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Klingen Type Eisenstein Series of Skew Holomorphic Jacobi Forms

by

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The theory of Jacobi forms plays important roles in the study of automorphic forms. There exists an isomorphism between Kohnen's plus space (the certain subspace of Siegel modular forms of half integral weight) and holomorphic Jacobi forms of index 1 (cf. Eichler-Zagier [4], Ibukiyama [8]). The notion of skew holomorphic Jacobi forms was introduced by Skoruppa [11], [12] as forms which have translation formula like Jacobi forms. These forms are real analytic functions on $\mathfrak{H}_1 \times \mathbb{C}$. This notion of skew holomorphic Jacobi forms was generalised by Arakawa [1] as real analytic functions on $\mathfrak{H}_n \times M_{n,l}(\mathbb{C})$. There also exists an isomorphism between Kohnen's plus space and skew holomorphic Jacobi forms of index 1. This isomorphism was shown by Skoruppa [11] in the case of degree 1 and was shown by the author [6] in the case of general degree.

Although the skew holomorphic Jacobi forms are not holomorphic functions, we expect that skew holomorphic Jacobi forms have good properties like holomorphic Jacobi forms.

The main purpose of this paper is to construct *Klingen type Eisenstein series* of skew holomorphic Jacobi forms (Theorem 2). Namely, we construct skew holomorphic Jacobi forms from skew holomorphic Jacobi cusp forms of lower degree. In order to show this, we use the fact that skew holomorphic Jacobi forms vanish under a certain differential operator. This fact characterises the shape of the Fourier expansion. As an application of theorem 2, we show the fact that if index of skew holomorphic Jacobi forms satisfy a certain condition (§2.3 (4.1)), then the space of skew holomorphic Jacobi forms is spanned by Klingen type Eisenstein series and cusp forms.

By using the isomorphism between plus space and skew holomorphic Jacobi forms of index 1, we can also construct the generalised Cohen type Eisenstein series of plus space. Cohen type Eisenstein series of degree 1 was obtained by linear combination of usual Eisenstein series of half integral weight (cf. Cohen [3]), and corresponds to the Eisenstein series of Jacobi forms of index 1 (cf. Eichler-Zagier [4]). Namely, we can regard the generalised Cohen type Eisenstein series as corresponding function with Klingen type Eisenstein series of skew holomorphic Jacobi forms of index 1. This construction is obtained by Arakawa in the case of holomorphic Jacobi forms. Here we have the skew holomorphic Jacobi forms version.

In section 1, we describe the definition of skew holomorphic Jacobi forms. In section 2, we construct the Klingen type Eisenstein series of skew holomorphic Jacobi forms, and we give some application.

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Notation: We let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers. Let $M_{m,n}(A)$ be the set of all $m \times n$ matrices over a commutative ring A with unity, and we put $M_n(A) = M_{n,n}(A)$. We denote all symmetric matrices in $M_n(A)$ by $\mathrm{Sym}_n(A)$. For matrices $N \in M_n(A)$ and $M \in M_{n,m}(A)$, we define $N[M] ={}^t MNM$ where ${}^t M$ is the transposition of M. Let 1_n and 0_n be the identity matrix and the zero matrix respectively. Let $GL_n(A)$ be the group of invertible matrices in $M_n(A)$ and $SL_n(A)$ the subgroup consisting of matrices with determinant 1. We put the set of all half integral symmetric matrices by

$$L_n^* := \left\{ (a_{i,j})_{i,j} \in \mathrm{Sym}_n(\mathbb{Q}) \mid 2a_{i,j} \in \mathbb{Z}, a_{i,i} \in \mathbb{Z} \ (0 \le i, j \le n) \right\} .$$

We use the symbol $e(x)$ $(x \in \mathbb{C})$ as an abbreviation for $\exp(2\pi\sqrt{-1}x)$.

1. Skew holomorphic Jacobi forms

The aim of this section is to describe the skew holomorphic Jacobi forms of general degree following Arakawa [1].

For commutative ring A with unity, we denote the symplectic group by $Sp(n, A)$, namely

$$Sp(n, A) := \left\{ M \in GL_{2n}(A) \mid {}^t M \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \right\} .$$

We fix a positive integer l. Let $G^J_{n,l}$ be the subgroup of $Sp(n + l, \mathbb{R})$ consisting of all elements of the form $(g, [(\lambda, \mu), \kappa]) := \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_l & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_l \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 & \mu \\ {}^t\lambda & 1_l & {}^t\mu & \kappa \\ 0 & 0 & 1_n & -\lambda \\ 0 & 0 & 0 & 1_l \end{pmatrix}$,
where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, $\lambda, \mu \in M_{n,l}(\mathbb{R})$, and $\kappa \in M_l(\mathbb{R})$ such that $\kappa +^t \mu\lambda \in Sym(n, \mathbb{R})$.

For any elements $\gamma_i = (g_i, [(\lambda_i, \mu_i), \kappa_i]) \in G^J_{n,l}$ $(i = 1, 2)$, the composition rule of $G^J_{n,l}$ is given as follows:

$$\begin{aligned} \gamma_1\gamma_2 = (g_1 g_2, [({}^t A_2\lambda_1 +^t C_2\mu_1 + \lambda_2, {}^t B_2\lambda_1 +^t D_2\mu_1 + \mu_2), \\ \kappa_1 + \kappa_2 +^t ({}^t A_2\lambda_1 +^t C_2\mu_1)\mu_2 -^t ({}^t B_2\lambda_1 +^t D_2\mu_1)\lambda_2]) . \end{aligned}$$

We denote by $H_{n,l}(\mathbb{R})$ the Heisenberg group, a subgroup of $G^J_{n,l}$, consisting of all elements of the form $(1_{2n}, [(\lambda, \mu), \kappa])$. Then the Jacobi group $G^J_{n,l}$ is the semi-direct product of $H_{n,l}(\mathbb{R})$ and $Sp(n, \mathbb{R})$, namely

$$G^J_{n,l} = Sp(n, \mathbb{R}) \ltimes H_{n,l}(\mathbb{R}) .$$

The real symplectic group $Sp(n,\mathbb{R})$ acts on the Siegel upper half space $\mathfrak{H}_n$ of degree n in a usual manner; for $g=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in Sp(n,\mathbb{R})$ and $\tau\in\mathfrak{H}_n$, set

$$g\cdot\tau:=(A\tau+B)(C\tau+D)^{-1}\quad\text{and}\quad J(g,\tau):=C\tau+D\,.$$

Let $\mathfrak{D}_{n,l}$ denote the product space of the Siegel upper half space $\mathfrak{H}_n$ and the set $M_{n,l}(\mathbb{C})$:

$$\mathfrak{D}_{n,l}=\mathfrak{H}_n\times M_{n,l}(\mathbb{C})\,.$$

Then the Jacobi group $G_{n,l}$ naturally acts on the space $\mathfrak{D}_{n,l}$:

$$\gamma(\tau,z):=(g\cdot\tau,{}^t J(g,\tau)^{-1}(z+\tau\lambda+\mu))\,,$$

where $\gamma=(g,[(\lambda,\mu),\kappa])\in G^J_{n,l}$, and $(\tau,z)\in\mathfrak{D}_{n,l}$.

We take a semi-positive definite half integral symmetric matrix $\mathcal{M}$ of size l. Now we define a factor of automorphy for the Jacobi group $G^J_{n,l}$. Let k be a fixed positive integer. Set, for $\gamma=(g,[(\lambda,\mu),\kappa])\in G^J_{n,l}$, $(\tau,z)\in\mathfrak{D}_{n,l}$,

$$\begin{aligned}j^{sk}_{k,\mathcal{M}}(\gamma,(\tau,z))&:=\overline{\det(J(g,\tau))}^{k-j}\,|\det(J(g,\tau))|^j\\&\times e(-tr(\mathcal{M}(\kappa+{}^t\mu\lambda+\tau[\lambda]+2{}^tz\lambda-(J(g,\tau)^{-1}C)[(z+\tau\lambda+\mu)])))\,,\end{aligned}$$

where $g=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in Sp(n,\mathbb{R})$, and $j:=rank(\mathcal{M})$. This factor of automorphy has the following property:

$$j^{sk}_{k,\mathcal{M}}(\gamma_1\gamma_2,(\tau,z))=j^{sk}_{k,\mathcal{M}}(\gamma_1,\gamma_2(\tau,z))j^{sk}_{k,\mathcal{M}}(\gamma_2,(\tau,z))\qquad(\gamma_1,\gamma_2\in G^J_{n,l})\,.$$

For a function $F(\tau,z)$ on $\mathfrak{D}_{n,l}$, we denote the slash operator $|_{k,\mathcal{M}}$ by

$$(F|_{k,\mathcal{M}}\gamma)(\tau,z):=j^{sk}_{k,\mathcal{M}}(\gamma,(\tau,z))^{-1}F(\gamma(\tau,z))\,,\ \gamma\in G^J_{n,l}\,.$$

We set $\Gamma^J_n:=G^J_{n,l}\cap Sp(n+l,\mathbb{Z})$, then $\Gamma^J_n=Sp(n,\mathbb{Z})\ltimes H_{n,l}(\mathbb{Z})$, where $H_{n,l}(\mathbb{Z})=H_{n,l}(\mathbb{R})\cap Sp(n+l,\mathbb{Z})$. If $j=rank(\mathcal{M})\neq 0$, then we can choose $X=(U\ V)\in GL_l(\mathbb{Z})$, $(U\in M_{l,j}(\mathbb{Z})\,,\ V\in M_{l,l-j}(\mathbb{Z}))$ satifies $\mathcal{M}[X]=\left(\begin{smallmatrix}\tilde{\mathcal{M}}&0\\0&0\end{smallmatrix}\right)\in L^*_l$, $\tilde{\mathcal{M}}\in GL_j(\mathbb{Q})\cap L^*_j$. The skew holomorphic Jacobi forms was introduced by Skoruppa [12] in the case of degree 1 and generalised for higher degree by Arakawa [1]. We use above notations for the following definition.

DEFINITION 1 (Skew holomorphic Jacobi forms). Let $F(\tau,z)$ be a real analytic function on $\mathfrak{D}_{n,l}$, especially holomorphic on $z\in\mathfrak{H}_n$. This $F(\tau,z)$ is said to be a skew-holomorphic Jacobi form of weight k and index $\mathcal{M}$ with respect to Γ^J_n, if it satisfies the following two conditions:

(i) $F|_{k,\mathcal{M}}\gamma=F$, for every $\gamma\in\Gamma^J_n$.
(ii) $F(\tau,z)$ has a Fourier expansion of the following form:

$$F(\tau,z)=\sum_{\substack{N\in L^*_n,\,R\in M_{n,l}(\mathbb{Z})\\4N-\tilde{R}\tilde{\mathcal{M}}^{-1t}\tilde{R}\le 0,\,RV=0}}C(N,R)e\left(tr\left(N\tau-\frac{1}{2}i(4N-\tilde{R}\tilde{\mathcal{M}}^{-1t}\tilde{R})Y+R^tz\right)\right).$$

where $Y = Im\,\tau$. In the above sum, if $rank(\mathcal{M}) = j \neq 0$, then we put $\tilde{R} := RU$; if $rank(\mathcal{M}) = 0$ then $N \in L_n^*$, $R \in M_{n,l}(\mathbb{Z})$ run through $N \leq 0$, $R = 0$, and we regard $\tilde{R}\tilde{\mathcal{M}}^{-1t}\tilde{R}$ as 0.

Moreover we say $F(\tau, z)$ is a cusp form, if the Fourier coefficients $C(N, R)$ satisfy the following condition (iii),

(iii) If $\mathcal{M} \neq 0_l$, then $C(N, R) = 0$ unless $4N - \tilde{R}\tilde{\mathcal{M}}^{-1t}\tilde{R} < 0$.
If $\mathcal{M} = 0_l$, then $C(N, R) = 0$ unless $N < 0$.

We denote by $J_{k,\mathcal{M}}^{sk}(\Gamma_n^J)$ (resp. $J_{k,\mathcal{M}}^{sk,cusp}(\Gamma_n^J)$) the $\mathbb{C}$-vectorspace of all skew-holomorphic Jacobi forms (resp. cusp forms) of weight k and index $\mathcal{M}$ with respect to Γ_n^J.

When $n = 0$, we put $J_{k,\mathcal{M}}^{sk}(\Gamma_0^J) = J_{k,\mathcal{M}}^{sk,cusp}(\Gamma_0^J) = \mathbb{C}$, namely we regard the skew holomorphic Jacobi forms with respect to Γ_0^J as constant functions.

We have the following lemma.

LEMMA 1. *Let F be an element in $J_{k,\mathcal{M}}^{sk}(\Gamma_n^J)$. The condition* (iii) *is equivalent to the following condition* (iii$'$)

(iii$'$) *There exists a constant $C > 0$, such that*
$|F(\tau, z)| \det(Y)^{\frac{k}{2}} e(-tr(\mathcal{M}^t\beta(iY)^{-1}\beta)) < C$ *, where $\beta = Im\,z$.*

Proof. (iii) $\Rightarrow$ (iii$'$) It can be done like proof of lemma 2.6 of Ziegler [13].
(iii$'$) $\Rightarrow$ (iii) It can be done like proof of lemma 1.4 of Murase [10]. □

2. Main theorems

In this section, we shall construct the Klingen type Eisenstein series of skew holomorphic Jacobi forms. We use the same notations used in the previous section.

2.1. Isomorphism map between skew holomorphic Jacobi forms

Let $\mathcal{M}$ be a semi-positive definite half integral symmetric matirx of size l, and U, V, $\tilde{\mathcal{M}}$ be the same notations appeared in the definition of skew-holomorphic Jacobi forms, namely $\mathcal{M}[(U\ V)] = \begin{pmatrix} \tilde{\mathcal{M}} & 0 \\ 0 & 0 \end{pmatrix}$ and $\tilde{\mathcal{M}}$ is a invertible matrix of size j.

Let F be a function on $\mathfrak{D}_{n,l}$. We define $\phi_U(F)$ a function on $\mathfrak{D}_{n,j}$ by

$$\phi_U(F)(\tau, z_1) := F(\tau, z_1^t U)\,.$$

In the same way with holomorphic Jacobi forms case (cf. Ziegler [13] Theorem 2.4), we can deduce the following theorem.

THEOREM 1. *The map $\phi_U : J_{k,\mathcal{M}}^{sk}(\Gamma_n^J) \ni F(\tau, z) \to \phi_U(F)(\tau, z_1) \in J_{k,\tilde{\mathcal{M}}}^{sk}(\Gamma_n^J)$ induces the isomorphism, $J_{k,\mathcal{M}}^{sk}(\Gamma_n^J) \cong J_{k,\tilde{\mathcal{M}}}^{sk}(\Gamma_n^J)$, and cusp forms of the both spaces correspond with each other.*

Proof. This theorem proved by a straightforward calculation. Details are omitted here. □

2.2. Klingen type Eisenstein series

Let r be an integer $(0 \le r \le n)$. We prepare the following subgroups,

$$\Gamma_{n,r} := \left\{ g \in Sp(n,\mathbb{Z}) \mid g = \begin{pmatrix} A_1 & 0 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & D_4 \end{pmatrix}, A_1, B_1, C_1, D_1 \in M_r(\mathbb{Z}) \right\},$$

and

$$\Gamma^J_{n,r} := \{(g, [(\lambda,\mu),\kappa]) \in \Gamma^J_n \mid g \in \Gamma_{n,r}, \lambda = \left(\begin{smallmatrix} \lambda_1 \\ 0 \end{smallmatrix}\right) \in M_{n,l}(\mathbb{Z}), \lambda_1 \in M_{r,l}(\mathbb{Z})\}.$$

Let $F(\tau_1, z_1)$ be an element in $J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)$ and let k be an integer satisfies $k \equiv j \mod 2$ (j is the rank of $\mathcal{M}$). We define a function F^* on $\mathfrak{D}_{n,l}$ by

$$F^*(\tau, z) := F(\tau_1, z_1), \tag{2.1}$$

where $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ {}^t\tau_2 & \tau_3 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $(\tau_1, z_1) \in \mathfrak{D}_{n,l}$.

If $\mathcal{M}$ is an invertible matrix, namely if $j = l$, we define the Klingen type Eisenstein series of skew holomorphic Jacobi form of F by

$$E^{sk}_{n,r}(F;(\tau,z)) := \sum_{\gamma \in \Gamma^J_{n,r}\backslash\Gamma^J_n} (F^*|_{k,\mathcal{M}}\gamma)(\tau,z), \quad (\tau,z) \in \mathfrak{D}_{n.l}. \tag{2.2}$$

The above sum does not depend on the choice of the representative elements, because F satisfies the transform formula and integer k satisfies $k \equiv j \mod 2$. It is clear that $E^{sk}_{n,r}(F;(\tau,z))$ formally transforms like a skew holomorphic Jacobi form in $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$. Because F is a cusp form, there exists a constant C such that

$$|F(\tau_1,z_1)| \det(Y_1)^{\frac{k}{2}} e(-tr(\mathcal{M}^t\beta_1(iY_1)^{-1}\beta_1)) < C \quad \text{for every} \quad (\tau_1,z_1) \in \mathfrak{D}_{r,l},$$

where β_1 and Y_1 are the imaginary part of z_1 and τ_1 respectively. Hence, by the same calculation as Ziegler [13] Theorem 2.5, we can show the fact that if $k > n + l + r + 1$ then $E^{sk}_{n,r}$ is uniformly absolutely convergent in the wider sense on $\mathfrak{D}_{n,l}$.

If $\mathcal{M}$ is not a invertible matrix, namely if $j \ne l$, we define the Klingen type Eisenstein series of skew holomorphic Jacobi form by using Theorem 1, namely we define

$$E^{sk}_{n,r}(F;(\tau,z)) := \phi_U^{-1} \circ E^{sk}_{n,r}(\phi_U(F);(\tau,z)).$$

Hence, we consider only the case of index $\mathcal{M}$ is invertible matrix.

We set differential operators $\frac{\delta}{\delta\tau} := \left(\frac{1+\delta_{s,t}}{2}\frac{\delta}{\delta\tau_{s,t}}\right)$, $\frac{\delta}{\delta z} := \left(\frac{\delta}{\delta z_{s,t}}\right)$, where $\delta_{s,t}$ is the Kronecker's delta symbol, and $\frac{\delta}{\delta\tau_{s,t}} := \frac{1}{2}\left(\frac{\delta}{\delta X_{s,t}} - i\frac{\delta}{\delta Y_{s,t}}\right)$, where $X_{s,t}$ (resp. $Y_{s,t}$) is the real part (resp. the imaginary part) of $\tau_{s,t}$.

For $\mathcal{M} > 0$, we define a differential operator

$$\Delta_{\mathcal{M}} := 8\pi i \frac{\delta}{\delta\tau} - \frac{\delta}{\delta z}\mathcal{M}^{-1t}\frac{\delta}{\delta z}\,.$$

LEMMA 2. *Let F be an element in $J^{sk}_{k,\mathcal{M}}(\Gamma^J_r)$, k be an integer satisfies $k \equiv j$* mod 2. *We assume $\mathcal{M}$ is an invertible matrix. Then we have the following equation*,

$$\Delta_{\mathcal{M}}(E^{sk}_{n,r}(F;(\tau,z))) = 0_n\,. \tag{2.3}$$

Proof. For $\gamma = (g,[(\lambda,\mu),\kappa]) \in G^J_n$, $g = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$, we put $F_{2,\gamma} := \overline{\det(J(g,\tau))}^{l-k}|\det(J(g,\tau))|^{-l}$, and $F_{3,\gamma} := e(tr(\mathcal{M}(\tau[\lambda] + 2^t z\lambda - (J(g,\tau)^{-1}C)[(z+\tau\lambda+\mu)])))$, then

$$E^{sk}_{n,r}(F;(\tau,z)) = \sum_{\gamma\in\Gamma^J_{n,r}\backslash\Gamma^J_n}(F^*\circ\gamma)F_{2,\gamma}F_{3,\gamma}\,,$$

where F^* is in eq. (2.1). By a direct calculation, we have the following equations;

$$\frac{\delta}{\delta\tau}(F^*\circ\gamma) = J(g,\tau)^{-1}\left(\frac{\delta}{\delta\tau}F\right)\circ\gamma^t J(g,\tau)^{-1} + \frac{1}{2}(T + ^t T)\,,$$

where

$$\begin{aligned}
T &= \{\lambda - J(g,\tau)^{-1}C(z+\tau\lambda+\mu)\}\left(\tfrac{\delta}{\delta\tau}F\right)\circ\gamma^t J(g,\tau)^{-1},\\
&\frac{\delta}{\delta\tau}F_{2,\gamma} = -\frac{1}{2}lJ(g,\tau)^{-1}CF_{2,\gamma}\,,\\
&\frac{\delta}{\delta\tau}F_{3,\gamma} = 2\pi i\mathcal{M}[^t\{\lambda - J(g,\tau)^{-1}C(z+\tau\lambda+\mu)\}]F_{3,\gamma}\,,\\
&\frac{\delta}{\delta z}(F^*\circ\gamma) = J(g,\tau)^{-1}\left(\frac{\delta}{\delta z}F\right)\circ\gamma\,,\\
&\frac{\delta}{\delta z}\mathcal{M}^{-1t}\frac{\delta}{\delta z}(F^*\circ\gamma) = J(g,\tau)^{-1}\left(\frac{\delta}{\delta z}\mathcal{M}^{-1t}\frac{\delta}{\delta z}F\right)\circ\gamma^t J(g,\tau)^{-1}\,,\\
&\frac{\delta}{\delta z}F_{2,\gamma} = 0\,,\quad \frac{\delta}{\delta z}F_{3,\gamma} = 4\pi i\{\lambda - J(g,\tau)^{-1}C(z+\tau\lambda+\mu)\}\mathcal{M}F_{3,\gamma}\\
&\frac{\delta}{\delta z}\mathcal{M}^{-1t}\frac{\delta}{\delta z}F_{3,\gamma} = \{-16\pi^2\mathcal{M}[^t(\lambda - J(g,\tau)^{-1}C(z+\tau\lambda+\mu))]\\
&\qquad\qquad -4\pi i l J(g,\tau)^{-1}C\}F_{3,\gamma}\,.
\end{aligned}$$

Moreover, F is an element of $J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)$, so F has a Fourier expansion satisfies the condition (ii) of the definition of skew holomorphic Jacobi forms, hence by a easy calculation we can obtain $\Delta_{\mathcal{M}}F^* = 0_n$.

Above all, by a direct calculation, we conclude that

$$\Delta_{\mathcal{M}}(E^{sk}_{n,r}(F;(\tau,z))) = \sum_{\gamma\in\Gamma^J_{n,r}\backslash\Gamma^J_n}\Delta_{\mathcal{M}}((F^*\circ\gamma)F_{2,\gamma}F_{3,\gamma}) = 0_n$$

Hence, we showed this lemma. □

LEMMA 3. *Let F be an element in $J^{sk}_{k,\mathcal{M}}(\Gamma^J_r)$, k be an integer satisfies $k \equiv j$* mod 2, *and we assume $\mathcal{M}$ is an invertible matrix. Then $E^{sk}_{n,r}(F;(\tau,z))$ has the following Fourier expansion*,

$$E^{sk}_{n,r}(F;(\tau,z)) = \sum_{N\in L^*_n, R\in M_{n,l}(\mathbb{Z})} C(N,R)e(tr(N\tau - \frac{1}{2}i(4N - R\mathcal{M}^{-1t}R)Y + R^tz)) . \tag{2.4}$$

Proof. Because $E^{sk}_{n,r}(F;(\tau,z))$ satisfies the condition (i) of the definition of skew holomorphic Jacobi forms, hence, $E^{sk}_{n,r}(F;(\tau,z))$ has the Fourier expansion as follows,

$$E^{sk}_{n,r}(F;(\tau,z)) = \sum_{N\in L^*_n, R\in M_{n,l}(\mathbb{Z})} K(N,R;Y)e(tr(NX + R^tz)) , \tag{2.5}$$

where X (resp. Y) is the real part (resp. the imaginary part) of τ, and coefficients $K(N,R;Y)$ depend only on N, R, and Y.

By using eq. (2.3) and eq. (2.5), we have

$$\Delta_{\mathcal{M}}(K(N,R;Y)e(tr(NX + R^tz))) = 0_n ,$$

hence, we have $\frac{\delta}{\delta Y}K(N,R;Y) + \left(-2\pi N + \pi R\mathcal{M}^{-1t}R\right)K(N,R;Y) = 0_n$.

Therefore, we obtain a constant $C(N,R)$ which satisfies

$$K(N,R;Y) = C(N,R)e(tr((-N + \frac{1}{2}R\mathcal{M}^{-1t}R)iY)) .$$

Thus, we proved this lemma. □

LEMMA 4. *In the eq.* (2.4) *of Lemma* 3, *the Fourier coefficients $C(N,R)$ satisfy $C(N,R) = 0$ unless $4N - R\mathcal{M}^{-1t}R \le 0$.*

Proof. Because $E_{n,r}(F;(\tau,z))$ satisfies the condition (i) of the definition of skew holomorphic Jacobi forms, we have

$$E_{n,r}(F;(\tau, z + \tau\lambda + \mu)) = e(-tr(\mathcal{M}(\tau[\lambda] + 2^t\lambda z)))E_{n,r}(F;(\tau,z)) ,$$

for every $\lambda, \mu \in M_{n,l}(\mathbb{Z})$. Hence $C(4N_1 - R_1\mathcal{M}^{-1t}R_1) = C(4N_2 - R_2\mathcal{M}^{-1t}R_2)$ whenever the matrices N_1, N_2, R_1, R_2 satisfy $4N_1 - R_1\mathcal{M}^{-1t}R_1 = 4N_2 - R_2\mathcal{M}^{-1t}R_2$ and $R_1 \equiv R_2$ mod $2\mathcal{M}$ (cf. Eichler-Zagier [4], Ziegler [13]).

By using above fact, we can induce the following equation

$$E_{n,r}(F;(\tau,z)) = \sum_{R\in M_{n,l}(\mathbb{Z})/(2M_{n,l}(\mathbb{Z})\mathcal{M})} f_R(\tau)\vartheta_{R,\mathcal{M}}(\tau,z) ,$$

where $f_R(\tau) = \sum_{N\in L^*_n} C(N,R)e(\frac{1}{4}tr((4N - R\mathcal{M}^{-1t}R)\overline{\tau}))$ and $\vartheta_{R,\mathcal{M}}(\tau,z) = \sum_{\lambda\in M_{n,l}(\mathbb{Z})} e(tr(\mathcal{M}(\tau[(\lambda + R(2\mathcal{M})^{-1})] + 2^tz(\lambda + R(2\mathcal{M})^{-1}))))$.

Because $f_R(i1_n)$ is absolutely convergent and coefficients $C(N,R)$ satisfy $C(N,R) = C({}^tUNU, {}^tUR)$ for every $U \in GL_n(\mathbb{Z})$, so we have the following equation,

$$f_R(i1_n) = \sum_{N\in L_n^*} C(N,R)e\left(-\frac{1}{4}i\,tr((4N-R\mathcal{M}^{-1\,t}R))\right)$$
$$= \sum_{N\in L_n^*/GL_n(\mathbb{Z})} C(N,R) \sum_{U\in GL_n(\mathbb{Z})} e\left(-\frac{1}{4}i\,tr({}^tU(4N-R\mathcal{M}^{-1\,t}R)U)\right),$$

where $L_n^*/GL_n(\mathbb{Z})$ is a complete set of equivalence classes of L_n^* with respect to $GL_n(\mathbb{Z})$. We can deduce the fact that if degree $n \geq 2$, and if $4N - R\mathcal{M}^{-1\,t}R \not\leq 0$, then $\sum_{U\in GL_n(\mathbb{Z})} e(-\frac{1}{4}i\,tr({}^tU(4N-R\mathcal{M}^{-1\,t}R)U))$ does not converge. For this reason, if $4N - R\mathcal{M}^{-1\,t}R \not\leq 0$ then the coefficient $C(N,R)$ satisfies $C(N,R) = 0$.

In the remaining case of $n = 1$, this case can be done like Eichler-Zagier [4], Chapter 1, §2.

Hence if $C(N,R) \neq 0$ then $4N - R\mathcal{M}^{-1\,t}R \leq 0$. Thus, we proved this lemma. □

Consequently, we have the following theorem.

THEOREM 2. *Let* $\mathcal{M} \geq 0$ *and* $F \in J^{sk}_{k,\mathcal{M}}(\Gamma_r^J)$. *If* $k > n + rank(\mathcal{M}) + r + 1$ *satisfies* $k \equiv rank(\mathcal{M}) \mod 2$, *then* $E^{sk}_{n,r}(F;(\tau,z))$ *is an element of* $J^{sk}_{k,\mathcal{M}}(\Gamma_n^J)$.

Proof. By using Lemma 3, Lemma 4 and Theorem 1, we have this theorem. □

2.3. The Siegel operator and the space of skew holomorphic Jacobi forms

In this subsection, we consider the Siegel operator for skew holomorphic Jacobi forms. We shall show that the Siegel operator of skew holomorphic Jacobi forms have same properties with holomorphic Jacobi forms case.

For a function $F(\tau,z)$ on $\mathfrak{D}_{n,l}$, we define a function

$$\Phi_r^n(F)(\tau_1,z_1) := \lim_{t\to+\infty} F\left(\begin{pmatrix}\tau_1 & 0\\ 0 & it1_{n-r}\end{pmatrix}, \begin{pmatrix}z_1\\ 0\end{pmatrix}\right), \quad (\tau_1,z_1)\in\mathfrak{D}_{n,r}\,.$$

Then $\Phi_r^n(F)$ is a function on $\mathfrak{D}_{r,l}$. This Φ_r^n is called the Siegel operator.

PROPOSITION 3. *Let* $F(\tau,z) \in J^{sk}_{k,\mathcal{M}}(\Gamma_n^J)$ *be a skew holomorphic Jacobi form, then* $\Phi_r^n(F)$ *is also a skew holomorphic Jacobi form in* $J^{sk}_{k,\mathcal{M}}(\Gamma_r^J)$.

Proof. We have this proposition from a straightforward computation. □

THEOREM 4. *If integer* k *ratifies* $k > n + rank(\mathcal{M}) + r + 1$ *and* $k \equiv rank(\mathcal{M}) \mod 2$, *then we have* $\Phi_r^n(E^{sk}_{n,r}(F;(\tau,z))) = F(\tau_1,z_1)$ *for every* $F(\tau_1,z_1) \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma_r^J)$. *Hence, the Siegel operator* Φ_r^n *induces a surjective map from* $J^{sk}_{k,\mathcal{M}}(\Gamma_n^J)$ *to* $J^{sk,cusp}_{k,\mathcal{M}}(\Gamma_r^J)$.

Proof. It can be done like the proof of Theorem 2.8 of Ziegler [13]. □

Here, we imitate some Arakawa's work [2]. We assume the following condition on $\mathcal{M} > 0$.

(4.1) If $\mathcal{M}[x] \in \mathbb{Z}$ for $x \in (2\mathcal{M})^{-1}M_{l,1}(\mathbb{Z})$, then necessarily, $x \in M_{l,1}(\mathbb{Z})$.

By the same argument with Arakawa [2] (Proposition 4.1, Theorem 4.2 of [2]), we deduce the following Proposition 5 and Theorem 6.

PROPOSITION 5. *Let* $F \in J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$. *Under the condition* (4.1) *on* $\mathcal{M}$, *we have* $F \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_n)$ *if and only if* $\Phi^n_{n-1}(F) = 0$.

Proof. It can be done like Proposition 4.1 of Arakawa [2]. □

THEOREM 6. *Assume that* $\mathcal{M}$ *satisfies the condition* (4.1). *Let* k *be a positive integer with* $k > 2n + l + 1$, $k \equiv l \mod 2$. *Then we have the direct sum decomposition* $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n) = \bigoplus_{r=0}^{n} J^{sk,(r)}_{k,\mathcal{M}}(\Gamma^J_n)$, *where* $J^{sk,(r)}_{k,\mathcal{M}} := \{E^{sk}_{n,r}(F; (\tau, z)) | F \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)\}$.

Proof. It can be done like Theorem 4.2 of Arakawa [2]. □

In the paper [6], we have the isomorphism between the plus space (a certain subspace of Siegel modular forms of half-integral weight) and the skew holomorphic Jacobi forms of index 1. Hence, by using theorem 6, if k is an odd integer satisfies $k > 2n + 2$, we can also obtain a similar theorem for the plus space of degree n of weight $k - \frac{1}{2}$ with trivial character. Namely, under these conditions, we can deduce the fact that the plus space of weight $k - \frac{1}{2}$ is spanned by Klingen-Cohen type Eisenstein series and cusp forms.

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