Twenty-eight Double Tangent Lines of a Plane Quartic Curve with an Involution and the Mordell-Weil Lattices

by
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1. Introduction

A smooth quartic curve $C$ in a projective plane $\mathbb{P}^2$ has twenty-eight double tangent lines. If $V$ is a double cover of $\mathbb{P}^2$ ramified along $C$, then $V$ is a del Pezzo surface of degree 2. $V$ has exactly fifty-six exceptional curves of the first kind, which lie in pairs above the twenty-eight double tangent lines. Shioda [S2] constructs an elliptic surface by blowing up $V$ in a certain way such that the fifty-six exceptional curves generate a Mordell-Weil lattice of type $E_7^*$. 

In this paper we generalize Shioda’s construction by choosing different points of $V$ at which we blow up, and we obtain various different elliptic surfaces starting from one quartic curve $C$. More precisely, from a quartic equation $F_C(x, y, z) = 0$ of $C$ and a point $P \in \mathbb{P}^2$, we construct an elliptic surface $E_{F_C, P}$ (see §4). The goal of this paper is to classify all the elliptic surfaces obtained this way, and to construct various examples of arithmetic interest.

In addition to Shioda’s paper [S2], our original motivation comes from the following classic fact. In the nineteenth century Kovalevskaia [K] studied a problem of reduction of abelian integrals to elliptic integrals. Though she obtained results considering relations among the periods of integrals, she stated her results in terms of geometry as follows. (See also [P] [B].)

THEOREM 1.1 (Kovalevskaia [K]). Let $y = f(x)$ be an algebraic function satisfying a quartic equation $F(x, y) = 0$. Then there exists an abelian integral of the first kind

$$u = \int \Psi(x, f(x)) \, dx$$

that can be reduced to an elliptic integral using a change of variables of degree 2 if and only if the quartic curve $C$ defined by $F(x, y) = 0$ has four double tangent lines meeting at one point.

Here, a natural question arises: What is the structure of $E_{F_C, P}$ when $C$ has four double tangent lines meeting at one point $P$. It turns out that such a plane quartic $C$ has an
involution and this involution is induced by the addition of a 2-torsion section of $E_{F_c, P}$ (Theorem 5.1).

We then ask if there are other special classes of quartic curves $C$ and a point $P$ such that $E_{F_c, P}$ has a special structure. To answer this question, we classify, in §6, all the possible elliptic surfaces $E_{F_c, P}$ (Theorem 6.1) according to the list of Oguiso and Shioda [OS].

Suppose $C$ has four double tangent lines meeting at one point $P$, and construct an elliptic surface using Shioda’s method by choosing $P$ as an inflection point. In this case the pull-backs of the four concurrent double tangent lines all become a section. However, the Mordell-Weil lattice of $E_{F_c, P}$ is still of type $E_7$ since the intersection pairing cannot distinguish whether or not more than two lines meet at one point. The difference occurs when we specialize the sections to the fiber passing through the meeting point $P$. The pull-backs of the four concurrent lines are specialized to a pair of points, and thus we expect that the rank of the Mordell-Weil group of the fiber is unusually smaller than the rank of the Mordell-Weil group of the elliptic surface. We make this precise in §7.

In §8 we consider quartic curves that admit several commuting involutions. Among those curves we give a family of curves whose twenty-eight double tangent lines are defined over the base field $k$. We then give several examples of arithmetic interests.

Throughout the paper the base field $k$ assumed to be an arbitrary field of characteristic different from 2.

## 2. Plane quartics

Let $C$ be a smooth quartic curve in a projective plane $\mathbb{P}^2$ defined over $k$. We call it a smooth plane quartic for short. In this section we review geometric properties of $C$, and thus we assume that the base field $k$ is algebraically closed for simplicity.

For a point $P$ in $C$, let $T_P$ be the tangent line to $C$ at $P$. The intersection product $T_P \cdot C$ is, in general, of the form $T_P \cdot C = 2P + P' + P''$ ($P, P', P''$ are mutually distinct). There are three special cases.

(i) $T_P \cdot C = 2P + 2P'$ ($P \neq P'$),
(ii) $T_P \cdot C = 3P + P'$ ($P \neq P'$),
(iii) $T_P \cdot C = 4P$.

When $T_P \cdot C = 2P + 2P'$, we call $T_P$ a bitangent or a double tangent line. If $P \neq P'$ as in (i), then we call $T_P$ an ordinary double tangent line. If $T_P \cdot C = 4P$, then we still count $T_P$ as a double tangent line, and call it a special double tangent line.

When $T_P \cdot C = 3P + P'$, we call $P$ an inflection point. If $P \neq P'$ as in (ii), then we call $P$ an ordinary inflection point. If $T_P \cdot C = 4P$, then we call $P$ a special inflection point.

We note in passing that an inflection point of a plane quartic is a Weierstrass point. An ordinary inflection point has weight 1, and a special inflection point has weight 2.

Let $C^*$ be the dual curve of $C$. Classically, $C^*$ is the set of all tangent lines to $C$ viewed as points in $(\mathbb{P}^2)^*$. For a point $P \in C$, let $P^*$ be the point in $(\mathbb{P}^2)^*$ corresponding to
the tangent line $TP$. If $TP$ is an ordinary double tangent line, then $P^*$ is a simple node. If $P$ is an ordinary inflection point, then $P^*$ is a simple cusp. If $TP$ is a special double tangent line, then $P^*$ is a triple point which looks like the origin $(0, 0)$ on the curve $y^3 = x^4$.

It is known that $C^*$ is an irreducible curve of degree 12, and its arithmetic genus $p_a(C^*)$ equals $(12 - 1)(12 - 2)/4 = 55$. Since $C^*$ is birationally equivalent to $C$ itself, the geometric genus of of the normalization $\tilde{C}^*$ equals 3.

In general, an inflection point of a plane curve is a point of intersection between the Hessian of the curve and the curve itself. If $C$ is a plane quartic, its Hessian is of degree 6. Thus, by Bézout’s theorem the number of inflection points is 24, counting special inflection points with multiplicity 2. Let $i_o$ be the number of ordinary inflection points, and let $i_s$ be the number of special inflection points. Then we have

$$i_o + 2i_s = 24.$$  

(1)

The arithmetic genus $p_a(C^*)$ is related to the geometric genus $g(\tilde{C}^*)$ in the following way:

$$g(\tilde{C}^*) = p_a(C^*) - \sum_{P^* \in C^*} \delta_{P^*},$$

([H, IV Ex.1.8]). If $P^*$ is an ordinary node or an ordinary cusp, we have $\delta_{P^*} = 1$. Whereas, if $P^*$ is a triple point, we have $\delta_{P^*} = 3(3 - 1)/2 = 3$. Thus, we have

$$3 = 55 - (\tau_o + i_o + 3i_s),$$

(2)

where $\tau_o$ is the number of ordinary double tangent lines. Let $\tau$ be the total number of double tangent lines, then we have $\tau = \tau_o + i_s$. Now, we can rewrite (2) as

$$3 = 55 - (\tau_o + i_s) - (i_o + 2i_s) = 55 - \tau - (i_o + 2i_s).$$

Using (1), we obtain $\tau = 55 - 24 - 3 = 28$.

Summing all these up, we have

**Proposition 2.1.** Let $C$ be a smooth plane quartic curve. Then $C$ has exactly twenty-eight distinct double tangent lines. If $C$ has $i_s$ special inflection points, then $C$ has $(24 - 2i_s)$ ordinary inflection points. The dual curve $C^*$ has $(28 - i_s)$ nodes, $(4 - 2i_s)$ ordinary cusps, and $i_s$ triple points.

**Remark 2.2.** The number $i_s$ can be up to 12. The curves $x^4 + y^4 + z^4 = 0$ and $x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) = 0$ have exactly 12 special inflection points. These are known to be the only quartics that have 12 special inflection points (See [GT]).

### 3. Del Pezzo surfaces of degree 2

A smooth surface $V$ over $k$ is a del Pezzo surface if $V_k = V \times_{\text{Spec} k} \text{Spec} \bar{k}$ is birationally equivalent to $\mathbb{P}^2_k$ and its anti-canonical sheaf $(\wedge^2 \Omega_V)^{-1}$ is ample. The degree of a del Pezzo surface is the self-intersection number of the anti-canonical class $\omega_X^{-1} = [(\wedge^2 \Omega_V)^{-1}]$. 
Proposition 3.1. Let $C$ be a plane quartic curve, and $V$ a double cover of $\mathbb{P}^2$ ramified along $C$. Then $V$ is smooth if and only if $C$ is smooth. If $V$ is smooth, $V$ is a del Pezzo surface of degree 2.

Proof. The first assertion follows from the fact that a singular point of a double cover of $\mathbb{P}^2$ ramified along a curve of even degree lies exactly over a singular point of the curve. The second assertion follows from a calculation of the self-intersection number of the anti-canonical class. For details, see for example [Er].

We have the following well-known facts. See [M] for details.

Proposition 3.2. Let $V$ be a del Pezzo surface of degree 2. Then, we have the following.

(i) $V$ is isomorphic over $\bar{k}$ to a surface obtained by blowing up $\mathbb{P}^2$ at 7 points.
(ii) The map $\pi : V \to \mathbb{P}^2$ defined by the section of $(\wedge^2 \Omega_V)^{-1}$ is a double cover ramified along a smooth quartic curve.
(iii) $V$ has exactly fifty-six exceptional curves of the first kind (curves of self-intersection $(-1)$). These exceptional curves are pairwise glued together, and they are projected onto double tangent lines of the quartic curve of ramification.
(iv) $\text{Pic}(V_{/\bar{k}}) \simeq \mathbb{Z}^8$, and it is generated by the exceptional curves of the first kind.

4. Elliptic surfaces constructed from a smooth plane quartic

Let $C$ be a smooth plane quartic with an equation $F_C(x, y, z) = 0$, and let $P$ be a point in $\mathbb{P}^2$. $P$ may or may not be on the curve. Let $\pi : V \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ whose affine model is given by $w^2 = F_C(x, y, 1)$. Consider the pencil of lines $\Lambda_P = \{L_t\}$ centered at $P$. We regard $\Lambda_P$ as a line in $(\mathbb{P}^2)^\ast$. The pull-back $\tilde{L}_t = \pi^{-1}(L_t)$ of a general line $L_t$ is a double cover of $L_t$ ramified at four points of intersection $C \cap L_t$. Thus, $L_t$ is in general a curve of genus 1. Conversely, if $Q$ is a point in $\mathbb{P}^2$ different from $P$, the line passing through $P$ and $Q$ is a member of the pencil $\Lambda_P$. Thus, we obtain a map $\tilde{f}_P : \tilde{V} \to \tilde{\Lambda}_P$, and this map lifts to $f_P : V - \pi^{-1}(P) \to \Lambda_P$.

Blowing up at points in $\pi^{-1}(P)$ in a suitable manner (depending on the position of $P$), we obtain an elliptic surface $\tilde{f}_P : \tilde{V} \to \Lambda_P$. We denote this elliptic surface by $E_{F_C, P}$.

Note that if we replace the equation $F_C(x, y, z) = 0$ by $aF_C(x, y, z) = 0$ with $a \in k$, the resulting elliptic surface $E_{aF_C, P}$ is a quadratic twist of $E_{F_C, P}$ by $a$. Thus, when the base field $k$ is assumed to be algebraically closed, we may write $E_{C, P}$ instead of $E_{F_C, P}$. Also, when $C$ is given by an equation explicitly, we write $E_{C, P}$ instead of $E_{F_C, P}$ for simplicity.

4.1. The case where $P$ is not on the curve

In this case, $\pi^{-1}(P)$ consists of two distinct points, say $\tilde{P}_0$ and $\tilde{P}_1$. Then, $E_{F_C, P}$ is obtained by blowing up at each of $\tilde{P}_0$ and $\tilde{P}_1$. Two exceptional curves $E_0$ and $E_1$ become sections. The singular fibers of this elliptic surface can be determined by looking at how $\Lambda_P$ and the dual curve $C^\ast$ intersect. Note that the condition $P \notin C$ is equivalent to the fact that $\Lambda_P$ is not tangent to $C^\ast$. 
Table 1 shows the types of singular fibers depending how $A_P$ and $C^*$ intersect. Each row shows the Kodaira type of singular fiber when $A_P$ and $C^*$ intersect in the way the second column, which corresponds the situation in $\mathbb{P}^2$ in the first column. Note that $A_P$ and $C^*$ intersect twelve times counting multiplicities.

4.2. The case where $P$ is on the curve

In this case, $\pi^{-1}(P)$ consists of one point. We need to blow up once at the point $\pi^{-1}(P)$, and once more to obtain $E_{F_0,P}$. There are four cases depending on $P$.

(i) $T_P$ is an ordinary double tangent line.
(ii) $P$ is an ordinary inflection point.
(iii) $P$ is a special inflection point.
(iv) $P$ is none of the above.

<table>
<thead>
<tr>
<th>$C$ and the lines in $A_P$</th>
<th>$C^*$ and $A_P$</th>
<th>Type of fiber</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td><img src="image2.png" alt="Diagram 2" /></td>
<td>$I_1$</td>
</tr>
<tr>
<td><img src="image3.png" alt="Diagram 3" /></td>
<td><img src="image4.png" alt="Diagram 4" /></td>
<td>$I_2$</td>
</tr>
<tr>
<td><img src="image5.png" alt="Diagram 5" /></td>
<td><img src="image6.png" alt="Diagram 6" /></td>
<td>II</td>
</tr>
<tr>
<td><img src="image7.png" alt="Diagram 7" /></td>
<td><img src="image8.png" alt="Diagram 8" /></td>
<td>III</td>
</tr>
</tbody>
</table>
In each case, the elliptic surface $E_{F_C,P}$ has a singular fiber at $T_p^* \in \Lambda_P$. The type of singular fiber at $T_p^*$ is indicated in Table 2. Other singular fibers can be identified according to Table 1.

If $T_p$ is a double tangent line, then the pull-back of $T_p$ yields two components of the singular fiber at $T_p^* \in \Lambda_P$. Otherwise, the pull-backs of double tangent lines become sections of the elliptic surface $E_{F_C,P}$.

**REMARK 4.1.** Our construction is a generalization of Shioda’s ([S2]). He chooses $P$ as an inflection point, either ordinary or special. With Shioda’s choice of $P$, the pull-backs of the double tangent lines all become sections and they generate the Mordell-Weil lattice of type $E_7^*$ or $E_6^*$.

5. **Reduction of abelian integrals to elliptic integrals**

Let $y = f(x)$ be an algebraic function defined by an algebraic equation $F(x, y) = 0$. If the abelian integral of the first kind

$$u = \int \Psi(x, f(x))dx$$

can be reduced to an elliptic integral

$$\int \frac{d\xi}{\sigma}$$

using the change of variables

$$\xi = \phi(x, y), \quad \sigma = \sqrt{\xi(1-\xi)(1-c^2\xi)} = \psi(x, y),$$

we say that this abelian integral was reduced to an elliptic integral. Geometrically, this means that the algebraic curve $C$ defined by $F(x, y) = 0$ admits a map to the elliptic curve $E$ defined by the equation $\sigma^2 = \xi(1-\xi)(1-c^2\xi)$, and the pull-back of the differential form $d\xi/\sigma$ coincides with $\Psi(x, y)dx$.

Sofia Kovalevskaya studied the case where the genus of $C$ is 3, and the degree of the map $C \to E$ is 2. As we stated in Theorem 1.1 in the introduction, she stated her result in terms of double tangent lines of $C$. Here, we will prove the following expanded version of her theorem using the theory of Mordell-Weil lattices.

**THEOREM 5.1.** Let $C$ be a smooth plane quartic curve defined over an algebraically closed field $k$. The following are all equivalent.

(i) There exists a curve $E$ of genus 1, and a dominant map $\varphi : C \to E$ of degree 2.

(ii) $C$ admits an involution (an automorphism of order 2).

(iii) By a suitable change of coordinates, $C$ can be defined by an equation of the form

$$(z^2 + q_2(x, y))^2 = q_4(x, y),$$

where $q_d(x, y)$ ($d = 2, 4$) is a homogeneous polynomial of degree $d$. 

(iv) Among the twenty-eight double tangent lines of $C$, four of them meet at one point.

Proof. (i) $\Rightarrow$ (ii). The function field $k(C)$ of $C$ is a quadratic extension of the function field $k(E)$ of $E$. In particular, $k(C)/k(E)$ is a Galois extension. Thus, there is an automorphism of $k(C)/k(E)$ of order 2, which induces an involution of $C$.

(ii) $\Rightarrow$ (iii). Since the embedding $i$ of $C$ into $P^2$ as a quartic curve is a canonical embedding, it is unique up to an automorphism of $P^2$. If $\sigma$ is an involution of $C$, then $i \circ \sigma$ is once again an embedding of $C$ into $P^2$. Thus, there exists an involution $\tilde{\sigma}$ of $P^2$ such that $i \circ \sigma = \tilde{\sigma} \circ i$. Since any automorphism of $P^2$ is a linear transformation of coordinates, we see that $\tilde{\sigma}$ can be written as $(x : y : z) \mapsto (x : y : -z)$ after suitable change of coordinates.

Under this coordinate system, the equation $f(x, y, z) = 0$ of $C$ does not contain terms of

<table>
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<th>Type of fiber</th>
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<tbody>
<tr>
<td>$T_P$</td>
<td>$A_P$</td>
<td>$T_{i_P}$</td>
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odd degree in \( z \). Thus, it can be written as
\[
a z^4 + p_2(x, y)z^2 + p_4(x, y),
\]
where \( p_d(x, y) \) is a homogenous polynomial of degree \( d \). If \( a = 0 \), then \( C \) has a singular point. Thus we may assume \( a = 1 \). By completing a square with respect to \( z^2 \), we can transform it to the form in the assertion (iii).

(iii) \( \Rightarrow \) (i). Let \( E \) be the curve given by the equation \( Y^2 = q_4(X, 1) \). It is easy to see that the smoothness of \( C \) implies that \( q_4(X, 1) \) does not have a multiple root. Thus \( E \) is a curve of genus 1. Define a map \( C \to E \) by
\[
(x : y : z) \mapsto \left(\frac{x}{y}, \frac{z^2}{y^2} + q_2\left(\frac{x}{y}, 1\right)\right).
\]
This is a dominant map of degree 2.

(iii) \( \Rightarrow \) (iv). Factoring \( q_4(x, y) = \prod_{i=0}^{4}(a_ix + b_iy) \), we see that \( T_i : a_ix + b_iy = 0 \) \( (i = 1, 2, 3, 4) \) are double tangent lines of \( C \), and they meet at \((0 : 0 : 1)\).

(iv) \( \Rightarrow \) (ii). Suppose double tangent lines \( T_i \) \( (i = 1, 2, 3, 4) \) meet at one point \( P \). If \( P \) is on the curve \( C \), then only one \( T_i \) can be the tangent line at \( P \) and the others do not tangent to \( C \) at \( P \). The latter must intersect with \( C \) five times counting multiplicities, which is contradiction. Thus, \( P \) is not on the curve. Now we construct the elliptic surface \( \tilde{E}_{C,P} \) as in §4. \( \tilde{V} \) is obtained by blowing-up \( V \) at two points \( \pi^{-1}(P) = \{\tilde{P}_1, \tilde{P}_2\} \). Let \( E_i \) be the exceptional curve over \( \tilde{P}_i \). The curve \( \pi^{-1}(T_i) \) decomposes into two components. Let \( \tilde{T}_{i,1} \) be the component passing through \( \tilde{P}_1 \), and let \( \tilde{T}_{i,2} \) the component passing through \( \tilde{P}_2 \). \( \tilde{T}_{i,1} \cup \tilde{T}_{i,2} \) forms a singular fiber of \( S \to \mathbb{P}^1 \) of type I_2 or III. Choosing \( E_1 \) as the 0-section, we define a group structure on the set of sections of \( S \to \mathbb{P}^1 \). \( E_2 \) is a section, and \( E_2 \) does not meet with the nonidentity component of the singular fibers nor the 0-section. This implies that the height of \( E_2 \) equals 0, which in turn implies that \( E_2 \) is a torsion section. Since a torsion section of a rational elliptic surface having four fibers of type I_2 or III must be of order 2, the order of \( E_2 \) equals 2. Thus, \( E_2 \) induces an involution on \( \tilde{V} \) by translation. This in turn induces an involution \( \sigma \) on \( C \) since the translation by \( E_2 \) is equivariant with the covering map \( \pi : V \to \mathbb{P}^2 \).

REMARK 5.2. When the base field \( k \) is not algebraically closed, the map \( \varphi \) in (i), the involution in (ii), and the change of coordinates in (iii) may not be defined over \( k \) itself. For example, the Klein’s quartic \( x^3y + y^3z + z^3x = 0 \) admits at least three involutions, but they are defined over the cyclotomic field \( \mathbb{Q}(\mu_7) \). It is known that a change of coordinates defined over \( \mathbb{Q}(\mu_7) \) transforms Klein’s quartic to
\[
x^4 + y^4 + z^4 + 3(\zeta_7 + \zeta_7^2 + \zeta_7^4)(x^2y^2 + y^2z^2 + z^2x^2) = 0,
\]
where \( \zeta_7 \) is a primitive 7-th root of unity (see for example Elkies [El]).
6. Classification of Mordell-Weil lattices obtained from plane quartics

As we saw in the proof of Theorem 5.1, if $C$ has an involution, an elliptic surface obtained from the intersection point of four double tangent lines has four singular fibers of type $I_2$ or $III$, and admits a section of order 2. Such an elliptic surface is classified as the surface No. 13 in Oguiso-Shioda [OS]. Here, we classify all the different types of elliptic surfaces obtained by choosing $C$ and $P$ differently.

When we consider the lattice structure, two singular fibers of different Kodaira types may not be distinguished as a lattice with respect to intersection pairing. For example, a fiber of type $I_1$ and type $II$ both have the same lattice structure of type $A_0$. Fibers of type $I_2$ and type $III$ are lattices of type $A_1$, while fibers of type $I_3$ and type $IV$ are lattices of type $A_2$.

**THEOREM 6.1.** An elliptic surface obtained as $\mathcal{E}_{F_t, P}$ in §4 falls into one of the following classes. (In the table, we omit fibers of type $A_0$.)

<table>
<thead>
<tr>
<th>Oguiso-Shioda classification</th>
<th>Types of singular fibers</th>
<th>Mordell-Weil lattice</th>
<th>Rank</th>
<th>Torsion subgroup</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 1</td>
<td>0</td>
<td>$E_8$</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>No. 2</td>
<td>$A_1$</td>
<td>$E_7^*$</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>No. 3</td>
<td>$A_2$</td>
<td>$E_6^*$</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>No. 4</td>
<td>$A_1 \oplus A_1$</td>
<td>$D_6^*$</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>No. 7</td>
<td>$A_1^{\oplus 3}$</td>
<td>$D_4^* \oplus A_1^*$</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>No. 13</td>
<td>$A_1^{\oplus 4}$</td>
<td>$D_4^*$</td>
<td>4</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
</tbody>
</table>

**Proof.** As we have seen in §4, the only types of singular fibers our elliptic surfaces admit are of type $A_0$, $A_1$, or $A_2$. A singular fiber of type $I_3$ or $IV$ appear only when we choose $P$ as one of tangent points of a double tangent line $T$. Since $C$ is smooth, the only tangent line to $C$ at $P$ is $T$. Thus, in this case the pencil $A_P$ does not contain any other double tangent lines than $T$. This implies that if we have one singular fiber of type $A_2$, all other singular fibers are of type $A_0$. Looking at the classification table of Oguiso-Shioda, we can eliminate a majority of cases. Remaining cases to eliminate is No. 14 and No. 24.

If our elliptic surface has four fibers of type $A_0$, then the pencil $A_P$ has four double tangent lines. This falls into the case of Theorem 5.2. As we have seen in the proof of Theorem 5.2, the elliptic surface obtained has a 2-torsion section. This eliminates No. 14.

To show that we cannot have five $A_0$’s, we first write an equation of $C$ under the form in Theorem 5.2(iii), using four of five double tangent lines. Then, it is not difficult show that a fifth line passing through $(0 : 0 : 1)$ becomes a double tangent line only if $C$ is singular. This eliminates No. 24.

Conversely, we can construct elliptic surfaces in the list of theorem explicitly. Explicit examples of No. 2 are constructed in Shioda [S2]. The elliptic surfaces of type No. 13 are
studied in §5, and an explicit example is given by the equation (4) in §7, together with an example of No. 3 given by the equation (5). In §8 we construct explicit examples of No. 4 and No. 7, given by the equations (9) and (8) respectively.

**Remark 6.2.** Consider the quartic curve
\[ C : (z^2 + x^2 + 6xy - 4y^2)^2 - 12xy(x - y)(x + 4y) = 0. \]
A simple calculation shows that \( C \) has two nodes \((±2 : 1 : 0)\). Choose \((0 : 0 : 1)\) as \( P \), and consider the pencil \( x = ty \). Then, the resulting elliptic surface has 6 fibers of type I2.

Thus, if we allow \( C \) to be singular, the list in Theorem 6.1 will become longer.

**7. Different elliptic surfaces obtained from one plane quartic**

Let \( C \) be a smooth plane quartic with involution, and let \( T_1, \ldots, T_4 \) its concurrent tangent lines passing through \( P \). Consider the elliptic surface \( E_{F_C, P} \) obtained using an equation \( F_C \) and the pencil \( A_P \) centered at \( P \). We saw in §5 that the pull-backs of these four double tangent lines are singular fibers of \( E_{F_C, P} \), and that its Mordell-Weil group has a 2-torsion section.

Let \( Q \) be a point in \( \mathbb{P}^2 \) that is not on either of four lines \( T_1, \ldots, T_4 \), and consider the elliptic surface \( E_{F_C, Q} \).

**Proposition 7.1.** The subgroup of the Mordell-Weil group of the elliptic surface \( E_{F_C, Q} \) generated by \( M = \{ \tilde{T}_{i,1} | i = 1, 2, 3, 4, j = 1, 2 \} \) has rank 3. The specialization of \( M \) to the fiber of \( E_{F_C, Q} \) passing through \( \pi^{-1}(P) \) is a finite group isomorphic to \( \mathbb{Z}/4\mathbb{Z} \).

**Proof.** Since \( Q \) is on the quartic \( C \), the involution of \( \pi : V \to C \) coincides with the multiplication-by-\((-1)\) map if we choose the 0-section as indicated in §5. Since \( \tilde{T}_{1,1} \) and \( \tilde{T}_{i,2} \) become inverse to each other, it suffices to consider \( \{ \tilde{T}_{i,1} | i = 1, 2, 3, 4 \} \). We calculate the height pairing using Shioda’s method ([S1, Th. 8.6]). If \( Q \) is an ordinary inflection point, then \( \tilde{T}_{i,1} \) intersects with the component of type III fiber that does not intersects with the 0-section. Together with the fact that \( \tilde{T}_{i,1} \)’s intersect at \( P \), we have
\[
\langle \tilde{T}_{i,1} , \tilde{T}_{j,1} \rangle = \begin{cases} 
3/2 & i = j, \\
-1/2 & i \neq j.
\end{cases}
\]
From this we see that the rank of height pairing matrix \( (\langle \tilde{T}_{i,1} , \tilde{T}_{j,1} \rangle)_{1 \leq i, j \leq 4} \) is 3. This shows that the subgroup generated by \( M \) has rank 3. Also from the height matrix we see the relation
\[
\tilde{T}_{1,1} + \tilde{T}_{2,1} + \tilde{T}_{3,1} + \tilde{T}_{4,1} = O.
\]
Since \( \tilde{T}_{i,1} \) are all specialized to the point \( \tilde{P}_1 \) on the fiber passing through \( \pi^{-1}(P) \), (3) implies that \( 4\tilde{P}_1 = O \). On the other hand, since we have \( -\tilde{P}_1 = \tilde{P}_2 \neq \tilde{P}_1 \), \( \tilde{P}_1 \) is not a point of order 2. This shows that the specialization of \( M \) generates a cyclic group of order 4. \( \square \)
EXAMPLE 7.2. Let $C$ be the smooth plane quartic given by

$$C : (2z^2 - x^2 - xy)^2 - xy(x - y)(x + 3y) = 0.$$ 

Then $x = 0$, $y = 0$, $x = y$, and $y = -x/3$ are double tangent lines of $C$. They meet at $P = (0 : 0 : 1)$. (See Figure 1.) Using the pencil $x = ty$ centered at $P$, we obtain the elliptic surface whose Weierstrass equation is given by

$$y^2 = (x - t^2 - t)(x^2 - t^4 - t^3 + t^2 - 3t).$$

It has a fiber of type III at $t = 0$, and fibers of type I2 at $t = 1, -3, \infty$. Its Mordell-Weil group is isomorphic to $\mathbf{Z}^4 \oplus \mathbf{Z}/2\mathbf{Z}$. Its Mordell-Weil lattice is isomorphic to $D_4^*$, which is classified as No. 13 in the Oguiso-Shioda classification.

On the other hand, $Q = (-1 : 1 : 1)$ is a special inflection point of $C$. The elliptic surface obtained from the pencil $y - z = (1 + t)(x + z)$ is given by the Weierstrass equation

$$y^2 = x^3 - 2t(5t + 4)x^2 + 8t^2(3t + 4)(t - 2)x + 64t^2(3t^3 + 8t^2 + 8t + 4).$$

It has a fiber of type IV at $t = 0$, and all other singular fibers are of type I1. Its Mordell-Weil lattice is isomorphic to $E_8$ (No. 3 in the Oguiso-Shioda classification). From the four double tangent lines mentioned above we obtain eight sections:

$$\pm P_1 = (-8t, \pm 16t), \quad \pm P_2 = (8t(t + 1), \pm 8t(t^2 - 2)),$$

$$\pm P_3 = (4t^2, \mp 16t(t + 1)), \quad \pm P_4 = (4t(3t + 4), \pm 8t(3t^2 + 6t + 2)).$$
These sections generate a free abelian group of rank 3. The intersection point \( P \) of the four double tangent lines is located on the fiber at \( t = -2 \), whose equation is given by

\[
y^2 = x^3 - 24x^2 + 256x - 1024.
\]

The eight sections above are specialized to either \((16, 32)\) or \((16, -32)\), both of which are points of order 4. The fiber at \( t = -2 \) is isomorphic to \( y^2 = x^3 - 4x \), and its Mordell-Weil group over \( \mathbb{Q} \) is isomorphic to \( \mathbb{Z}/4\mathbb{Z} \).

8. Plane quartics with commuting involutions

In this section we consider plane quartics which admit a pair of commuting involutions. In other words, we consider plane quartics whose automorphism group contains \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) as a subgroup.

**Proposition 8.1.** Let \( C \) be a smooth plane quartic with a pair of commuting involutions. Then, by a suitable change of coordinates defined possibly over some extension of the base field, \( C \) can be defined by an equation of the form

\[
a_1x^4 + a_2y^4 + a_3z^4 + a_4x^2y^2 + a_5y^2z^2 + a_6z^2x^2 = 0.
\]

**Proof.** As we saw in the proof of Theorem 5.2, an involution on \( C \) is induced by an involution on \( \mathbb{P}^2 \). Representing involutions by matrices, and diagonalizing simultaneously, we may assume that the involutions are given by \((x : y : z) \mapsto (\pm x : \pm y : \pm z)\). A quartic curve is stable under this action if and only if its equation is given in the above form. \( \square \)

Completing the square with respect to each variable, we can write (6) in three ways:

\[
a_1(x^2 + q_{2,1}(y, z))^2 = q_{4,1}(y, z),
\]

\[
a_2(y^2 + q_{2,2}(z, x))^2 = q_{4,2}(z, x),
\]

\[
a_3(z^2 + q_{2,3}(x, y))^2 = q_{4,3}(x, y),
\]

where \( q_{d,i} \) is a homogeneous polynomial of degree \( d \). A linear factor of \( q_{4,i} \) determines a double tangent line of \( C \), so we would like to choose \( a_1, \ldots, a_6 \in k \) such that \( q_{4,1}(y, z), q_{4,2}(z, x) \) and \( q_{4,3}(x, y) \) all factorize into linear terms over \( k \). It turns out that this is possible, and we found the following family of plane quartics.

**Theorem 8.2.** Let \( \lambda, \mu, \nu \) be three elements in \( k \) satisfying

\[
(1 - \lambda^2)(1 - \mu^2)(1 - \nu^2)(1 - \mu^2 \lambda^2)(1 - \nu^2 \lambda^2)(1 - \mu^2 \nu^2)(1 - \lambda^2 \mu^2 \nu^2) \neq 0.
\]

Let \( C_{\lambda,\mu,\nu} \) be the plane quartic given by

\[
C_{\lambda,\mu,\nu} : ((1 - \mu^2 \nu^2)x^2 + (1 - \nu^2 \lambda^2)y^2 + (1 - \lambda^2 \mu^2)z^2)^2
- 4(1 - \lambda^2 \mu^2 \nu^2)((1 - \nu^2)x^2y^2 + (1 - \lambda^2)y^2z^2 + (1 - \mu^2)z^2x^2) = 0.
\]
Let \( \delta = (1 - \lambda^2 \mu^2)(1 - \lambda^2)(1 - \mu^2) \), and let \( E_{\lambda \mu}, E_{\mu \nu}, E_{\nu \lambda} \) be the three elliptic curves given by the quartic equations respectively:

\[
E_{\lambda \mu} : Y^2 = -\delta (X^2 - \lambda^2)(\mu^2 X^2 - 1), \\
E_{\mu \nu} : Y^2 = -\delta (X^2 - \mu^2)(\nu^2 X^2 - 1), \\
E_{\nu \lambda} : Y^2 = -\delta (X^2 - \nu^2)(\lambda^2 X^2 - 1).
\]

Then, there exist three dominant maps \( \varphi_{\lambda \mu} : C_{\lambda \mu \nu} \to E_{\lambda \mu}, \varphi_{\mu \nu} : C_{\lambda \mu \nu} \to E_{\mu \nu}, \) and \( \varphi_{\nu \lambda} : C_{\lambda \mu \nu} \to E_{\nu \lambda} \). The map \( \varphi_{\lambda \mu} \) is given by

\[
\varphi_{\lambda \mu} : (x : y : z) \mapsto (X, Y) = \left( \frac{x}{y}, \frac{b_1 x^2 + b_2 y^2 + b_3 z^2}{2 y^2} \right),
\]

where

\[
b_1 = (1 - \lambda^2 \mu^2)(1 - \mu^2)(1 - \lambda^2(1 - \mu^2)), \\
b_2 = (1 - \lambda^2 \mu^2)(1 - \nu^2)(1 - \lambda^2(1 - \mu^2)), \\
b_3 = (1 - \mu^2)^2.
\]

The other maps, \( \varphi_{\mu \nu} \) and \( \varphi_{\nu \lambda} \), are obtained by permuting \( (\lambda, \mu, \nu) \) and \( (x, y, z) \) suitably.

The Jacobian \( J(C_{\lambda \mu \nu}) \) is isogenous to \( E_{\lambda \mu} \times E_{\mu \nu} \times E_{\nu \lambda} \). The twenty-eight double tangent lines of \( C_{\lambda \mu \nu} \) are given by

\[
x \pm \lambda y = 0, \quad \mu x \pm y = 0, \\
y \pm \mu z = 0, \quad \nu y \pm z = 0, \\
z \pm \nu x = 0, \quad \lambda z \pm x = 0, \\
(1 + \mu \nu)x \pm (1 + \nu \lambda)y \pm (1 - \lambda \mu)z = 0, \\
(1 + \mu \nu)x \pm (1 - \nu \lambda)y \pm (1 + \lambda \mu)z = 0, \\
(1 - \mu \nu)x \pm (1 + \nu \lambda)y \pm (1 + \lambda \mu)z = 0, \\
(1 - \mu \nu)x \pm (1 - \nu \lambda)y \pm (1 - \lambda \mu)z = 0.
\]

Proof. Once \( C_{\lambda \mu \nu} \) is found, each statement can be verified by straightforward calculations. To find all the twenty-eight double tangent lines, we use the elliptic surface \( \tilde{E}_{C_{\lambda \mu \nu}, P} \), where \( P = (0 : 0 : 1) \). The pull-back of a double tangent line becomes either a component of a singular fiber or a section. Finding sections for a rational elliptic surface is straightforward using Shioda’s description of generators ([S1]).

Remark 8.3. An equation of the elliptic curve \( E_{\lambda \mu} \) may be written as

\[
-\delta Y^2 = X(X + (1 - \lambda \mu)^2)(X + (1 + \lambda \mu)^2).
\]

Note that the elliptic surface \( Y^2 = X(X + (1 - \lambda \mu)^2)(X + (1 + \lambda \mu)^2) \) has a point of order 4 given by \( (X, Y) = (1 - \lambda^2 \mu^2, 2(1 - \lambda^2 \mu^2)) \).
REMARK 8.4. If we let \( \lambda = \mu = \nu \), then \( C_{\lambda, \mu, \nu} \) is given by the equation

\[
x^4 + y^4 + z^4 = (1 + \lambda')(x^2 y^2 + y^2 z^2 + z^2 x^2), \quad \text{where} \quad \lambda' = \frac{(1 - \lambda^2)^2}{(1 + \lambda^2)^2}.
\]

The curve \( C_{\lambda, \mu, \nu} \) is isogenous to the self-product \( E_{\lambda, \mu} \times E_{\lambda, \nu} \). By a suitable change of coordinates \( E_{\lambda, \nu} \) can be given by the equation

\[
(3 + \lambda') y^2 = X(X - 1)(X - \lambda').
\]

This is shown in [T]. (See also [AT].)

In the following examples the base field \( k \) is taken to be \( \mathbb{Q} \).

EXAMPLE 8.5. Letting \( \lambda = 2, \mu = 3/2 \) and \( \nu = 1/4 \), we obtain the quartic

\[
C_{2, \frac{3}{2}, \frac{1}{4}} : \left( \frac{55}{64} x^4 + \frac{3}{4} y^2 - 8z^2 \right)^2 = \frac{105}{64} x^2 y^2 + \frac{35}{16} x^2 z^2 + \frac{21}{4} y^2 z^2 = 0.
\]

The set of real points \( C_{2, \frac{3}{2}, \frac{1}{4}}(\mathbb{R}) \) together with the twenty-eight double tangent lines are shown in Figure 2. In affine coordinates \((x, y)\), twenty-eight double tangent lines are:

\[
x = \pm 2, \quad x = \pm 4, \quad y = \pm \frac{3}{2}, \quad y = \pm \frac{1}{2} x, \quad y = \pm \frac{3}{2} x, \\
y = \frac{5}{4} x \pm 4, \quad y = \frac{11}{4} x \pm 8, \quad y = \frac{5}{12} x \pm \frac{8}{3}, \quad y = \frac{11}{12} x \pm \frac{4}{3}.
\]

Note that the lines \( y = \pm (5/4)x \pm 4 \) are tangent to the quartic curve at points not defined over \( \mathbb{R} \), and thus they do not touch to \( C_{2, \frac{3}{2}, \frac{1}{4}}(\mathbb{R}) \).

Using \( C_{2, \frac{3}{2}, \frac{1}{4}} \), we can construct some interesting elliptic surfaces.

EXAMPLE 8.6. Consider the double cover of \( \mathbb{P}^2 \) ramified along \( C_{2, \frac{3}{2}, \frac{1}{4}} \) defined by the affine equation

\[
w^2 = \left( \frac{55}{64} x^4 + \frac{3}{4} y^2 - 8 \right)^2 = \frac{105}{64} x^2 y^2 + \frac{35}{16} x^2 z^2 + \frac{21}{4} y^2 z^2.
\]

It has a rational point \((x, y, w) = (0, 64/9, 920/27)\). The point \((x, y) = (0, 64/9)\) in the plane is not on any of the twenty-eight double tangent lines. Using the pencil \(5x - 2t(9y - 64) = 0\), we obtain an elliptic surface

\[
y^2 = x^3 - 9(1109827t^4 - 29342t^2 + 283)x + 54(168253527t^6 - 1133417t^4 - 58451t^2 + 741).
\]

The Mordell-Weil lattice of this elliptic surface is of type \( E_8 \), and it is generated by the following eight sections:

\[
P_1 = (2529t^2 - 57, 9072t^2), \\
P_2 = (-3543t^2 + 39, 9072t^2), \\
P_3 = (2529t^2 + 15, 27288t^2 - 72), \\
P_4 = (2601t^2 - 57, 144t(184t^2 + 5)).
\]
 Twenty-eight Double Tangents of a Plane Quartic

\[ P_5 = (5121t^2 - 57, 432r(703r^2 - 10)) , \]
\[ P_6 = (-1527r^2 + 39, 1008r(143r^2 - 2)) , \]
\[ P_7 = (3541r^2 - 532r - 17, 28(19r - 2)(253r^2 - 76t - 5)) , \]
\[ P_8 = (-3221r^2 + 110r + 43, 88550r^3 + 13128r^2 - 1650r - 100) . \]

**Example 8.7.** Looking at Figure 2, we see that three double tangent lines \( y = 4 \) and \( y = \pm (5/4)x + 4 \) meet at the point \((0 : 4 : 1)\). Using the pencil \( y = tx + 4 \), we obtain the elliptic surface given by

\[
y^2 = x^3 - 108 (30564t^4 - 82350r^2 + 75625)x + 108 (17286048t^6 - 22177800t^4 - 68900625t^2 + 83187500) .
\]

(8)

This elliptic surface has singular fibers of type I₂ at \( t = 0 \) and \( t = \pm 5/4 \). Its Mordell-Weil lattice is isomorphic to \( D_4^* \oplus A_4^* \) (No. 7 in the Oguiso-Shioda classification).

![Figure 2. Quartic curve \( C_{2, \frac{1}{4}, \frac{1}{4}} \) and its twenty-eight double tangents.](image-url)
Also using the intersection point $(0 : 4/3 : 1)$ of two double tangent lines, we obtain

\[
y^2 = x^3 - 36(91692t^4 - 46890t^2 + 120025)x + 324(5762016t^6 + 3994920t^4 - 3186675t^2 + 10660000).
\]

This elliptic surface has singular fibers of type $I_2$ at $t = \pm 11/12$. Its Mordell-Weil lattice is isomorphic to $D_4^*$ (No. 4 in the Oguiso-Shioda classification).

In both examples, all the sections are defined over $\mathbb{Q}$. It is not difficult to write them down.

References


[H] Robin Hartshorne, Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.


