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A Complete Characterization of Quasi-p-Pure-Injective Groups

by

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An abelian group G is said to be *quasi-injective* with respect to its subgroup H, if every homomorphism from H to G can be extended to an endomorphism of G. A group is said to be *quasi-p-pure-injective* if it is quasi-injective with respect to all its p-pure subgroups. The idea of considering groups which are quasi-injective with respect to various families of their subgroups originated from Problem 17 in [3] p. 134. The first author and A. Laroche introduced quasi-p-pure-injective groups (q.p.p.i.) in [1] and succeeded in characterizing the p-reduced q.p.p.i. groups with torsion p-basic subgroups. They also gave a complete description of those p-reduced q.p.p.i. groups with finite torsion free rank p-basic subgroups. We study in the present article the remaining cases: namely, the p-divisible reduced q.p.p.i. groups and those p-reduced ones with infinite torsion free rank p-basic subgroups. We give a complete characterization of all cases. In particular, the p-reduced q.p.p.i. groups with infinite torsion free rank p-basic subgroups turn out to be complete in their p-adic topology and thus simply p-pure-injective. All groups considered here are abelian groups. For notation and terminology we follow the standards set in [3]. Throughout this article p stands for an arbitrary but fixed prime number.

1. p-Divisible q.p.p.i. reduced groups

Let G be a reduced q.p.p.i. group and let D be its largest p-divisible subgroup. D is a p-pure fully invariant subgroup of G and, as such, it is itself a q.p.p.i. group. The next proposition characterizes p-divisible reduced q.p.p.i.-groups.

PROPOSITION 1.1. *Let G be a reduced p-divisible group. G is q.p.p.i. if and only if*

$$G=\bigoplus_{q\neq p} G_q$$

(q is a prime number) and G_q is a direct sum of cyclic q-groups of same order.

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Proof. G being p-divisible, the family of p-pure subgroups of G coincides with the family of p-neat subgroups of G. Thus G is, in fact, quasi-p-neat-injective. These were studied in [2] and the result follows immediately from Theorem 3.1 of [2]. The converse follows from Proposition 1.1 of [1] and that, in this case, G_q is, in fact, already quasi-injective.

We show next that reduced q.p.p.i. groups which contain non-zero p-divisible subgroups must be torsion groups. We need the following lemmas:

LEMMA 1.2. *Let G be a reduced q.p.p.i. groups and suppose that its largest p-divisible subgroup D is non-zero. Then for every prime number $q \neq p$, the D-high subgroup of G containing G_p are q-divisible.*

Proof. Clearly $D_p = 0$, since G is reduced. Therefore, $G_p \cap D = 0$. Let H be a D-high subgroup of G containing G_p. H is a p-pure subgroup of G. Indeed, H is automatically p-neat as a D-high subgroup of G. Let $p^n g \in H$, for some $g \in G$, then exists $h \in H$, such that $p(p^{n-1}g) = ph$, i.e. $p^{n-1}g - h \in G_p \subset H$. Therefore $p^{n-1}g \in H$. Repeating this procedure at most n-times we see that $g \in H$ and H is, in fact, p-pure in a strong sense. Let $q \neq p$, be a prime number. Then qH is also a p-pure subgroup of G and since $H_q = H \cap G_q = H \cap D_q = 0$, the map $qh \to h$ is a well defined homomorphism between qH and H. This homomorphism extends to an endomorphism φ of G. Therefore $\varphi(qh) = q\varphi(h) = h \in qG$, and $H \subset qG$. However H is q-neat, being a D-high subgroup of G. Therefore $H = qH$.

LEMMA 1.3. *Let G and D be as in the preceding lemma. Then $(D \oplus G_p)/G_p$ is an absolute summand of G/G_p.*

Proof. Let H/G_p be a $(D \oplus G_p)/G_p$-high subgroup of G/G_p. Then H is a D-high subgroup of G containing G_p. We need only show that $G = H \oplus D$. Let $x \in G$, if $x \notin H$ then $\langle H, x\rangle \cap D \neq 0$, and there exists $a \in Z$, $h \in H$ and $d \neq 0$, $d \in D$, such that $ax + h = d$. Let $a = p^i m$ where $(m, p) = 1$. Since D is p-divisible there is $d' \in D$ such that $d = p^i d'$. Since H is p-pure, there is $h' \in H$ such that $h = p^i h'$. Thus $p^i(mx + h' - d') = 0$. It follows that $mx + h' - d' \in H$. Say: $mx + h' - d' = h'' \in H$. However, H is m-divisible since $(m, p) = 1$, and D is pure in G, since

$$D = \bigoplus_{q \neq p} D_q = \bigoplus_{q \neq p} G_q .$$

Therefore there exists $h_0 \in H$ and $d_0 \in D$ such that

$$h' - h'' = mh_0 , \qquad d' = md_0 \qquad \text{and} \qquad m(x + h_0 - d_0) = 0 .$$

In other words, $x + h_0 - d_0 \in D$. Therefore $x \in H \oplus D$, and $G = H \oplus D$.

PROPOSITION 1.4. *Let G be a reduced group with non-zero maximal p-divisible subgroup D, then G is q.p.p.i. if and only if G is torsion and G_q is the direct sum of cyclic groups of same order for each $q \neq p$ while G_p is a torsion complete group.*

Proof. If G is q.p.p.i. then from the previous lemmas $(D\oplus G_p)/G_p$ is an absolute summand of G/G_p and if H/G_p is a $(D\oplus G_p)/G_p$-high subgroup then $G/G_p = (D\oplus G_p)/G_p \oplus H/G_p$. We want to show that H/G_p is the largest divisible subgroup of G/G_p. It suffices to show that H/G_p is p-divisible since, from Lemma 1.2, H is already q-divisible for all prime $q\neq p$. This is true if and only if the p-basic subgroup of G is torsion. If such was not the case then there would exist in G a cyclic p-pure subgroup of infinite order $\langle x\rangle$. Then $\langle qx\rangle$ is also a p-pure subgroup of G for every $q\neq p$. Now $D\neq 0$, say: $D_q\neq 0$. Let $y\in D_q$, $y\neq 0$, then the application $qx\to y$ defines a homomorphism from $\langle qx\rangle$ onto $\langle y\rangle$ which extends to an endomorphism φ of G therefore $y=\varphi(qx)=q\varphi(x)$ and D_q is q-divisible. This is a contradiction. Therefore H/G_p is the divisible part of G/G_p. This means that H is unique, but this is possible if and only if $H=G_p$. Therefore

$$G=\left(\bigoplus_{q\neq p} G_q\right)\oplus G_p$$

and the result follows from Propositon 1.1 and Theorem 1.2 of [1]. In view of the preceding result, if G is a reduced q.p.p.i. group and G is not torsion then G must be p-reduced. These groups were studied in [1] and in particular it was shown that they are q-divisible for every $q\neq p$ and $G^1=\bigcap nG=\bigcap p^nG=0$. ([1], Theorem 2.5 and corollary, p. 580–581.)

Before closing this section we investigate how the q.p.p.i. reduced p-divisible groups can combine with divisible groups and remain q.p.p.i.

From [1], if G is q.p.p.i. and we write $G=D\oplus R$ where D is divisible and R is reduced, then R is q.p.p.i. Furthermore if R is p-reduced then the converse is true ([1], Theorem 2.6). What happens if R is q.p.p.i. not p-reduced?

THEOREM 1.5. *Let G be a reduced q.p.p.i. group with a non-zero p-divisible maximal subgroup. The $G\oplus D$ is a* q.p.p.i. group where *D is divisible if and only if the following conditions hold for D*:

1) $D_q=0$ *if* $G_q\neq 0$, $q\neq p$;
2) *D is torsion if $G_q\neq 0$ for some prime $q\neq p$.*

Proof. If $G_q\neq 0$ then $G_q\oplus D$ is a q.p.p.i. group and since it is p-divisible it is in fact a q.p.n.i. group. Write D as $D_p\oplus D'$ then $G_q\oplus D'$ is also q.p.n.i. whose p-primary component is zero. Therefore from Theorem 3.1 in [2], $D_q=0$ and D is torsion. Conversely, suppose

$$G=\left(\bigoplus_{q\neq p} G_q\right)\oplus G_p$$

is a non-p-reduced q.p.p.i. reduced group and let D be a divisible group satisfying condition 1 and 2. Then $G\oplus D$ is q.p.p.i. Indeed, since G is non-p-reduced, at least one G_q is non-trivial. Let $A=\{q\in P \mid G_q\neq 0\}$. Then

$$G=\left(\bigoplus_{q\in A} G_q\right) \quad \text{and} \quad D=\bigoplus_{q\notin A} D_q\,.$$

Then $G \oplus D$ is a torsion group all of whose primary components are q.p.p.i. Therefore $G \oplus D$ is q.p.p.i.

2. p-Reduced q.p.p.i. groups

In [1] the case where the groups had a p-basic subgroup with finite torsion free rank was in fact completely settled. For the sake of completeness we recall here that result:

THEOREM 2.1. *Let G be a p-reduced group whose p-basic subgroups have finite torsion free rank. The G is a q.p.p.i. group if and only if $G = H \oplus K$ where:*

a) *H has torsion p-basic subgroup B_p and H is isomorphic to a p-pure fully invariant subgroup of $\hat{B}_p$ the p-adic completion of B_p.*

b) *K is a free finite dimensional R-module where R is a p-pure subring of J_p such that $U_R = R \cap U_{J_p}$ ($U_R = \{x \in R \mid \exists y \in R,\ xy = 1\}$).*

This statement is a combination of Theorems 2.5, 3.1, Corollary to Theorem 4.5 and Theorem 6.3 of [1].

In order to complete this to the case where the torsion free rank of p-basic subgroup is infinite we need the following lemma.

LEMMA 2.2. *Let G be a group containing a subgroup $B = B_p \oplus B_0$ such that $G/B \simeq Q$ and B_0 is a free group of infinite rank and B_p is a primary direct sum of cyclic groups, then there exists an epimorphism from B onto G.*

Proof. Clearly G/B_p is a torsion free group whose rank is the same as the rank of B_0. Since B_0 is a free group, there exists φ: $B_0 \twoheadrightarrow G/B_p$ and since B_0 is projective, we can lift φ to $\theta : B_0 \to G$ such that $\nu_{B_p} \circ \theta = \varphi$. Therefore, $G = \theta(B_0) + \ker \nu_{B_p} = \theta(B_0) + B_p$. Define $\psi : B \to G$, by $\psi(b_0 + b_p) = \theta(b_0) + b_p$. Then ψ is an epimorphism of B onto G.

THEOREM 2.3. *Let G be a p-reduced group with a p-basic subgroup of infinite torsion free rank. Then G is q.p.p.i. if and only if G is complete in its p-adic topology. In other words if and only if G is p-pure-injective.*

Proof. Let G be q.p.p.i. then $G^1 = \bigcap p^n G = 0$ ([1], Corollary to Theorem 2.5) therefore we can embed G as a pure subgroup in its p-adic completion $\hat{G}$. Furthermore $\hat{G}/G$ is divisible. Let B be a p-basic subgroup of G then $\hat{G}/B$ is also divisible. Let $x \in \hat{G}$. Suppose $o(x) = \infty$. Then consider $x + B$. Two cases may occur: first $x + B$ is of infinite order in which case there exists a subgroup R of G containing x and B such that $R/B \simeq Q$: or $x + B$ is of finite order, say, of order $m = p^i a$, $(a, p) = 1$, then there exists $b \in B$ such that $p^i ax = p^i b$ and $ax - b \in \hat{G}_p$, but, from Theorem 2.5 in [1], $\hat{G}_p = G_p$, therefore $ax \in G$. Now G is pure in $\hat{G}$ and $\hat{G}/G$ is in fact torsion free. This means $x \in G$. So let us return to the first case. There we have $x \in R$ and $R/B \simeq Q$. From Lemma 2.2 there exists an epimorphism $\theta : B \twoheadrightarrow R$. Let $A = \theta^{-1}(B)$ then A is a pure subgroup of B and in fact $B/A \cong Q$. Therefore A is also a p-basic subgroup of G. Let φ

be an extension of $\theta|_A : A \to B$ to G and φ' an extension of φ to $\hat{G}$. Since $\hat{G}$ is p-pure-injective (see [3], p. 166, Ex. 8), then there also exists an extension θ' of θ to $\hat{G}$. Now $(\theta' - \varphi')(a) = 0\ \forall a \in A$, therefore $\ker(\theta' - \varphi') \supset A$. Thus $G/\ker(\theta' - \varphi')$ is a homomorphism image of $\hat{G}/A$ which is divisible and further $\hat{G}/\ker(\theta' - \varphi') \cong \mathrm{Im}(\theta' - \varphi') \subset \hat{G}$ which is reduced. Therefore $\mathrm{Im}(\theta' - \varphi') = 0$, in other words $\theta' = \varphi'$. From this it follows that $\varphi'(B) = \varphi(B) \subset G$, but on the other hand $\varphi'(B) = \theta'(B) = \theta(B) = R$. Therefore $R \subset G$ and $x \in G$. We see that G contains all the torsion free elements of $\hat{G}$ and consequently $G = \hat{G}$. The converse is clear.

Finally from Theorem 2.6 in [1], the p-reduced case combines freely with divisible groups to give further q.p.p.i. groups. This completes the characterization of this class of groups. The pertinent results are Proposition 1.1, Proposition 1.4, Theorem 2.1, Theorem 2.3, for the reduced case. Furthermore, the non-p-reduced q.p.p.i. groups combine with divisible group in a restricted manner given in Theorem 1.5.

Bibliography

[1] BENABDALLAH, K. and LAROCHE, A.; Quasi-p-pure-injective groups, *Can. J. Math.*, **29** (1977), 578–586.

[2] BENABDALLAH, K. BRADLEY, R. and LAROCHE, A.; Sur les groupes quasi-p-nets-injectifs et projectifs, Leucture Notes in Mathematics No. 616, *Abelian Group Theory*, Springer Verlag, 1977, pp. 121–134.

[3] FUCHS, L.; *Infinite Abelian Groups*, Vol. I, Academic Press, New York, 1970.

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