)$$

where $x^\nu = (x^1)^{v_1} \cdots (x^n)^{v_n}$ when $\nu = (v_1, \ldots, v_n)$. These power series together with their formal derivatives converge uniformly on $\mathbb{R}^n$, and hence their sums $u^\alpha(x)$ are smooth functions. Let $u_0$ be the section of $\mathcal{E}^0$ which has the description $u_0 = (u^1(x), \ldots, u^{m_0}(x))$. Observing that $j_{l+k}(u_0)(x_0) = P_k (k \geq 0)$, we see that $u_0$ is a desired section. Q.E.D.

5. Linearizations and a supplementary fact

In local problems concerning a nonlinear differential operator between fibered manifolds, using fibered charts, one may assume that fibered manifolds admit vector bundle structures. Besides the notions and notations already stated, we shall also use the following
ones. If \( u \) is a section of a real vector bundle \( E \) over a manifold \( X \), by \( V(E)|_u \) we shall mean the vertical bundle \( V(E) \) restricted to the image of \( u \) which is a submanifold of \( E \).

One has a canonical vector bundle isomorphism \( \iota : E \rightarrow V(E)|_u \) defined as follows: Let \( e \in E_x \). The mapping \( \gamma(t) = u(x) + te\ (t \in (-\infty, \infty)) \) defines a curve in \( E_x \) passing through the point \( u(x) \). \( \iota(e) \) is defined to be the tangent vector of \( \gamma(t) \) at \( u(x) \).

Let \( E^0, E^1 \) be real vector bundles over a manifold \( X \). Let \( \Phi \) be a nonlinear differential operator of order \( l \) from \( E^0 \) to \( E^1 \), and \( \psi : J_l(E^0) \rightarrow E^1 \) be the fibered morphism associated with it; \( \Phi = \psi \circ j_l \). We shall denote the symbol of \( \Phi \) by \( G_l \). Let \( w \) be a section of \( J_l(E^0) \). We have the canonical isomorphisms

\[
\iota^{(0)} : J_l(E^0) \rightarrow V(J_l(E^0))|_w, \quad \iota^{(1)} : E^1 \rightarrow V(E^1)|_{\psi \circ w}.
\]

We define the vector bundle morphism \( D_w \psi : J_l(E^0) \rightarrow E^1 \) by \( D_w \psi = (\iota^{(1)})^{-1} \circ \psi \circ \iota^{(0)} \), \( \psi_w \) being the differential of \( \psi \). Let us extend the notion of linearization of single nonlinear differential operators (cf. Taylor [30], Rauch [25]) to general nonlinear operators.

**Definition.** The linearization \( D_u \Phi \) of \( \Phi \) at a section \( u \) of \( E^0 \) is the linear differential operator of order \( l \) from \( E^0 \) to \( E^1 \) of which associated vector bundle morphism is \( D_j_l(u) \psi \).

**Lemma 5.1.**
(i) If \( \Phi \) is quasi-involutive, then the linearization \( D_u \Phi \) is quasi-involutive.
(ii) If \( \Phi \) is elliptic, so is \( D_u \Phi \).
(iii) If the symbol morphism \( \sigma(\Phi) \) is surjective, so is \( \sigma(D_u \Phi) \).

**Proof.** Observe that the symbol morphism \( \sigma(D_u \Phi) \) is canonically identified with the morphism \( \sigma(\Phi) \) restricted to the submanifold \( j_l(u) \subset J_l(E^0) \). Hence the symbol \( g_l \) of \( D_u \Phi \) is canonically identified with the symbol \( G_l \) of \( \Phi \) restricted to \( j_l(u) \). Since the first prolongation \( \sigma_1(D_u \Phi) \) is canonically identified with the first prolongation \( \sigma_1(\Phi) \) restricted to \( j_l(u) \), the first prolongation \( g_{l+1} \) of \( g_l \) is canonically identified with the first prolongation \( G_{l+1} \) of \( G_l \) restricted to \( j_l(u) \). The bundle \( R_l = \ker D_j_l(u) \psi \) is a vector bundle if and only if \( D_j_l(u) \psi \) is of constant rank. Bearing in mind these facts, we see at once that (i),(ii),(iii) are valid.

Q.E.D.

In what follows we shall assume that \( \Phi \) is of the first order, that is, \( l = 1 \). To clarify one feature of involutiveness, we make some preliminary discussions. Identifying the vertical bundles \( V(E^1) \) with the bundles induced from \( E^1 \) by the respective projections, the first prolongation \( \sigma_1(\Phi) \) of \( \sigma(\Phi) \) is described as

\[
\sigma_1(\Phi) : S^2 T^* \otimes_j l(E^0) E^0 \rightarrow T^* \otimes E^1 E^1.
\]

Assume that \( \Phi \) is quasi-involutive and \( \sigma(\Phi) \) is surjective. Then \( \sigma_1(\Phi) \) is of constant rank, since \( G_2 = \ker \sigma_1(\Phi) \) is a vector bundle. Setting \( \hat{E}^2 \) to be the cokernel of the morphism \( \psi^{-1}(\sigma_1(\Phi)) \) induced from \( \sigma_1(\Phi) \) by \( \psi \), we have an exact sequence of vector bundles over \( J_1(E^0) \)

\[
S^2 T^* \otimes E^0 \xrightarrow{\psi^{-1}(\sigma_1(\Phi))} \psi^{-1}(T^* \otimes E^1) \xrightarrow{\hat{\iota}} \hat{E}^2 \rightarrow 0,
\]

(5.1)
whee $\tilde{\tau}$ is the canonical projection.

Given a point $P_0 \in J_1(E^0)$, we can choose a neighborhood $U_0$ of $P_0$ in $J_1(E^0)$ such that the vector bundle $\hat{E}^2$ restricted to $U_0$ is isomorphic to the bundle $(\pi_{1})^{-1}E^2$ induced from a product vector bundle $E^2$ over $U_0 = \pi_{1}(U_0) \subset X$. Thus the vector bundle morphism $\hat{\tau}$ in (5.1) is locally described as

$$\varphi^{-1}(T^{*} \otimes E^{1})|_{U_0} \xrightarrow{\hat{\tau}} (\pi_{1})^{-1}E^2 = \hat{E}^2|_{U_0}. \tag{5.2}$$

In the following lemma, besides the above circumstances, we also assume that $E^0$ and $E^1$ are product vector bundles over $U_0$. Thus we may use usual matrix representations of linear differential operators.

**Lemma 5.2.** Let $\Phi$ be a differential operator of the first order, and let $P_0 \in J_1(E^0)$, $x_0 = \pi_{1}(P_0)$. Let $u_0$ and $v_0$ be smooth sections of $E^0$ and $E^1$, respectively, such that $j_1(u_0)(x_0) = P_0$ and that $\Phi(u_0)(x_0) = v_0(x_0)$. Assume that $\Phi = \psi \circ j_1$ is involutive over $v_0$, and that $\sigma(\Phi)$ is surjective. Let $D^1$ be a linear differential operator of order $1$ satisfying the conditions (i)–(iii) of Lemma 3.1 for the linearization $D^0 = D_{u_0}\Phi : E^0 \to E^1$. Then we can choose a neighborhood $\mathcal{U}$ of $P_0$ in $J_1(E^0)$ with $\mathcal{U}$ being contained in the neighborhood $U_0$ chosen above such that the following (I), (II) hold for any sufficiently small neighborhood $U$ of $x_0$:

(I) The operator $D^1$ restricted to $U$ is a linear differential operator from $E^1|_{U}$ to $E^2|_{U}$, and its symbol morphism $\sigma(D^1)$ coincides with the morphism $(j_1(u_0)|_{U})^{-1}(\tilde{\tau})$ induced from the morphism $\tilde{\tau}$ in (5.2) by the mapping $j_1(u_0)|_{U} : U \to J_1(E^0)$.

(II) For any section $u$ of $E^0|_{U}$ differentiable of class $C^2$ and satisfying

$$j_1(u)(x) \in \mathcal{U} \quad \text{for all } x \in U, \tag{5.3}$$

one can construct a linear differential operator $\hat{D}_u$ of order $1$ from $E^1|_{U}$ to $E^2|_{U}$ possessing the following properties (i)–(iii):

(i) The section $v(x) = \Phi(u)(x) - v_0(x)$ satisfies $\hat{D}_u(v) = 0$ in $U$;

(ii) The symbol morphism $\sigma(\hat{D}_u)$ coincides with the morphism $j_1(u)^{-1}(\tilde{\tau})$ induced by the mapping $j_1(u)$ from $\tilde{\tau}$ in (5.2);

(iii) The coefficients of $\hat{D}_u$ are continuous, and are bounded on $U$ uniformly with respect to all $C^2$ sections $u$ and neighborhoods $U$ satisfying (5.3).

**Proof.** We may assume that $U_0$ is the $n$-dimensional Euclidean space $\mathbb{R}^n$ with standard coordinates $(x^1, \ldots, x^n)$, and that $E^1|_{U_0} = U_0 \times \mathbb{R}^m_i. \ (0 \leq i \leq 2)$. Let $y = (y^\alpha; \ 1 \leq \alpha \leq m_0)$ be the standard coordinates of the fiber $\mathbb{R}^m_0$, and $z = (z^\beta; \ 1 \leq \beta \leq m_1)$ be those of $\mathbb{R}^{m_1}$. Associated with the coordinates $(x, y)$ of $E^0$, we have a coordinate system

$$(x, y, p) \quad \text{where} \quad p = (p^\alpha_0; \ 1 \leq \alpha \leq m_0, \ |v| = 1)$$

of $J_1(E^0)$ and a coordinate system

$$(x, y, p, p') \quad \text{where} \quad p' = (p^\alpha_i; \ 1 \leq \alpha \leq m_0, \ |v| = 2)$$

of $J_2(E^0)$ (cf. section 2). We also have a coordinate system

$$(x, z, q) \quad \text{where} \quad q = (q^\beta_0; \ 1 \leq \beta \leq m_0, \ |\mu| = 1)$$
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Let the fibered morphism \( \phi : J_1(E) \rightarrow E \) be locally described by

\[
z^\beta = \phi^\beta(x, y, p) \quad (\beta = 1, 2, \ldots, m_1),
\]

where \( \phi^\beta \)'s are smooth functions. Then the vector bundle morphism \( D_{j_1(u_0)}\phi : J_1(E) \rightarrow E \) associated with the linearization \( Du_0\Phi \) is described by

\[
z^\beta = \sum_{\alpha=1}^{m_0} \sum_{0 \leq |\nu| \leq 1} \frac{\partial \phi^\beta}{\partial p^\alpha}(j_1(u_0)(x))p^\alpha \quad (\beta = 1, 2, \ldots, m_1),
\]

and the prolongation \( p_1(\phi) \) is described by the system consisting of (5.4) and

\[
q^\beta_{i_1} = \frac{d\phi^\beta}{dx^i}(x, y, p, p') \quad (i = 1, \ldots, n, \beta = 1, \ldots, m_1),
\]

where

\[
\frac{d\phi^\beta}{dx^i} = \frac{\partial \phi^\beta}{\partial x^i}(x, y, p) + \sum_{\alpha=1}^{m_0} \sum_{0 \leq |\nu| \leq 1} \frac{\partial \phi^\beta}{\partial p^\alpha}(x, y, p)p^\alpha_0 + \frac{1}{i_i}(i = 1, \ldots, n, \beta = 1, \ldots, m_1).
\]

Using the representations (in which each fiber \( V(x, y, p)(E^i) \) is identified with \( E^i \))

\[
\zeta = \sum_{\alpha=1}^{m_0} \sum_{|\nu|=2} \zeta^\alpha \nu (dx^\nu) \otimes \frac{\partial}{\partial y^\alpha} \in S^2T^*_x \otimes E^0_x,
\]

\[
\eta = \sum_{\beta=1}^{m_1} \sum_{i=1}^{n} \eta^\beta_i dx^i \otimes \frac{\partial}{\partial z^\beta} \in T^*_x \otimes E^1_x,
\]

the vector bundle morphism \( \phi^{-1}(\sigma_1(\Phi)) \) in (5.1) is described by

\[
\eta^\beta_i = \sum_{\alpha=1}^{m_0} \sum_{|\nu|=1} \frac{\partial \phi^\beta}{\partial p^\alpha}(x, y, p)\zeta^\alpha_0 \nu_{i+1} \quad (i = 1, \ldots, n, \beta = 1, \ldots, m_1).
\]

Set \( T^i_{\beta}(x, y, p) = \tilde{\tau}(dx^i \otimes (\partial/\partial z^\beta)); T^i_{\beta}'s \) are smooth sections of \( E^*_{x} |_{U_0} \). Then, the morphism \( \tilde{\tau} \) in (5.2) is described by

\[
\tilde{\tau}(\eta) = \sum_{i=1}^{n} \sum_{\beta=1}^{m_2} \eta^\beta_i T^i_{\beta}(x, y, p).
\]

Comparing the description (5.5) of \( D^\beta = D_{u_0}\Phi \) and the description (5.8) of \( \phi^{-1}(\sigma_1(\Phi)) \), we know that the morphism \( \sigma_1(D^\beta) \) coincides with the morphism induced from \( \phi^{-1}(\sigma_1(\Phi)) \) in (5.1) by the mapping \( j_1(u_0) \). Hence, bearing in mind (5.1),(5.2) together with Lemma 3.1, we see that (I) is valid.

To prove (II), let us consider the \( \mathbb{R}^{m_2} \)-valued function on \( J_2(E^0)|_{U_0} \):

\[
\tilde{T}(x, y, p, p') = \sum_{\beta=1}^{m_1} \sum_{i=1}^{n} \left( \frac{d\phi^\beta}{dx^i}(x, y, p, p') - \frac{\partial \phi^\beta}{\partial x^i}(x) \right) T^i_{\beta}(x, y, p),
\]
where \( v_0(x) = (v_0^1(x), \ldots, v_0^{m_1}) \) is the representation of \( v_0 \). For brevity we denote \( R_1 = \ker v \), \( R_2 = \ker f_0(v_0) p_1(\varphi) \). From (5.6), we know that \( \hat{T}(x, y, p, p') \) vanishes everywhere on \( R_2 \). The exactness of the sequence (5.1) together with (5.2) implies that all the coefficients of the variables \( p' \) of the function on the right side of (5.10) vanish everywhere (cf. (5.7) and (5.9)). Hence \( \hat{T} \) is the pull-back of a function \( T \) defined on an open set in \( J_1(E^0) \). Since \( \pi_1^1 : R_2 \rightarrow R_1 \) is surjective by Lemma 4.1, \( T \) vanishes everywhere on \( R_1 \).

Now \( \varphi \) is a surjective submersion, for it is a fibered morphism of constant rank and \( \sigma(\varphi) = \sigma(\Phi) \) is surjective. Hence the equation \( \varphi^\beta(x, y, p) - v^\beta_0(x) = 0 \) (1 \( \leq \beta \leq m_1 \)) gives a regular local equation of \( R_1 \) around the point \( P_0 = j_1(v_0(x)) \). Therefore the \( \mathbb{R}^{m_2} \)-valued function \( T \) is expressed on an open neighborhood \( U \subset U_0 \) of \( P_0 \) in \( J_1(E^0) \) as follows:

\[
T(x, y, p) = - \sum_{\beta=1}^{m_1} (\varphi^\beta(x, y, p) - v^\beta_0(x)) T^0_{\beta}(x, y, p),
\]

where \( T^0_{\beta} \) ’s are smooth \( \mathbb{R}^{m_2} \)-valued functions on \( U \). Let \( U \) be a neighborhood of \( P_0 \) in \( J_1(E^0) \) such that its closure \( \overline{U} \) is a compact subset of \( U \). Assume that \( U \) is a neighborhood of \( x_0 \) with \( U \subset \pi_1^1(U) \). Let \( u \) be a \( C^2 \) section of \( E^0 \) over \( U \) satisfying (5.3). We define a vector bundle morphism \( \varphi_u : J_1(E^0)|_U \rightarrow E^2|_U \) by

\[
\varphi_u(x, z, q) = \sum_{\beta=1}^{m_1} \sum_{i=0}^n q^\beta_i j_i^0 T^0_{\beta}(j_1(u)(x)) + \sum_{\beta=1}^{m_1} z^\beta T^0_{\beta}(j_1(u)(x)).
\]

Let \( \hat{D}_u \) be the linear differential operator defined by \( \hat{D}_u = \varphi_u \circ j_1 \). From the construction and (5.10), we see at once that (i) and (ii) hold true. The coefficients of \( \hat{D}_u \) are continuous on \( U \), and their bounds are estimated by the bounds of the components of \( T^0_{\beta} \) and \( T^0_{\beta} \) on the compact set \( \overline{U} \) which do not depend on \( u \) and \( U \). Thus (iii) holds true.

Q.E.D.

6. Nonlinear differential operators in Sobolev spaces

We fix some notations concerning function spaces. Let \( E \) be a real vector bundle over a manifold \( X \). Let \( U \) be a coordinate neighborhood of \( X \) with coordinates \( (x^1, \ldots, x^n) \) over which \( E \) is trivial. In terms of the coordinates \( (x^1, \ldots, x^n) \), we shall regard \( U \) as an open set of the Euclidean space \( \mathbb{R}^n \), and fixing a trivialization, we shall identify the bundle \( E \) restricted to \( U \) with the product vector bundle \( U \times \mathbb{R}^m \). Then a (not necessarily smooth) section \( u \) of \( E \) over \( U \) may be represented by an ordered set \( (u^1(x), \ldots, u^m(x)) \) of \( m \) real-valued functions on \( U \). We shall also use the abbreviated notation \( \partial^v_x f = (\partial/\partial x^1)^{v_1} \cdots (\partial/\partial x^n)^{v_n} f \), where \( v = (v_1, \ldots, v_m) \) is a multi-index. When \( |v| = 0 \), \( \partial^v_x f \) means \( f \) by convention.

Let \( k \) be a non-negative integer. Let \( H^k(U) \) denote the Sobolev space consisting of locally integrable real-valued functions \( f \) on \( U \) such that the distribution derivatives \( \partial^v_x f \) (0 \( \leq |v| \leq k \)) are functions which are measurable and square-integrable on \( U \), and
∥f∥k denote the norm

\[ \|f\|_k = \|f\|_{k,U} = \left( \sum_{0 \leq |\nu| \leq k} \int_U |\partial^\nu_x f(x)|^2 \, dx \right)^{1/2}. \]

We denote by \( C^k(U) \) the Banach space of all \( k \) times differentiable functions \( f \) such that \( \partial^\nu_x f (0 \leq |\nu| \leq k) \) are bounded and uniformly continuous on \( U \) equipped with the following norm:

\[ \|f\|_k = \|f\|_{k,U} = \left( \sum_{0 \leq |\nu| \leq k} \sup_{x \in U} |\partial^\nu_x f(x)| \right)^{1/2}. \]

We shall denote \( H^0(U, E) \) also by \( L^2(U, E) \), and \( \|u\|_0 \) by \( \|u\| \). The inner product in \( L^2(U, E) \) will be denoted by \( (u, v) \). The spaces \( H^k(U, E) \) \((k = 0, 1, 2, \ldots)\) are real Hilbert spaces.

Let \( C^k(U, E) \) be the space of sections \( u = (u^1(x), \ldots, u^m(x)) \) of \( E \) over \( U \) with \( u^\alpha(x) \in C^k(U) \) \((1 \leq \alpha \leq m)\) with the following norm:

\[ \|u\|_k = \|u\|_{k,U} = \left( \sum_{\alpha=1}^m \|u^\alpha\|^2_{k,U} \right)^{1/2}. \]

Let us recall some fundamental facts.

**Lemma 6.1.** Assume that \( U \) is a bounded open set in \( \mathbb{R}^n \) with smooth boundary. Then the following (I)–(III) are valid:

(I) Let \( F = F(x, y) \) be a smooth function on \( \overline{U} \times \mathbb{R}^m \supset E|_U \), and let \( k > n/2 \). If \( u \in H^k(U, E) \), then the composite function \( F(x, u(x)) \) belongs to \( H^k(U) \). Moreover if \( \|u\|_k \) is bounded by a given constant \( M > 0 \), then \( \|F(x, u(x))\|_k \) is estimated by a constant \( C > 0 \) depending only on \( U, k, M \) and \( |F(x,y)|_{k,U \times K_0} \), where \( K_0 \subset \mathbb{R}^m \) is a compact set chosen depending only on \( M \).

(II) (Sobolev embedding theorem) Let \( k, l \) be non-negative integers with \( k > n/2+1 \). Then \( u \in H^k(U, E) \) implies \( u \in C^l(\overline{U}, E) \). In addition, there exists a constant \( C > 0 \) such that

\[ \|u\|_l \leq C \|u\|_k. \quad u \in H^k(U, E). \]

(III) Let \( k \) be an integer with \( k > n/2 \).

(i) Let \( v^1, \ldots, v^p \) be multi-indices with \( |v^1| + \cdots + |v^p| = l \leq k \). Then there exists a constant \( C > 0 \) such that

\[ \|(\partial_{v^1} f_1)(\partial_{v^2} f_2)\cdots(\partial_{v^p} f_p)\| \leq C \sum_{j=1}^p \prod_{i \neq j} |f_i|_0 \|f_j\|_l \]

holds for any \( f_1, \ldots, f_p \in H^k(U) \).
(ii) There exists a constant $C > 0$ such that
\[ \|fg\|_k \leq C \|f\|_k \|g\|_k, \quad f, g \in H^k(U). \]

**Proof.** See the following references. (I): Bony [2], (II): Rauch [25] or Gilbarg and Trudinger [6], (III): Taylor [30], Chapter 13; For the proof of (i), use the continuous extension operator $E: H^k(U) \to H^0_0(U_1)$ where $U_1$ is a bounded open set containing $U$ such that $E: C^0(U) \to C^0(U_1)$ is also continuous, and apply Lemma 3.10 in Chapter 13. We note that (ii) and the second assertion of (I) can be proved by using (II) and (III)–(i), since the chain rule and Leibniz formula are valid even for functions of Sobolev spaces (cf. Gilbarg and Trudinger [6], Chapter 6). Q.E.D.

Let us now consider differential operators. Let $E_0, E_1$ be real vector bundles over a manifold $X$ of dimension $n$. Let $\Phi$ be a nonlinear differential operator of order $l$ from $E_0$ to $E_1$, and $\phi: J_l(E_0) \to E_1$ be the fibered morphism associated with $\Phi$. Let $U_0$ be a coordinate neighborhood of $X$ with coordinates $(x^1, \ldots, x^n)$ over which $E_0, E_1$ are trivial. Let
\[
(x, y) = (x^i, y^\alpha; 1 \leq i \leq n, 1 \leq \alpha \leq m_0),
\]
\[
(x, z) = (x^i, z^\beta; 1 \leq i \leq n, 1 \leq \beta \leq m_1)
\]
be coordinate systems in the vector bundle charts of $E_0, E_1$ over $U_0$, respectively. We have a coordinate system
\[
(x, y, p) = (x^i, y^\alpha, p^\alpha; 1 \leq i \leq n, 1 \leq \alpha \leq m_0, 1 \leq |\nu| \leq l).
\]
of $J_l(E_0)$ defined as in (2.2). In terms of these coordinates, let the morphism $\phi$ be described by
\[
z^\beta = \phi^\beta(x, y, p) \quad (\beta = 1, 2, \ldots, m_1),
\]
where $\phi$'s are smooth functions on $J_l(E_0)|_{U_0}$. Then the morphism $D_\beta(\phi): J_l(E_0)|_{U_0} \to E^1|_{U_0}$ which defines the linearization $D_\beta(\Phi)$ of $\Phi$ at a section $u$ is described by
\[
z^\beta = \sum_{\alpha=1}^{m_0} \sum_{\nu=1}^{|\nu|} \frac{\partial \phi^\beta}{\partial p^\nu}(jt(u)(x)) p^\alpha \quad (\beta = 1, 2, \ldots, m_1).
\]

Let $U$ be an open set with smooth boundary such that its closure is a compact subset of $U_0$, and $k$ be an integer greater than $n/2$. By virtue of (I) in Lemma 6.1, one can define a nonlinear operator
\[ \Phi: H^{l+k}(U, E_0) \to H^k(U, E_1), \quad \Phi(u) = (\varphi^1(jt(u)(x)), \ldots, \varphi^{m_1}(jt(u)(x))). \]
As we know from the description (6.2), we may define the linearization $D_u \Phi$ as a differential operator between Sobolev spaces even when $u$ is not smooth.

Let us show that $\Phi$ is differentiable. (For the notions used here, refer to Lang [20], Chapter 1, or Schwartz [26].) We shall denote by $L(B_1, B_2)$ the Banach space of all bounded linear operators from a Banach space $B_1$ to another one $B_2$ with the usual operator norm.
**Proposition 6.2.** Let $U$ be as above, and let $k > n/2 + 1$. The mapping $\Phi : H^{l+k}(U, E^0) \to H^k(U, E^1)$ is differentiable of class $C^1$. More precisely, $\Phi$ is Fréchet differentiable at each $u \in H^{l+k}(U, E^0)$, its Fréchet derivative at $u$ coincides with the linear operator $D_u \Phi : H^{l+k}(U, E^0) \to H^k(U, E^1)$ induced by the linearization $D_u \Phi$, and the mapping from $H^{l+k}(U, E^0)$ to $L(H^{l+k}(U, E^0), H^k(U, E^1))$ sending $u$ to the Fréchet derivative $D_u \Phi$ is continuous.

**Proof.** The operator $\Phi$ is the composition of the operators $j_l : H^{l+k}(U, E^0) \to H^k(U, J_l(E^0))$ and $\varphi : H^k(U, J_l(E^0)) \to H^k(U, E^1)$. Since the operator $j_l$ is a continuous linear operator, it is Fréchet differentiable at each $u$, and its Fréchet derivative is equal to $j_l$ itself. The operator $\varphi$ is also Fréchet differentiable at each $w \in H^k(U, J_l(E^0))$, and its Fréchet derivative at $w$ is equal to the operator induced by the vector bundle morphism $D_\varphi$ (cf. section 5). In fact, bearing in mind the description (6.1) of $\varphi$, we know that this assertion follows from the following Sublemma.

**Sublemma.** Let $a(x, y)$ be a real-valued smooth function on the product bundle $E = U_0 \times \mathbb{R}^N$ over $U_0$. Let $k, U$ be as above, and let $H^k(U, \mathbb{R}^N)$ denote $H^k(U, E)$. 

(i) The operator $A : H^k(U, \mathbb{R}^N) \to H^k(U)$ defined by $Af = a(x, f(x))$ is Fréchet differentiable at each $f_0 \in H^k(U, \mathbb{R}^N)$, and its Fréchet derivative is the operator defined by 

$$
(D_f A)g = \sum_{j=1}^N (\partial y_j a)(x, f_0(x)) g^j(x),
$$

where $g = (g^1, \ldots, g^N) \in H^k(U, \mathbb{R}^N)$.

(ii) The mapping $H^k(U, \mathbb{R}^N) \to L(H^k(U, \mathbb{R}^N), H^k(U))$ sending $f$ to $D_f A$ is continuous.

From what we have stated, it follows that $\Phi$ is differentiable at each $u$, and its Fréchet derivative at $u$ is equal to the composition $D_{\varphi(j_l(u))} \circ j_l$, which is nothing but the operator $D_u \Phi$. It also follows the last assertion. Thus the proof of Proposition 6.2 is complete once we have proved the Sublemma.

In proving it, to make the descriptions simpler, we shall assume that $N = 1$. For each $f, g \in H^k(U, \mathbb{R}^1)$, we set 

$$
\Psi(f; g) = a(x, f(x) + g(x)) - a(x, f(x)) - (\partial_y a)(x, f(x))g(x).
$$

To prove (i), it suffices to show that we can find a real-valued function $\psi$ on $H^k(U)$ with $\lim_{g \to 0} \psi(g) = 0$ such that 

$$
\|\Psi(f_0; g)\|_k \leq \psi(g)\|g\|_k, \quad g \in H^k(U).
$$

We first note the following fact: Let $b(x, y)$ be a smooth function on $U_0 \times \mathbb{R}^1$, and $M_0$ be a positive constant. Then, for any $f, g \in H^k(U)$ with $\|f\|_k, \|g\|_k \leq M_0$, the estimate 

$$
\|b(x, f(x)) - b(x, g(x))\|_k \leq C \|f - g\|_k
$$

is satisfied.
holds with a constant \( C > 0 \) depending only on \( U, k, M_0, |b|_{k+1, K_0} \), where \( K_0 \subset \mathbb{R}^1 \) is a compact set chosen depending on \( M_0 \). In fact, if we define a smooth function \( h(x, y, z) \) on \( U_0 \times \mathbb{R} \times \mathbb{R} \) by

\[
h(x, y, z) = \int_0^1 (\partial_j b)(x, z + t(y - z)) dt,
\]

the function \( b(x, y) - b(x, z) \) can be written as \( h(x, y, z)(y - z) \). Hence we have

\[
b(x, f(x)) - b(x, g(x)) = h(x, f(x), g(x))(f(x) - g(x)), \quad x \in U.
\]

We can apply the chain rule and Leibniz formula to calculate the derivatives of the function \( b(x, f(x)) - b(x, g(x)) \) (cf. the proof of Lemma 6.1). The conditions \( \|f\|_k, \|g\|_k \leq M_0 \) implies that \( |f|_0, |g|_0 \) are estimated by a constant multiple of \( M_0 \) in virtue of the Sobolev embedding theorem. Using the estimates (ii) of (III) and (I) in Lemma 6.1, we can readily prove the required estimate (6.4).

From this fact, we know that \( \Psi(f; g) \) is continuous in the following sense: If \( \lim_{j \to \infty} f_j = f, \lim_{j \to \infty} g_j = g \) in \( H^k(U) \), then \( \lim_{j \to \infty} \Psi(f_j; g_j) = \Psi(f; g) \).

We now assert that for any \( f, g \in H^k(U) \) with \( \|f\|_k \leq \|f_0\|_k + 1, \|g\|_k \leq 1 \), the estimate

\[
\|\Psi(f; g)\|_k \leq C_0 \|g\|_k^2
\]

holds with a constant \( C_0 > 0 \). By the continuity of \( \Psi(f; g) \) and by the density of the set \( C^k(U) \) in \( H^k(U) \), it suffices to prove (6.5) when \( f, g \in C^k(U) \). By the fundamental theorem of calculus, we have \( \Psi(f; g) = I(x)g(x) \) with

\[
I(x) = \int_0^1 [(\partial_\mu a)(x, f(x) + tg(x)) - (\partial_\mu a)(x, f(x))] dt.
\]

By (ii) of (III) in Lemma 6.1, \( \|\Psi(f; g)\|_k \) is estimated by a constant multiple of \( \|I\|_k \|g\|_k \).

Using the Sobolev embedding theorem and (6.4), we see that there exist constants \( C_1, C_2 > 0 \) such that, for any multi-index \( \mu \) and any integer \( j \geq 0 \) with \( |\mu| + j \leq k + 1 \), the estimate

\[
|\partial_\mu^j \partial_\nu^j a(x, f(x) + tg(x))] \leq C_1, \quad (t, x) \in [0, 1] \times U,
\]

holds, and

\[
\sup\{(\partial_\mu^j \partial_\nu^j a)(x, f(x) + t g(x)) - (\partial_\mu^j \partial_\nu^j a)(x, f(x))\} \leq C_2, \quad (t, x) \in [0, 1] \times U
\]

is estimated by \( C_2 \|g\|_k \). Calculating the derivatives of \( I(x) \), and using these estimates, we can deduce that \( \|I\|_k \leq C_3 \|g\|_k \) with a constant \( C_3 > 0 \). Thus we have proved (6.5).

We now define \( \psi(g) \) to be \( C_0 \|g\|_k \) if \( \|g\|_k < 1 \) and to be \( \|\Psi(f_0; g)\|_k/\|g\|_k \) if \( \|g\|_k \geq 1 \). Then it is apparent that (6.3) holds. Thus (i) is proved.

We can readily prove (ii) by using (6.4). Q.E.D.

7. Linear differential operators in two independent variables

Let \( E^0, E^1 \) be real vector bundles over a manifold \( X \). Let \( D^0 \) be a linear differential operator from \( E^0 \) to \( E^1 \). We shall always assume that
(A.1) \( \dim X = 2 \), and \( D^0 \) is of the first order,
(A.2) \( D^0 \) is quasi-involutive (cf. section 3), and
(A.3) the symbol morphism \( \sigma(D^0) : T^\ast \otimes E^0 \to E^1 \) is surjective.

Let \( g_1 \) denote the symbol of \( D^0 \); \( g_1 = \ker \sigma(D^0) \). We shall denote by \( s_1, s_2 \) the Cartan characters of \( g_1 \) (cf. section 1).

We have an exact sequence of vector bundles

\[
0 \to g_1 \to T^\ast \otimes E^0 \xrightarrow{\sigma(D^0)} E^1 \to 0.
\]

Let \( E^2 \) be the cokernel of the prolongation \( \sigma_1(D^0) \) of \( \sigma(D^0) \), and \( \tau : T^\ast \otimes E^1 \to E^2 \) be the natural projection. Then, we have also an exact sequence

\[
0 \to g_2 \to S^2 T^\ast \otimes E^0 \xrightarrow{\sigma_1(D^0)} T^\ast \otimes E^1 \xrightarrow{\tau} E^2 \to 0,
\]

where \( g_2 \) is the first prolongation of \( g_1 \).

Let \( D^1 \) be a linear differential operator of order 1 from \( E^1 \) to \( E^2 \) constructed from \( D^0 \) and possessing the properties (i)–(iii) of Lemma 3.1. It yields a sequence

\[
E^0 \xrightarrow{D^0} E^1 \xrightarrow{D^1} E^2,
\]

which is not necessarily a differential complex.

In what follows the fiber dimension of \( E^0 \) will be denoted by \( m_0 \), and the two-fold exterior product of \( T^\ast \) by \( \wedge^2 T^\ast \).

7.1. Structures of the differential operators \( D^0, D^1 \)

We begin to clarify the structures of the operators \( D^0, D^1 \) in (7.3). It is by virtue of the assumption \( \dim X = 2 \) that they have simple structures.

LEMMA 7.1. Assume that (A.1)–(A.3) hold, and that \( s_2 = 0 \). Then, for any point \( x_0 \in X \), there exist a neighborhood \( U \) of \( x_0 \) and product vector bundles \( E_I = U \times \mathbb{R}^{m_0 - s_1} \), \( E_{II} = U \times \mathbb{R}^{s_1} \) over \( U \) such that we can construct vector bundle isomorphisms

\[
\kappa^0 : E^0|_U \to E_I \oplus E_{II},
\]

\[
\kappa^1 : E^1|_U \to (T^\ast|_U \otimes E_I) \oplus E_{II},
\]

\[
\kappa^2 : E^2|_U \to (\wedge^2 T^\ast|_U) \otimes E_I
\]

in terms of which the morphisms \( \sigma = \sigma(D^0) \) in (7.1) and \( \tau \) in (7.2) are described by

\[
\sigma(\xi \otimes (u_I, u_{II})) = (\xi \otimes u_I, \lambda(\xi \otimes u_{II})),
\]

\[
\tau(\xi \otimes (\eta \otimes e_I), \xi \otimes e_{II}) = (\xi \wedge \eta) \otimes e_I,
\]

where \( \xi, \eta \in T^\ast_x \), \( u_I, e_I \in E_{I,x} \), \( u_{II}, e_{II} \in E_{II,x} \) (\( x \in U \)), and \( \lambda \) is a vector bundle morphism from \( T^\ast|_U \otimes E_{II} \) to \( E_{II} \).

Proof. Since \( g_1 \) is involutive and \( s_2 = 0 \) by assumptions, We have \( \dim g_{1,x} = \dim g_{2,x} = s_1 \). Let \( (x^1, x^2) \) be a coordinate system of \( X \) around \( x_0 \) such that \( \{ \partial/\partial x^1, \partial/\partial x^2 \} \) is a regular basis for \( g_{1,x} \). By Lemma 1.2, we can choose a neighborhood \( U \) of \( x_0 \) contained in the coordinate neighborhood in such a way that \( \{ \partial/\partial x^1, \partial/\partial x^2 \} \) is regular for any
\(g_{1,t}\) with \(x \in U\). In the following discussion, we shall assume that the vector bundles denote their restrictions to \(U\), and that we shrink \(U\) if necessary. Let \(i : g_1 \to E^0\) be the vector bundle morphism sending \(\zeta \in g_1\) to \(\zeta (\partial/\partial x^1) \in E^0\). We readily see that \(i\) is injective. Hence the image \(i(g_1)\) is a subbundle of \(E^0\) with fiber dimension \(s_1\). Set \(E_{11} = i(g_1)\). Choose a subbundle \(E_I\) of \(E^0\) such that \(E^0 = E_I \oplus E_{11}\) (direct sum); \(E_I\) is of fiber dimension \(m_0 - s_1\). The bundles \(E_I, E_{11}\) may be identified with product vector bundles. We define \(\kappa^0\) by this direct sum decomposition together with the identifications.

We show that the following inclusion and decomposition hold:

\[(7.6) \quad g_1 \subset T^* \otimes E_{11}, \quad g_1 \oplus (dx^2 \otimes E_{11}) = T^* \otimes E_{11}.\]

Fixing \(x \in U\), we assume that the bundles such as \(T^*, E_{11}\) denote their fibers over \(x\). Let \(\{e_\alpha^I; 1 \leq \alpha \leq m_0 - s_1\}\) be a basis of \(E_I\), and \(\{e_\beta^I; 1 \leq \beta \leq s_1\}\) be a basis of \(E_{11}\). We can choose a basis \(\{\xi^\beta; 1 \leq \beta \leq s_1\}\) of \(g_1\) such that each \(\xi^\beta\) takes the form

\[\xi^\beta = dx^1 \otimes e_\alpha^I + \sum_{\alpha = 1}^{m_0 - s_1} I e_\alpha^\beta dx^2 \otimes e_\alpha^I + \sum_{\gamma = 1}^{s_1} \gamma e_\beta^\gamma dx^2 \otimes e_\beta^I.\]

By definition, the prolongation \(g_2\) consists of those elements \(\phi\) in \((\otimes^2 T^*) \otimes E^0\) which take the form

\[\phi = \sum_{\beta = 1}^{s_1} (a_\beta dx^1 \otimes \xi^\beta + b_\beta dx^2 \otimes \xi^\beta)\]

and which belong to \(S^2 T^* \otimes E^0\). The condition \(\phi \in S^2 T^* \otimes E^0\) is equivalent to the one

\[(7.7) \quad \sum_{\beta = 1}^{s_1} a_\beta \left( \sum_{\alpha = 1}^{m_0 - s_1} I e_\alpha^\beta e_\alpha^I + \sum_{\gamma = 1}^{s_1} \gamma e_\beta^\gamma e_\beta^I \right) = \sum_{\beta = 1}^{s_1} b_\beta e_\beta^I.\]

Since \(\{e_\alpha^I, e_\beta^I\}\) are linearly independent, this implies that each \(b_\beta\) must be a linear combination of the \(a_\beta\)'s, and hence that \(\dim g_2 \leq s_1\). In order that \(\dim g_2\) is equal to \(s_1\), it is necessary (and sufficient) that all the \(I e_\alpha^\beta\) vanish. In fact, if some \(I e_\alpha^\beta\) is not equal to zero, the equality (7.7) indicates that some \(a_\beta\) must be zero, whence \(\dim g_2 < s_1\). From the vanishing of \(I e_\alpha^\beta\), it follows (7.6).

Set \(\sigma^{(I)} = \sigma|_{T^* \otimes E_I}, \sigma^\dagger = \sigma|_{dx^2 \otimes E_{11}}\). From the exactness of (7.1) together with (7.6), we know that \(\sigma^{(I)}, \sigma^\dagger\) are injective, and \(E^1\) is the direct sum of their images \(\im \sigma^{(I)}, \im \sigma^\dagger\). We define an isomorphism \(\kappa^1\) by

\[\kappa^1(\sigma^{(I)}(\xi \otimes u_I) + \sigma^\dagger(dx^2 \otimes u_{11})) = (\xi \otimes u_I, u_{11}).\]

In terms of the isomorphisms \(\kappa^0, \kappa^1, \sigma\) is described as a direct sum of the identity map \(\sigma^{(I)} = \id : T^* \otimes E_I \to T^* \otimes E_I\) and a surjective morphism \(\sigma^{(II)} = \sigma|_{T^* \otimes E_{11}} : T^* \otimes E_{11} \to E_{11}\).

Let us investigate the structure of \(\coker \sigma_1\), where \(\sigma_1 = \sigma_1(D^0)\). The morphism \(\sigma_1\) is the direct sum of the first prolongations \(\sigma_1^{(I)}, \sigma_1^{(II)}\) of \(\sigma^{(I)}, \sigma^{(II)}\), and \(\sigma_1^{(I)}\) is the canonical inclusion map \(S^2 T^* \otimes E_I \to T^* \otimes (T^* \otimes E_I)\). Since the fiber dimension of \(g_2\) equals
$s_1$, it follows from the exactness of (7.1), (7.2) that $\sigma_1^{(I)}$ has the rank $2s_1$, and hence that
$\sigma_1^{(I)}$ is surjective. By virtue of the direct sum decomposition $\otimes^2 T^* = S^2 T^* \oplus \wedge^2 T^*$, we find that coker $\sigma_1^{(I)}$ is canonically isomorphic with $\wedge^2 T^* \otimes E_I$, and the projection $\rho_I : S^2 T^* \otimes E_I \to \text{coker } \sigma_1^{(I)}$ is described by $\rho_I(\xi \otimes \eta \otimes e_I) = (\xi \wedge \eta) \otimes e_I$. Combining these, we have a canonical isomorphism

$$\kappa^2 : E^2 \cong \text{coker } \sigma_1 \to \text{coker } \sigma_1^{(I)} \cong \wedge^2 T^* \otimes E_I$$

such that the natural projection $\tau : T^* \otimes E^1 \to E^2$ is described by the morphism $\kappa^2 \circ \tau = \rho_I$. Then, in terms of $\kappa^1, \kappa^2$, $\tau$ has the desired description (7.5). Q.E.D.

**Proposition 7.2.** Assume that (A.1)–(A.3) hold, and that $s_2 = 0$. For any point $x_0 \in X$, there exists a neighborhood $U$ of $x_0$ over which one can construct vector bundle isomorphisms $\kappa^0, \kappa^1, \kappa^2$ as in Lemma 7.1 in terms of which the differential operators $D^0, D^1$ in (7.3) are described over $U$ as follows:

$$D^0 u = (d u_I + A_I(u), \left(\sum_{i=1}^2 A_i \frac{\partial}{\partial x^i}\right) u_{II} + A_{II}(u)),$$

where $u = (u_I, u_{II})$ is a section of $E^0|_U = E_I \oplus E_{II}$, $d$ is the exterior differentiation tensored with the identity map, $A_i$'s are vector bundle morphisms from $E_{II}$ to itself, and $A_I, A_{II}$ are linear differential operators of order 0;

$$D^1 v = d v_I + B(v),$$

where $v = (v_I, v_{II})$ is a section of $E^1|_U = (T^*|_U \otimes E_I) \oplus E_{II}$, $d$ is the exterior differentiation, $B$ is a linear differential operator of order 0.

If moreover $D^0$ is elliptic, then the determined operator $\sum_{i=1}^2 A_i \frac{\partial}{\partial x^i}$ in (7.8) is elliptic.

**Proof.** Choose a neighborhood $U$ of $x_0$ in the same way as in Lemma 7.1. Since the symbol morphism $\sigma(d)$ of the exterior differentiation $d : \wedge^i T^* \to \wedge^{i+1} T^*$ is the exterior multiplication $T^* \otimes \wedge^i T^* \to \wedge^{i+1} T^*$, it follows that $D^0, D^1$ admit the descriptions (7.8), (7.9), respectively. The last assertion is obvious. Q.E.D.

### 7.2. Reduction to differential operators on the unit disc

Let us assume that (A.1)–(A.3) hold, and that $s_2 = 0$. Then we can apply Proposition 7.2. Let $U$ be a coordinate neighborhood of $X$ around a point $x_0 \in X$ over which the differential operators $D^0, D^1$ admit the descriptions (7.8), (7.9), respectively, where the identifications

$$(7.10) \quad E^0|_U = E_I \oplus E_{II}, \quad E^1|_U = (T^*|_U \otimes E_I) \oplus E_{II}, \quad E^2|_U = (\wedge^2 T^*|_U) \otimes E_I$$

in terms of the isomorphisms $\kappa^0, \kappa^1, \kappa^2$ in Lemma 7.1 are used. Let $(x^1, x^2)$ be a coordinate system in $U$ with $x^i(x_0) = 0$ ($i = 1, 2$). We endow the cotangent bundle $T^*|_U$ the metric for which $\{dx^1, dx^2\}$ forms an orthonormal basis in each fiber, and the product bundles $E_I, E_{II}$ the Euclidean metrics. We shall assume that the bundles $E^1|_U$ are endowed with metrics induced from those of $T^*|_U, E_I, E_{II}$.
In terms of the identifications of (7.10), a section $u$ of $E^0$ over $U$ is represented by an ordered set $(u_1, \ldots, u_{m_0})$ of functions $u_\alpha$ on $U$, a section $v$ of $E^1$ over $U$ by an ordered set $(v_1, \ldots, v_{m_0})$ where the first $m_0 - s_1$ components $v_\alpha$ are 1-forms on $U$ and the remaining $s_1$ components $v_\beta$ are functions on $U$, and a section $w$ of $E^2$ over $U$ by an ordered set $(w_1, \ldots, w_{m_0-s_1})$ of 2-forms $w_\gamma$ on $U$. We shall call this representation the representation with differential form components. Besides this representation, we shall use other representation of sections of $E^1, E^2$. Using the (fixed) coordinates $(x^1, x^2)$, describe the components of 1-forms $v_\alpha$ as

$$v_\alpha = f_{\alpha,1}(x) \, dx^1 + f_{\alpha,2}(x) \, dx^2 \quad (1 \leq \alpha \leq m_0 - s_1),$$

and the components of 2-forms $w_\gamma$ as

$$w_\gamma = g_{\gamma}(x) \, dx^1 \wedge dx^2 \quad (1 \leq \gamma \leq m_0 - s_1).$$

Then a section $v$ of $E^1|_U$ is also represented by an ordered set

$$(f_{\alpha,1}, f_{\alpha,2} (1 \leq \alpha \leq m_0 - s_1), v_\beta (m_0 - s_1 < \beta \leq m_0))$$

of functions on $U$, and a section $w$ of $E^2|_U$ by an ordered set $(g_1, \ldots, g_{m_0-s_1})$ of functions on $U$. We shall call this representation the product bundle representation in terms of the coordinates $(x_1, x_2)$. Note that for any other coordinate system $(\tilde{x}_1, \tilde{x}_2)$, we have the product bundle representation defined similarly.

Let $E$ be one of $E^0, E^1, E^2$, and $U'$ an open set in $U$. Define $L^2(U', E)$ to be the set of measurable sections $u$ of $E|_{U'}$ which satisfy

$$\|u\| = \|u\|_{U'} = \left( \int_{U'} |u| (x) dx^1 dx^2 \right)^{1/2} < \infty,$$

$\langle , \rangle$ being the metric of $E$. This is a real Hilbert space with inner product

$$\langle u, v \rangle = \langle u, v \rangle_{U'} = \int_{U'} \langle u, v \rangle (x) dx^1 dx^2.$$

The norm $\|u\|$ clearly coincides with the norm defined canonically by using product bundle representations of sections of $E$ (cf. section 6). We denote by $D(U', E)$ the set of smooth sections of $E|_{U'}$ with compact supports in $U'$, and by $D'(U', E)$ the set of all sections of $E|_{U'}$ which have product bundle representations with components being distributions in $U'$. The space $L^2(U', E)$ may be canonically regarded as a subspace of $D'(U', E)$. For each $T \in D'(U', E)$, we define its derivative $(\partial / \partial x)^T$ to be the element of $D'(U', E)$ obtained from $T$ by applying the operation $(\partial / \partial x)^T$ to each of its components.

Let $\Omega_\epsilon$ denote the open disc of radius $\epsilon > 0$ and with center $x_0$;

$$\Omega_\epsilon = \{(x^1, x^2) \in \mathbb{R}^2; (x^1)^2 + (x^2)^2 < \epsilon^2\}.$$
has the product bundle representation \((u_1, \ldots, u_m)\) with \(u_a\)'s being functions, we define \(\tau^*_\epsilon u \in \mathcal{D}(\Omega_1, E)\) by
\[
\tau^*_\epsilon u = (\tau^*_\epsilon u_1, \ldots, \tau^*_\epsilon u_m), \quad (\tau^*_\epsilon u_a)(x) = u_a(\epsilon x).
\]
We emphasize that \(\tau^*_\epsilon\) does not mean the action on differential forms in the representation with differential form components.

Let \(D\) be a linear differential operator of order \(l\) from \(E\) to \(E'\), where \(E, E'\) are vector bundles in \([E^i|_U; 0 \leq i \leq 2]\). For each \(\epsilon \in (0, 1]\), let \(D(\epsilon)\) be the linear differential operator from \(E|_{\Omega_1}\) to \(E'|_{\Omega_1}\) defined by
\[
D(\epsilon) u = \tau^*_\epsilon \circ D \circ (\tau^*_\epsilon)^{-1} u, \quad u \in \mathcal{D}(\Omega_1, E).
\]
We set \(D_\epsilon = \epsilon^l D(\epsilon), l\) being the order of \(D\) (cf. Sweeney [29]). Using the product bundle representations of \(E\) and of \(E'\), a linear differential operator \(D\) is described by
\[
D = \sum_{|\nu| \leq l} A_\nu(x) \left( \frac{\partial}{\partial x} \right)^\nu,
\]
where \(A_\nu(x)\)'s are matrices with function entries on \(U\). Then the differential operator \(D_\epsilon\) has the following description:
\[
D_\epsilon = \sum_{|\nu| \leq l} \epsilon^{l-|\nu|} A_\nu(\epsilon x) \left( \frac{\partial}{\partial x} \right)^\nu.
\]
It is clear that \(D_\epsilon\) has the same order as \(D\), and the coefficients in the representation of \(D_\epsilon\) are bounded on \(\Omega_1\) uniformly with respect to \(\epsilon \in (0, 1]\). If \(D^\dagger\) is a linear differential operator from \(E\) to \(E'\) having the same order as \(D\), then \((D + D^\dagger)_\epsilon = D_\epsilon + D^\dagger_\epsilon\). If \(D'\) is a linear differential operator from \(E'\) to \(E''\), and if the order of \(D' \circ D\) is equal to the sum of the orders of \(D, D'\), then \((D' \circ D)_\epsilon = D'_\epsilon \circ D_\epsilon\). Let \(D^\dagger\) denote the formal adjoint of \(D\) defined using the inner products in \(L^2(U, E')\). Then \((D^\dagger)_\epsilon = (D_\epsilon)^\dagger\).

Let \(k\) be a non-negative integer. The Sobolev space \(H^k(\Omega_\epsilon, E')\) is defined as the set of all sections \(u\) of \(E\) satisfying \(\partial^n u \in L^2(\Omega_\epsilon, E')\) for all multi-index \(\nu\) with \(0 \leq |\nu| \leq k\), where \(\partial^\nu = (\partial/\partial x)^\nu\). We endow it the norm
\[
\|u\|_k = \|u\|_{k, \Omega_\epsilon} = \left( \sum_{|\nu| \leq k} \|\partial^\nu u\|^2 \right)^{1/2}.
\]
We note that this space is a real Hilbert space, and coincides with the Sobolev space defined as in section 6 in terms of the product bundle representation. The mapping \(\tau^*_\epsilon\) canonically induces a mapping \(\tau^*_\epsilon : H^k(\Omega_\epsilon, E') \to H^k(\Omega_1, E')\). We shall state some of its properties.

**Lemma 7.3.** Let \(E, E'\) be vector bundles in \([E^i|_U; 0 \leq i \leq 2]\), and let \(D\) be a linear differential operator of order \(l\) from \(E\) to \(E'\). For any integer \(k \geq 0\), the following are valid:

(i) \(\tau^*_\epsilon : H^k(\Omega_\epsilon, E) \to H^k(\Omega_1, E)\) is a bijective isomorphism, and moreover
\[
\epsilon^{-1+k} \|v\|_{k, \Omega_\epsilon} \leq \|\tau^*_\epsilon v\|_{k, \Omega_1} \leq \epsilon^{-1} \|v\|_{k, \Omega_\epsilon}, \quad v \in H^k(\Omega_\epsilon, E).
\]
(ii) For any \( v \in H^{k+l}(\Omega_\epsilon, E) \),
\[
\epsilon^{-1+k+l} \| Dv \|_{k, \Omega_\epsilon} \leq \| D_\epsilon (\tau_\epsilon^* v) \|_{k, \Omega_1} \leq \epsilon^{-1+l} \| Dv \|_{k, \Omega_\epsilon}.
\]

Proof. Since \( \tau_\epsilon \) is a diffeomorphism, the first part of (i) is obvious. The change of variables formula yields the formula
\[
\| \partial_\nu x u \|_{\Omega_\epsilon} = \epsilon^{-|\nu|} \| \partial_\nu x (\tau_\epsilon^* u) \|_{\Omega_1}, \quad u \in H^k(\Omega_\epsilon, E), \quad |\nu| \leq k.
\]
Using this formula, we can readily prove the estimates in (i) and (ii). Q.E.D.

Applying the above procedure to the differential operators \( D_0, D_1 \), we have operators \( D_0_\epsilon, D_1_\epsilon \) on the unit disc. We find it convenient to use the same descriptions of the differential operators \( D_0_\epsilon, D_1_\epsilon \) on \( \Omega_1 \) as in (7.8), (7.9) of Proposition 7.2. They have the following descriptions in terms of the representations with differential form components
\[
(7.11) \quad D_0^0 u = (d u_I + \epsilon A_I(\epsilon x)(u), \left( \sum_{i=1}^2 A_i(\epsilon x) \frac{\partial}{\partial x^i} \right) u_{II} + \epsilon A_{II}(\epsilon x)(u)).
\]
\[
(7.12) \quad D_1^0 v = dv_I + \epsilon B(\epsilon x)(v).
\]
We note that the coefficients of the differential operators \( D_0^0, D_1^0 \) together with their derivatives of arbitrary order are bounded on \( \Omega_1 \) uniformly with respect to \( \epsilon \in (0, 1) \).

Let us introduce two differential operators by setting \( \epsilon = 0 \) in (7.11) and (7.12);
\[
(7.13) \quad D_0^0 u = (d u_I, P_0 u_{II}), \quad D_1^0 v = dv_I,
\]
where \( P_0 = \sum_{i=1}^2 A_i(0, 0) \frac{\partial}{\partial x^i} \) is a determined linear differential operator of order 1 in \( E_{II} \) with constant coefficients. We note that \( D_0^0, D_1^0 \) converge, respectively, to \( D_0^0, D_1^0 \) as \( \epsilon \) tends to zero in the sense that, in terms of the product bundle representations, the coefficients of \( D_0^0, D_1^0 \) converge to the corresponding coefficients of \( D_0^0, D_1^0 \) in \( C^k(\Omega_1) \) topology as \( \epsilon \) tends to zero, \( k \) being any positive integer.

7.3. A differential complex on the unit disc
In this subsection we denote by \( \Omega \) the unit disc; \( \Omega = \Omega_1 \subset \mathbb{R}^2 \). We shall consider the sequence on the unit disc
\[
(7.14) \quad E^0|\Omega \longrightarrow D_0^0|\Omega \longrightarrow E^1|\Omega \longrightarrow D_1^0|\Omega \longrightarrow E^2|\Omega.
\]
From the representations in (7.13), we know that this is a differential complex; \( D_1^0 \circ D_0^0 = 0 \). In what follows, we shall assume that \( D_0^0 \) is elliptic, equivalently, the determined operator \( P_0 \) is elliptic.

Let \( \partial \Omega \) denote the boundary of \( \Omega \); \( \partial \Omega \) is the unit circle. Let \( L^2(\partial \Omega) \) be the Hilbert space of real-valued measurable functions \( f \) on \( \partial \Omega \) satisfying
\[
\| f \|_{\partial \Omega}^2 = \int_{\partial \Omega} |f|^2 ds = \int_0^{2\pi} |f(\cos \theta, \sin \theta)|^2 d\theta < +\infty,
\]
where \( ds \) denotes the line element of the circle. For each bundle \( E^i \), we define the Hilbert space \( L^2(\partial \Omega, E^i) \) to be the space consisting of all sections \( u = (u_1, \ldots, u_m) \) of \( E^i \) over
\( \partial \Omega \) satisfying \( u_\alpha \in L^2(\partial \Omega) \) \((1 \leq \alpha \leq m_1)\) and endowed with the norm
\[
\|u\|_{\partial \Omega} = \left( \sum_{\alpha=1}^{m_1} |u_\alpha|_{\partial \Omega}^2 \right)^{1/2},
\]
\( u = (u_1, \ldots, u_{m_1}) \) being the product bundle representations of sections. If \( u \in H^1(\Omega, E^1) \), one can define its trace \( u|_{\partial \Omega} \in L^2(\partial \Omega, E^1) \) to the boundary; the mapping sending \( u \) to \( u|_{\partial \Omega} \) is a continuous linear operator from \( H^1(\Omega, E^1) \) to \( L^2(\partial \Omega, E^1) \) (cf. Rauch [25], Chapter 5, Taylor [30], Chapter 4).

Let \( \eta \) denote the outward-pointing unit normal to the boundary \( \partial \Omega \) of \( \Omega \). Identifying the field \( \eta \) of tangent vectors with the field of cotangent vectors in terms of the Euclidean metric of the plane \( \mathbb{R}^2 \), we have morphisms \( \sigma_\eta((D^0_0)^t) : E^1_\alpha \rightarrow E^0_\alpha \) and \( \sigma_\eta(D^1_0) : E^2_\alpha \rightarrow E^1_\alpha \) for each point \( x \in \partial \Omega \) (cf. section 3). For sections \( u \in H^1(\Omega, E^0) \), \( v \in H^1(\Omega, E^1) \), \( w \in H^1(\Omega, E^2) \), we have Green's formulas

\[
(D^0_0 u, v) = (u, (D^0_0)^t v) + \int_{\partial \Omega} (u, \sigma_\eta((D^0_0)^t) v) ds, \tag{7.15}
\]
\[
(v, (D^1_0)^t w) = (D^1_0 v, w) - \int_{\partial \Omega} (v, \sigma_\eta((D^1_0)^t) w) ds \tag{7.16}
\]

(cf. Taylor [30], Chapter 9, Proposition 9.1).

Set \( \Delta_0 = -\Delta_0 \cap (D^0_0 \circ (D^0_0)^t) + (D^1_0)^t \circ D^1_0) \); \( \Delta_0 \) is a linear differential operator of order 2 in \( E^1 \). We have one more Green's formula derived from \( 7.15 \), \( 7.16 \):

\[
(\Delta_0 v, v') = (\sigma_\eta(D^0_0)^t v, (D^0_0)^t v') + (D^1_0 v, D^1_0 v) \tag{7.17}
\]

\[
+ \int_{\partial \Omega} \{ (\sigma_\eta(D^0_0)^t v, \sigma_\eta((D^0_0)^t) v') + \langle \sigma_\eta((D^1_0)^t) D^1_0 v, v' \rangle \} ds, \]

where \( v \in H^2(\Omega, E^1) \), \( v' \in H^1(\Omega, E^1) \). These formulas lead us to introduce natural boundary conditions (cf. the \( D \)-Neumann problem in Sweeney [28]). We set

\[
H^k_0(\Omega, E^1) = \{ v \in H^k(\Omega, E^1) ; \sigma_\eta((D^0_0)^t) v|_{\partial \Omega} = 0 \} \quad (k \geq 1), \]
\[
H^k_0(\Omega, E^1) = \{ v \in H^k(\Omega, E^1) ; \sigma_\eta((D^1_0)^t)(D^1_0 v)|_{\partial \Omega} = 0 \} \quad (k \geq 2). \]

In accordance with the direct sum decomposition in \( 7.10 \), we have the direct sum decomposition

\[
H^k_0(\Omega, E^1) = H^k_0(\Omega, T^* \otimes E_1) \oplus H^k_0(\Omega, E_{II}) \tag{7.18}
\]

where

\[
H^k_0(\Omega, T^* \otimes E_1) = \{ v_1 \in H^k(\Omega, T^* \otimes E_1) ; \sigma_\eta(d^* v_1)|_{\partial \Omega} = 0 \}, \]
\[
H^k_0(\Omega, E_{II}) = \{ v_{II} \in H^k(\Omega, E_{II}) ; \sigma_\eta(P_0^* v_{II})|_{\partial \Omega} = 0 \}. \]

Observe that \( \sigma_\eta(d^* v_1) = \eta \cdot v_1 \) (the interior product), and that, since \( P_0^* \) is elliptic, the condition \( \sigma_\eta(P_0^* v_{II})|_{\partial \Omega} = 0 \) is equivalent to the Dirichlet boundary condition \( v_{II}|_{\partial \Omega} = 0 \). Thus, using the usual notation, we have

\[
H^k_0(\Omega, E_{II}) = H^k(\Omega, E_{II}) \cap H^1_0(\Omega, E_{II}).
\]
Around the boundary \( \partial \Omega \), it is convenient to use also the polar coordinates \((r, \theta)\). Let \( \Omega_\theta \) denote the domain \( \Omega \) excluded the closure of \( \Omega_1/\Omega \), and \( K \) denote the product manifold \((1/2, 1) \times (\mathbb{R}/2\pi \mathbb{Z}), \mathbb{Z} \) being the module of integers. The mapping \( x^1 = r \cos \theta, x^2 = r \sin \theta \) defines a diffeomorphism from \( K \) onto \( \Omega_\theta \). We shall use \((r, \theta)\) as a coordinate system of \( \Omega_\theta \), and use the simplified notations \( \partial_r = \partial/\partial r, \partial_\theta = \partial/\partial \theta \).

Let \( v_I \in H^1_0(\Omega, T^* \otimes E_I) \). It is described on \( \Omega_\theta \) by
\[
(7.19) \quad v_I = a^{(r)}(r, \theta) \, dr + a^{(0)}(r, \theta) \, d\theta ,
\]
where \( a^{(r)}, a^{(0)} \) are sections of \( E_I \). Then the boundary condition \( \sigma_0(d^* v_I)|_{\partial \Omega} = 0 \) is stated as the condition \( a^{(r)}(r, \theta)|_{r = 1} = 0 \) in \( L^2(\partial \Omega, E_I) \).

**Sublemma 7.4.** Concerning the differential operators \( d' \)'s in (7.13), we have the estimate
\[
(7.20) \quad \|v_I\|^2 \leq 4\|d' v_I\|^2 + \|d v_I\|^2 , \quad v_I \in H^1_0(\Omega, T^* \otimes E_I) ,
\]
and if \( P_0 \) is elliptic, we have the estimate with a constant \( C > 0 \)
\[
(7.21) \quad \|v_{II}\|^2 \leq C \|P_0^1 v_{II}\|^2 , \quad v_{II} \in H^1_0(\Omega, E_{II}) .
\]

**Proof.** To prove (7.20), we first note that any \( v_I \in H^1_0(\Omega, T^* \otimes E_I) \) is a limit of a sequence of 1-forms belonging to both \( C^2(\overline{\Omega}, T^* \otimes E_I) \) and \( H^1_0(\Omega, T^* \otimes E_I) \). In fact, \( v_I \) may be expressed as a sum of a 1-form \( v^{(1)}_I \in H^1(\Omega, T^* \otimes E_I) \) with compact support in \( \Omega \) and a 1-form \( v^{(2)}_I \in H^1_0(\Omega, T^* \otimes E_I) \) with \( \text{supp} \, v^{(2)}_I \) being contained in \( \Omega_\theta \). Clearly \( v^{(1)}_I \) is a limit of a sequence of smooth 1-forms with compact supports in \( \Omega \). Let \( v_I = v^{(2)}_I \) be described by (7.19). Since \( a^{(r)}(r, \theta)|_{r = 1} = 0, a^{(r)} \) can be approximated by smooth sections on \( \Omega_\theta \) vanishing near \( \partial \Omega \), and clearly \( a^{(0)} \) by sections of \( C^2(\overline{\Omega}, E_I) \). From this we see that \( v^{(2)}_I \) is a limit of a sequence of 1-forms satisfying the required conditions. Thus we have the desired fact. This fact indicates that it suffices to prove (7.20) for all \( v_I \) belonging also to \( C^2(\overline{\Omega}, T^* \otimes E_I) \). Moreover we may also assume that the fiber dimension of \( E_I \) is equal to 1, and hence that \( v_I \) is a 1-form \( \omega \). Let
\[
\omega = a_1(x) \, dx^1 + a_2(x) \, dx^2 .
\]
For brevity we denote \( \partial_{x^i} = \partial/\partial x^i \), and use the notations
\[
\|\omega\|^2_{(1)} = \sum_{i,j=1}^2 \|\partial_{x^i} a_{j}\|^2 , \quad \|\omega\|^2_{\partial \Omega} = \sum_{i=1}^2 \|a_i\|^2_{\partial \Omega} .
\]
We have the identity
\[
(7.22) \quad \|\omega\|^2_{\partial \Omega} + \|\omega\|^2_{(1)} = \|d' \omega\|^2 + \|d \omega\|^2 .
\]
In fact, setting \( J = (\partial_{x^1} a_1, \partial_{x^2} a_2) - (\partial_{x^2} a_1, \partial_{x^1} a_2) \), we have
\[
\|d' \omega\|^2 + \|d \omega\|^2 = \|\omega\|^2_{(1)} + 2J .
\]
Moreover, integrating by part in two different ways yields the equalities
\[
J = ((x^1 \partial_{x^1} - x^1 \partial_{x^2})a_1, a_2)_{\partial \Omega} = (a_1, (x^1 \partial_{x^2} - x^2 \partial_{x^1})a_2)_{\partial \Omega} .
\]
The boundary condition for \( \omega \in H^1_b(\Omega, T^* \otimes E_1) \) may be stated as
\[
F(x) = x^1 a_1(x) + x^2 a_2(x) = 0 \quad (x = (x^1, x^2) \in \partial \Omega).
\]
Let \( V \) denote the vector field \( a_1(x) \partial x_1 + a_2(x) \partial x_2 \). By virtue of \( F|_{\partial \Omega} = 0 \), \( V \) is tangential to the boundary \( \partial \Omega \). Hence \( V(F)|_{\partial \Omega} = V(F|_{\partial \Omega}) = 0 \). From this it follows that
\[
a_1^2 + a_2^2 + \{(x^1 \partial x_2 - x^2 \partial x_1) a_1 \} a_2 + a_1(x^2 \partial x_1 - x^1 \partial x_2) a_2 = 0 \quad \text{on} \quad \partial \Omega.
\]
Integrating this on \( \partial \Omega \) gives rise to \( 2J = \| \omega \|_{\partial \Omega}^2 \). Thus (7.22) is proved.

We have also the estimate
\[
\| \omega \|_{\partial \Omega}^2 \leq 4(\| \omega \|_{\partial \Omega}^2 + \| \omega \|_{(1)}^2).
\]
Indeed, using the polar coordinates, we obtain
\[
| a_i(r, \theta) |^2 \leq 2(| a_i(1, \theta) |^2 + \int_1^r |(\partial r a_i)(t, \theta) |^2 dt).
\]
Multiplying this by \( r \) and integrating on \((0, 1] \times [0, 2\pi] \), we get
\[
\| a_i \|_{\partial \Omega}^2 \leq \| a_i \|_{\partial \Omega}^2 + 2\| \partial r a_i \|_{\partial \Omega}^2.
\]
Since \( \| \partial_r a_i \|_{\partial \Omega}^2 \leq 2(\| \partial_x a_i \|_{\partial \Omega}^2 + \| \partial_x a_i \|_{\partial \Omega}^2) \), it follows (7.23). Combining (7.22) with (7.23) yields (7.20).

Let us next prove (7.21). Since \( P^0_0 \) is an elliptic homogeneous linear differential operator with constant coefficients, using the Fourier transformation and the Parseval formula, we can easily deduce that
\[
\sum_{i=1}^2 \| \partial_i v_{II} \|_{\partial \Omega}^2 \leq C \| P^0_0 v_{II} \|_{\partial \Omega}^2
\]
with a constant \( C > 0 \). On the other hand, the Poincaré inequality (cf. e.g. Rauch [25], Chapter 5) asserts that \( \| v_{II} \|_{\partial \Omega}^2 \leq 4\| v_{II} \|_{(1)}^2 \). The estimate (7.21) follows from these two inequalities.

**Lemma 7.5.** Assume that \( D^0_0 \) is elliptic. Then, we can find a constant \( C > 0 \) such that
\[
\| v \|_{\partial \Omega}^2 \leq C(\| D^0_0 v \|_{\partial \Omega}^2 + \| D^1_0 v \|_{\partial \Omega}^2), \quad v \in H^1_b(\Omega, E_1).
\]

**Proof.** Let \( \kappa \) be the reflection with respect to the unit circle \( \partial \Omega \). Choose a smooth function \( \chi \) on \( \Omega_2 \) such that \( \chi(x) = 1 \) if \( x \in \Omega_{5/4} \), \( \chi(x) = 0 \) if \( x \notin \Omega_{7/4} \). Let \( v = (v_I, v_{II}) \in H^1_b(\Omega, E_1) \) (cf. (7.18)). We define
\[
(v_I)(x) = \begin{cases} v_I(x), & x \in \Omega, \\ \chi(x)(\kappa^* v_I)(x), & x \in \Omega_2 \setminus M, \end{cases}
\]
where \( \kappa^* v_I \) is the pull-back of the 1-form \( v_I \) by \( \kappa \). Then \( v_I \in H^1_b(\Omega_2, T^* \otimes E_1) \). In fact, assume that \( v_I \) has the description (7.19). Then \( \kappa^* v_I \) has the description
\[
(\kappa^* v_I)(r, \theta) = -a(\gamma)(r^{-1}, \theta) r^{-2} dr + a(\gamma)(r^{-1}, \theta) d\theta.
\]
for \( r > 1 \). Bearing in mind the boundary condition, we can readily verify that all coefficients of \( \kappa^* u_I \) have distribution derivatives of order 1 with respect to \( r, \theta \) (and hence with respect to \( x^1, x^2 \)) which are square-integrable on a neighborhood of \( \partial \Omega \). This fact implies the required assertion. Moreover we note that \( \| u_I \|_1 \leq \| u_I \|_{1, \partial \Omega} \leq C \| u_I \|_1 \). Here and below \( C \) denotes a generic positive constant. Since \( d^* \oplus d \) is an elliptic differential operator, we may use the Fourier transformation to obtain the estimate

\[
\| u_I \|_1^2 \leq C (\| d^* (u_I) \|_2^2 + \| d (u_I) \|_2^2 + \| u_I \|_2^2).
\]

Since \( d (\kappa^* v_I) = \kappa^* (d v_I) \), and \( d^* (\kappa^* v_I) (x) = |x|^4 \kappa^* (d^* v_I) (x) \), we have also the estimates

\[
\| d (v_I) \|_2^2 \leq C (\| d v_I \|_2^2 + \| v_I \|_2^2), \quad \| d^* (v_I) \|_2^2 \leq C (\| d^* (v_I) \|_2^2 + \| v_I \|_2^2).
\]

From the obtained estimates, we have

\[
\| u_I \|_1^2 \leq C (\| d^* u_I \|_2^2 + \| d v_I \|_2^2 + \| v_I \|_2^2).
\]

Bearing in mind the descriptions of \( D_0^1, D_1^1 \) in (7.13), from this estimate together with (7.20), we can readily deduce the required estimate. Q.E.D.

Let \( H^1_b (\Omega, E^1)^* \) denote the dual space of the real Banach space \( H^1_b (\Omega, E^1) \). The norm of the Banach space \( H^1_b (\Omega, E^1)^* \) will be denoted by \( \| f \|_{-1} \). Let us define a linear operator \( \mathcal{L} : H^1_b (\Omega, E^1) \to H^1_b (\Omega, E^1)^* \) by

\[
\langle \mathcal{L} v, v' \rangle = (\langle D_0^1 v, v' \rangle + \langle D_1^1 v, D_0^1 v' \rangle), \quad v, v' \in H^1_b (\Omega, E^1),
\]

where \( \langle , \rangle \) denotes the duality pairing, and \( \langle , \rangle \) the inner products in \( L^2 (\Omega, E^1) \). Clearly \( \mathcal{L} \) is a bounded linear operator and \( \mathcal{L} \) is self-adjoint, that is, the adjoint \( \mathcal{L}^* \) coincides with \( \mathcal{L} \). Let us consider the problem of finding a solution \( u \) to the equation \( \mathcal{L} u = f \), where \( f \in H^1_b (\Omega, E^1)^* \).

**Proposition 7.6.** Assume that \( D_0^1 \) is elliptic. Then \( \mathcal{L} : H^1_b (\Omega, E^1) \to H^1_b (\Omega, E^1)^* \) is bijective.

**Proof.** From the estimate in Lemma 7.5 and (7.24), we have the estimate

\[
\| v \|_1 \leq C \| \mathcal{L} v \|_{-1} \text{ for all } v \in H^1_b (\Omega, E^1).
\]

This implies that \( \mathcal{L} \) has the closed range and \( \ker \mathcal{L}^* = \{ 0 \} \). Hence we can apply the closed range theorem due to Banach (see Yosida [32]) to conclude that \( \mathcal{L} \) is bijective. Q.E.D.

We shall now investigate the regularity of solutions. We regard \( L^2 (\Omega, E^1) \) as a subspace of \( H^1_b (\Omega, E^1)^* \) in terms of the injection \( \iota_0 : L^2 (\Omega, E^1) \to H^1_b (\Omega, E^1)^* \) defined by

\[
\langle \iota_0 v, v' \rangle = (v, v'), \quad v, v' \in L^2 (\Omega, E^1),
\]

and also regard \( L^2 (\Omega, E^1) \) as a subspace of \( D^* (\Omega, E^1) \) in the usual manner.

**Proposition 7.7.** Assume that \( D_0^1 \) is elliptic. If \( v \in H^1_b (\Omega, E^1) \) satisfies \( \mathcal{L} v \in L^2 (\Omega, E^1) \), then \( v \in H^2_b (\Omega, E^1) \). Moreover there exists a constant \( C > 0 \) such that

\[
\| v \|_2 \leq C (\| \Delta v \|_1^2 + \| v \|_1^2), \quad v \in H^2_b (\Omega, E^1).
\]
To give a proof of this proposition, we need to introduce an operator associating each section of $E^i$ its difference quotient in a certain sense. Let $t_h : \Omega \rightarrow \Omega$ be the rotation defined by $t_h(r, \theta) = (r, \theta + h)$. Let $E$ be one of the vector bundles $E^i$ ($0 \leq i \leq 2$). An element $u \in L^2(\Omega, E)$ is represented by an ordered set $(u_1, \ldots, u_m)$ of differential forms $u_a$ on $\Omega$ (the representation with differential form components; cf. also (7.10)). We define $t_h^* u$ to be an element $(t_h^* u_1, \ldots, t_h^* u_m) \in L^2(\Omega, E^1)$ where $t_h^* u_a$'s are pull-backs of differential forms $u_a$ by the mapping $t_h$. For each real number $h \neq 0$, we define an operator $\Theta_h$ by

$$\Theta_h u = \frac{t_h^* u - u}{h} \in L^2(\Omega, E), \quad u \in L^2(\Omega, E).$$

Let $u$ be a section of $E$ over $\Omega$, and $u = (u_1, \ldots, u_m)$ be its representation with differential form components. Using the description of each differential form $u_a$ in terms of the polar coordinates $(r, \theta)$, we define $\partial_r u_a$, $\partial_\theta u_a$ to be the differential forms obtained by differentiating the coefficients of the description of $u_a$ with respect to the variables $r, \theta$, respectively, and for $u$ we define

$$\partial_r u = (\partial_r u_1, \ldots, \partial_r u_m), \quad \partial_\theta u = (\partial_\theta u_1, \ldots, \partial_\theta u_m).$$

We also define higher order derivatives such as $\partial_r \partial_\theta u$ inductively on the orders. Observe that, when acting on differentiable sections, the operator $\Theta_h$ commutes with $\partial_r$, $\partial_\theta$, and $\partial_r \partial_\theta u = \lim_{h \to 0} \Theta_h u$.

We now extend the operator $\Theta_h$ to $H^1_b(\Omega, E^1)^*$. Bearing in mind the decomposition (7.18) and describing the boundary condition in terms of $(r, \theta)$ (see the discussion around (7.19)), we can verify that this important fact: If $v \in H^1_b(\Omega, E^1)$, then $t_h^* v$, $\Theta_h v \in H^1_b(\Omega, E^1)$. (We have defined the operator $\Theta_h$ so that this is valid.) Let $\Theta_h^*$ denote the adjoint of the operator $\Theta_h$ in $L^2(\Omega, E^1)$. We readily see that $\Theta_h^* = -\Theta_{-h}$. For $f \in H^1_b(\Omega, E^1)^*$, we define $\Theta_h f \in H^1_b(\Omega, E^1)^*$ by

$$(\Theta_h f, v) = (f, \Theta_h^* v), \quad v \in H^1_b(\Omega, E^1).$$

**Lemma 7.8.** Let $k$ be a non-negative integer, and $H^k_b(\Omega, E)$ denote the set of all $u \in H^k(\Omega, E)$ with $\sup u \in \Omega$.

(i) There exist constants $C, C' > 0$ such that

$$(7.26) \quad \|\Theta_h u\|_k \leq C' \|\partial_r u\|_k \leq C \|u\|_{k+1}, \quad u \in H^{k+1}_b(\Omega, E), \; h \neq 0.$$  

(ii) Let $v \in H^k_b(\Omega, E)$. If $\|\Theta_h v\|_{k; h \neq 0}$ are bounded by a constant $M > 0$, then the distribution derivative $\partial_r v$ belongs to $H^1_b(\Omega, E)$ and $\|\partial_r v\|_k \leq C_0 M$ with $C_0$ being a positive constant not depending on $v$.

(iii) Let $D$ be a linear differential operator of order $l$ form $E$ to $E'$, where $E, E'$ are bundles in $E^i; 0 \leq i \leq 2$, and let $k \geq l$. Then we can find a constant $C > 0$ such that

$$(7.27) \quad \|D, \Theta_h v\|_{k-l} \leq C \|v\|_k, \quad v \in H^k_b(\Omega, E), \; h \neq 0.$$  

**Proof.** Since the Sobolev space may be equivalently defined using any coordinate system, in proving these assertions, we may use the representation in terms of the polar
coordinates \((r, \theta)\), although the constants must be carefully chosen in deducing the estimates. To prove (i),(ii), we may also assume that \(u, v\) are functions of \((r, \theta) \in K\). The estimate (7.26) can be easily proved by using the Fourier transformation. Let us prove (ii) when \(k = 0\). We can choose a sequence \(\{h_j\}_{j=1,2,3,...}\) converging to zero in such a way that \(\{h_ju\}_{j=1,2,3,...}\) converges to an element \(w\) weakly in \(L^2(K)\). On the other hand, for any \(\varphi \in D(K)\), \(\langle \partial h v, \varphi \rangle = \langle \partial_\Omega v, \varphi \rangle\) converges to \(\langle \partial_\Omega v, \varphi \rangle\) as \(h \to 0\). Thus \(\partial_\Omega v = w\), and hence \(\partial_\Omega v \in L^2(K)\). Moreover \(\|\partial_\Omega v\| \leq \liminf_{j \to \infty} \|\partial h v\| \leq M\). This proves (ii) when \(k = 0\). The assertion (ii) when \(k > 0\) can be proved by applying what we have proved to the derivatives \(\partial_\theta \partial_\varphi v\) with \(p + q \leq k\).

To prove (iii), we shall use the product bundle representations of sections of \(E\) over \(\Omega\) in terms of the coordinates \((r, \theta)\), and those of \(E'\). Let \(D\) have the description
\[
D = \sum_{p+q=0}^I A_{p,q}(r, \theta) \partial_\varphi^p \partial_\Theta^q ,
\]
where \(A_{p,q}\)'s are matrices whose entries together with their derivatives are bounded on \(K\). then we have the description
\[
[D, \partial_\Omega]v = \sum_{p+q=0}^I (-\partial_\Omega A_{p,q}) t^*_x (\partial_\varphi^p \partial_\Theta^q v) .
\]
From this, we see that any derivative \(\partial_\varphi^p \partial_\Theta^q[D, \partial_\Omega]v\) of order \(p' + q' \leq k - l\) has the \(L^2\) norm bounded by a constant multiple of \(\sum_{p+q \leq k} \|\partial_\varphi^p \partial_\Theta^q v\|\). Hence we have (7.27). Q.E.D.

**Proof of Proposition 7.7.** In the proof, \(C\) will denote a generic positive constant. Let \(\chi_i\) \((i = 1,2)\) be real-valued smooth functions on \(\mathbb{R}^2\) such that
\[
\chi_1(x) + \chi_2(x) = 1 \quad (x \in \mathbb{R}^2), \quad \text{supp} \chi_1 \subset \Omega_{3/4}, \quad \text{supp} \chi_2 \cap \overline{\Omega_{1/2}} = \emptyset .
\]
Given \(v \in H^1_0(\Omega, E^1)\), we set \(v_i(x) = \chi_i(x)v(x)\). Then \(v = v_1 + v_2\) with \(v_1 \in H^1_0(\Omega, E^1)\), \(v_2 \in H^1_\delta(\Omega, E^1)\), and \(\text{supp} v_2 \subset \Omega_\delta\). We assert that if \(L v \in L^2(\Omega, E^1)\), then \(L v_i \in L^2(\Omega, E^1)\), \(\text{supp} L v_i \subset \text{supp} \chi_i (i = 1,2)\), and
\[
(7.28) \quad \|L v_i\| \leq C(\|L v\| + \|v_1\|) .
\]
To show this, fixing \(i\), we write \(\chi = \chi_i\). The function \(\chi\) is constant on a neighborhood of \(\partial \Omega\). Hence the commutators \([D_0^\alpha, \chi]\) and \([D_0^\alpha, \chi]\) (here \(\chi\) denotes the multiplication operator by the function \(\chi\)) are differential operators of order 0 vanishing identically near \(\partial \Omega\). Consider the pairing \(\langle L(\chi v), v' \rangle\) where \(v' \in H^1_0(\Omega, E^1)\) defined by (7.25). By integration by parts or by (7.15),(7.16), we readily see that \(\langle L(\chi v), v' \rangle\) can be expressed as the sum of \(\langle L v, \chi v' \rangle\) and a term \(R(v, v')\) satisfying the estimate \(|R(v, v')| \leq C\|v_1\|\|v'\|\).

Therefore we have
\[
|\langle L(\chi v), v' \rangle| \leq |\langle L v, \chi v' \rangle + R(u, v')| \leq C(\|L v\| + \|v_1\|)\|v'\|.
\]
Thus we can apply the Riesz representation theorem to conclude that there exists an element \(f \in L^2(\Omega, E^1)\) with \(\|f\| \leq C(\|L v\| + \|v_1\|)\) such that \(\langle L(\chi v), v' \rangle = (f, v')\) for all
Let us show that \( v_1 \in H^2(\Omega, E^i) \) (\( i = 1, 2 \)). We shall state only the proof of \( v_2 \in H^2(\Omega, E^1) \), since the proof of \( v_1 \in H^2(\Omega, E^1) \) is much more easy (cf. the interior regularity theorem).

Applying (7.25) to \( v' = \Theta_h v_2 \) gives rise to the estimate
\[
\|\Theta_h v_2\|_1 \leq C \|L(\Theta_h v_2)\|_{-1} \quad \text{for all } h \neq 0.
\]

Here we observe the following two estimates:
\[
\|\Theta_h v'\|_{-1} \leq C \|v'\|, \quad v' \in H_0^1(\Omega, E^1), \quad h \neq 0;
\]
\[
\||\mathcal{L}, \Theta_h v_2\| \leq C\|v_2\|_1 \leq C\|v\|_1, \quad h \neq 0.
\]

The estimate (7.30) is equivalent to the one that \( \|\Theta_h v', v''\| \) is equal to or less than \( C\|v''\|/\|v\| \) for all \( v'' \in H_0^1(\Omega, E^1) \), which is readily verified by using (7.26). The estimate (7.31) can be shown by using (7.30).

Using (7.30), from (7.29) and (7.31), we obtain
\[
\|\Theta_h v_2\|_1 \leq C(\|\mathcal{L} v\| + \|v\|_1) \quad \text{for all } h \neq 0.
\]

By (ii) of Lemma 7.8, this implies that \( \partial_0 v_2 \in H_0^1(\Omega, E^1) \), and \( \|\partial_0 v_2\|_1 \) is estimated by a constant multiple of \( M = \|\mathcal{L} v\| + \|v\|_1 \). This implies in turn that the derivatives \( \partial_0^p \partial_0^q v_2 \) (\( p + q \leq 2, p < 2 \)) belong to \( L^2(\Omega, E^1) \), and their \( L^2 \) norms are estimated by a constant multiple of \( M \). Since \( v_2 \in H_0^1(\Omega, E^1) \) and \( f_2 = \mathcal{L} v_2 \in L^2(\Omega, E^1) \), it follows from (7.24) that \( \Delta_0 v_2 = f_2 \in \mathcal{D}'(\Omega, E^1) \). The differential operator \( \Delta_0 \) is determined elliptic, and hence the equality \( \Delta_0 v_2 = f_2 \) implies the equality
\[
\partial_0^2 v_2 = \sum_{p+q \leq 2, p < 2} A_{p,q} \partial_0^p \partial_0^q v_2 + A f_2,
\]
where \( v_2 \) denotes its product bundle representations in terms of \((r, \theta)\), and \( A \)'s are square matrices whose entries together with their derivatives of any order are bounded on \( \Omega \).

From this equality we know that \( \partial_0^2 v_2 \in L^2(\Omega, E^1) \) and \( \|\partial_0^2 v_2\| \) is estimated by a constant multiple of \( M + \|\mathcal{L} v_2\| \). Thus we have \( v_2 \in H^2(\Omega, E^1) \), and bearing in mind (7.28), we also see that \( \|v_2\|_2 \) is estimated by a constant multiple of \( \|\mathcal{L} v\| + \|v\|_1 \).

We next show that \( v \) satisfies the boundary condition, that is, \( v_0 = \sigma(\partial_0^1) v = \sigma(\partial_0^1) \) vanishes. Since \( \mathcal{L} v = \Delta_0 v \) in \( \mathcal{D}'(\Omega, E^1) \) and \( v \in H^2(\Omega, E^1) \), as we have already shown, we can apply Green’s formula (7.17) to obtain
\[
\int_{\partial \Omega} \langle v_0, v' \rangle ds = 0 \quad \text{for all } v' \in H_0^1(\Omega, E^1).
\]

Let \( w \) be any smooth section of \( E^2 \) over \( \partial \Omega \). Construct a smooth extension \( v' \) of \( \sigma(\partial_0^1) w \) to \( \Omega \). Since \( \sigma(\partial_0^1) v \circ \sigma(\partial_0^1) = 0 \), this \( v' \) belongs to \( H_0^1(\Omega, E^1) \). Hence the above equality implies that
\[
\int_{\partial \Omega} \langle v_0, \sigma(\partial_0^1) w \rangle ds = 0 \quad \text{for all } w \in C_0^\infty(\partial \Omega, E^2).
\]
Observing that $\sigma_D(D_0^1v) = \sigma_D(D_0^1v)'$ and that $C^\infty(\partial \Omega, E^2)$ is dense in $L^2(\partial \Omega, E^2)$, we can deduce that $\sigma_D(D_0^1v)' = 0$. From this it follows that $v_\theta = 0$, for the mapping $\sigma_D(D_0^1)$ is injective in the image of $\sigma_D(D_0^1)'$.

Let $v \in H^2_B(\Omega, E^1)$. Using Green’s formula (7.17), we can show that $L v = \Delta_0 v \in L^2(\Omega, E^1)$. Hence the above argument indicates that the required estimate is valid. Q.E.D.

**Proposition 7.9.** Assume that $D_0^1$ is elliptic. Then, for each integer $k \geq 1$, the following are valid.

(i) If $v \in H^k_B(\Omega, E^1)$ and $L v = \Delta_0 v \in H^{k-1}(\Omega, E^1)$, then $v \in H^{k+1}_B(\Omega, E^1)$;

(ii) There exists a constant $C > 0$ such that

$$
||v||^2_{k+1} \leq C(||\Delta_0 v||^2_{k-1} + ||v||^2_1), \quad v \in H^{k+1}_B(\Omega, E^1).
$$

**Proof.** We shall prove the assertions by induction on $k$. When $k = 1$, the assertions are nothing else but those of Proposition 7.7. For each $k > 1$, assuming that the assertions with $k$ replaced by $1, 2, \ldots, k-1$ are valid, we shall prove the assertions (i),(ii). Let $v \in H^k_B(\Omega, E^1)$ satisfy $\Delta_0 v \in H^{k-1}(\Omega, E^1)$. Then by induction hypothesis, we have $v \in H^k_B(\Omega, E^1)$. We first observe that $\theta_{\alpha} v \in H^k_B(\Omega, E^1)$ for all $\alpha \neq 0$. In fact, we already see that $\theta_{\alpha} v \in H^k_B(\Omega, E^1)$. Let $v = (v_1, v_2)$ (cf. (7.18)), and (7.19) be the description of $v_1$ in terms of the coordinates $(r, \theta)$. Then the boundary condition $\sigma_D((D_0^1v)'(D_0^1v))_{\partial \Omega} = 0$ is stated as follows:

$$(\partial_r a^{\alpha}) - \partial_\theta a^{(r \alpha)} + a^{\alpha(\theta)})|_{r=1} = 0.$$ 

From this description, we know that $f^*_h v$ satisfies the boundary condition for any $h$, and hence so does $\theta_{\alpha} v$. Now let $v = v_1 + v_2$ be the decomposition given at the beginning of the proof of Proposition 7.7. It suffices to prove that, for each $i = 1, 2$, $v_i \in H^{k+1}(\Omega, E^1)$, and $||v_i||^2_{k+1}$ is estimated by a constant multiple of $M^2 = ||\Delta_0 v||^2_{k-1} + ||v||^2_1$. As before $C, C'$ will denote generic positive constants. Applying the induction hypothesis, we have

$$||\theta_{\alpha} v_i||^2_{k+1} \leq C(||\Delta_0 \theta_{\alpha} v_2||^2_{k-2} + ||\theta_{\alpha} v_2||^2_{k+1}) \quad \text{for all} \quad \alpha \neq 0.$$ 

Combining this with the estimates (7.26) and (7.27), we have

$$||\theta_{\alpha} v_2||^2_{k+1} \leq C(||\Delta_0 v_2||^2_{k-1} + ||v_2||^2_1) \quad \text{for all} \quad \alpha \neq 0.$$ 

Clearly the term on the right side is estimated by a constant multiple of $M^2$. Applying (ii) of Lemma 7.8, we conclude that $\theta_{\alpha} v_2 \in H^2(\Omega, E^1)$, and $||\theta_{\alpha} v_2||_k \leq CM$. In other words, the derivatives $\partial^p \theta_{\alpha} v_2 \ (q \geq 1, p + q \leq k + 1)$ belong to $L^2(\Omega, E^1)$, and their $L^2$ norms are estimated by $CM$. Observe also that, for each $p \leq k$, $||\partial^p \theta_{\alpha} v_2||_k$ is estimated by $C' ||v_2||_k \leq CM$. Using the equality obtained by operating $\partial^{k-1}$ to the equality (7.32), we also know that $\partial^{k+1} v_2 \in L^2(\Omega, E^1)$, and $||\partial^{k+1} v_2|| \leq CM$. Thus we have proved the required result for $v_2$. By a more simple argument, we have the same result for $v_1$. Thus we have proved (i), and the estimate (7.33) holds for such $v$. If $v \in H^{k+1}_B(\Omega, E^1)$, then $v$ satisfies the assumption of (i). Hence from the above result, we have the estimate (7.33). Q.E.D.
THEOREM 7.10. Assume that $D^0_0$ is elliptic. Then, for any integer $k \geq 0$, the differential operator
\begin{equation}
\Delta_0 : H^{k+2}_B(\Omega, E^1) \rightarrow H^k(\Omega, E^1)
\end{equation}
is bijective (topologically isomorphic), and the estimates
\begin{equation}
C_0 \|v\|_{k+2} \leq \|\Delta_0 v\|_k \leq C_1 \|v\|_{k+2}, \quad v \in H^{k+2}_B(\Omega, E^1)
\end{equation}
hold with constants $C_0, C_1 > 0$.

Proof. From Green’s formula (7.17) and Schwarz’s inequality, it follows that if $v \in H^2_B(\Omega, E^1)$, \[
\|(D^0_0)^t v\|^2 + \|D^1_0 v\|^2 = (\Delta_0 v, v) \leq \delta^{-1}\|\Delta_0 v\|^2 + \delta \|v\|^2
\]
for any number $\delta > 0$. Combining this estimate in which $\delta$ is chosen to be sufficiently small with the estimate in Lemma 7.5, we have \[
\|v\|_1 \leq C \|\Delta_0 v\|, \quad v \in H^2_B(\Omega, E^1)
\]
with a constant $C > 0$. From this estimate and the estimates (7.33) with $k$ being replaced by 1, 2, \ldots, $k+1$, it follows the inequality on the left side of (7.35). The inequality on the right side of (7.35) is obviously valid.

From the estimates (7.35), we know that $\Delta_0$ is continuous and injective. Let $f \in H^k(\Omega, E^1)$. Then, by Proposition 7.6, there exists a (unique) element $v \in H^k_1(\Omega, E^1)$ such that $Lv = f$. By Proposition 7.7, $v \in H^2_B(\Omega, E^1)$ and $Lv = \Delta_0 v$. Hence we can apply Proposition 7.9, (i) to conclude that $v \in H^{k+2}_B(\Omega, E^1)$. Thus we have proved that $\Delta_0$ is surjective. Q.E.D.

Let $G_0 : H^k(\Omega, E^1) \rightarrow H^{k+2}_B(\Omega, E^1)$ be the inverse mapping of the operator $\Delta_0$ in (7.34); $G_0$ is a bijective linear operator satisfying
\begin{equation}
C_0^{-1} \|v\|_k \leq \|G_0 v\|_{k+2} \leq C_0^{-1} \|v\|_k, \quad v \in H^k(\Omega, E^1).
\end{equation}

Clearly we have
\begin{equation}
v = D^0_0 \circ (D^0_0)^t \circ G_0 v + (D^1_0)^t \circ D^1_0 \circ G_0 v, \quad v \in H^k(\Omega, E^1).
\end{equation}

Let us introduce the following real Hilbert spaces:
\begin{itemize}
\item $Z^k(\Omega) = \{v \in H^k(\Omega, E^1); D^1_0 v = 0\} \quad (k \geq 0)$,
\item $W^k_B(\Omega) = \{v \in H^k_B(\Omega, E^1); (D^0_0)^t v = 0\} \quad (k \geq 1)$.
\end{itemize}

LEMMA 7.11. Assume that $D^0_0$ is elliptic. Let $k$ be a positive integer. Let $\rho : H^k(\Omega, E^1) \rightarrow H^k(\Omega, E^1)$ be the linear operator defined by
\[
\rho(v) = D^0_0 \circ (D^0_0)^t \circ G_0 v.
\]

(i) $\rho$ is a projection from $H^k(\Omega, E^1)$ onto $Z^k(\Omega)$, that is, $\rho$ is a bounded linear operator from $H^k(\Omega, E^1)$ onto $Z^k(\Omega)$ satisfying
\begin{equation}
\rho(v) = D^0_0 \circ (D^0_0)^t \circ G_0 v = v \quad \text{if} \quad v \in Z^k(\Omega).
\end{equation}
Theorem 7.12. Assume that $D_0^1$ is elliptic. Let $k$ be a positive integer. Then we can find a closed subspace $Y^{k+1}(\Omega)$ of $H^{k+1}(\Omega, E^0)$ for which the linear differential operator

$$D_0^1 : Y^{k+1}(\Omega) \rightarrow Z^k(\Omega)$$

is bijective, and satisfies the estimates

$$C_0 \|u\|_{k+1} \leq \|D_0^1 u\|_k \leq C_1 \|u\|_{k+1}, \quad u \in Y^{k+1}(\Omega)$$

with constants $C_0, C_1 > 0$.

Proof. Set $Y^{k+1}(\Omega) = (D_0^1)^t \circ G_0(Z^k(\Omega))$. Clearly $Y^{k+1}(\Omega)$ is a subspace of $H^{k+1}(\Omega, E^0)$. Let $\{u_j\}_{j=1,2,3,\ldots}$ be a Cauchy sequence in $Y^{k+1}(\Omega)$; $u_j = (D_0^1)^t \circ G_0 z_j$ ($j = 1, 2, 3, \ldots$) with $z_j \in Z^k(\Omega)$. By (7.38), $D_0^1 u_j = z_j$ ($j = 1, 2, \ldots$), and hence $\{z_j\}$ is a Cauchy sequence in $Z^k(\Omega)$. Since $Z^k(\Omega)$ is complete, the limit $z^t = \lim_{j \rightarrow \infty} z_j \in Z^k(\Omega)$ exists. It follows that the limit $u^t = \lim_{j \rightarrow \infty} u_j$ is equal to $\lim_{j \rightarrow \infty}(D_0^1)^t \circ G_0 z_j = (D_0^1)^t \circ G_0 z^t$, and hence that $u^t \in Y^{k+1}(\Omega)$. Thus $Y^{k+1}(\Omega)$ is closed.

By virtue of (7.38), the operator $D_0^1$ of (7.40) is surjective. The injectivity of this operator is a consequence of (7.41), which we shall now prove. Let $u \in Y^{k+1}(\Omega); u = (D_0^1)^t \circ G_0 z, z \in Z^k(\Omega)$. Then $D_0^1 u = z$. Hence, by continuity of the operators, we have $\|u\|_{k+1} \leq C' \|G_0 z\|_{k+2} \leq C'' \|z\|_k, C', C'' > 0$ being constants. This proves the inequality on the left side of (7.41). The one on the right side of (7.41) is obvious. Q.E.D.
7.4. Operators $D^0$, $D^1$ on $\Omega_\epsilon$ and some auxiliary operators

We are now in a position to deduce a result concerning the differential operators $D^0$, $D^1$ in (7.3) on $\Omega_\epsilon$. This result is crucial in the proof of our existence theorem which will be carried out in section 8.

**Proposition 7.13.** Assume that (A.1)–(A.3) hold, and that $D^0$ is elliptic. Let $k$ be a positive integer; then we can find a number $\epsilon_0 \in (0, 1]$ such that, for each $\epsilon \in (0, \epsilon_0]$, there exist a closed subspace $Y^{k+1}(\Omega_\epsilon)$ of $H^{k+1}(\Omega_\epsilon, E^0)$ and a closed subspace $Z^k(\Omega_\epsilon)$ of $H^k(\Omega_\epsilon, E^1)$ together with an operator $\rho_\epsilon : H^k(\Omega_\epsilon, E^1) \to Z^k(\Omega_\epsilon)$ which possess the following properties (i)–(iii) in which by constants we mean positive constants which do not depend on $\epsilon \in (0, \epsilon_0]$:

(i) $\rho_\epsilon$ is a projection; $\rho_\epsilon$ is a bounded linear operator with $\rho_\epsilon(z) = z$ for all $z \in Z^k(\Omega_\epsilon)$. Moreover the operator norm $\|\rho_\epsilon\|$ is estimated by $C\epsilon^{-k}$, where $C$ is a constant.

(ii) There exists a constant $C$ such that

$$
\epsilon^{-2} \|v\|^2 + \|v\|_1^2 \leq C\|D^1v\|^2, \quad v \in \ker \rho_\epsilon ;
$$

(iii) Let $T_\epsilon$ be the composition of $D^0 : Y^{k+1}(\Omega_\epsilon) \to H^k(\Omega_\epsilon, E^1)$ and $\rho_\epsilon$. Then $T_\epsilon : Y^{k+1}(\Omega_\epsilon) \to Z^k(\Omega_\epsilon)$ is bijective, and there exist constants $C_0, C_1$ such that

$$
C_0 \epsilon^k \|u\|_{k+1} \leq \|T_\epsilon u\|_k \leq C_1 \epsilon^{-k} \|u\|_{k+1}, \quad u \in Y^{k+1}(\Omega_\epsilon).
$$

**Proof.** The ellipticity of $D^0$ implies that there exist non-characteristic real cotangent vectors for the symbol $g_1$ (cf. sections 1.3). By (IV) of Lemma 1.1, we see that $s_2 = 0$. Hence we can apply Proposition 7.2 to the operators $D^0$, $D^1$. From the construction we know that the differential operator $D^0_\epsilon$ is elliptic. Thus we can use all the results obtained in subsection 7.3.

Let $\epsilon \in (0, 1]$. Using the topological isomorphisms $\tau_\epsilon^* : H^k(\Omega_\epsilon, E^i) \to H^k(\Omega_1, E^i)$ (cf. subsection 7.2), we set

$$
Y^{k+1}(\Omega_\epsilon) = (\tau_\epsilon^*)^{-1}(Y^{k+1}(\Omega_1)), \quad Z^k(\Omega_\epsilon) = (\tau_\epsilon^*)^{-1}(Z^k(\Omega_1)),
$$

and $\rho_\epsilon = (\tau_\epsilon^*)^{-1} \circ \rho \circ \tau_\epsilon^*$, where $Y^{k+1}(\Omega_1)$, $Z^k(\Omega_1)$, $\rho$ are the spaces and the operator in Lemma 7.11 and Theorem 7.12. Clearly $Y^{k+1}(\Omega_\epsilon)$, $Z^k(\Omega_\epsilon)$ are closed subspaces, and $\rho_\epsilon : H^k(\Omega_\epsilon, E^1) \to Z^k(\Omega_\epsilon)$ is a projection for each $\epsilon \in (0, 1]$. Bearing in mind (i) of Lemma 7.3, we see that $\|\rho_\epsilon(v)\|_k \leq C\epsilon^{-k} \|v\|_k$ for $v \in H^k(\Omega_\epsilon, E^1)$. This proves (i).

By (ii) of Lemma 7.11, we have the estimate

$$
\|\tau_\epsilon^* v\|_1 \leq C\|D^1_\epsilon(\tau_\epsilon^* v)\|, \quad v \in \ker \rho_\epsilon.
$$

Therefore, bearing in mind (7.12), we find that there exists a number $\epsilon_1 \in (0, 1]$ such that, for each $\epsilon \in (0, \epsilon_1]$, we have the estimate

$$
\|\tau_\epsilon^* v\|_1 \leq C'\|D^1_\epsilon(\tau_\epsilon^* v)\|, \quad v \in \ker \rho_\epsilon
$$

with another constant $C'$. Applying Lemma 7.3, we can deduce the estimate (7.42).
a number $\epsilon_2 \in (0, 1)$ such that, for each $\epsilon \in (0, \epsilon_2]$, $L_\epsilon$ is bijective and the estimates

$$C_0 \|u\|_{k+1} \leq \|L_\epsilon u\|_k \leq C_1 \|u\|_{k+1}, \quad u \in Y^{k+1}(\Omega_1)$$

hold with constants $C_0, C_1 > 0$. In fact, the estimate on the right side is obvious. To show the estimate on the left side, observe that

$$\|\rho \circ (D_\epsilon^0 - D_0^0)u\|_k \leq C(\epsilon)\|u\|_{k+1},$$

where $C(\epsilon)$ is a positive constant with $C(\epsilon) \to 0$ as $\epsilon \to +0$. Hence, using (7.41), for any sufficiently small $\epsilon$, we have

$$\|L_\epsilon u\|_k = \|\rho D_\epsilon^0 u\|_k \geq \|D_0^0 u\|_k - \|\rho (D_\epsilon^0 - D_0^0)u\|_k \geq C_0^0 \|u\|_{k+1} - C(\epsilon)\|u\|_{k+1} \geq \frac{C_0^0}{2} \|u\|_{k+1},$$

where $C_0^0$ denotes the constant $C_0$ in (7.41). Let us prove that $L_\epsilon$ is bijective. The estimate just proved implies that $L_\epsilon$ is injective. By Theorem 7.12, the operator $D_0^0$ of (7.40) is bijective. From (7.44), it follows that, for each sufficiently small $\epsilon$, $\|L_\epsilon - D_0^0\| \leq C_0^0/2$. Let $z \in Z^k(\Omega_1)$. We show that there is $u \in Y^{k+1}(\Omega_1)$ such that $L_\epsilon u = z$. In fact, this equation is equivalent to the equation

$$u + (D_0^0)^{-1} \circ (L_\epsilon - D_0^0)u = (D_0^0)^{-1}z.$$

Since $\|(D_0^0)^{-1} \circ (L_\epsilon - D_0^0)\| \leq 1/2$, one can solve this by the method of iteration. Thus we can find a number $\epsilon_2 \in (0, 1]$ such that, for each $\epsilon \in (0, \epsilon_2]$, $L_\epsilon$ is injective and surjective. By virtue of (i) of Lemma 7.3, what we have just proved implies that, if $\epsilon < \epsilon_2$, $T_\epsilon$ is bijective and the estimates in (7.43) hold.

Set $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. Then (i)–(iii) hold true for each $\epsilon \in (0, \epsilon_2]$. Q.E.D.

8. Existence of smooth solutions

We shall first treat a nonlinear differential operator. Let $E^0, E^1$ be fibered manifolds over a manifold $X$.

**Theorem 8.1.** Let $\Phi$ be a nonlinear differential operator of the first order from $E^0$ to $E^1$, and $\psi : J_1(E^0) \to E^1$ be the fibered morphism associated with $\Phi$. Let $v_0$ be a smooth section of $E^1$. Assume that

(i) $\dim X = 2$,

(ii) $\Phi = \psi \circ j_1$ is involutive over $v_0$,

(iii) the symbol morphism $\sigma(\Phi)$ is surjective, and

(iv) $\Phi$ is elliptic.

Let $x_0 \in X$, and $P_0$ be a point of $J_1(E^0)$ with $\psi(P_0) = v_0(x_0)$. Then, for any neighborhood $U$ of $P_0$ in $J_1(E^0)$, there exists a smooth section $u$ of $E^0$ defined over a neighborhood $U$ of $x_0$ such that

$$\Phi(u)(x) = v_0(x) \text{ in } U \quad \text{and} \quad j_1(u)(x) \in U \quad (x \in U).$$
Proof. Since the assertion is local, using fibered charts over a coordinate neighborhood of $X$ around $x_0$, we may and shall assume that $X$ is the plane $\mathbb{R}^2$, and each $E^i$ is the product vector bundle $E^i = X \times \mathbb{R}^{m_i}$, $i = 0, 1$. By Lemma 4.3, we can choose a smooth section $u_0$ of $E^0$ such that

\[ j_1(u_0)(x_0) = p_0, \quad j_k(\Phi(u_0))(x_0) = j_k(v_0)(x_0) \quad \text{for all } k > 0. \]

Denote by $D^0$ the linearization of $\Phi$ at $u_0$; $D^0 = D_{u_0} \Phi$ is a linear differential operator of order 1 from $E^0$ to $E^1$. By Lemma 5.1, $D^0$ is quasi-involutive, the symbol morphism $\sigma(D^0)$ is surjective, and $D^0$ is elliptic. Hence, as we noted at the beginning in the proof of Proposition 7.13, this operator $D^0$ satisfies all the assumptions imposed in section 7, and hence we may use the results obtained in section 7.

In what follows we shall keep the same notations and conventions as in section 7. In particular $D^1$ will denote the linear differential operator of order 1 from $E^1$ to $E^2$ constructed from $D^0$ by applying Lemma 3.1 (cf. the beginning of section 7), and $\Omega_\epsilon$ will denote the disc of radius $\epsilon > 0$ and with center $x_0$. Let us consider the nonlinear differential operator

\[ \Phi : H^{k+1}(\Omega_\epsilon, E^0) \rightarrow H^k(\Omega_\epsilon, E^1) \]

where $k$ is an integer greater than $\max\{m_0, 3\}$. By Proposition 6.2, the operator $\Phi$ of (8.1) is differentiable of class $C^1$, and its Fréchet derivative coincides with the continuous linear operator

\[ D^0 : H^{k+1}(\Omega_\epsilon, E^0) \rightarrow H^k(\Omega_\epsilon, E^1). \]

By Proposition 7.13, we can choose a number $\epsilon_0 \in (0, 1]$ such that the following are valid for each $\epsilon \in (0, \epsilon_0]$:

1. There exist a closed subspace $Y^{k+1}(\Omega_\epsilon)$ of $H^{k+1}(\Omega_\epsilon, E^0)$ and a closed subspace $Z^k(\Omega_\epsilon)$ of $H^k(\Omega_\epsilon, E^1)$ together with a projector $\rho_\epsilon : H^k(\Omega_\epsilon, E^1) \rightarrow Z^k(\Omega_\epsilon)$ which possess the following properties:

   - The linear operator
     \[ T_\epsilon = \rho_\epsilon \circ D^0 : Y^{k+1}(\Omega_\epsilon) \rightarrow Z^k(\Omega_\epsilon) \]
     is a topological isomorphism, and satisfies the estimates

     \[ C_0 \epsilon^k \|u\|_{k+1} \leq \|T_\epsilon u\|_k \leq C_1 \epsilon^{-k} \|u\|_{k+1}, \quad u \in Y^{k+1}(\Omega_\epsilon); \]

   \[ \|\rho_\epsilon\| \leq C_2 \epsilon^{-k}; \]

   \[ \epsilon^{-2} \|v\|^2 + \|v\|_1^2 \leq C_3 \|D^1 v\|^2, \quad v \in \ker \rho_\epsilon. \]

2. Here and below, $C_0, C_1, \ldots$ denote positive constants which do not depend on $\epsilon$, and $C$ denotes a generic positive constant not depending on $\epsilon$.

Let $\tau_0 : Y^{k+1}(\Omega_\epsilon) \rightarrow H^{k+1}(\Omega_\epsilon, E^0)$ be the translation defined by $\tau_0(u) = u_0 + u$, where $u_0$ denotes $u_0|_{\Omega_\epsilon} \in H^{k+1}(\Omega_\epsilon, E^0)$. Let us consider the nonlinear operator $F$ defined by

\[ F = \rho_\epsilon \circ \Phi \circ \tau_0 : Y^{k+1}(\Omega_\epsilon) \rightarrow Z^k(\Omega_\epsilon). \]
Lemma 6.1 to obtain by virtue of (i) of Lemma 7.3. Hence we can use the above estimate and (ii) of (III) in valid for all $f$

$$
(8.8)
$$

holds for any $w$

$$
(8.6)
$$

Since $\rho_k$ is a continuous linear operator and $\tau_0$ is a translation, $Dv_1 \rho_k$ equals $\rho_k$ and $Dv_\tau 0$ is the inclusion $\iota : Y^{k+1}(\Omega_\epsilon) \to H^{k+1}(\Omega_\epsilon, E^0)$. Thus $D_F = \rho_k \circ D_{u_1} \Phi \circ \iota$. In particular the Fréchet derivative $D_{u_1} F$ at the zero element $0 \in Y^{k+1}(\Omega_\epsilon)$ coincides with $T_\epsilon$ of (8.2).

Let us discuss the continuity of $D_u F$ with respect to $u$.

(I) Let $M_0 > 0$ be a given arbitrary constant. We can choose a constant $C_4$ not depending on $\epsilon \in (0, 1]$ in such a way that the operator norm of the linear operator $D_{u_1} F - D_{u_1} F : Y^{k+1}(\Omega_\epsilon) \to Z^k(\Omega_\epsilon)$ admits the estimate

$$
(8.6)
$$

$$
\|D_{u_2} F - D_{u_1} F\| \leq C_4 \epsilon^{-(2k+1)}\|u_2 - u_1\|_{k+1}
$$

for any $u_1, u_2 \in Y^{k+1}(\Omega_\epsilon)$ with $\|u_1\|_{k+1}, \|u_2\|_{k+1} \leq M_0 \epsilon$.

Proof of (I). The operator $D_{u_2} F - D_{u_1} F$ is equal to $\rho_k \circ (D_{u_2} \Phi - D_{u_1} \Phi) \circ \iota$ where $w_1 = u_0 + u_1$. We shall prove that the estimate

$$
(8.7)
$$

$$
\| (D_{u_2} \Phi - D_{u_1} \Phi) \phi \|_k \leq C_5 \epsilon^{-(k+1)}\|u_2 - u_1\|_{k+1} \| \phi \|_{k+1}
$$

holds for any $u_1, u_2 \in H^{k+1}(\Omega_\epsilon, E^0)$ satisfying $\|u_1\|_{k+1}, \|u_2\|_{k+1} \leq M_0 \epsilon$ and for any $\phi \in H^{k+1}(\Omega_\epsilon, E^0)$.

The argument in the proof of Proposition 6.2 indicates that it suffices to show that, under the circumstances of the Sublemma in that proof with $U$ being replaced by $\Omega_\epsilon$, the estimate

$$
(8.8)
$$

$$
\| (D_{f_2} A - D_{f_1} A) g \|_k \leq C \epsilon^{-(k+1)}\|f_2 - f_1\|_k \| g \|_k
$$

holds for any $f_1, f_2 \in H^k(\Omega_\epsilon, \mathbb{R}^N)$ satisfying $\|f_1\|_k, \|f_2\|_k \leq M_0 \epsilon$ and for any $g \in H^k(\Omega_\epsilon, \mathbb{R}^N)$. In proving (8.8), as in the proof of the Sublemma, we assume for simplicity that $N = 1$. Denote $\Gamma = (D_{f_2} A - D_{f_1} A) g \in H^k(\Omega_\epsilon)$. We shall use the notation

$$
Q(b; f_1, f_2) = b(x, f_2(x)) - b(x, f_1(x))
$$

where $b = b(x, \epsilon)$ is a smooth function defined on $U_\epsilon \times \mathbb{R}^1$. The Sublemma indicates that $\Gamma = Q(\delta; a_1, f_1, f_2) g(x)$. We shall use the isomorphism $\tau_\epsilon^* : H^k(\Omega_\epsilon) \to H^k(\Omega_\epsilon)$ introduced in subsection 7.2. For a given function $b$, we denote $b_\epsilon(x, y) = b(\epsilon x, y)$, $\epsilon \in (0, 1]$. Clearly $[b_\epsilon]_{k, \Omega_\epsilon \times K_\epsilon}$ is estimated by $C \frac{|b|}{k_{\Omega_\epsilon \times K_\epsilon}}$, where $K_\epsilon \subset \mathbb{R}^1$ is a compact set and $C$ is a constant not depending on $\epsilon \in (0, 1]$. Therefore we can apply the estimate (6.4) to obtain the estimate

$$
\| Q(b_\epsilon; f_1^\epsilon, f_2^\epsilon) \|_k \leq C \| f_1^\epsilon - f_2^\epsilon \|_k
$$

valid for all $f_1^\epsilon, f_2^\epsilon \in H^k(\Omega_\epsilon)$ satisfying $\|f_1^\epsilon\|_k, \|f_2^\epsilon\|_k \leq M_0$. Now it is clear that $\tau_\epsilon^* \Gamma = Q(\delta; b_\epsilon, f_1^\epsilon, f_2^\epsilon) \tau_\epsilon^* g$. The condition $\|f_1\|_{k, \Omega_\epsilon} \leq M_0 \epsilon$ implies $\|\tau_\epsilon^* f_1 \|_{k, \Omega_\epsilon} \leq M_0$ by virtue of (i) of Lemma 7.3. Hence we can use the above estimate and (ii) of (III) in Lemma 6.1 to obtain

$$
\| \tau_\epsilon^* \Gamma \|_k \leq C \| \tau_\epsilon^* (f_2 - f_1) \|_k \| \tau_\epsilon^* g \|_k.
$$
Using (i) of Lemma 7.3, from this we obtain the required estimate (8.8). From (8.7) and (8.4) it follows the estimate (8.6). This completes the proof of (I).

(II) For any integer \( q > 0 \), there exists a constant \( C_6 \) such that

\[
\|F(0) - \rho_k (v_0)\| \leq C_6 \epsilon^q, \quad \epsilon \in (0, \epsilon_0).
\]

Proof of (II). By definition, \( F(0) = \rho_k \circ \Phi(u_0) \). Set \( q_1 = q + k \). Since \( j_{k+q_1-1}(\Phi(u_0)) (x_0) \) is equal to \( j_{k+q_1-1}(v_0)(x_0) \), applying the Taylor’s formula, we can show that \( \|\Phi(u_0) - v_0\| \leq C \epsilon^{q_1-1} \). This implies that \( \|\Phi(u_0) - v_0\|_k,\Omega \leq C \epsilon^{q_1} \). From this and (8.4), it follows the required estimate.

(III) Let \( F : Y \to Z \) be a nonlinear mapping between Banach spaces. Denote by \( B(u_0; r) \) the closed ball \( \{u \in Y; \|u - u_0\| \leq r\} \) with center \( u_0 \) and of radius \( r \). Assume that

(i) \( F \) is differentiable of class \( C^1 \),
(ii) the Fréchet derivative \( D_{u_0} F : Y \to Z \) at \( u_0 \) is bijective,
(iii) \( \|(D_{u_0} F)u\| \geq a_0 \|u\| \quad (u \in Y) \), \( a_0 \) being a positive constant, and
(iv) \( \|D_k F - D_{u_0} F\| \leq \delta \quad (u \in B(u_0; r)) \), where \( \delta \) is a positive number less than \( a_0 \).

Then, the image of \( B(u_0; r) \) by \( F \) contains the ball

\[
\{z \in Z; \|z - F(u_0)\| \leq r(a_0 - \delta)\}.
\]

Proof of (III). Though this is essentially a known implicit function theorem (cf. Schwartz [26], Lang [20]), we give a proof for completeness. We may assume that \( u_0 = 0 \), \( F(u_0) = 0 \). Let \( z \in Z \), and \( \|z\| \leq r(a_0 - \delta) \). Set \( u_0 = 0 \), and define \( u_n \) \((n = 1, 2, \ldots)\) inductively by

\[
(D_{u_0} F)u_n + (F(u_{n-1}) - (D_{u_0} F)u_{n-1}) = z.
\]

Denoting \( u(n-1; t) = u_{n-2} + t(u_{n-1} - u_{n-2}) \), we have

\[
F(u_{n-1}) - F(u_{n-2}) = \int_0^1 (D_{u(n-1; t)} F)(u_{n-1} - u_{n-2}) dt.
\]

Using this and the assumptions (iii),(iv), we find no difficulty in proving by induction on \( n = 1, 2, \ldots \) that \( \|u_n\| \leq r \) \((n \geq 1)\), and

\[
\|u_{n} - u_{n-1}\| \leq r \left(1 - \frac{\delta}{a_0}\right) \left(\frac{\delta}{a_0}\right)^{n-1} \quad (n \geq 1).
\]

It follows that the sequence \( \{u_n\}_{n=0,1,2,\ldots} \) is a Cauchy sequence in \( B(u_0; r) \). Set \( u^* = \lim_{n \to \infty} u_n \in B(u_0; r) \). Letting \( n \) tend to infinity in (8.9), by continuity, we have \( F(u^*) = z \). This completes the proof of (III).

(IV) There exist a number \( \epsilon_1 \in (0, \epsilon_0] \) and constants \( r_0, r_1 > 0 \) such that, for any \( \epsilon \in (0, \epsilon_1] \), the image of the closed ball

\[
\{u \in Y^{k+1}(\Omega_\epsilon); \|u\|_{k+1} \leq r_0 \epsilon^{3k+1}\}
\]

by \( F \) contains the closed ball

\[
\{z \in Z^k(\Omega_\epsilon); \|z - F(0)\| \leq r_1 \epsilon^{4k+1}\}.
\]
Proof of (IV). As we already stated, the mapping $F$ is differentiable of class $C^1$. Since $D_0F$ is equal to $T_e$ of (8.2), $D_0F$ is bijective, and (8.3) implies that
$$
\| (D_0F)u \|_k \geq C_0 \epsilon^k \| u \|_{k+1}, \quad u \in Y^{k+1}(\Omega_e).
$$
By (I), given a constant $M_0 > 0$, we have the estimate
$$
\| D_0F - D_0F \| \leq C_4 \epsilon^{-(2k+1)} \| u \|_{k+1}, \quad \text{for } u \in Y^{k+1}(\Omega_e) \text{ with } \| u \|_{k+1} \leq M_0 \epsilon.
$$
To apply the theorem (III) to the operator $F$ with $Y = Y^{k+1}(\Omega_e)$, $Z = Z^k(\Omega_e)$, $u_0 = 0$, we set
$$
a_0 = C_0 \epsilon^k, \quad \delta = \frac{1}{2} C_0 \epsilon^k, \quad r = \frac{C_0}{2 C_4} \epsilon^{3k+1}.
$$
Choose $\epsilon_1 \in (0, \epsilon_0]$ in such a way that $r \leq M_0 \epsilon$ for all $\epsilon \in (0, \epsilon_1]$ (cf. (I)). Then, for each $\epsilon \in (0, \epsilon_1]$, we can apply (III) to obtain the desired result with $r_0 = C_0/(2C_4)$, $r_1 = C_0^2/(4C_4)$. This completes the proof of (IV).

(V) Given any number $\delta > 0$, we can choose a number $\epsilon_2 \in (0, \epsilon_1]$ in such a way that, for any $\epsilon \in (0, \epsilon_2]$, there exists $u \in Y^{k+1}(\Omega_e)$ such that $F(u) = \rho_\epsilon(v_0)$ and $|u|_{1, \Omega_e} < \delta$.

Proof of (V). By applying (II) with $q = 4k + 2$, we have
$$
\| F(0) - \rho_\epsilon(v_0) \|_k \leq C_6 \epsilon^{4k+2}, \quad \epsilon \in (0, \epsilon_0].
$$
We also have
$$
(8.10) \quad \| u \|_{1, \Omega_e} \leq C_7 \epsilon^{-2} \| u \|_{k, \Omega_e}, \quad u \in H^k(\Omega_e, E^0).
$$
In fact, when $\epsilon = 1$, this holds true by the Sobolev embedding theorem ((II) of Lemma 6.1).

Bearing in mind (i) of Lemma 7.3 and the estimate $|u|_{1, \Omega_e} \leq \epsilon^{-1} |r_0^* u|_{1, \Omega_e}$, we can use the isomorphism $r_0^* : H^k(\Omega_e, E^0) \to H^k(\Omega_1, E^0)$ to obtain the desired estimate (8.10).

Let us choose $\epsilon_2 \in (0, \epsilon_1]$ so that, for any $\epsilon \in (0, \epsilon_2]$, the inequalities
$$
C_6 \epsilon^{4k+2} \leq r_1 \epsilon^{4k+1}, \quad (r_0 \epsilon^{3k+1})(C_7 \epsilon^{-2}) < \delta
$$
hold, where $r_0, r_1$ are constants in (IV). Then, for any $\epsilon \in (0, \epsilon_2]$, we can apply (IV) to conclude that there exists an element $u \in Y^{k+1}(\Omega_e)$ with $|u|_{k+1} \leq r_0 \epsilon^{3k+1}$ such that $F(u) = \rho_\epsilon(v_0)$. By (8.10), we have $|u|_{1, \Omega_e} < \delta$ as required, whence the proof of (V) is complete.

(VI) There exists a number $\epsilon_3 \in (0, 1]$ and a number $\delta_0 > 0$ such that the following is valid for any $\epsilon \in (0, \epsilon_3]$; if $u_1 \in H^{k+1}(\Omega_e, E^3)$ satisfies $F(u_1) = \rho_\epsilon(v_0)$ and $|u_1|_{1, \Omega_e} < \delta_0$, then $\Phi(u_0 + u_1) = v_0$ in $H^k(\Omega_e, E^1)$.

Proof of (VI). Set $v_1 = \Phi(u_0 + u_1) - v_0 \in H^k(\Omega_e, E^1)$. The hypothesis implies that $\rho_\epsilon(v_1) = 0$. Put $u = u_0 + u_1$. By (II) of Lemma 6.1, $u_1$ (and hence $u$) is differentiable of class $C^2$. Thus if $\epsilon$ and $\delta_0$ are sufficiently small, we can apply Lemma 5.2 to construct a linear differential operator $D_\epsilon$ of order 1 from $E^1|_U \to E^2|_U$ ($U$ being a neighborhood of $x_0$) which satisfies the conditions $D_\epsilon(v_1) = 0$, (ii),(iii) in (II) of Lemma 5.2. Bearing in mind also (I) of Lemma 5.2, we see at once that the coefficients of $\sigma(D_\epsilon)$ converge to the corresponding coefficients of $\sigma(D^1)$ in $C^0(\overline{\Omega_e})$-topology as $|u_1|_{1, \Omega_e}$ tends to zero. Choose a number $\gamma > 0$ such that $2C_3 \gamma < 1$ (cf. (8.5)). We can find a number $\epsilon' \in (0, \epsilon_0]$ and a
number $\delta_0 > 0$ such that, for any $\epsilon \in (0, \epsilon')$, if $u_1 \in H^{k+1}(\Omega_\epsilon, E^0)$ satisfies $|u_1|_{1,\Omega_\epsilon} < \delta_0$, then the estimate

$$\|D^1v\|^2 \leq \|\hat{D}_u v\|^2 + \gamma \|v\|^2 + C_7\|v\|^2, \quad v \in H^1(\Omega_\epsilon, E^1)$$

holds with a constant $C_7$ not depending on $\epsilon$ and also on $u_1$. Choose $\epsilon_3 \in (0, \epsilon')$ so that $\epsilon^{-2} \geq C_3 C_7$ for all $\epsilon \in (0, \epsilon_3]$. Then, if $\epsilon \in (0, \epsilon_3]$, from (8.5) and (8.11) we obtain $\|v\|^2 \leq 2\|\hat{D}_u v\|^2$ for $v \in \ker \rho_\epsilon$. Applying this to the section $v_1$ yields $v_1 = 0$. This proves (VI).

We are now in a position to complete the proof of Theorem 8.1. Given a neighborhood $\mathcal{U}$ of $P_0$, we can choose a number $\epsilon_4 \in (0, 1]$ and a number $\delta > 0$ with $\delta < \delta_0$ in such a way that, for any $\epsilon \in (0, \epsilon_4]$, if a continuously differentiable section $u$ satisfies $|u - u_0|_{1,\Omega_\epsilon} < \delta$, then $j_1(u)(x) \in \mathcal{U}$ (x $\in \Omega_\epsilon$). For this $\delta$, choose a number $\epsilon_2 \in (0, \epsilon_4]$ as in (V). Let $\epsilon > 0$ be any number less than $\epsilon_2, \epsilon_3,$ and $\epsilon_4$. Then we can apply (V) and (VI) to deduce that there exists an element $u \in H^{k+1}(\Omega_\epsilon, E^0)$ such that $\Phi(u) = v_0 = v_0$ in $H^k(\Omega_\epsilon, E^1)$ and $|u - u_0|_{1,\Omega_\epsilon} < \delta$. Since $k$ is chosen to be large enough, we can apply Proposition 4.2 to conclude that $u$ is a smooth section. This completes the proof of Theorem 8.1.

Let us next treat a nonlinear differential equation. Let $\mathcal{E}$ be a fibered manifold over $X$.

**Theorem 8.2.** Let $\mathcal{R}_i$ be a system of nonlinear partial differential equations of order $l$ on $\mathcal{E}$. Assume that $\dim X = 2$, and that $\mathcal{R}_i$ is involutive and elliptic. Let $P_0$ be a point of $\mathcal{R}_i$. Then, for any neighborhood $\mathcal{U}$ of $P_0$ in $J_1(\mathcal{E})$, there exists a smooth solution $u$ of $\mathcal{R}_i$ defined on a neighborhood $U$ of $x_0 = \pi_1^{-1}(P_0)$ in $X$ such that $j_1(u)(x) \in \mathcal{U}$ (x $\in U$).

**Proof.** The proof consists of showing the way we apply Theorem 8.1 to obtain Theorem 8.2. By virtue of Theorem 2.3, it suffices to prove the assertion under the following additional assumptions: The order $l$ is equal to 1; The projection $\pi_1^0 : \mathcal{R}_1 \to \mathcal{E}$ is a surjective submersion; $\mathcal{R}_1$ is an affine bundle over $\mathcal{E}$ modeled on a vector bundle $W$, and is an affine subbundle of the affine bundle $J_1(\mathcal{E})$ modeled on the vector bundle $T^* \otimes V(\mathcal{E})$. Observe that the symbol $G_1$ of $\mathcal{R}_1$ coincides with $(\pi_1^0|_{\mathcal{R}_1})^{-1}W$. Let $(x, y) = (x^i, y^a; 1 \leq i \leq 2, 1 \leq a \leq m)$ be local coordinates in a fibered chart of $\mathcal{E}$ around $\pi_1^0(P_0)$ with coordinate neighborhood $\mathcal{E}^0$. $\mathcal{E}^0$ is a fibered manifold over $X_0 = \pi(\mathcal{E}^0)$. Associated with $(x, y)$ we have the coordinates

$$(x, y, p) = (x^i, y^a, p^\alpha_i; 1 \leq i \leq 2, 1 \leq \alpha \leq m)$$

of $J_1(\mathcal{E}^0)$. In what follows $\mathcal{R}_1$ will denote $\mathcal{R}_1 \cap J_1(\mathcal{E}^0)$. Bearing in mind the additional assumptions, and shrinking $\mathcal{E}^0$ if necessary, we readily see that the system $\mathcal{R}_1$ is described by a system of equations

$$\phi^\beta(x, y, p) = 0 \quad (\beta = 1, 2, \ldots, m_1)$$

where $\phi^\beta$ are functions of the form

$$\phi^\beta(x, y, p) = \sum_{i=1}^2 \sum_{\alpha=1}^m a^\beta_i(x, y) p^\alpha_i + b^\beta(x, y)$$
with \(a\)'s, \(b\)'s being smooth functions on \(\mathcal{E}^0\), and the Jacobian matrix of \(\varphi^1, \ldots, \varphi^{m_1}\) with respect to the variables \(p = (p^\alpha_i; 1 \leq i \leq 2, 1 \leq \alpha \leq m)\) has the rank \(m_1\) at each point of \(\mathcal{E}^0\).

Let \(\mathcal{E}^1\) be the product fibered manifold \(X_0 \times \mathbb{R}^{m_1}\), and \((x, z) = (x^i, z^\beta; 1 \leq i \leq 2, 1 \leq \beta \leq m_1)\) be its standard coordinates. Let \(v_0\) denote the distinguished section of \(\mathcal{E}^1\) on \(X_0\) defined by \(z^\beta = 0\) \((\beta = 1, 2, \ldots, m_1)\). We define a fibered morphism \(\varphi : J_1(\mathcal{E}^0) \to \mathcal{E}^1\) over \(X_0\) by

\[
\begin{align*}
(\mathbf{8.12}) & \quad z^\beta = \varphi^\beta(x, y, p) \quad (\beta = 1, 2, \ldots, m_1), \\
\end{align*}
\]

and a nonlinear differential operator \(\Phi\) of order 1 from \(\mathcal{E}^0\) to \(\mathcal{E}^1\) by \(\Phi = \varphi \circ j_1\). Obviously \(\varphi\) has the constant rank \(m_1\). The symbol morphism \(\sigma(\Phi) = \sigma(\varphi) : T^* \otimes_{J_1(\mathcal{E}^0)} V(\mathcal{E}^0) \to V(\mathcal{E}^1)\) is described by

\[
\sigma(\varphi)
\left(
\sum_{i=1}^{2} \sum_{a=1}^{m} e^a_i \, dx^i \otimes \frac{\partial}{\partial x^a}\right)
= \sum_{\beta=1}^{m_1} \left(
\sum_{i=1}^{2} \sum_{a=1}^{m} \varphi^\beta \varphi^a_i (x, y) \, e^a_i \right) \frac{\partial}{\partial z^\beta},
\]

and the prolongation \(p_1(\varphi) : J_2(\mathcal{E}^0) \to J_1(\mathcal{E}^1)\) is described by the system consisting of \((\mathbf{8.12})\) and

\[
q^\beta_{ij} = \left(\frac{d\varphi^\beta}{dx^i}\right)(x, y, p, p') \quad (i = 1, 2; \beta = 1, \ldots, m_1),
\]

where the same notations as in the proof of Lemma 5.2 are used. The description of \(\sigma(\Phi)\) indicates that it is surjective, and that \(\ker(\varphi)\) may be considered as the induced family \((\pi^1_0)^{-1}W\) of a family \(\tilde{W} \subset T^* \otimes V(\mathcal{E}^0)\) over \(\mathcal{E}^0\). We know at once that \(\tilde{W} = W|_{\mathcal{E}^0}\). Since \(G_1 = (\pi^1_0|_{R_1})^{-1}W\) is involutive by assumption, it follows that the symbol \(\ker(\varphi) \subset T^* \otimes_{J_1(\mathcal{E}^0)} V(\mathcal{E}^0)\) is involutive. We also see that \(\ker\sigma(\varphi)\) is a family induced from a family on \(\mathcal{E}^0\) by \(\pi^1_0\), and that \(\ker\sigma(\varphi)\) restricted to \(R_1\) coincides with the prolongation \(G_2\) of \(G_1\) (cf. sections 1, 2 and 4). Since \(R_1\) is involutive, \(G_2\) is a vector bundle over \(R_1\). From these facts, it follows that \(\ker\sigma(\varphi)\) is a vector bundle over \(J_1(\mathcal{E}^0)\). Thus we have proved that \(\Phi\) is quasi-involutive. The construction indicates that \(R_1 = \ker v_0. \varphi\). From the above description of \(\sigma(\varphi)\), we know that the first prolongation \(R_2\) of \(R_1\) is equal to \(\ker j_1(v_0)p_1(\varphi)\) (cf. e.g. Pommaret [24], Chapter 2, section 2). Hence the involutivity of \(R_1\) implies that the mapping \(\pi^1_1 : \ker j_1(v_0)p_1(\varphi) \to \ker v_0. \varphi\) is surjective. It is clear that \(v_0 \subset \text{im} (\varphi|_{R_1.x})\) is not empty for any \(x \in X\), and hence \(\Phi\) is involutive over \(v_0\). From the fact already stated we see also that \(\Phi\) is elliptic.

We can now apply Theorem 8.1 to the differential operator \(\Phi\) to see that there exists a smooth section \(u\) of \(\mathcal{E}^0\) over a neighborhood \(U\) of \(x_0\) such that \(\Phi(u)(x) = v_0(x), j_1(u)(x) \in U\) (\(x \in U\)). This section \(u\) is indeed a smooth solution of \(R_1\). This completes the proof of Theorem 8.2.

**Remark.** Theorem 8.1 can be extended to higher order differential operators. The extended result is proved by reducing a higher order operator to a first order operator and then applying Theorem 8.1, or more easily by using Theorem 8.2.
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References


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