On the Zeros of Dirichlet L -Functions II

by

Akio FUJII

(Received September 18, 2003)

§1. Introduction

Here we are concerned with the distribution of the zeros of Dirichlet L -functions. Corresponding to ζ -function theorems proved in our previous work [3], analogous theorems will be presented for L -functions when we consider their conductors as a variable.

Let $L(s, \chi)$ be the Dirichlet L -function attached to a primitive Dirichlet character $\chi \bmod q (> 1)$. We denote the non-trivial zeros of $L(s, \chi)$ by $\rho(\chi) = \beta(\chi) + i\gamma(\chi)$ with real numbers $\beta(\chi)$ and $\gamma(\chi)$. When $T \neq \gamma(\chi)$ for any $\gamma(\chi)$, we put

$$S(T, \chi) = \frac{1}{\pi} \arg L\left(\frac{1}{2} + iT, \chi\right),$$

where the argument is obtained by the continuous variation along the straight line $s = \sigma + iT$, when σ varies from $+\infty$ to $\frac{1}{2}$, starting with the value zero. When $T = \gamma(\chi)$, then we put

$$S(T, \chi) = \frac{1}{2}(S(T+0, \chi) + S(T-0, \chi)).$$

It is known that

$$S(T, \chi) = O(\log q(1+T))$$

and that under the Generalized Riemann Hypothesis (cf. Theorem 6 on p. 330 of Selberg [5])

$$S(T, \chi) = O\left(\frac{\log q(1+T)}{\log \log q(1+T)}\right).$$

In Fujii [2], we have been concerned with the average behavior of $S(T, \chi)$ and defined

$$\tilde{S}_m(T, \chi)$$

for $m \geq 0$ as follows. When $T \neq \gamma(\chi)$, then we put

$$\tilde{S}_0(T, \chi) = S(T, \chi)$$

and

$$\tilde{S}_m(T, \chi) = \int_0^T \tilde{S}_{m-1}(t, \chi) dt + C_m(\chi)$$

for any integer $m \geq 1$, where $C_m(\chi)$'s are the constants defined by

$$C_{2k-1}(\chi) = (-1)^{k-1} \frac{1}{\pi} \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k-1} \log |L(\sigma, \chi)| (d\sigma)^{2k-1}$$

for $m = 2k - 1$ and

$$C_{2k}(\chi) = (-1)^k \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k} \Im \log L(\sigma, \chi) (d\sigma)^{2k}$$

for $m = 2k$, respectively. When $T = \gamma(\chi)$, then we put

$$\tilde{S}_m(T, \chi) = \frac{1}{2}(\tilde{S}_m(T + 0, \chi) + \tilde{S}_m(T - 0, \chi)).$$

We shall first describe the relation between $\tilde{S}_m(T, \chi)$ and the integral $I_m(T, \chi)$ which will be defined as follows. When $T \neq \gamma(\chi)$, then we put for $k \geq 1$

$$I_{2k-1}(T, \chi) = \frac{1}{\pi} (-1)^{k-1} \Re \left\{ \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k-1} \log L(\sigma + iT, \chi) (d\sigma)^{2k-1} \right\}$$

and

$$I_{2k}(T, \chi) = \frac{1}{\pi} (-1)^k \Im \left\{ \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k} \log L(\sigma + iT, \chi) (d\sigma)^{2k} \right\}.$$

When $T = \gamma(\chi)$, then we put for $m \geq 1$

$$I_m(T, \chi) = \frac{1}{2}(I_m(T + 0, \chi) + I_m(T - 0, \chi)).$$

For $\sigma \geq \frac{1}{2}$ and $T > T_o$, let $N(\sigma, T, \chi)$ be the number of the zeros $\beta(\chi) + i\gamma(\chi)$ of $L(s, \chi)$ such that $\beta(\chi) > \sigma$ and $0 \leq \gamma(\chi) < T$ when $T \neq \gamma(\chi)$, the zeros with $\gamma(\chi) = 0$ counting one half only. When $T = \gamma(\chi)$, then we put

$$N(\sigma, T, \chi) = \frac{1}{2}((N(\sigma, T + 0, \chi) + N(\sigma, T - 0, \chi)).$$

In Fujii [2], we have shown the following results. We suppose in (I) and (II) below that $T > 0$ and χ is a primitive Dirichlet character mod $q > 1$.

I. We have

$$\tilde{S}_1(T, \chi) = I_1(T, \chi)$$

and for any integer $m \geq 2$

$$\tilde{S}_m(T, \chi) = I_m(T, \chi) + 2 \sum_{\substack{h+2r=m \\ r \geq 1, h \geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(T, \chi),$$

where we put for $h \geq 1$ and $r \geq 1$

$$\tilde{N}_{h,2r}(T) = \underbrace{\int_0^T \cdots \int_0^t}_h \underbrace{\int_{\frac{1}{2}}^1 \int_{\sigma}^1 \cdots \int_{\sigma}^1}_{2r} N(\sigma, t, \chi) (d\sigma)^{2r} (dt)^h$$

and for $h = 0$ and $r \geq 1$

$$\tilde{N}_{0,2r}(T, \chi) = \underbrace{\int_{\frac{1}{2}}^1 \int_{\sigma}^1 \cdots \int_{\sigma}^1}_{2r} N(\sigma, T, \chi) (d\sigma)^{2r}.$$

II-(i). We have for any integer $m \geq 1$,

$$I_m(T, \chi) \ll \log q(1 + |T|),$$

where the constant involved in the upper bound may depend on m .

II-(ii). We have

$$\tilde{S}_1(T, \chi) = O(\log q(1 + T))$$

and for any integer $m \geq 2$,

$$\tilde{S}_m(T, \chi) = 2 \sum_{\substack{h+2r=m \\ r \geq 1, h \geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(T, \chi) + O(\log q(1 + T)).$$

II-(iii). Suppose further that $q \ll T^{\frac{1}{4}-\varepsilon}$ with an arbitrarily small positive ε . Then we have for any integer $m \geq 2$

$$\tilde{S}_m(T, \chi) \ll \frac{T^{m-1}}{\log T}.$$

II-(iv). (On the Generalized Riemann Hypothesis) We have for any integer $m \geq 1$,

$$\tilde{S}_m(T, \chi) = O\left(\frac{\log q(1 + T)}{(\log \log q(1 + T))^{m+1}}\right).$$

II-(v). The following statement is equivalent to the Generalized Riemann Hypothesis for $L(s, \chi)$. For any integer $m \geq 3$, we have

$$\tilde{S}_m(T, \chi) = o(T^{m-2}) \quad (T \rightarrow \infty).$$

We now turn our attentions to q -analogues. When we consider q -analogue in this article, we always assume, for simplicity as in pp. 281–340 of Selberg [5], that q runs over the odd prime numbers. Selberg (cf. Theorem 8 on p. 336 of [5]) showed that for $|T| \leq q^{\frac{1}{4}-\varepsilon}$, ε being a fixed arbitrarily small positive number $\leq \frac{1}{4}$,

$$\sum_{\chi \neq \chi_0} S(T, \chi) = O(q),$$

where χ runs over the Dirichlet characters mod q and χ_0 is the principal character mod q . Moreover, Selberg showed that for $|T| \leq q^{\frac{1}{4}-\varepsilon}$

$$\sum_{\chi \neq \chi_0} (S(T, \chi))^{2k} = \frac{(2k)!}{(2\pi)^{2k} k!} q (\log \log q)^k + O(q (\log \log q)^{k-\frac{1}{2}}).$$

It has been our problem to extend these to $\tilde{S}_m(T, \chi)$ for $m \geq 1$. Concerning $\sum_{\chi \neq \chi_0} \tilde{S}_m(T, \chi)$, we [2] have shown the following results.

III-(i). For any integer $m \geq 1$ and for $|T| \leq q^{\frac{1}{4}-\varepsilon}$, we have

$$\sum_{\chi \neq \chi_0} I_m(T, \chi) = O(q).$$

III-(ii). Suppose that $0 < T \leq q^{\frac{1}{4}-\varepsilon}$ with an arbitrarily small positive number $\varepsilon \leq \frac{1}{4}$. Then we have

$$\sum_{\chi \neq \chi_0} \tilde{S}_1(T, \chi) = O(q).$$

For any integer $m \geq 2$, we have

$$\begin{aligned} \sum_{\chi \neq \chi_0} \tilde{S}_m(T, \chi) &= 2 \sum_{\substack{h+2r=m \\ r \geq 1, h \geq 0}} (-1)^{r-1} \sum_{\chi \neq \chi_0} \tilde{N}_{h,2r}(T, \chi) + O(q) \\ &= O\left(\frac{qT^{m-1}}{\log q} + q\right). \end{aligned}$$

III-(iii). For any integer $m \geq 2$ and for $0 < T \ll (\log q)^{\frac{1}{m-1}}$, we have

$$\sum_{\chi \neq \chi_0} \tilde{S}_m(T, \chi) = O(q).$$

III-(iv). (On the Generalized Riemann Hypothesis) For any integer $m \geq 0$ and for $0 < T \ll e^{A(\log q)^{m+1}}$ with any positive constant A , we have

$$\sum_{\chi \neq \chi_0} \tilde{S}_m(T, \chi) = O(q).$$

Here we are concerned with the mean value theorems on high powers of $\tilde{S}_m(t, \chi)$ and shall prove the following theorem which is a q -analogue of Theorem 2 in our previous work [3].

THEOREM. Suppose that $|T| \leq q^{\frac{1}{4}-\varepsilon}$, where ε is any arbitrarily small positive number $\leq \frac{1}{4}$. Then for any integer $m \geq 1$ and $k \geq 1$, we have

$$\begin{aligned}
& \sum_{\chi \neq \chi_0} (\tilde{S}_m(T, \chi) - 2 \sum_{\substack{h+2r=m \\ r \geq 1, h \geq 0}} (-1)^{r-1} \tilde{N}_{h,2r}(T, \chi))^{2k} \\
&= \sum_{\chi \neq \chi_0} (I_m(T, \chi))^{2k} \\
&= \frac{q}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)} \\
&\quad \sum_{\substack{n_l=2 \\ n_1 \cdots n_j = n_{j+1} \cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1) \cdots \Lambda(n_{2k})}{n_1 \cdots n_j (\log n_1 \cdots \log n_{2k})^{m+1}} + O\left(\frac{q}{\log q} R_m(q)\right),
\end{aligned}$$

where $\Lambda(n)$ is the von-Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and we put

$$R_m(q) = \begin{cases} \sqrt{\log \log q} & \text{if } m = 1 \\ 1 & \text{if } m \geq 2. \end{cases}$$

We may notice some direct consequences of this theorem.

COROLLARY 1. Suppose that $|T| \leq q^{\frac{1}{4}-\varepsilon}$, where ε is any arbitrarily small positive number $\leq \frac{1}{4}$. Then for any integer $k \geq 1$, we have

$$\begin{aligned}
& \sum_{\chi \neq \chi_0} (\tilde{S}_1(T, \chi))^{2k} = \frac{q}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} \\
& \cdot \sum_{\substack{n_l=2 \\ n_1 \cdots n_j = n_{j+1} \cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1) \cdots \Lambda(n_{2k})}{n_1 \cdots n_j (\log n_1 \cdots \log n_{2k})^2} + O\left(\frac{q \sqrt{\log \log q}}{\log q}\right).
\end{aligned}$$

COROLLARY 2. (On the Generalized Riemann Hypothesis) Suppose that m is an integer ≥ 2 and $|T| \leq e^{A(\log q)^m}$, where A is some positive absolute constant. Then for any integer $k \geq 1$, we have

$$\begin{aligned}
& \sum_{\chi \neq \chi_0} (\tilde{S}_m(T, \chi))^{2k} = \frac{q}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)} \\
& \cdot \sum_{\substack{n_l=2 \\ n_1 \cdots n_j = n_{j+1} \cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1) \cdots \Lambda(n_{2k})}{n_1 \cdots n_j (\log n_1 \cdots \log n_{2k})^{m+1}} + O\left(\frac{q}{\log q}\right).
\end{aligned}$$

§2. Proof of Theorem.

In the rest of this article, we shall prove Theorem . Hereafter we suppose that $|t| \leq q^{\frac{1}{4}-\varepsilon}$, where ε is any arbitrarily small positive number $\leq \frac{1}{4}$. We put

$$X = q^{\frac{\varepsilon C}{k}}$$

with some positive absolute constant C . We start with the following lemma which is proved in Fujii [2].

LEMMA. *For any integer $m \geq 1$, we have*

$$\begin{aligned} I_m(t, \chi) = & \frac{1}{\pi} \Im \left\{ i^m \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{\sigma_\chi + it} (\log n)^{m+1}} \sum_{v=0}^m \frac{((\sigma_\chi - \frac{1}{2}) \log n)^v}{v!} \right\} \\ & + O \left(\left(\sigma_\chi - \frac{1}{2} \right)^{m+1} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{\sigma_\chi + it}} \right| \right) + O \left(\left(\sigma_\chi - \frac{1}{2} \right)^{m+1} \log q (1 + |t|) \right), \end{aligned}$$

where $\Lambda_X(n)$ is defined by

$$\Lambda_X(n) = \begin{cases} \Lambda(n) & \text{for } 1 \leq n \leq X \\ \Lambda(n) \frac{\log^2 \frac{X^3}{n} - 2 \log^2 \frac{X^2}{n}}{2 \log^2 X} & \text{for } X \leq n \leq X^2 \\ \Lambda(n) \frac{\log^2 \frac{X^3}{n}}{2 \log^2 X} & \text{for } X^2 \leq n \leq X^3 \end{cases}$$

and we put, as in pp. 323–325 of Selberg [5],

$$\sigma_\chi = \frac{1}{2} + 2 \max_{\rho(\chi)} \left(\beta(\chi) - \frac{1}{2}, \frac{2}{\log X} \right),$$

$\rho(\chi)$ running over all zeros $\beta(\chi) + i\gamma(\chi)$ of $L(s, \chi)$ for which

$$|t - \gamma(\chi)| \leq \frac{X^{3|\beta(\chi) - \frac{1}{2}|}}{\log X}.$$

For convenience, we shall decompose the right hand side further as follows.

$$\begin{aligned} I_m(t, \chi) = & \frac{1}{\pi} \Im \left\{ i^m \sum_{n < X} \frac{\Lambda(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right\} \\ & + \frac{1}{\pi} \Im \left\{ i^m \sum_{X \leq n < X^3} \frac{\Lambda_X(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\pi} \Im \left\{ i^m \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{it} (\log n)^{m+1}} \left(\frac{1}{n^{\sigma_\chi}} - \frac{1}{\sqrt{n}} \right) \right\} \\
& + \frac{1}{\pi} \Im \left\{ i^m \sum_{v=1}^m \frac{(\sigma_\chi - \frac{1}{2})^v}{v!} \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{\sigma_\chi + it} (\log n)^{m+1-v}} \right\} \\
& + O \left(\left(\sigma_\chi - \frac{1}{2} \right)^{m+1} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{\sigma_\chi + it}} \right| \right) \\
& + O \left(\left(\sigma_\chi - \frac{1}{2} \right)^{m+1} \log q(1 + |t|) \right).
\end{aligned}$$

The last three sums are rewritten in the following form as in p. 334 of Selberg [5].

$$\begin{aligned}
& \left| \frac{1}{\pi} \Im \left\{ i^m \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{it} (\log n)^{m+1}} \left(\frac{1}{n^{\sigma_\chi}} - \frac{1}{\sqrt{n}} \right) \right\} \right| \\
& = \left| \frac{1}{\pi} \Im \left\{ i^m \int_{\frac{1}{2}}^{\sigma_\chi} \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{\sigma' + it} (\log n)^m} d\sigma' \right\} \right| \\
& \ll \left(\sigma_\chi - \frac{1}{2} \right) X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^m} \right| d\sigma, \\
& \left| \frac{1}{\pi} \Im \left\{ i^m \sum_{v=1}^m \frac{(\sigma_\chi - \frac{1}{2})^v}{v!} \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{\sigma_\chi + it} (\log n)^{m+1-v}} \right\} \right| \\
& \ll \sum_{v=1}^m \left(\sigma_\chi - \frac{1}{2} \right)^v X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^{m+1-v}} \right| d\sigma
\end{aligned}$$

and

$$\begin{aligned}
& \left(\sigma_\chi - \frac{1}{2} \right)^{m+1} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n)}{n^{\sigma_\chi + it}} \right| \\
& \ll \left(\sigma_\chi - \frac{1}{2} \right)^{m+1} X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it}} \right| d\sigma.
\end{aligned}$$

Thus we have

$$\begin{aligned}
I_m(t, \chi) & = \frac{1}{\pi} \Im \left\{ i^m \sum_{n < X} \frac{\Lambda(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right\} \\
& + \frac{1}{\pi} \Im \left\{ i^m \sum_{X \leq n < X^3} \frac{\Lambda_X(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right\}
\end{aligned}$$

$$\begin{aligned}
& + O\left(\left(\sigma_\chi - \frac{1}{2}\right) X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^m} \right| d\sigma\right) \\
& + O\left(\sum_{v=1}^m \left(\sigma_\chi - \frac{1}{2}\right)^v X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^{m+1-v}} \right| d\sigma\right) \\
& + O\left(\left(\sigma_\chi - \frac{1}{2}\right)^{m+1} X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it}} \right| d\sigma\right) \\
& + O\left(\left(\sigma_\chi - \frac{1}{2}\right)^{m+1} \log q (1 + |t|)\right).
\end{aligned}$$

We shall estimate the $2k$ -th moment of each term of this expression. We notice first that

$$\begin{aligned}
\sum_{\chi} \left(\sum_{X \leq p < X^3} \frac{\Lambda_X(p) \chi(p)}{\sqrt{p} \cdot p^{it} (\log p)^{m+1}} \right)^{2k} &= \sum_{\chi} \left| \sum_{X^k \leq n < X^{3k}} \frac{\chi(n) b(n)}{\sqrt{n} n^{it}} \right|^2 \\
&\ll q \sum_{X^k \leq n < X^{3k}} \frac{|b(n)|^2}{n} \ll q \frac{1}{(\log X)^{km}} \left(\sum_{X \leq p < X^3} \frac{1}{p (\log p)^m} \right)^k \ll \frac{q}{(\log X)^{2km}},
\end{aligned}$$

where we put

$$b(n) = \sum_{n=p_1 \cdots p_j \cdots p_k, X \leq p_j < X^3} \frac{\Lambda_X(p_1) \cdots \Lambda_X(p_k)}{(\log p_1)^{m+1} \cdots (\log p_k)^{m+1}}$$

and use the estimate

$$b(n) \ll \frac{1}{(\log X)^{km}}.$$

In a similar manner, we get

$$\sum_{\chi} \left| \sum_{X \leq p^2 < X^3} \frac{\Lambda_X(p^2) \chi(p^2)}{p \cdot p^{2it} (\log p^2)^{m+1}} \right|^{2k} \ll \frac{q}{(\log X)^{2km}}.$$

Thus we get

$$\sum_{\chi} \left| \sum_{X \leq n < X^3} \frac{\Lambda_X(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right|^{2k} \ll \frac{q}{(\log X)^{2km}}.$$

Applying Lemma 17 on p. 331 of Selberg [5], we get

$$\sum_{\chi \neq \chi_0} \left(\sigma_\chi - \frac{1}{2} \right)^{2k(m+1)} (\log q (1 + |t|))^{2k} \ll \frac{q \log^{2k} q}{(\log X)^{2k(m+1)}}.$$

Combining these two types of the estimates, we can estimate the rest as follows.

$$\begin{aligned}
& \sum_{\chi \neq \chi_0} \left(\left(\sigma_\chi - \frac{1}{2} \right) X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^m} \right| d\sigma \right)^{2k} \\
& \ll \left(\sum_{\chi \neq \chi_0} \left(\sigma_\chi - \frac{1}{2} \right)^{4k} X^{4k(\sigma_\chi - \frac{1}{2})} \right)^{\frac{1}{2}} \\
& \quad \cdot \left(\sum_{\chi} \left(\int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^m} \right| d\sigma \right)^{4k} \right)^{\frac{1}{2}} \\
& \ll \left(\sum_{\chi \neq \chi_0} \left(\sigma_\chi - \frac{1}{2} \right)^{4k} X^{4k(\sigma_\chi - \frac{1}{2})} \right)^{\frac{1}{2}} \\
& \quad \cdot \left(\frac{1}{(\log X)^{4k-1}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \sum_{\chi} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^m} \right|^{4k} d\sigma \right)^{\frac{1}{2}} \\
& \ll \frac{q}{(\log X)^{2k}} \begin{cases} (\log \log X)^k & \text{if } m = 1 \\ 1 & \text{if } m \geq 2, \end{cases}
\end{aligned}$$

because we have

$$\begin{aligned}
& \sum_{\chi} \left| \sum_{p < X^3} \frac{\Lambda_X(p) \chi(p) \log(Xp)}{p^{\sigma + it} (\log p)^m} \right|^{4k} = \sum_{\chi} \left| \sum_{n < X^{6k}} \frac{\chi(n) b(n)}{n^{\sigma + it}} \right|^2 \\
& \ll q (\log X)^{4k} \cdot \left(\sum_{p < X^3} \frac{\log p}{p \log^m p} \right)^{2k} \ll q (\log X)^{4k} \begin{cases} (\log \log X)^{2k} & \text{if } m = 1 \\ 1 & \text{if } m \geq 2 \end{cases}
\end{aligned}$$

with

$$\begin{aligned}
b(n) &= \sum_{n = p_1 \cdots p_j \cdots p_{2k}, p_j \leq X^3} \frac{\Lambda_X(p_1) \cdots \Lambda_X(p_{2k}) \log(Xp_1) \cdots \log(Xp_{2k})}{(\log p_1 \cdots \log p_{2k})^m} \\
&\ll (\log X)^{2k}.
\end{aligned}$$

In the same manner, we have, for each $1 \leq \nu \leq m$,

$$\begin{aligned}
& \sum_{\chi \neq \chi_0} \left(\left(\sigma_\chi - \frac{1}{2} \right)^{\nu} X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^{m+1-\nu}} \right| d\sigma \right)^{2k} \\
& \ll \left(\sum_{\chi \neq \chi_0} \left(\sigma_\chi - \frac{1}{2} \right)^{4k\nu} X^{4k(\sigma_\chi - \frac{1}{2})} \right)^{\frac{1}{2}} \\
& \quad \cdot \left(\frac{1}{(\log X)^{4k-1}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2} - \sigma} \sum_{\chi} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma + it} (\log n)^{m+1-\nu}} \right|^{4k} d\sigma \right)^{\frac{1}{2}}
\end{aligned}$$

$$\ll \frac{q}{(\log X)^{2kv}} \begin{cases} (\log \log X)^k & \text{if } v = m \\ 1 & \text{if } 1 \leq v \leq m-1. \end{cases}$$

Hence, we get

$$\begin{aligned} & \sum_{\chi \neq \chi_0} \left(\sum_{v=1}^m \left(\sigma_\chi - \frac{1}{2} \right)^v X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2}-\sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma+it} (\log n)^{m+1-v}} \right| d\sigma \right)^{2k} \\ & \ll \frac{q}{(\log X)^{2km}} (\log \log X)^k + \frac{q}{(\log X)^{2k}} \\ & \ll \frac{q}{(\log X)^{2k}} \begin{cases} (\log \log X)^k & \text{if } m = 1 \\ 1 & \text{if } m \geq 2. \end{cases} \end{aligned}$$

Finally, we have in the same manner

$$\begin{aligned} & \sum_{\chi \neq \chi_0} \left(\left(\sigma_\chi - \frac{1}{2} \right)^{m+1} X^{\sigma_\chi - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2}-\sigma} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma+it}} \right| d\sigma \right)^{2k} \\ & \ll \left(\sum_{\chi \neq \chi_0} \left(\sigma_\chi - \frac{1}{2} \right)^{4k(m+1)} X^{4k(\sigma_\chi - \frac{1}{2})} \right)^{\frac{1}{2}} \\ & \quad \cdot \left(\frac{1}{(\log X)^{4k-1}} \int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2}-\sigma} \sum_{\chi} \left| \sum_{n < X^3} \frac{\Lambda_X(n) \chi(n) \log(Xn)}{n^{\sigma+it}} \right|^{4k} d\sigma \right)^{\frac{1}{2}} \\ & \ll \frac{q}{(\log X)^{k(2m+1)}}. \end{aligned}$$

Consequently, we get

$$\begin{aligned} & \sum_{\chi \neq \chi_0} \left| I_m(t, \chi) - \frac{1}{\pi} \Im \left\{ i^m \sum_{n < X} \frac{\Lambda(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right\} \right|^{2k} \\ & \ll \frac{q}{(\log q)^{2k}} \begin{cases} (\log \log q)^k & \text{if } m = 1 \\ 1 & \text{if } m \geq 2. \end{cases} \end{aligned}$$

We notice that

$$\begin{aligned} & \sum_{\chi \neq \chi_0} \left(\frac{1}{\pi} \Im \left\{ i^m \sum_{n < X} \frac{\Lambda(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right\} \right)^{2k} \\ & = \frac{(-1)^{(m+1)k}}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j(m+1)} \sum_{\chi} \left(\sum_{n < X} \frac{\Lambda(n) \bar{\chi}(n)}{\sqrt{n} n^{-it} (\log n)^{m+1}} \right)^j \\ & \quad \cdot \left(\sum_{n < X} \frac{\Lambda(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right)^{2k-j} - \left(\frac{1}{\pi} \Im \left\{ i^m \sum_{n < X} \frac{\Lambda(n) \chi_0(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right\} \right)^{2k} \end{aligned}$$

$$\begin{aligned}
&= \frac{q-1}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)} \sum_{\substack{n_1=2 \\ n_1 \cdots n_j = n_{j+1} \cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1) \cdots \Lambda(n_{2k})}{n_1 \cdots n_j (\log n_1 \cdots \log n_{2k})^{m+1}} \\
&\quad + O\left(\frac{q}{\log q}\right).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\sum_{\chi \neq \chi_0} (I_m(t, \chi))^{2k} &= \sum_{\chi \neq \chi_0} (I_m(t, \chi) - M + M)^{2k} \\
&= \sum_{\chi \neq \chi_0} \sum_{j=0}^{2k} \binom{2k}{j} M^j (I_m(t, \chi) - M)^{2k-j} \\
&= \sum_{\chi \neq \chi_0} M^{2k} + \sum_{\chi \neq \chi_0} \sum_{j=0}^{2k-1} \binom{2k}{j} M^j (I_m(t, \chi) - M)^{2k-j} \\
&= \sum_{\chi \neq \chi_0} M^{2k} + O\left(\sum_{j=0}^{2k-1} \binom{2k}{j} \left(\sum_{\chi \neq \chi_0} M^{2k}\right)^{\frac{j}{2k}}\right) \cdot \left(\sum_{\chi \neq \chi_0} (I_m(t, \chi) - M)^{2k}\right)^{\frac{2k-j}{2k}} \\
&= \frac{q}{(2\pi)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{(j+k)(m+1)} \\
&\quad \sum_{\substack{n_1=2 \\ n_1 \cdots n_j = n_{j+1} \cdots n_{2k}}}^{\infty} \frac{\Lambda(n_1) \cdots \Lambda(n_{2k})}{n_1 \cdots n_j (\log n_1 \cdots \log n_{2k})^{m+1}} + O\left(\frac{q}{\log q} R_m(q)\right),
\end{aligned}$$

where we put

$$M = \frac{1}{\pi} \Im \left\{ i^m \sum_{n < X} \frac{\Lambda(n) \chi(n)}{\sqrt{n} n^{it} (\log n)^{m+1}} \right\}$$

and

$$R_m(q) = \begin{cases} \sqrt{\log \log q} & \text{if } m = 1 \\ 1 & \text{if } m \geq 2. \end{cases}$$

Thus we have obtained our theorem.

References

- [1] A. Fujii, On the zeros of the Riemann zeta function, *Comment. Math. Univ. St. Pauli* **51** (2002), 1–17.
- [2] A. Fujii, On the zeros of Dirichlet L -functions, to appear.
- [3] A. Fujii, On the zeros of the Riemann zeta function II, *Comment. Math. Univ. St. Pauli* **52** (2003), 165–190.

- [4] J. E. Littlewood, On the zeros of Riemann's zeta function, Proc. of Camb. Phil. Soc., **22** (1924), 295–318.
- [5] A. Selberg, Collected Papers, Springer Verlag, vol. 1, 1989.
- [6] E. C. Titchmarsh, The theory of the Riemann zeta function (2nd ed. rev. by D. R. Heath-Brown), Oxford Univ. Press, 1951 (1988).

Department of Mathematics
Rikkyo University
Nishi-Ikebukuro, Toshimaku
Tokyo, 171–8501 Japan