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On the Spin L-function of Ikeda's Lifts

by

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Abstract. The recent construction of Siegel modular forms of degree $2n$ from elliptic cusp forms by Ikeda is interpreted as a special case of Langlands functoriality. This is then used to express the (full) spin L-function of an Ikeda lift as a product of symmetric power L-functions for GL(2). As a corollary one obtains the desired analytic properties of these spin L-functions provided the analogous properties of the symmetric power L-functions are known.

Introduction

Let F be a classical Siegel modular form of degree n, assumed to be a cuspidal eigenform. There are (at least) two different L-functions attached to F, the *standard* L-function, which is an Euler product of degree $2n+1$, and the *spin* L-function, an Euler product of degree 2^n. It is explained in [AS] how these two L-functions are related to the two "smallest" irreducible finite-dimensional representations of the group $\mathrm{Spin}(2n+1, \mathbb{C})$.

While the expected analytic properties of the standard L-function, in particular the analytic continuation and functional equation, were shown to hold by Böcherer [Bö], very little seems to be known about the spin L-function beyond degree 2, where Andrianov has complete results (see [An]). One goal of this note is to get insight into the analytic properties of the spin L-function at least for a small class of Siegel modular forms, namely the lifts constructed by Ikeda in [Ik].

Let $f \in S_{2k}(\mathrm{SL}(2, \mathbb{Z}))$ be a cuspidal elliptic eigenform. Given an integer $n \equiv k \bmod 2$, Ikeda constructs a cuspidal eigenform $F \in S_{k+n}(\mathrm{Sp}(4n, \mathbb{Z}))$ of degree $2n$ such that the (finite part of the) standard L-function of F is given by

$$\zeta(s) \prod_{i=1}^{2n} L(s+k+n-i, f),$$

where $L(s, f)$ is the usual Hecke L-function of f. If $n = 1$ then Ikeda's lifting coincides with the Saito–Kurokawa lifting.

Our first goal is to interpret Ikeda's construction in terms of Langlands' principle of functoriality. In the Saito–Kurokawa case this has been done by Langlands in [La2]. In this lowest-dimensional case one has a conjectured lifting

$$\text{from}\quad \mathrm{PGL}(2,\mathbb{A})\times\mathrm{PGL}(2,\mathbb{A})\quad\text{to}\quad \mathrm{PGSp}(4,\mathbb{A})\,, \tag{1}$$

coming from the embedding $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\to\mathrm{Sp}(4,\mathbb{C})$ of L-groups. If we put on the first $\mathrm{PGL}(2,\mathbb{A})$ the automorphic representation π_1 associated to the classical eigenform f, and on the second $\mathrm{PGL}(2,\mathbb{A})$ a certain "anomalous" representation π_2 (independent of f), then the automorphic representation Π of $\mathrm{PGSp}(4,\mathbb{A})$ associated to the Saito–Kurokawa lift of f is the expected Langlands lift of $\pi_1\otimes\pi_2$.

In higher dimensions this generalizes as follows. There is a conjectured Langlands lifting

$$\text{from}\quad \mathrm{PGL}(2,\mathbb{A})\times\mathrm{SO}(2n+1,\mathbb{A})\quad\text{to}\quad \mathrm{PGSp}(4n,\mathbb{A})\,, \tag{2}$$

coming from an embedding $\mathrm{SL}(2,\mathbb{C})\times\mathrm{Sp}(2n,\mathbb{C})\to\mathrm{Spin}(4n+1,\mathbb{C})$ of the L-groups. We shall show that if we put on $\mathrm{PGL}(2,\mathbb{A})$ the automorphic representation π_1 associated to a cuspform f of weight k, and on $\mathrm{SO}(2n+1,\mathbb{A})$ a certain anomalous representation independent of f, then the representation of $\mathrm{PGSp}(4n,\mathbb{A})$ associated to the Ikeda lift F of f coincides with the expected Langlands lift of $\pi_1\otimes\pi_2$, at least if $k>n$. This can be checked place by place since we have all the information at infinity and at each finite place.

We shall then put the 2^{2n}-dimensional spin representation ρ on the L-group $\mathrm{Spin}(4n+1,\mathbb{C})$ and compute the resulting L-function, which is the spin L-function of F. Our main result (Theorem 2.4.1) is the formula

$$L(s,\ \Pi,\ \rho)=\delta(s)\prod_{j=0}^{n}\ \prod_{\substack{r=j(j-2n)\\ \text{(step 2)}}}^{j(2n-j)} L(s+r/2,\ \pi_1,\ \mathrm{Sym}^{n-j})^{\beta(r,j,n)} \tag{3}$$

which expresses the spin L-function of the lift in terms of symmetric power L-functions of the original cusp form π_1 on $\mathrm{PGL}(2,\mathbb{A})$. Here $\delta(s)$ is a certain polynomial and $\beta(k,\ j,n)$ are certain positive integers defined in a combinatorial way. The L-function $L(s,\ \pi_1,\ \mathrm{Sym}^0)$ is understood to be the completed Riemann zeta function.

From this formula the analytic properties of $L(s,\ \Pi,\ \rho)$ can be derived provided those of the symmetric power L-functions are known. For example, the analytic continuation and functional equation of $L(s,\ \pi_1,\ \mathrm{Sym}^m)$ for $m\in\{1,\cdots,n\}$ would imply the analytic continuation and functional equation for $L(s,\ \Pi,\ \rho)$.

We remark that Kohnen and Choie have independently found a formula similar to (3), thereby also relating the spin L-functions of Ikeda's lifts to symmetric power L-functions for elliptic modular forms.

Note that the L-functions in our formula are *completed* L-functions (including the archimedean place). In fact, the polynomial $\delta(s)$ comes from the Euler factor at infinity which accounts for most of the technical difficulties. But once the archimedean factors are included we get more precise information. For example, if the expected holomorphy properties of the symmetric power L-functions hold, we can conclude that $L(s,\ \Pi,\ \rho)$ has poles.

It is not surprising that the L-functions of Ikeda's lifts have poles since they are CAP representations, see section 1.3. Another special property of the lifts is that they are *non-tempered* at almost every place, thus providing counterexamples to the generalized Ramanujan conjecture. This is well-known in the Saito–Kurokawa case.

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Notations

We define

$$J_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in \mathrm{GL}(n)\,, \tag{4}$$

and let

$$\mathrm{SO}(n) = \{A \in \mathrm{SL}(n) :\ A J_n {}^tA = J_n\}\,,$$

$$\mathrm{Sp}(2n) = \{A \in \mathrm{SL}(2n) :\ A J\, {}^tA = J\}\,, \quad J = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}.$$

These and other groups will be considered as algebraic groups defined over some number field F. We shall be mainly concerned with the following groups:

$$\text{type } B_n: \qquad \mathrm{Spin}(2n+1) \to \mathrm{SO}(2n+1)\,,$$

$$\text{type } C_n: \qquad \mathrm{Sp}(2n) \to \mathrm{PGSp}(2n)\,.$$

The spin group is the unique non-trivial algebraic covering of $\mathrm{SO}(2n+1)$, and $\mathrm{PGSp}(2n)$ is obtained from $\mathrm{Sp}(2n)$ by moding out the 2–element center. The groups on the left are simply connected, those on the right are of adjoint type. We shall write down the root data for the groups of type C_n. The root datum of $\mathrm{Spin}(2n+1)$ is dual to that of $\mathrm{PGSp}(2n)$, and the root datum of $\mathrm{SO}(2n+1)$ is dual to that of $\mathrm{Sp}(2n)$ (cf. [Sp] for basic facts about root data).

Root datum of $\mathrm{Sp}(2n)$:

$$\begin{array}{ll} P = \langle e_1, \cdots, e_n\rangle & P^\vee = \{\sum c_i f_i :\ c_i \in \mathbb{Z}\ \forall i \text{ or } c_i \in \mathbb{Z} + \frac{1}{2}\ \forall i\} \\ \| & {}_2| \\ X & X^\vee = \langle f_1, \cdots, f_n\rangle \\ {}_2| & \| \\ Q = \langle e_1 - e_2, \cdots, e_{n-1} - e_n,\ 2e_n\rangle & Q^\vee = \langle f_1 - f_2, \cdots, f_{n-1} - f_n,\ f_n\rangle \end{array}$$

Root datum of $\mathrm{PGSp}(2n)$:

$$\begin{array}{ll} P = \langle e_1, \cdots, e_n \rangle & P^\vee = \{\sum c_i f_i : \ c_i \in \mathbb{Z} \ \ \forall i \text{ or } c_i \in \mathbb{Z} + \frac{1}{2} \ \ \forall i\} \\ {\scriptstyle 2}| & \| \\ X = \{\sum c_i e_i : \sum c_i \in 2\mathbb{Z}\} & X^\vee \\ \| & {\scriptstyle 2}| \\ Q = \langle e_1 - e_2, \cdots, e_{n-1} - e_n, \ 2e_n \rangle & Q^\vee = \langle f_1 - f_2, \cdots, f_{n-1} - f_n, \ f_n \rangle \end{array}$$

Here X (P, Q) denotes the character lattice (weight lattice, root lattice), and $X^\vee$ ($P^\vee$, $Q^\vee$) the cocharacter lattice (coweight lattice, coroot lattice).

1. Ikeda's lifting and functoriality

In this first part we shall interpret Ikeda's lifting as a special case of Langlands' principle of functoriality. This involves the notion of *anomalous* automorphic representation as defined in [La2].

1.1. An anomalous representation on $\mathrm{SO}(2n+1)$

Let F be a number field and $\mathbb{A}$ its ring of adeles. The standard maximal torus of the group $\mathrm{SO}(2n+1)$ is

$$T = \{t = \mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1}) : \ t_i \in \mathrm{GL}(1)\} \simeq \mathrm{GL}(1)^n .$$

As a Borel subgroup B we can choose upper triangular matrices. Consider the adelic points of these groups. If $\chi_1, \cdots, \chi_n$ are characters of $F^*\backslash\mathbb{A}^*$, then let

$$I = I(\chi_1, \cdots, \chi_n) = \mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{SO}(2n+1,\mathbb{A})}(\chi)$$

(normalized induction), where the character

$$\chi(t) = \chi_1(t_1) \cdot \cdots \cdot \chi_n(t_n)$$

is considered in the usual way as a character on $B(\mathbb{A})$. The irreducible constituents of $I(\chi_1, \cdots, \chi_n)$ are the representations $\pi = \otimes \pi_v$, where π_v is an irreducible constituent of the local induced representation

$$I_v = I_v(\chi_{1,v}, \cdots, \chi_{n,v}) = \mathrm{Ind}_{B(F_v)}^{\mathrm{SO}(2n+1,F_v)}(\chi_v) \quad (\chi_i = \otimes \chi_{i,v}),$$

and π_v is the unique spherical constituent for almost all v (see [La1], Lemma 1). If we are in the fundamental Weyl chamber, then each local induced representation has a unique irreducible quotient, the Langlands quotient, and if this one is taken for π_v at every place, then the resulting constituent $\pi = \otimes \pi_v$ is called an *isobaric representation*. All the other constituents of I are called *anomalous* (see [La2], sections 2 and 3).

Now consider the special case $\chi = |\delta_B|^{1/2}$, i.e.,

$$\chi_1 = |\,|^{(2n-1)/2}, \ \chi_2 = |\,|^{(2n-3)/2}, \ \cdots, \ \chi_n = |\,|^{1/2}, \tag{5}$$

where $|\,|$ denotes the adelic absolute value. The condition on the fundamental Weyl chamber is clearly fulfilled, so we have a unique Langlands quotient at every place. Indeed, this

Langlands quotient is just the trivial representation. Thus the isobaric constituent of I is the global trivial representation.

1.1.1. PROPOSITON. *Assume that v is a real place and let the character χ of $B(\mathbb{R})$ be defined by* (5). *Then the induced representation I_v has as one of its constituents the discrete series representation with Harish-Chandra parameter $\varrho = \sum_{i=1}^{n}(n-i+1/2)e_i$.*

The proof of this proposition is postponed to the end of this section.

For simplicity we shall now assume that the ground field is $\mathbb{Q}$. Given the character χ as in (5), we shall single out a constituent π_2 of the global induced representation I by taking the Langlands quotient (the trivial representation) at every finite place, but the discrete series representation mentioned in Proposition 1.1.1 at the infinite place. Then π_2 is an anomalous representation.

We shall describe the local parameters for the components $\pi_{2,v}$ of our anomalous representation π_2. At the archimedean place the local parameter is a homomorphism

$$W_{\mathbb{R}} = \mathbb{C}^* \sqcup j\mathbb{C}^* \to \mathrm{Sp}(2n, \mathbb{C})$$

of the real Weil group $W_{\mathbb{R}}$ to the complex points of the dual group of $\mathrm{SO}(2n+1)$, which is $\mathrm{Sp}(2n, \mathbb{C})$. Let $e_1, \cdots, e_n$ be the natural basis for the character lattice of $\mathrm{SO}(2n+1)$, and consider the element

$$\varrho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = \frac{2n-1}{2}e_1 + \frac{2n-3}{2}e_2 + \cdots + \frac{1}{2}e_n\,,$$

where Δ_+ is the standard set of positive roots of $\mathrm{SO}(2n+1)$. Then our induced representation $I_\infty = I_\infty(|\,|^{(2n-1)/2}, \cdots, |\,|^{1/2})$ at the archimedean place coincides with

$$U(MAN, \mathbf{1}, \varrho)$$

in the notation of [Kn1], Proposition 8.22 (MAN is the Langlands decomposition of $B(\mathbb{R})$). It follows from this proposition that I_∞ has infinitesimal character χ_ϱ, also in the notation of [Kn1]. Consequently the discrete series representation $\pi_{2,\infty}$ has Harish-Chandra parameter ϱ.

The local parameters for real discrete series representations are described in [Bo]. The element j may be assumed to have image in the normalizer of the maximal torus, and then the element of the Weyl group determined by this image is necessarily the longest one. An element $z \in \mathbb{C}^* \subset W_{\mathbb{R}}$ is mapped to

$$z^{\varrho}\bar{z}^{-\varrho} \in \mathrm{Sp}(2n, \mathbb{C})\,,$$

where ϱ, the Harish-Chandra parameter of our $\pi_{2,\infty}$, is identified with a cocharacter of $\mathrm{Sp}(2n, \mathbb{C})$. In terms of matrices, the local parameter attached to $\pi_{2,\infty}$ is thus explicitly given by

$$z = re^{i\theta} \mapsto \begin{pmatrix} e^{i(2n-1)\theta} & & & & & 0 \\ & \ddots & & & & \\ & & e^{i\theta} & & & \\ & & & e^{-i\theta} & & \\ & & & & \ddots & \\ 0 & & & & & e^{-i(2n-1)\theta} \end{pmatrix}, \tag{6}$$

$$j \mapsto \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \quad J_n \text{ as in (4)}. \tag{7}$$

For finite places p our local parameters are given by semisimple conjugacy classes in $\mathrm{Sp}(2n, \mathbb{C})$, since $\pi_{2,p}$, the trivial representation, is unramified. It is clear that for our Langlands quotient the associated conjugacy class is represented by

$$t_p = \mathrm{diag}(|p|^{(2n-1)/2}, \cdots, |p|^{1/2}, |p|^{-1/2}, \cdots, |p|^{-(2n-1)/2}).$$

In other words, our *Satake parameter* at the finite place p is

$$t_p = \begin{pmatrix} p^{-(2n-1)/2} & & & & & 0 \\ & \ddots & & & & \\ & & p^{-1/2} & & & \\ & & & p^{1/2} & & \\ & & & & \ddots & \\ 0 & & & & & p^{(2n-1)/2} \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{C}). \tag{8}$$

Proof of Proposition 1.1.1

To prove Proposition 1.1.1 we shall need some preparations. First note that the real points of our group $\mathrm{SO}(2n+1)$ are isomorphic to the classical real group $\mathrm{SO}(n+1, n)$ defined by the quadratic form

$$J_{n+1,n} = (\underbrace{1, \cdots, 1}_{n+1}, \underbrace{-1, \cdots, -1}_{n}).$$

An explicit isomorphism is given as follows. For the matrix

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \iddots & \\ & & 1 & & 1 & & \\ & & & 1 & & & \\ & & -1 & & 1 & & \\ & \iddots & & & & \ddots & \\ -1 & & & & & & 1 \end{pmatrix}$$

we have ${}^t S J_{n+1,n} S = J_{2n+1}$ and therefore

$$\varphi : \ \mathrm{SO}(2n+1, \mathbb{R}) \xrightarrow{\sim} \mathrm{SO}(n+1, n)\,,$$
$$g \mapsto SgS^{-1}\,.$$

The same map $X \mapsto SXS^{-1}$ identifies the Lie algebras. The Lie algebra $\mathfrak{a}$ of the split torus

$$A = \{\mathrm{diag}(a_1, \cdots, a_n, 1, a_n^{-1}, \cdots, a_1^{-1}) : \ a_i > 0\} \subset \mathrm{SO}(2n+1, \mathbb{R})$$

becomes a Lie algebra of matrices whose non-zero elements are on the “second” diagonal. The maximal compact subgroup of $\mathrm{SO}(n+1, n)$ we shall work with is

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \ A \in \mathrm{O}(n+1), \ B \in \mathrm{O}(n), \ \det(A) = \det(B) \right\} = S(\mathrm{O}(n+1) \times \mathrm{O}(n))\,.$$

This group has two connected components, the identity component being $K_0 = \mathrm{SO}(n+1) \times \mathrm{SO}(n)$. Consequently $\mathrm{SO}(n+1, n)$ also has two connected components, and for this proof we shall denote the identity components by $\mathrm{SO}(2n+1, \mathbb{R})_0$ and $\mathrm{SO}(n+1, n)_0$.

Let M be the centralizer of $\mathfrak{a}$ in K_0. Explicitly, in both realizations $\mathrm{SO}(2n+1, \mathbb{R})_0$ and $\mathrm{SO}(n+1, n)_0$ we have

$$M = \{\mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n, 1, \varepsilon_n, \cdots, \varepsilon_1) : \ \varepsilon_i \in \{\pm 1\}, \ \textstyle\prod \varepsilon_i = 1\}\,.$$

We shall write the typical element of M as $s(\varepsilon_1, \cdots, \varepsilon_n)$. The characters of M shall be written $\sigma = (\mathrm{sgn}^{\delta_1}, \cdots, \mathrm{sgn}^{\delta_n})$, $\delta_i \in \{0, 1\}$, where

$$\sigma(s(\varepsilon_1, \cdots, \varepsilon_n)) = \prod_{i=1}^{n} \mathrm{sgn}(\varepsilon_i)^{\delta_i}\,.$$

If N denotes the unipotent upper triangular matrices in $\mathrm{SO}(2n+1, \mathbb{R})_0$, then $B = MAN$ is the Langlands decomposition of the standard minimal parabolic subgroup. Using the notation $U(\cdots)$ of [Kn1] for normalized parabolic induction on $\mathrm{SO}(2n+1, \mathbb{R})_0$, we shall show that the principal series representation

$$U(MAN, \mathbf{1}, \varrho^+) \tag{9}$$

contains the discrete series representation with Harish-Chandra parameter ϱ as one of its constituents. Here $\mathbf{1}$ is the trivial representation of M and $\varrho^+ = \sum_{i=1}^{n} (n - i + 1/2) f_i$, where f_i are linear forms on $\mathfrak{a}$ such that

$$\begin{aligned} & f_i \pm f_j\,, && 1 \le i < j \le n\,, \\ & f_i\,, && 1 \le i \le n\,, \end{aligned}$$

are the positive restricted roots. The discrete series representation desired in Proposition 1.1.1 is then obtained by inducing to $\mathrm{SO}(2n+1, \mathbb{R})$ and combines the K_0-types of *two* discrete series representations of $\mathrm{SO}(2n+1, \mathbb{R})_0$.

We shall now assume that $n = 2m$ is <u>even</u>. The argument in the odd case is very similar, but notations are a little different.

1.1.2. LEMMA. *The discrete series representation π_ϱ is a subrepresentation of $U(MAN, \sigma, \nu')$, where*

$$\sigma = (\mathrm{sgn}, 1, 1, \mathrm{sgn}, \mathrm{sgn}, 1, 1, \mathrm{sgn}, \mathrm{sgn}, \cdots)$$

and

$$\nu' = \sum_{j=1}^{m} ((m + j - 1/2) f_{2j-1} + (j - m - 1/2) f_{2j}) .$$

Proof. This is an application of the explicit subrepresentation theorem of [KW1], actually of its corrected version in [KW2]. In this paper, given a discrete series representation π_Λ with Harish-Chandra parameter Λ, a representation σ of M and a linear form ν on $\mathfrak{a}$ are constructed such that there exists a non-zero intertwining map (given also explicitly)

$$S : \ U(MAN, \sigma, \nu) \to \pi_\Lambda .$$

This exhibits π_Λ as a quotient of $U(MAN, \sigma, \nu)$, and by applying some Weyl group conjugation one can make it a subrepresentation of a certain principal series representation.

We shall now apply this with $\Lambda = \varrho$. Actually our group $\mathrm{SO}(n+1, n)_0$ is among the counterexamples mentioned in [KW2], thus we need to use the corrected version Theorem A of this paper.

Let (τ, V) be the minimal K_0-type of π_ϱ and let $v_0 \in V$ be a highest weight vector. Let $\tilde{\sigma}$ be the representation of M on the M-cyclic subspace $\tilde{H}$ of V generated by v_0. Let

$$\tilde{\sigma} = \sum_{j=1}^{r} \sigma_j$$

be a decomposition of $\tilde{\sigma}$ into irreducibles. Then Theorem A of [KW2] says that *every* σ_j can be used to exhibit π_ϱ as a constituent of $U(MAN, \sigma_j, \nu)$ for some ν.

As a maximal compact torus $T \subset K_0$ we choose the set of matrices

$$\begin{pmatrix} r(\theta_1) & & & & & & 0 \\ & \ddots & & & & & \\ & & r(\theta_m) & & & & \\ & & & 1 & & & \\ & & & & r(\theta_{m+1}) & & \\ & & & & & \ddots & \\ 0 & & & & & & r(\theta_n) \end{pmatrix} , \quad \theta_1, \cdots, \theta_n \in \mathbb{R}/2\pi\mathbb{Z} ,$$

where $r(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. It is easy to see that the positive compact roots are

$$\begin{aligned} & e_i \pm e_j , \quad && 1 \le i < j \le m , \\ & e_i , && 1 \le i \le m , \\ & e_i \pm e_j , && m+1 \le i < j \le n . \end{aligned}$$

The first two types of compact roots belong to $\mathrm{SO}(n+1)$ and constitute a root system of type B_m. The third type of compact roots belong to $\mathrm{SO}(n)$ and make a root system of type

D_m. For the half sum of positive compact roots one computes

$$\varrho_c = \sum_{i=1}^{m}(m-i+1/2)e_i + \sum_{i=m+1}^{n}(n-i)e_i\,,$$

and consequently the minimal K_0-type (τ, V) of π_ϱ has highest weight (see [Kn1] Theorem 9.20)

$$\lambda = \varrho + \varrho - 2\varrho_c = 2(\varrho - \varrho_c) = n\sum_{i=1}^{m} e_i + \sum_{i=m+1}^{n} e_i\,. \tag{10}$$

We shall define two subgroups of M as

$$M_1 = M \cap T\,,$$
$$M' = \{s(\varepsilon_1, \cdots, \varepsilon_n) : \ \varepsilon_i = 1 \text{ for } i \text{ even}, \ \textstyle\prod_{i=1}^n \varepsilon_i = 1\}\,.$$

It is then easy to see that $M = M_1 \times M'$. Since $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} r(\theta) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = r(-\theta)$, each element of M' takes a vector $v \in V$ of weight $a_1e_1 + \cdots + a_ne_n$ to a vector of weight $\pm a_1e_1 \pm \cdots \pm a_ne_n$. In particular, if $v_0 \in V$ is a highest weight vector, we have

$$M_1 = M_2 := \{m \in M : \ \tau(m)\mathbb{C}v_0 = \mathbb{C}v_0\}\,.$$

Since

$$\tau(r(\theta_1), \cdots, r(\theta_n))v_0 = e^{in(\theta_1+\cdots+\theta_m)+i(\theta_{m+1}+\cdots+\theta_n)}v_0$$

by (10), and since n is even, it is clear how M_1 acts on v_0:

$$\tau(s(\varepsilon_1, \cdots, \varepsilon_n))v_0 = (-1)^d v_0 \quad \text{for all } \ s(\varepsilon_1, \cdots, \varepsilon_n) \in M_1\,,$$

where d is half the number of indices i with $\varepsilon_i = -1$.

By [KW2], the characters σ_j of M have a common formula on M_2 (that we just computed) and can be chosen arbitrarily on M'. Choosing the trivial character on M' would yield the character

$$\sigma = (1, \mathrm{sgn}, 1, \mathrm{sgn}, \cdots)$$

of M. Choosing other characters of M' amounts to switching a pair $(1, \mathrm{sgn})$ to $(\mathrm{sgn}, 1)$. Thus there is one $\sigma = \sigma_j$ as claimed in the lemma.

It is a computational matter to determine ν, and we shall only make some indications. As a "fundamental sequence" of positive non-compact roots in the sense of [KW1] one can choose

$$\alpha_1 = e_m - e_{m+1},\ \cdots,\ \alpha_m = e_1 - e_n\,,\ \alpha_{m+1} = e_m + e_{m+1},\ \cdots,\ \alpha_n = e_1 + e_n\,.$$

The numbers n_j in [KW1] Theorem 6.1 compute to

$$n_j = \begin{cases} m+1-j\,, & \text{if } \ 1 \le j \le m\,, \\ 0\,, & \text{if } \ m+1 \le j \le n\,. \end{cases}$$

The linear form ν in [KW1] Theorem 6.1 is given by

$$\nu(E_{\alpha_j}+E_{-\alpha_j})=\begin{cases}2n-2j+1\,, & \text{if } 1\le j\le m\,,\\ n+1\,, & \text{if } m+1\le j\le n\,,\end{cases}$$

on the normalized basis $E_{\alpha_j}+E_{-\alpha_j}$ of $\mathfrak{a}$. The lexicographic ordering on the restricted roots determined by this basis is such that the positive roots are

$$\begin{array}{lll} \pm f_{2k}-f_{2k-1}\,, & 1\le k\le m\,, & \\ f_{2k}\pm f_i\,, & 2\le k\le m\,, & 1\le i\le 2k-2\,,\\ -f_{2k-1}\pm f_i\,, & 2\le k\le m\,, & 1\le i\le 2k-2\,,\\ f_{2k}\,, & 1\le k\le m\,, & \\ -f_{2k1}\,, & 1\le k\le m\,. & \end{array}$$

The half sum of those positive roots is

$$\varrho^+=\sum_{j=1}^m((-2j+1/2)f_{2j-1}+(2j-3/2)f_{2j})\,.$$

By [KW1] Theorem 6.1$'$ our π_ϱ is a subrepresentation of $U(MAN_1,\ \sigma,\ \nu-\varrho^+)$ (note that this paper works with non-normalized induction), where N_1 is the analytic subgroup whose Lie algebra $\mathfrak{n}_1$ is spanned by the root spaces for the set of positive restricted roots from above. To get to the standard set of upper triangular matrices (in $\mathrm{SO}(2n+1,\mathbb{R})_0$) we have to conjugate by the Weyl group element $w=w_1w_2$, where $w_1,w_2\in W$ are defined by

$$\begin{aligned} w_1:&\ f_{2j-1}\mapsto -f_{2j-1}\,, && f_{2j}\mapsto f_{2j}\,,\\ w_2:&\ f_{2j-1}\mapsto f_{n-(2j-1)}\,, && f_{2j}\mapsto f_{n+2-2j}\,.\end{aligned}$$

This w moves our set of positive roots to the standard set of positive roots, and

$$U(MAN_1,\sigma,\nu-\varrho^+)\simeq U(MAN,\sigma^w,(\nu-\varrho^+)^w)\,.$$

We have $(\nu-\varrho^+)^w=\nu'$ with ν' as desired. Note that any change that w might cause to σ can be compensated by choosing a different σ. Hence we are done. ∎

We shall now prove Proposition 1.1.1. Let σ and ν' be as in the lemma. Essentially we have to get rid of all the sgn's appearing in σ. In $\mathrm{SO}(2n+1,\mathbb{R})_0$ let H be the centralizer of the kernel of the restricted root f_1-f_4. We have $H\simeq \mathrm{GL}(2,\mathbb{R})\times\mathrm{GL}(1,\mathbb{R})^{n-2}$. If we induce from MAN to the standard parabolic P whose Levi component is H we get reducibility due to the GL(2)-factor. Namely, just looking at $\mathrm{GL}(2,\mathbb{R})$, we get

$$\begin{aligned}&\mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}(2,\mathbb{R})}(\mathrm{sgn}\,|\,|^{m+1/2},\mathrm{sgn}\,|\,|^{-m+3/2})\\ &=\mathrm{sgn}\,|\,|\otimes\mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}(2,\mathbb{R})}(|\,|^{m-1/2},|\,|^{-m+1/2})\\ &=\underbrace{\mathrm{sgn}\,|\,|\,V_m}_{\text{quot}}+\underbrace{\mathrm{sgn}\,|\,|\,\mathcal{D}}_{\text{sub}}\,,\end{aligned}$$

where V_m is a finite-dimensional representation and $\mathcal{D}$ is a discrete series representation. Since π_ϱ is a subrepresentation of $U(MAN, \sigma, \nu')$, the constituent containing $\mathrm{sgn}\,|\,|\,\mathcal{D}$ is the relevant part, meaning that if we induce this constituent all the way up to $\mathrm{SO}(2n+1, \mathbb{R})_0$ we still get π_ϱ as a subrepresentation. The point now is that discrete series representations of $\mathrm{GL}(2, \mathbb{R})$ are stable under twisting with the non-trivial character:

$$\mathrm{sgn}\,|\,|\,\mathcal{D} \simeq |\,|\,\mathcal{D}\,.$$

It follows that π_ϱ is also a subrepresentation of $U(MAN, \sigma', \nu')$, where σ' is obtained from σ by deleting the first two sgn's. We can continue like this and remove all the sgn's except possibly one. But this one can also be removed by a similar argument, considering the parabolic subgroup corresponding to the short root f_{2n-1}, whose Levi component contains a factor $\mathrm{SO}(2, 1) \simeq \mathrm{PGL}(2, \mathbb{R})$.

In the end we obtain π_ϱ as a subrepresentation of $U(MAN, \mathbf{1}, \nu')$. Since ν' is W-conjugate to ϱ^+, we see that π_ϱ is a constituent of $U(MAN, \mathbf{1}, \varrho^+)$. ■

1.2. A case of functoriality

Since the tensor product of two symplectic vector spaces is an orthogonal space, there exists a homomorphism $\mathrm{SL}(2) \times \mathrm{Sp}(2n) \to \mathrm{SO}(4n)$ such that the image of $\mathrm{Sp}(2n)$ is the centralizer of the image of $\mathrm{SL}(2)$ and vice versa. In coordinates, this homomorphism can be realized as follows. With J_n as in (4), we define the involutive automorphism

$$A \mapsto A' := J_n\, {}^t\!A^{-1} J_n$$

of $\mathrm{GL}(n)$. Then we have an embedding

$$\mathrm{GL}(2n) \to \mathrm{SO}(4n), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}. \tag{11}$$

There is another embedding

$$\mathrm{SL}(2) \to \mathrm{SO}(4n)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a\mathbf{1} & bS \\ cS & d\mathbf{1} \end{pmatrix}, \tag{12}$$

where $\mathbf{1}$ is the $2n \times 2n$ identity matrix, and

$$S = \mathrm{diag}(\underbrace{1, \cdots, 1}_{n}, \underbrace{-1, \cdots, -1}_{n})\,.$$

It is easily checked that the matrix $\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}$ in (11) commutes with the image of the map (12) if and only if $A \in \mathrm{Sp}(2n)$. Consequently there is a homomorphism

$$\mathrm{SL}(2) \times \mathrm{Sp}(2n) \to \mathrm{SO}(4n), \tag{13}$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, A \right) \mapsto \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \begin{pmatrix} a\mathbf{1} & bS \\ cS & d\mathbf{1} \end{pmatrix}.$$

The kernel of this homomorphism clearly has two elements, the non-trivial one being $(-\mathbf{1}, -\mathbf{1})$. Thus our map induces an injection

$$(\mathrm{SL}(2) \times \mathrm{Sp}(2n))/\{\pm 1\} \to \mathrm{SO}(4n)\,. \tag{14}$$

We now compose the map (14) with the embedding

$$\mathrm{SO}(4n) \to \mathrm{SO}(4n+1)\,, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B \\ 0 & 1 & 0 \\ C & 0 & D \end{pmatrix}.$$

Pulling the image of $(\mathrm{SL}(2) \times \mathrm{Sp}(2n))/\{\pm 1\}$ in $\mathrm{SO}(4n+1)$ back to $\mathrm{Spin}(4n+1)$, we obtain the following diagram:

$$\begin{tikzcd} \mathrm{Spin}(4n+1) \arrow[r] & \mathrm{SO}(4n+1) \\ \mathrm{SL}(2) \times \mathrm{Sp}(2n) \arrow[u] \arrow[r] & (\mathrm{SL}(2) \times \mathrm{Sp}(2n))/\{\pm 1\} \arrow[u] \end{tikzcd} \tag{15}$$

(To see that the map on the left exists, one can look at the Lie algebras and use the fact that $\mathrm{SL}(2) \times \mathrm{Sp}(2n)$ is simply connected.) Taking complex points on the left hand side we get a homomorphism of complex Lie groups

$$\mathrm{SL}(2, \mathbb{C}) \times \mathrm{Sp}(2n, \mathbb{C}) \to \mathrm{Spin}(4n+1, \mathbb{C})\,. \tag{16}$$

We interpret this as a homomorphism of L-groups (all the groups are split, so this can be done). Langlands' principle of functoriality then predicts the existence of a lifting of automorphic forms

$$\text{from} \quad \mathrm{PGL}(2, \mathbb{A}) \times \mathrm{SO}(2n+1, \mathbb{A}) \quad \text{to} \quad \mathrm{PGSp}(4n, \mathbb{A})\,, \tag{17}$$

where $\mathbb{A}$ is the ring of adeles of any number field. We shall see shortly that Ikeda's lifting is a very special case of this conjectured general lifting.

To clarify the meaning of "lifting" let $\pi_1 = \bigotimes \pi_{1,v}$ be an automorphic form on $\mathrm{PGL}(2, \mathbb{A})$ and let $\pi_2 = \bigotimes \pi_{2,v}$ be an automorphic form on $\mathrm{SO}(2n+1, \mathbb{A})$. To say that an automorphic form $\Pi = \bigotimes \Pi_v$ is a lift of $\pi_1 \otimes \pi_2$ means that for each place v the local component Π_v is a local Langlands lift of $\pi_{1,v} \otimes \pi_{2,v}$. This in turn means the following. To $\pi_{1,v}$ there is associated a local parameter

$$\phi_1 : \; W_v' \to \mathrm{SL}(2, \mathbb{C})$$

from the local Weil–Deligne group to (the connected component of) the local L-group by the local Langlands correspondence. Similarly, to $\pi_{2,v}$ there is associated a parameter

$$\phi_2 : \; W_v' \to \mathrm{Sp}(2n, \mathbb{C})$$

(of course, at present it is not known how to do this in generality except for $n = 1$, but in our application we shall only have real and unramified places, where the local Langlands correspondence is known). Then

$$\phi_1 \times \phi_2 : \; W_v' \to \mathrm{SL}(2, \mathbb{C}) \times \mathrm{Sp}(2n, \mathbb{C})$$

is the local parameter of $\pi_{1,v} \otimes \pi_{2,v}$. Composing $\phi_1 \times \phi_2$ with the L-morphism (16) we get a local parameter

$$W_v' \to \mathrm{Spin}(4n+1, \mathbb{C})\,.$$

For Π_v to be a local Langlands lift of $\pi_{1,v} \otimes \pi_{2,v}$ it is required that this local parameter is associated to Π_v by the local Langlands correspondence for $\mathrm{PGSp}(4n, F_v)$.

Supposing the lifting (17) exists, we shall now describe what is happening on the local level. We shall describe the unramified case and the case of real discrete series representations. This suffices for the application to classical modular forms.

The local lift at infinity

1.2.1. LEMMA. *Let $\lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n$ be the Harish-Chandra parameter for a discrete series representation π_λ of $\mathrm{SO}(2n+1, \mathbb{R})$, with $e_1, \cdots, e_n$ being the natural basis for the character lattice of $\mathrm{SO}(2n+1)$. Let $\mathcal{D}(l)$ be the discrete series representation of $\mathrm{PGL}(2, \mathbb{R})$ with a lowest weight vector of weight $l+1$ (necessarily l is odd). Assuming that $\frac{l}{2} > \lambda_1$, the local image of the representation*

$$\mathcal{D}(l) \otimes \pi_\lambda \quad \textit{on} \quad \mathrm{PGL}(2, \mathbb{R}) \times \mathrm{SO}(2n+1, \mathbb{R})$$

under the lifting (17) is a discrete series representation of $\mathrm{PGSp}(4n, \mathbb{R})$ with infinitesimal character χ_ν, where

$$\nu = \left(\frac{l}{2} + \lambda_1\right)e'_1 + \cdots + \left(\frac{l}{2} + \lambda_n\right)e'_n + \left(\frac{l}{2} - \lambda_n\right)e'_{n+1} + \cdots + \left(\frac{l}{2} - \lambda_1\right)e'_{2n}\,, \quad (18)$$

with $e'_1, \cdots, e'_{2n}$ a basis for the character lattice of $\mathrm{PGSp}(4n)$.

Proof. By definition our embedding $\mathrm{Sp}(2n, \mathbb{C}) \to \mathrm{Spin}(4n+1, \mathbb{C})$ in (16) is such that the element

$$e_i(z) \quad \text{maps to} \quad (e'_i - e'_{2n-i+1})(z)\,, \quad z \in \mathbb{C}^*, \;\; i \in \{1, \cdots, n\} \qquad (19)$$

(the e_i identify with cocharacters of $\mathrm{Sp}(2n, \mathbb{C})$, and the e'_i identify with cocharacters of $\mathrm{Spin}(4n+1, \mathbb{C})$). It follows that

$$z^\lambda \bar{z}^{-\lambda} \quad \text{maps to} \quad z^{\nu_1} \bar{z}^{-\nu_1}\,, \quad \nu_1 = \lambda_1 e'_1 + \cdots + \lambda_n e'_n - \lambda_n e'_{n+1} - \cdots - \lambda_1 e'_{2n}\,. \quad (20)$$

Again by definition, the embedding $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{Spin}(4n+1, \mathbb{C})$ in (16) is such that

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto (e'_1 + \cdots + e'_{2n})(t)\,, \quad t \in \mathbb{C}^*\,, \qquad (21)$$

where as before the e'_i are considered as cocharacters of $\mathrm{Spin}(4n+1, \mathbb{C})$. The local parameter $W_\mathbb{R} \to \mathrm{SL}(2, \mathbb{C})$ corresponding to $\mathcal{D}(l)$ is given by

$$z \mapsto \begin{pmatrix} z^{l/2}\bar{z}^{-l/2} & 0 \\ 0 & z^{-l/2}\bar{z}^{l/2} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and its composition with the map $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{Spin}(4n+1, \mathbb{C})$ therefore maps

$$z \in \mathbb{C}^* \quad \text{to} \quad z^{\nu_2}\bar{z}^{-\nu_2}\,, \quad \nu_2 = \frac{l}{2}(e'_1 + \cdots + e'_{2n})\,. \qquad (22)$$

Multiplying the images in (20) and (22), we see that the composition

$$W_\mathbb{R} \to \mathrm{Sp}(2n, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Spin}(4n+1, \mathbb{C})$$

is such that

$$z \in \mathbb{C}^* \quad \text{maps to} \quad z^{\nu}\bar{z}^{-\nu}$$

with ν as in the lemma. It is also clear by (13) that (representatives for) the longest Weyl group elements combine to the longest Weyl group element. Thus the resulting parameter $W_{\mathbb{R}} \to \mathrm{Spin}(4n+1, \mathbb{C})$ belongs to the L-packet containing the discrete series representations of $\mathrm{PGSp}(4n, \mathbb{R})$ with infinitesimal character χ_ν (see [Bo] 10.5). ∎

REMARK. In this lemma we have to assume $\frac{l}{2} > \lambda_1$ for ν as in (18) to be in the positive Weyl chamber. If this condition is not fulfilled, then the local parameter $W_{\mathbb{R}} \to \mathrm{Spin}(4n+1, \mathbb{C})$ is still as in the proof of the lemma, but the corresponding representation does not belong to the discrete series.

The local lifts at finite places

1.2.2. LEMMA. *Let F be a $\mathfrak{p}$-adic field, let $e_1, \cdots, e_n$ be the natural basis for the character lattice of* $\mathrm{SO}(2n+1)$, *and let π_2 be the spherical representation of* $\mathrm{SO}(2n+1, F)$ *with Satake parameter $e_1(b_1) \cdot \cdots \cdot e_n(b_n) \in \mathrm{Sp}(2n, \mathbb{C})$, where $b_1, \cdots, b_n \in \mathbb{C}^*$. Then, with χ being an unramified character, the local image of the representation*

$$\pi(\chi, \chi^{-1}) \otimes \pi_2 \quad \textit{on} \quad \mathrm{PGL}(2, F) \times \mathrm{SO}(2n+1, F)$$

under the lifting (17) *is the spherical representation of* $\mathrm{PGSp}(4n, F)$ *with Satake parameter*

$$e'_1(bb_1) \cdot \cdots \cdot e'_n(bb_n)\, e'_{n+1}(bb_n^{-1}) \cdot \cdots \cdot e'_{2n}(bb_{2n}^{-1}) \in \mathrm{Spin}(4n+1, \mathbb{C}), \quad b = \chi(\omega), \tag{23}$$

with $\omega \in F^*$ a prime element, and $e'_1, \cdots, e'_{2n}$ a basis for the weight lattice of $\mathrm{PGSp}(4n)$ (*identified with the coweight lattice of* $\mathrm{Spin}(4n+1)$).

Proof. This is easy to see from (19) and (21). ∎

From this lemma the local unramified Euler factors of the lift can be computed. Let

$$\rho_1 : \mathrm{Spin}(4n+1, \mathbb{C}) \to \mathrm{SO}(4n+1, \mathbb{C}) \tag{24}$$

be the "projection representation" of the L-group, and let ρ_2 be the 2^{2n}-dimensional spin representation of $\mathrm{Spin}(4n+1, \mathbb{C})$. With b, b_i as in the lemma, and Π being the lift, it is immediate from (23) that

$$L(s, \Pi, \rho_1) = (1 - q^{-s})^{-1} \prod_{i=1}^{n} (1 - bb_i q^{-s})^{-1} (1 - bb_i^{-1} q^{-s})^{-1}. \tag{25}$$

It is known that ρ_2 has weight structure

$$\frac{c_1 f'_1 + \cdots + c_{2n} f'_{2n}}{2}, \quad c_i \in \{\pm 1\}, \tag{26}$$

where $f'_1, \cdots, f'_{2n}$ are the characters of (the maximal torus of) $\mathrm{Spin}(4n+1)$ dual to $e'_1, \cdots, e'_{2n}$. If we put

$$b'_1 = b_1, \cdots, b'_n = b_n, \quad b'_{n+1} = b_n^{-1}, \cdots, b'_{2n} = b_1^{-1},$$

then it follows from (23) and (26) that

$$L(s,\Pi,\rho_2)=\prod_{j=0}^{2n}\prod_{1\le i_1<\cdots<i_j\le 2n}(1-b^{j-n}b'_{i_1}\cdot\cdots\cdot b'_{i_j}q^{-s})^{-1}\,. \tag{27}$$

1.3. Ikeda's lifting

In [Ik] Ikeda proved the existence of a lifting from classical elliptic cusp forms to Siegel modular forms of degree $2n$, for each $n=1,2,\cdots$. In case $n=1$ one obtains the classical Saito–Kurokawa lifting. We shall now explain why Ikeda's construction is a special case of the general lifting (17).

The idea is as follows. The ground field being $\mathbb{Q}$, we put on $\mathrm{PGL}(2,\mathbb{A})$ the automorphic representation π_f corresponding to a classical cuspidal eigenform $f\in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$. On $\mathrm{SO}(2n+1,\mathbb{A})$ we put the anomalous representation π_2 defined in 1.1.

Then we have the representation

$$\pi_f\otimes\pi_2\quad\text{on}\quad \mathrm{PGL}(2,\mathbb{A})\times\mathrm{SO}(2n+1,\mathbb{A})\,. \tag{28}$$

If $F\in S_{k+n}(\mathrm{Sp}(4n,\mathbb{Z}))$ is the Ikeda lift of f in degree $2n$, and Π_F is the associated automorphic representation of $\mathrm{PGSp}(4n,\mathbb{A})$ (see [AS]), then we shall show that Π_F is the representation predicted by the functorial lifting (17), by comparing local parameters at all places.

Let $\Pi=\bigotimes\Pi_p$ be the predicted image of the automorphic representation (28) under the lift (17). We have to show that the local parameter of Π_∞ is that of a holomorphic modular form of weight $k+n$ and degree $2n$, and that for finite p the local Euler factors of Π_p coincide with those given in [Ik].

First we have a look at Π_∞. In the notation of Lemma 1.2.1, the Harish-Chandra parameter of $\pi_{2,\infty}$ is

$$\varrho=\frac{2n-1}{2}e_1+\frac{2n-3}{2}e_2+\cdots+\frac{1}{2}e_n\,,$$

see section 1.1. By Lemma 1.2.1, if $k>n$, the lift Π_∞ is a discrete series representation with infinitesimal character χ_ν, where

$$\nu=(k+n-1)e'_1+\cdots+(k-n)e'_{2n}$$

(let $l=2k-1$ in Lemma 1.2.1). This is indeed the Harish-Chandra parameter for a holomorphic discrete series representation of $\mathrm{PGSp}(4n,\mathbb{R})$ attached to a Siegel modular form of weight $k+n$, see [AS]. If $k\le n$, the resulting parameter $W_{\mathbb{R}}\to\mathrm{Sp}(4n+1,\mathbb{C})$ should still be the parameter for a lowest weight representation as it appears for holomorphic Siegel modular forms, but this is not completely clear since these representations do not belong to the discrete series.

We now consider a finite place p. The Satake parameter of the lift Π_p is given by Lemma 1.2.2, with b the Satake parameter of $\pi_{f,p}$, and

$$b_1=p^{-(2n-1)/2},\cdots,b_n=p^{-1/2}\,, \tag{29}$$

see (8). Thus, by (25), the "standard" local Euler factor for Π_p is

$$L(s, \Pi_p, \rho_1) = (1 - p^{-s})^{-1} \prod_{i=1}^{2n} L(\pi_{f,p}, s + n - i + 1/2)\,, \tag{30}$$

where ρ_1 is as in (24) and where

$$L(s, \pi_{f,p}) = (1 - bp^{-s})^{-1}(1 - b^{-1}p^{-s})^{-1} \quad (\pi_{f,p} = \pi(\chi, \chi^{-1}),\ b = \chi(p))\,.$$

The relation with the classical L-factor is

$$L(s, \pi_{f,p}) = L_p(s', f)\,, \quad s' = s + \frac{2k-1}{2}\,,$$

and consequently

$$L(s, \Pi_p, \rho_1) = (1 - p^{-s})^{-1} \prod_{i=1}^{2n} L_p(f, s + k + n - i)\,.$$

This is precisely the Euler factor given by Ikeda in [Ik], showing that Ikeda's lifting and the special case of the lifting (17) we discussed are compatible. See also the remarks in §15 of [Ik].

However, the "standard" Euler factor (30) does not completely characterize the local representation $\Pi_{F,p}$. Let $a_0, a_1, \cdots, a_{2n}$ be the Satake parameters of the modular form F, normalized such that

$$a_0^2 a_1 \cdot \cdots \cdot a_{2n} = 1 \tag{31}$$

(cf. [AS] Lemma 10). The "standard" Euler factor only gives information about $a_1, \cdots, a_{2n}$; in our case it says that

$$a_j = bp^{-n+j-1/2}\,, \quad j = 1, \cdots, 2n\,.$$

This leaves the two possibilities $a_0 = \pm b^{-n}$. For the Ikeda lift to really coincide with our functorial construction we need to have the positive sign, see (27). The following lemma therefore completes the proof of our claim that Ikeda's lifts provide examples for the general (hypothetical) lifting (17).

1.3.1. LEMMA. *Let f be a cuspidal elliptic eigenform and F the Ikeda lift of f in degree $2n$. Then the Satake parameters of F, normalized as in* (31), *are given by*

$$a_0 = b^{-n} \quad \textit{and} \quad a_j = bp^{-n+j-1/2}\,, \quad j = 1, \cdots, 2n\,.$$

Here b is the Satake parameter of f.

Proof. As usual, let $T(p)$ denote the Hecke operator $\Gamma \mathrm{diag}(1, \cdots, 1, p, \cdots, p)\Gamma$ (the number of p's is $2n$). Let $E_{2n,k}$ be the holomorphic Eisenstein series of weight k in degree $2n$. By [Ik], §14, there exists a $\Phi \in \mathbb{C}[X + X^{-1}]$ such that

$$E_{2n,k'+n} \mid T(p) = \Phi(p^{k'-1/2}) E_{2n,k'+n}$$

for k' large enough. Indeed, with $c_j = p^{-n+j-1/2}$, $j = 1, \cdots, 2n$, and provided the correct normalization of Hecke operators, we can take

$$\Phi(X) = \prod_{\nu=1}^{n} (X + c_\nu + c_\nu^{-1} + X^{-1}) \,.$$

This follows from [Fr] IV.4.6, where the $T(p)$–eigenvalue of $E_{n,k}$ is given explicitly. Note that

$$\Phi(X) = X^{-n} \sum_{i=0}^{2n} E_i(c_1, \cdots, c_{2n}) X^i \,, \tag{32}$$

where E_i are the elementary symmetric polynomials in $2n$ variables. Now, by [Ik], §14,

$$F \mid T(p) = \Phi(b) F \,. \tag{33}$$

On the other hand, the Satake isomorphism $\mathcal{S}$, as defined in [AS] 3.1, maps $T(p)$ to $X_0(E_0 + \cdots + E_{2n})$ (cf. [Fr] IV.3.14). Thus

$$F \,\big|\, T(p) = \mathcal{S}(T(p))\big|_{a_0, \cdots, a_{2n}} F = a_0 \Bigg(\sum_{i=0}^{2n} E_i(a_1, \cdots, a_{2n}) \Bigg) F \,. \tag{34}$$

Since $a_i = bc_i$, it follows from (32), (33) and (34) that $a_0 = b^{-n}$. ∎

We remark that by (23) and (29) the local lifts at finite places are *non-tempered* representations. Thus Ikeda's lifts provide counterexamples to the generalized Ramanujan conjecture for the groups $\mathrm{PGSp}(4n, \mathbb{A})$.

The Saito–Kurokawa case

Things become a little simpler in case $n = 1$ which is the classical Saito–Kurokawa situation. Since $\mathrm{Spin}(5) \simeq \mathrm{Sp}(4)$, our homomorphism (16) of L-groups becomes the embedding

$$\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Sp}(4, \mathbb{C}) \,,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & b \\ 0 & a' & b' & 0 \\ 0 & c' & d' & 0 \\ c & 0 & 0 & d \end{pmatrix} .$$

The lifting (17) becomes the functoriality

$$\text{from} \quad \mathrm{PGL}(2, \mathbb{A}) \times \mathrm{PGL}(2, \mathbb{A}) \quad \text{to} \quad \mathrm{PGSp}(4, \mathbb{A}) \,. \tag{35}$$

Our anomalous representation π_2 is a constituent of $\mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{GL}(2,\mathbb{A})}(|\,|^{1/2}, |\,|^{-1/2})$, meaning we put the global character

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto |a|^{1/2} |d|^{-1/2}$$

on the Borel subgroup. All the local induced representations have two constituents, with the Langlands quotient being the trivial representation. The subrepresentation is the Steinberg representation at finite places, and the lowest discrete series representation (lowest weight

2) at the archimedean place. The anomalous representation π_2 is the one where we do not choose the Langlands quotient precisely at infinity.

Let π_1 be the representation associated to a classical cuspform of weight $2k-2$ with even k. Then the classical Saito–Kurokawa lift F of f, which is a modular form of degree 2 and weight k, has an associated representation that is the expected functorial lift of $\pi_1 \otimes \pi_2$ under the lifting (35). This was already noted by Langlands in [La2].

Remark on CAP representations

Consider the torus T_1 in $\mathrm{SO}(4n+1)$ consisting of all elements of the form

$$d(t_1, \cdots, t_n) := \mathrm{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}, 1, t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}), \quad t_i \in \mathrm{GL}(1).$$

This is the common kernel of the roots $f_1' + f_{2n}', \cdots, f_n' + f_{n+1}'$. It is also the standard maximal torus of $\mathrm{Sp}(2n)$ under our embedding $\mathrm{Sp}(2n) \to \mathrm{SO}(4n+1)$ (see (15)). Let $\hat{P} = \hat{M}\hat{N}$ be the standard parabolic subgroup of $\mathrm{SO}(4n+1)$ whose Levi component $\hat{M}$ centralizes the torus T_1. A direct computation shows

$$\hat{M} \simeq \mathrm{GL}(2) \times \cdots \times \mathrm{GL}(2) \quad (n \text{ factors}).$$

The order of the factors can be so arranged that

$$\{d(1, \cdots, 1, t_i, 1, \cdots, 1) : \ t_i \in \mathrm{GL}(1)\}$$

is the center of the i-th factor $\mathrm{GL}(2)$. Our embedding $\mathrm{SL}(2) \to \mathrm{SO}(4n+1)$ (see (15)) has image in $\hat{M}$. In fact, it is the diagonal embedding

$$\mathrm{SL}(2) \to \hat{M}, \quad A \mapsto (A, \cdots, A).$$

Consider a local field F and any irreducible, admissible representation π_1 of $\mathrm{PGL}(2, F)$ with local parameter $\varphi : W_F' \to \mathrm{SL}(2, \mathbb{C})$. Consider further the trivial representation π_2 of $\mathrm{SO}(2n+1, F)$ which has local parameter

$$\begin{aligned} \psi :\ & W_F' \to \mathrm{Sp}(2n, \mathbb{C}), \\ & w \mapsto \mathrm{diag}(|w|^{n-1/2}, \cdots, |w|^{1/2}, |w|^{-1/2}, \cdots, |w|^{-n+1/2}). \end{aligned}$$

The composition

$$W_F' \to \mathrm{SL}(2, \mathbb{C}) \times \mathrm{Sp}(2n, \mathbb{C}) \to \mathrm{SO}(4n+1, \mathbb{C}), \tag{36}$$

where the first map is $\varphi \times \psi$, obviously has image in $\hat{M}$. Explicitly, by the above remarks, it is given by

$$w \mapsto (|w|^{n-1/2}\varphi(w), \cdots, |w|^{1/2}\varphi(w)). \tag{37}$$

The parabolic $P = MN$ of $\mathrm{Sp}(4n)$ which is dual to $\hat{P}$ has Levi component $M \simeq \mathrm{GL}(2) \times \cdots \times \mathrm{GL}(2)$ (n factors); let us write $g(A_1, \cdots, A_n)$ for a typical element of M, where $A_i \in \mathrm{GL}(2)$. Since the parameter (36) has image in $\hat{M}$, it corresponds to the Langlands quotient of a representation induced from P. More precisely, looking at (37), the induced representation is

$$\mathrm{Ind}_P^{\mathrm{Sp}(4n,F)}(\sigma)\,, \quad \text{where } \sigma = |\,|^{n-1/2}\pi_1 \otimes \cdots \otimes |\,|^{1/2}\pi_1 \tag{38}$$

(because tensoring the parameter with a character of W_F corresponds to twisting the representation). But we are really on $\mathrm{PGSp}(4n, F)$ and not on $\mathrm{Sp}(4n, F)$, and instead of the parameter (36) we should consider

$$W'_F \to \mathrm{SL}(2,\mathbb{C}) \times \mathrm{Sp}(2n,\mathbb{C}) \to \mathrm{Spin}(4n+1,\mathbb{C})\,. \tag{39}$$

Instead of (38) we are then talking about the representation

$$\mathrm{Ind}_{P'}^{\mathrm{GSp}(4n,F)}(\sigma)\,, \quad \text{where } \sigma' = |\,|^{n-1/2}\pi_1 \otimes \cdots \otimes |\,|^{1/2}\pi_1 \otimes |\,|^{-n^2/2}\,. \tag{40}$$

Here $P' = M'N'$ and $M' \simeq \mathrm{GL}(2) \times \cdots \times \mathrm{GL}(2) \times \mathrm{GL}(1)$. After a conjugation one can realize M' as

$$M' = \left\{ \begin{pmatrix} A_1 & & & & & 0 \\ & \ddots & & & & \\ & & A_n & & & \\ & & & uA'_n & & \\ & & & & \ddots & \\ 0 & & & & & uA'_n \end{pmatrix} : \; A_i \in \mathrm{GL}(2),\; u \in \mathrm{GL}(1) \right\}$$

Note that the representation (40) really descends to a representation of $\mathrm{PGSp}(4n)$.

If $\pi_1 = \pi_{1,v}$ is a local component of a cuspidal automorphic representation of $\mathrm{PGL}(2,\mathbb{A})$, then σ as in (38) is a local component of a cuspidal automorphic representation of $M(\mathbb{A})$. Thus an Ikeda lift, which is a cuspform, looks at every finite place like a non-cuspform, namely like a representation globally induced from $P(\mathbb{A})$ to $\mathrm{PGSp}(4n,\mathbb{A})$. This is the CAP phenomenon (cuspidal associated to parabolics). The spin L-function of an Ikeda lift can therefore be expected to have poles, and we will see in section 2.4 that this is indeed the case. In the Saito–Kurokawa case it is shown in [PS] that the spin L-function having poles characterizes CAP representations.

2. Computation of the spin L-function

In this second part we shall compute the spin L-function of Ikeda's lifts. It turns out that these L-functions can be expressed by symmetric power L-functions for $\mathrm{GL}(2)$.

2.1. Decomposition of the spin representation

We view $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$ as subgroups of $\mathrm{Spin}(4n+1,\mathbb{C})$ via our embedding

$$\mathrm{SL}(2,\mathbb{C}) \times \mathrm{Sp}(2n,\mathbb{C}) \to \mathrm{Spin}(4n+1,\mathbb{C})\,. \tag{41}$$

Let ρ be the 2^{2n}-dimensional spin representation of $\mathrm{Spin}(4n+1,\mathbb{C})$. In this section we shall decompose ρ under the action of the above subgroups. For the material in this section compare [GW] Theorem 4.5.8 and Section 5.1.3.

2.1.1. LEMMA. *As a representation of* $\mathrm{Sp}(2n,\mathbb{C})$ *the spin representation* ρ *is isomorphic to the exterior algebra* $\bigwedge \mathbb{C}^{2n}$, *where* $\mathbb{C}^{2n}$ *denotes the standard representation of* $\mathrm{Sp}(2n,\mathbb{C})$.

Proof. Let W be the 2^{2n}-dimensional spin representation. It is known that W has weight structure

$$\frac{c_1 f_1' + \cdots + c_{2n} f_{2n}'}{2}\,, \quad c_i \in \{\pm 1\}\,, \tag{42}$$

where $f_1', \cdots, f_{2n}'$ are the characters of (the maximal torus of) $\mathrm{Spin}(4n+1)$ dual to $e_1', \cdots, e_{2n}'$. We let $w_{c_1,\cdots,c_{2n}} \in W$ be a vector spanning the one-dimensional weight space for the weight $(c_1 f_1' + \cdots + c_{2n} f_{2n}')/2$. For α any root of $\mathrm{Sp}(2n,\mathbb{C})$, let $X_\alpha \in \mathfrak{sp}(2n)$ be a vector spanning the corresponding root space. Now consider the embedding (41) on the Lie-algebra level:

$$\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sp}(2n,\mathbb{C}) \to \mathfrak{so}(4n+1,\mathbb{C})\,. \tag{43}$$

From its explicit form we can see that under this map

$$X_{e_i - e_j} \mapsto X_{f_i' - f_j'} + X_{f_{2n+1-j}' - f_{2n+1-i}'} \quad \text{for all } 1 \le i < j \le n\,, \tag{44}$$

where the vectors on the right span root spaces in $\mathfrak{so}(4n+1,\mathbb{C})$. Similarly,

$$X_{e_i + e_j} \mapsto X_{f_i' - f_{2n+1-j}'} \quad \text{for all } 1 \le i \le j \le n\,. \tag{45}$$

From (44) and (45) it is easy to see that the highest weight vectors in W for the action of $\mathrm{Sp}(2n,\mathbb{C})$ are precisely the $2n+1$ vectors

$$w_{-1,-1,\cdots,-1,-1}\,, \quad w_{1,-1,\cdots,-1,-1}\,, \quad \cdots\,, \quad w_{1,1,\cdots,1,-1}\,, \quad w_{1,1,\cdots,1,1}\,.$$

Consider the subspaces

$$W_j := \langle w_{c_1,\cdots,c_{2n}} :\ \#\{i :\ c_i = 1\} = j \rangle\,, \quad j = 0, 1, \cdots, 2n\,.$$

We see from (44) and (45) that each W_j is invariant under $\mathrm{Sp}(2n,\mathbb{C})$. Since there are only $2n+1$ highest weight vectors in W, the vector space direct sum

$$W = \bigoplus_{j=0}^{2n} W_j$$

is also the decomposition of W into irreducible $\mathrm{Sp}(2n,\mathbb{C})$-modules. It remains to identify W_j with $\bigwedge^j \mathbb{C}^{2n}$. The highest weight vector in W_j is

$$w_{1,\cdots,1,-1,\cdots,-1} \quad \text{(the number of 1's is } j)\,. \tag{46}$$

Our embedding $\mathfrak{sp}(2n,\mathbb{C}) \to \mathfrak{so}(2n+1,\mathbb{C})$ induces the following mapping on the character lattice:

$$f_1' \mapsto e_1, \cdots,\ f_n' \mapsto e_n,\ f_{n+1}' \mapsto -e_n,\ \cdots,\ f_{2n}' \mapsto -e_1\,.$$

It is then immediate that under the $\mathrm{Sp}(2n,\mathbb{C})$-action the vector (46) has weight $e_1+\cdots+e_j$ if $j \le n$, and weight $e_1+\cdots+e_{2n-j}$ if $j \ge n$. But this is also the highest weight of $\bigwedge^j \mathbb{C}^{2n}$:

If $b_1, \cdots, b_{2n}$ is the canonical basis of $\mathbb{C}^{2n}$, then a highest weight vector is $b_1 \wedge \cdots \wedge b_j$, and it has the same weight. ∎

Next we consider the restriction of ρ to $\mathrm{SL}(2,\mathbb{C})$. If e_1 spans the character lattice of $\mathfrak{sl}(2,\mathbb{C})$, then our embedding $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(4n+1,\mathbb{C})$ induces on the character lattices the map

$$f_1' \mapsto e_1, \ \cdots, \ f_{2n}' \mapsto e_1 .$$

Consequently, the weight of $w_{c_1,\cdots,c_{2n}}$ (notation as in the previous proof) under the action of $\mathfrak{sl}(2,\mathbb{C})$ is

$$\frac{c_1 e_1 + \cdots, c_{2n} e_1}{2} = \frac{c_1 + \cdots + c_{2n}}{2} e_1 .$$

It follows that W_j is a weight space for the $\mathrm{SL}(2,\mathbb{C})$*-action for the weight* $(j-n)e_1$. Next consider the action of the root vector X_α, where α is the positive root of $\mathfrak{sl}(2,\mathbb{C})$. This element raises the weight by 2, thus maps W_j to W_{j+2}. This is also clear from the explicit form of the embedding $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(4n+1,\mathbb{C})$ which maps

$$X_\alpha \mapsto X_{f_1'+f_{2n}'} + \cdots + X_{f_n'+f_{n+1}'} .$$

Thus, for the combined action of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$ on W we get the following picture:

$$\begin{array}{cc} W_0 & \\ \downarrow & W_1 \\ W_2 & \downarrow \\ \downarrow & W_3 \\ \vdots & \vdots \\ W_{2n-2} & \downarrow \\ \downarrow & W_{2n-1} \\ W_{2n} & \end{array}$$

$\mathrm{Sp}(2n,\mathbb{C})$ acts "horizontally" on each W_j,
while $\mathrm{SL}(2,\mathbb{C})$ acts "vertically".

Let $(V_{2n}, \mathrm{Sym}^{2n-1})$ be the $2n$-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. It is known that this is a symplectic representation, meaning that V_{2n} carries an $\mathrm{SL}(2,\mathbb{C})$-invariant skew symmetric form (see [GW] Lemma 5.1.22). Thus we can consider Sym^{2n-1} a homomorphism $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(2n,\mathbb{C})$.

Now let G_1 and G_2 be two copies of $\mathrm{SL}(2,\mathbb{C})$, and consider the composition

$$G_1 \times G_2 \to \mathrm{SL}(2,\mathbb{C}) \times \mathrm{Sp}(2n,\mathbb{C}) \to \mathrm{Spin}(4n+1,\mathbb{C}) ,$$

where the first map is $\mathrm{id} \times \mathrm{Sym}^{2n-1}$. We would like to describe the restriction of the 2^{2n}-dimensional spin representation ρ to $G_1 \times G_2$. We already know that G_1 acts "vertically" and G_2 acts "horizontally".

The lowest weight for the action of G_2 on $W_j = \bigwedge^j V_{2n}$ is

$$\sum_{i=0}^{j-1}(1-2n+2i) = j(j-2n)\,.$$

Let

$$W_j = \bigoplus_{\substack{r=j(j-2n)\\ \text{(step 2)}}}^{j(2n-j)} W_{j,r}$$

be the decomposition of W_j into weight spaces for the action of G_2. Thus we get a picture as follows.

$$\begin{array}{ccccccc}
 & & & W_0 & & & \\
 & & W_{1,1-2n} & \cdots & W_{1,2n-1} & & \\
 & W_{2,4(1-n)} & \cdots & \cdots & \cdots & W_{2,4(n-1)} & \\
 & & & \vdots & & & \\
W_{n,-n^2} & \cdots & \cdots & \cdots & \cdots & \cdots & W_{n,n^2} \\
 & & & \vdots & & & \\
 & & W_{2n-1,1-2n} & \cdots & W_{2n-1,2n-1} & & \\
 & & & W_{2n} & & &
\end{array}$$

Define

$$\alpha(r,j,n) := \dim_{\mathbb{C}}(W_{j,r})\,, \quad \beta(r,j,n) := \alpha(r,j,n) - \alpha(r,j-2,n)\,. \tag{47}$$

Then obviously

$$\sum_{\substack{r=j(j-2n)\\ \text{(step 2)}}}^{j(2n-j)} \alpha(r,j,n) = \binom{2n}{j}, \quad \sum_{\substack{r=j(j-2n)\\ \text{(step 2)}}}^{j(2n-j)} \beta(r,j,n) = \binom{2n}{j} - \binom{2n}{j-2}. \tag{48}$$

Note that the element $X_\alpha \in \mathfrak{g}_1$ acts injectively on W_j for $j \le n-2$, and that consequently $\beta(r,j,n) \ge 0$ for $j \le n$. Actually we have

$$\beta(2r,j,n) > 0 \quad \text{for} \quad j \le n \quad \text{and} \quad j(j-2n)/2 \le r \le j(2n-j)/2\,,$$

since W_j contains the G_2-representation with lowest weight $j(2n-j)$, but W_{j-2} does not. Consider the “vertical” spaces

$$\tilde{W}_r := \bigoplus_{j=0}^{2n} W_{j,r}\,.$$

Since the actions of G_1 and G_2 commute, these spaces are G_1-invariant. The numbers $\beta(r,j,n)$ were defined in such a way that the decomposition of $\tilde{W}_r$ into irreducible G_1-modules is given by

$$\tilde{W}_r = \bigoplus_{j=0}^{n} (\mathrm{Sym}^{n-j})^{\beta(r,j,n)} . \tag{49}$$

This yields the decomposition of $W = \bigoplus \tilde{W}_r$ as a G_1-representation. We shall use this decomposition in the next section to compute local L-factors for the spin L-function of Ikeda's lifts. Note that the numbers $\alpha(r, j, n)$ have the following combinatorial description:

$$\begin{aligned}\alpha(r, j, n) = {} & \text{number of possibilities to choose } j \text{ different numbers}\\ & \text{from the set } \{1-2n, 3-2n, \cdots, 2n-1\} \text{ (not observing}\\ & \text{the order) such that their sum equals } r.\end{aligned} \tag{50}$$

The appendix gives some impression of how large these numbers are.

2.2. Computation of spin L-factors

Let F be a number field and $\mathbb{A}$ its ring of adeles. Let $\pi_1 = \bigotimes \pi_{1,v}$ be any automorphic representation of $\mathrm{PGL}(2, \mathbb{A})$, and $\pi_2 = \bigotimes \pi_{2,v}$ a constituent of the induced representation on $\mathrm{SO}(2n+1, \mathbb{A})$ whose isobaric constituent is the trivial representation, see section 1.1. Let Π be an automorphic representation of $\mathrm{PGSp}(4n, \mathbb{A})$ that is a Langlands lift of $\pi_1 \otimes \pi_2$ with respect to the embedding (16) of L-groups. In this section we shall compute the local Euler factors

$$L_v(s, \Pi_v, \rho),$$

where ρ is the 2^{2n}-dimensional spin representation of $\mathrm{Spin}(4n+1, \mathbb{C})$, for the following cases:

- v is a finite place and $\pi_{2,v}$ is the trivial representation;
- v is a real place, $\pi_{1,v}$ is a discrete series representation, and $\pi_{2,v}$ is the discrete series representation of $\mathrm{SO}(2n+1, \mathbb{R})$ considered in Proposition 1.1.1.

This is enough to compute the full spin L-function of Ikeda's lifts.

The finite places

First assume that v is a finite place. Let W_v' be the local Weil–Deligne group of F_v. Let

$$\phi_1 : W_v' \to G_1 ,$$

be the local parameter of $\pi_{1,v}$, where $G_1 = \mathrm{SL}(2, \mathbb{C})$. Let G_2 be another copy of $\mathrm{SL}(2, \mathbb{C})$, and define

$$\begin{aligned}\phi_2 : W_v' &\to G_2,\\ w &\mapsto \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}.\end{aligned}$$

Then $\mathrm{Sym}^{2n-1} \circ \phi_2$ is the local parameter for the trivial representation of $\mathrm{SO}(2n+1, F_v)$. The fact that Π is a Langlands lift of $\pi_1 \otimes \pi_2$ means that Π_v has the local parameter Φ obtained by composing the following maps:

$$W_v' \to G_1 \times G_2 \to \mathrm{SL}(2, \mathbb{C}) \times \mathrm{Sp}(2n, \mathbb{C}) \to \mathrm{Spin}(4n+1, \mathbb{C}) .$$

Here the first map is (ϕ_1, ϕ_2), the second is id $\times$ Sym^{2n-1}, and the third is our embedding of L-groups. For $i = 1, 2$ define Φ_i to be the composition

$$\begin{aligned} W'_v \xrightarrow{\phi_i} G_i \longrightarrow G_1 \times G_2 \longrightarrow \mathrm{SL}(2, \mathbb{C}) \times \mathrm{Sp}(2n, \mathbb{C}) \\ \longrightarrow \mathrm{Spin}(4n+1, \mathbb{C}) \xrightarrow{\rho} \mathrm{GL}(2^{2n}, \mathbb{C}) \,, \end{aligned} \tag{51}$$

so that $(\rho \circ \Phi)(w) = \Phi_1(w)\Phi_2(w)$. We shall now use the notation of the previous section, where we determined the decomposition of the spin representation ρ under the action of G_1. It follows from (49) that the representation Φ_1 of W'_v decomposes as follows:

$$\Phi_1 = \bigoplus_{r=-n^2}^{n^2} \tilde{W}_r = \bigoplus_{j=0}^{n} \bigoplus_{\substack{r=j(j-2n)\\ (\text{step } 2)}}^{j(2n-j)} (\mathrm{Sym}^{n-j} \circ \phi_1)^{\beta(r,j,n)} \,.$$

By definition, Φ_2 acts on $\tilde{W}_r$ as the character $w \mapsto |w|^{r/2}$. Thus

$$\rho \circ \Phi = \bigoplus_{j=0}^{n} \bigoplus_{\substack{r=j(j-2n)\\ (\text{step } 2)}}^{j(2n-j)} ((\mathrm{Sym}^{n-j} \circ \phi_1) \otimes |\ |^{r/2})^{\beta(r,j,n)} \,.$$

Taking L-factors, the twist $|\ |^{r/2}$ results in a shift $s \mapsto s + r/2$ (note that this would be wrong for v a real place: on a one-dimensional representation of the real Weil group $W_{\mathbb{R}}$ the twist $|\ |^{r/2}$, with $|\ |$ being the usual real absolute value, would result in a shift by $r/2$, but on a two-dimensional irreducible representation of $W_{\mathbb{R}}$ the shift would be $r/4$; see [Kn2] (3.6)). Therefore we get the following result.

2.2.1. PROPOSITION. *Let v be a finite place of the number field F. Let Π_v be a representation of* $\mathrm{PGSp}(4n, F_v)$ *which is a local Langlands lift of the representation $\pi_{1,v} \otimes \mathbf{1}_v$ on* $\mathrm{PGL}(2, F_v) \times \mathrm{SO}(2n+1, F_v)$ *under the embedding* (16) *of L-groups. Let ρ be the 2^{2n}–dimensional spin representation of* $\mathrm{Spin}(4n+1, \mathbb{C}) = {}^L\mathrm{PGSp}(4n, F_v)^0$. *Then*

$$L_v(s,\ \Pi_v,\ \rho) = \prod_{j=0}^{n} \prod_{\substack{r=j(j-2n)\\ (\text{step } 2)}}^{j(2n-j)} L(s + r/2,\ \pi_{1,v},\ \mathrm{Sym}^{n-j})^{\beta(r,j,n)} \,. \tag{52}$$

The exponents $\beta(r, j, n)$ are all positive integers and are of the form $\beta(r, j, n) = \alpha(r, j, n) - \alpha(r, j-2, n)$ with $\alpha(r, j, n)$ given by (50).

This proposition says that the spin L-function of the global lift Π is essentially a product of symmetric power L-functions for the original representation π_1 on $\mathrm{PGL}(2, \mathbb{A})$. Thus any known analytic properties of these symmetric power L-functions will imply the corresponding analytic properties for the spin L-function (and vice versa). See section 2.4 for more precise statements.

Note that Sym^0 is the trivial representation, and that $L(s, \pi_{1,v}, \mathrm{Sym}^0)$ is a local Euler factor of the Dedekind zeta function of F (the Riemann zeta function if $F = \mathbb{Q}$). Thus

$L(s, \Pi, \rho)$ potentially has many poles. We will however see that the archimedean Euler factor cancels most of them (but not all of them).

In general the local factor starts like this:

$$L_v(s,\ \Pi_v,\ \rho) = L(s,\ \pi_{1,v},\ \mathrm{Sym}^n) \prod_{r=0}^{2n-1} L\left(s + \frac{2n-1}{2} - r,\ \pi_{1,v},\ \mathrm{Sym}^{n-1}\right) \cdot\ \cdots$$

We shall give some explicit formulas in low-dimensional cases, for which we abbreviate

$$\zeta(s) = L(s,\ \pi_{1,v},\ \mathrm{Sym}^0)\,.$$

This is independent of $\pi_{1,v}$, and for a finite place is given by $\zeta(s) = (1 - q^{-s})^{-1}$, where q is the number of elements of the residue field.

$\underline{n = 1}$:

$$L_v(s,\ \Pi_v,\ \rho) = L(s,\ \pi_{1,v})\,\zeta(s + 1/2)\,\zeta(s - 1/2)\,. \tag{53}$$

This is the local factor familiar from the classical Saito–Kurokawa lifting, see [EZ] §6.

$\underline{n = 2}$:

$$L_v(s,\ \Pi_v,\ \rho) = L(s,\ \pi_{1,v},\ \mathrm{Sym}^2) \prod_{i=0}^{3} L(s + 3/2 - i,\ \pi_{1,v}) \prod_{i=0}^{4} \zeta(s + 2 - i)\,.$$

$\underline{n = 3}$:

$$\begin{aligned} L_v(s,\ \Pi_v,\ \rho) = L(s,\ \pi_{1,v},\ \mathrm{Sym}^3) &\prod_{i=0}^{5} L(s + 5/2 - i,\ \pi_{1,v},\ \mathrm{Sym}^2) \\ &\cdot \prod_{i=0}^{8} L(s + 4 - i,\ \pi_{1,v})^{\varepsilon_i} \prod_{i=0}^{9} \zeta(s + 9/2 - i)^{\varepsilon'_i} \end{aligned}$$

with

$$\varepsilon_i = \begin{cases} 1 & i \in \{0, 1, 7, 8\}\,, \\ 2 & i \in \{2, \cdots, 6\}\,, \end{cases} \qquad \varepsilon'_i = \begin{cases} 1 & i \in \{0, 1, 2, 7, 8, 9\}\,, \\ 2 & i \in \{3, 4, 5, 6\}\,. \end{cases}$$

The real case

We now assume that v is a real place and that $\pi_{1,v}$ is a discrete series representation of $\mathrm{PGL}(2, \mathbb{R})$. We shall compute the local spin L-factor $L_v(s,\ \Pi_v,\ \rho)$ for the case that $\pi_{2,v}$ is the discrete series representation of $\mathrm{SO}(2n + 1, \mathbb{R})$ mentioned in Proposition 1.1.1.

The local parameter $W_{\mathbb{R}} \to \mathrm{Sp}(2n, \mathbb{C})$ of $\pi_{2,v}$ is explicitly given by (6) and (7). Note that it is the composition

$$\begin{aligned} W_{\mathbb{R}} &\stackrel{\phi_2}{\longrightarrow} \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{Sp}(2n,\mathbb{C})\,, \\ re^{i\theta} &\longmapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \\ j &\longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned}$$

where the second map is Sym^{2n-1}, and where ϕ_2 is the parameter for the lowest discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$.

We will assume that $\pi_{1,v}$ is a discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$, say with lowest weight $l+1$. The odd number $l \geq 1$ is the Harish-Chandra parameter, and the local parameter of $\pi_{1,v}$ is explicitly given by

$$\begin{aligned} \phi_1 :\ W_{\mathbb{R}} &\longrightarrow \mathrm{SL}(2,\mathbb{C})\,, \\ re^{i\theta} &\longmapsto \begin{pmatrix} e^{il\theta} & 0 \\ 0 & e^{-il\theta} \end{pmatrix}, \\ j &\longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{54}$$

We define $\Phi_1, \Phi_2 :\ W_{\mathbb{R}} \to \mathrm{GL}(2^{2n},\mathbb{C})$ as in (51). What we have to consider is the decomposition of the representation $\Phi(w) := \Phi_1(w)\Phi_2(w)$ into irreducibles. As any representation of the real Weil group, Φ decomposes into one- and two-dimensional irreducible representations that need to be treated separately.

Since $\pi_{1,v}$ is a discrete series representation, we have $\phi_1(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G_1$. Thus $\Phi_1(j)$ inverts the weights for the action of G_1 on W. In other words,

$$\Phi_1(j) \ \text{ maps } \ W_{j,r} \ \text{ to } \ W_{2n-j,r}\,. \tag{55}$$

Thus we see that the two-dimensional constituents of Φ_1 are spanned by a vector in $W_{j,r}$ and a vector in $W_{j,2n-r}$ for $j \neq n$, while the one-dimensional constituents are contained in W_n. The latter one are either trivial representations or sgn, since W_n is the weight–0 space for the action of G_1 (here sgn is the character of $W_{\mathbb{R}}$ that is trivial on $\mathbb{C}^*$ and has $\mathrm{sgn}(j) = -1$).

Recall that Φ_2 acts "horizontally". Because $\phi_2(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G_2$, we see that

$$\Phi_2(j) \ \text{ maps } \ W_{j,r} \ \text{ to } \ W_{j,-r}\,. \tag{56}$$

Various cases and subcases have to be considered, and they will be numbered in a systematic way.

1. First we shall consider some non-zero vector $v_1 \in W_{j,r}$ for $\underline{j < n}$.

1.1. Also assume that $\underline{r < 0}$ and define $v_1' := \Phi_1(j)v_1$, $v_2 := \Phi_2(j)v_1$, $v_2' := \Phi_2(j)v_1'$. By (55) and (56), these vectors are arranged as follows:

$$\begin{tikzcd}
W_{j,r} \ni v_1 \arrow[d, dotted, no head] \arrow[dr, no head] & v_2 \in W_{j,-r} \arrow[d, dotted, no head] \arrow[dl, no head] \\
W_{2n-j,r} \ni v_1' & v_2' \in W_{2n-j,-r}
\end{tikzcd}$$

What we shall do is compare the two actions Φ_1 and Φ_2 of $W_{\mathbb{R}}$ on the four-dimensional vector space $\langle v_1, v_2, v_1', v_2' \rangle$. Without the action of Φ_2 the span $\langle v_1, v_1' \rangle$ (and also $\langle v_2, v_2' \rangle$) would be an irreducible constituent of W. In terms of matrices, the representation of $W_{\mathbb{R}}$ on this space would be given by

$$\tau_1 : re^{i\theta} \mapsto \begin{pmatrix} e^{il(j-n)\theta} & 0 \\ 0 & e^{il(n-j)\theta} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & (-1)^{l(n-j)} \\ 1 & 0 \end{pmatrix} \tag{57}$$

(recall from the previous section that W_j is a weight space for the action of $G_1 = \mathrm{SL}(2, \mathbb{C})$ for the weight $j - n$). In general, the L-factor associated to a representation φ of $W_{\mathbb{R}}$ of the form

$$re^{i\theta} \mapsto \begin{pmatrix} r^{2t}e^{im\theta} & \\ & r^{2t}e^{-im\theta} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & (-1)^m \\ 1 & 0 \end{pmatrix}$$

is given by $L(s, \varphi) = 2(2\pi)^{-(s+t+|m|/2)}\Gamma(s + t + |m|/2)$, see [Kn2], (3.6). Thus

$$L(s, \tau_1) = 2(2\pi)^{-(s+l(n-j)/2)}\Gamma(s + l(n - j)/2)\,. \tag{58}$$

But now that $\pi_{2,v}$ is a discrete series representation and $\Phi_2(j)$ acts non-trivially, we see that $\langle v_1, v_2' \rangle$ and $\langle v_2, v_1' \rangle$ are the irreducible constituents of our four-dimensional space.

1.1.1. First consider the representation τ_2 on $\langle v_1, v_2' \rangle$. Explicitly, it is given by

$$\tau_2 : re^{i\theta} \mapsto \begin{pmatrix} e^{i(l(j-n)+r)\theta} & 0 \\ 0 & e^{-i(l(j-n)+r)\theta} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & (-1)^{l(j-n)+r} \\ 1 & 0 \end{pmatrix} \tag{59}$$

(note that $\Phi_2(re^{i\theta})$ acts on $W_{j,r}$ by multiplication with $e^{ir\theta}$). Since $l(j - n) + r < 0$, the associated L-factor is

$$L(s, \tau_2) = 2(2\pi)^{-(s+(l(n-j)-r)/2)}\Gamma(s + (l(n - j) - r)/2) = L(s - r/2, \tau_1)\,. \tag{60}$$

Therefore *the contribution of τ_2 to the L-factor is the same as that of τ_1 except for a shift in the argument by $r/2$.* Note that we computed the same shift of $r/2$ for finite places and $\pi_{2,v}$ trivial, see (52).

1.1.2. Similarly we now compare the L-factor for the constituent $\langle v_2, v_1' \rangle$ (Φ-action) to the one for $\langle v_2, v_2' \rangle$ (Φ_1-action). The latter representation is again τ_1 as in (57), while the former one is

$$\tau_3 : re^{i\theta} \mapsto \begin{pmatrix} e^{i(l(j-n)-r)\theta} & 0 \\ 0 & e^{-i(l(j-n)-r)\theta} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & (-1)^{l(j-n)-r} \\ 1 & 0 \end{pmatrix}. \tag{61}$$

Unlike in 1.1.1, the sign of the exponents is not determined, and we have to distinguish three cases.

1.1.2.1. Assume that $l(n - j) + r > 0$. In this case we have

$$L(s,\tau_3)=2(2\pi)^{-(s+(l(n-j)+r)/2)}\Gamma(s+(l(n-j)+r)/2)=L(s+r/2,\tau_1)\,,$$

and we get a similar effect as in (60).

1.1.2.2. Assume that $l(n-j)+r=0$. Then our representation (61) becomes $\tau_3=\mathrm{triv}\oplus\mathrm{sgn}$, the sum of two one-dimensional representations. By [Kn2], (3.6), the associated L-factor is

$$L(s,\tau_3)=\pi^{-s/2}\Gamma(s/2)\pi^{-(s+1)/2}\Gamma((s+1)/2)=2(2\pi)^{-s}\Gamma(s)=L(s+r/2,\tau_1)\,.$$

Hence this case is analogous to 1.1.2.1.

1.1.2.3. Assume that $l(n-j)+r<0$. Then

$$L(s,\tau_3)=2(2\pi)^{-(s-(l(n-j)+r)/2)}\Gamma(s-(l(n-j)+r)/2)=L(s-l(n-j)-r/2,\tau_1)\,.$$

But also in this case we would like to have a contribution like $L(s+r/2,\tau_1)$. Therefore we write

$$L(s,\tau_3)=L(s+r/2,\tau_1)\,\delta_r^{(j)}(s)$$

with the correcting factor

$$\delta_r^{(j)}(s)=\frac{L(s-l(n-j)-r/2,\tau_1)}{L(s+r/2,\tau_1)}\,. \tag{62}$$

Note that our notation does not reflect the fact that $\delta_r^{(j)}(s)$ does depend on $\pi_{1,v}$ (because of the l). However, using the property $s\Gamma(s)=\Gamma(s+1)$, it is immediately computed that

$$\delta_r^{(j)}(s)=(2\pi)^{-m}\prod_{i=1}^{m}\left(s+\frac{m}{2}-i\right),\quad m=l(j-n)-r>0\,. \tag{63}$$

Thus $\delta_r^{(j)}(s)$ is just a polynomial. Moreover, it has the "functional equation"

$$\frac{\delta_r^{(j)}(s)}{\delta_r^{(j)}(1-s)}=(-1)^{l(j-n)-r}\,. \tag{64}$$

1.2. Now consider the case $\underline{r=0}$ and some $v\in W_{j,0}$ (we are still assuming $j<n$). Put $v':=\Phi_1(j)v$, so that the representation τ_1 of $W_{\mathbb{R}}$ on $\langle v,v'\rangle$, neglecting the Φ_2-action, is explicitly given by (57). We may assume that v and v' belong to irreducible "horizontal" G_2-representations. Then Φ_2 acts on $\langle v\rangle$ and $\langle v'\rangle$ either trivially or through sgn, the non-trivial character of $W_{\mathbb{R}}$ sending $\mathbb{C}^*$ to 1 and j to -1. Therefore the representation Φ on $\langle v,v'\rangle$ is either as in (57) again, or differs from that by taking the negative of the matrix that j is mapped to. In either case the representation is equivalent to τ_1, so that the L-factor is given by (58).

To summarize the discussion in 1.1 and 1.2, let $j<n$ and $r\le 0$ and define polynomials $\delta_r^{(j)}$ by (63) if $l(j-n)-r>0$, and $\delta_r^{(j)}(s)=1$ otherwise. Define further

$$\delta^{(j)}(s) = \prod_{\substack{r=j(j-2n)\\ \text{(step 2)}}}^{p} \delta_r^{(j)}(s)^{\alpha(r,j,n)}\,, \tag{65}$$

where $p = 0$ if j is even and $p = -1$ if j is odd. Then the contribution of the space $W_j \oplus W_{2n-j}$ to the archimedean L-factor $L_v(s,\ \Pi_v,\ \rho)$ is given by

$$\delta^{(j)}(s) \prod_{\substack{r=j(j-2n)\\ \text{(step 2)}}}^{j(2n-j)} L(s + r/2,\ \tau_1)^{\alpha(r,j,n)} \tag{66}$$

with $L(s, \tau_1)$ as in (58). The polynomial $\delta^{(j)}(s)$ has a functional equation $\delta^{(j)}(s) = \pm\delta^{(j)}(1 - s)$.

2. Next we consider the contribution to $L_v(s,\ \Pi_v,\ \rho)$ of the irreducible constituents of W which are contained in W_n, i.e., we are now in the case $\underline{j = n}$.

2.1. Assume that $\underline{r < 0}$, let $v_1 \in W_{n,r}$ be some non-zero vector and put $v_2 := \Phi_2(j)v_1$. By (56), these vectors are arranged as follows:

$$W_{n,r} \ni v_1 \text{——} v_2 \in W_{n,-r}\ .$$

Without the action of Φ_2 both $\langle v_1\rangle$ and $\langle v_2\rangle$ would be invariant subspaces on which $W_{\mathbb{R}}$ acts by $\mathrm{sgn}^{\varepsilon}$, $\varepsilon \in \{0, 1\}$. The L-factor for the representation $\tau_1 = \mathrm{sgn}^{\varepsilon}$ is

$$L(s, \tau_1) = \pi^{-(s+\varepsilon)/2}\,\Gamma((s + \varepsilon)/2)\,. \tag{67}$$

But with Φ_2 acting non-trivially we now have an irreducible two-dimensional representation τ_2 of $W_{\mathbb{R}}$ on $\langle v_1, v_2\rangle$ which is explicitly given as follows:

$$\tau_2 :\ re^{i\theta} \mapsto \begin{pmatrix} e^{ir\theta} & 0\\ 0 & e^{-ir\theta}\end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & (1-2\varepsilon)(-1)^r\\ 1-2\varepsilon & 0\end{pmatrix}. \tag{68}$$

The L-factor associated to this representation is

$$L(s, \tau_2) = 2(2\pi)^{-s+r/2}\Gamma(s - r/2) \tag{69}$$

(independent of ε). On the other hand, what we would like to have is a contribution to $L_v(s,\ \Pi_v,\ \rho)$ that fits with the contributions (66) to yield an L-factor as in (52) (so that the full L-function is a product of *completed* symmetric power L-functions for π_1, times a polynomial). Such a contribution would come from the reducible representation

$$re^{i\theta} \mapsto \begin{pmatrix} r^{r/2} & 0\\ 0 & r^{-r/2}\end{pmatrix}, \quad j \mapsto \begin{pmatrix} 1-2\varepsilon & 0\\ 0 & 1-2\varepsilon\end{pmatrix}, \tag{70}$$

because its L-factor $L(s + r/2, \tau_1)L(s - r/2, \tau_1)$ is a product of L-factors of the $W_{\mathbb{R}}$-representation $\mathrm{sgn}^{\varepsilon}$ with the appropriate shift of $\pm r/2$. We therefore write our L-factor as

$$L(s, \tau_2) = L(s + r/2, \tau_1)L(s - r/2, \tau_1)\delta_{r,\varepsilon}^{(n)}(s) \tag{71}$$

with

$$\delta^{(n)}_{r,\varepsilon}(s) := \frac{L(s,\tau_2)}{L(s+r/2,\tau_1)L(s-r/2,\tau_1)}\,. \tag{72}$$

We will study the factors $\delta^{(n)}_{r,\varepsilon}$ in the next section. For odd n they will be polynomials, similar to the factors defined in (62).

2.2. In case n is even there is also a contribution from the space $W_{n,0}$. The operators $\Phi_1(j)$ and $\Phi_2(j)$ define commuting, involutive endomorphisms of this space, and accordingly we decompose it as

$$W_{n,0} = W_{n,0,0,0} \oplus W_{n,0,1,0} \oplus W_{n,0,0,1} \oplus W_{n,0,1,1}\,, \tag{73}$$

where $\Phi_i(j)$ acts on $W_{n,0,\varepsilon_1,\varepsilon_2}$ with eigenvalue $1-2\varepsilon_i$. Without the action of Φ_2 present $W_{\mathbb{R}}$ would act as $\mathrm{sgn}^{\varepsilon}$ on $W_{n,0,\varepsilon,1}$; the contribution to the L-factor would be

$$(\pi^{-(s+\varepsilon)/2}\Gamma((s+\varepsilon)/2))^{\dim(W_{n,0,\varepsilon,1})}\,.$$

But with $\Phi_2(j)$ acting non-trivially, the actual contribution is

$$(\pi^{-(s+1-\varepsilon)/2}\Gamma((s+1-\varepsilon)/2))^{\dim(W_{n,0,\varepsilon,1})}\,.$$

Similarly as in (72) we introduce factors

$$\delta^{(n)}_{0,0}(s) = \frac{\pi^{-(s+1)/2}\Gamma((s+1)/2)}{\pi^{-s/2}\Gamma(s/2)}\,,\qquad \delta^{(n)}_{0,1}(s) = \frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{-(s+1)/2}\Gamma((s+1)/2)}\,, \tag{74}$$

so that we can write the contribution to the L-factor as

$$\begin{aligned}(\pi^{-(s+\varepsilon)/2}&\Gamma((s+\varepsilon)/2))^{\dim(W_{n,0,\varepsilon,1})}\delta^{(n)}_{0,\varepsilon}(s)^{\dim(W_{n,0,\varepsilon,1})}.\\ &= L(s,\tau_1)^{\dim(W_{n,0,\varepsilon,1})}\delta^{(n)}_{0,\varepsilon}(s)^{\dim(W_{n,0,\varepsilon,1})}\,.\end{aligned}$$

Now what is the total contribution of the space W_n to $L_v(s,\ \Pi_v,\ \rho)$? Define a factor $\delta^{(n)}(s)$ as follows. If n is odd, then

$$\delta^{(n)}(s) = \prod_{\substack{r=-n^2\\ \text{(step 2)}}}^{-1}\ \prod_{\substack{j=1\\ \text{(step 2)}}}^{n} \delta^{(n)}_{r,\varepsilon(j)}(s)^{\beta(r,j,n)} \tag{75}$$

with

$$\varepsilon(j) = \begin{cases} 0 & \text{if } n-j\equiv 0 \mod 4\,,\\ 1 & \text{if } n-j\equiv 2 \mod 4\,. \end{cases} \tag{76}$$

and $\delta^{(n)}_{r,\varepsilon}(s)$ as in (72) and $\beta(r,\,j,\,n)$ as in (47). If n is even, then

$$\delta^{(n)}(s) = \Bigg(\prod_{\substack{r=-n^2\\ \text{(step 2)}}}^{-2}\ \prod_{\substack{j=0\\ \text{(step 2)}}}^{n} \delta^{(n)}_{r,\varepsilon(j)}(s)^{\beta(r,j,n)}\Bigg)\delta^{(n)}_{0,0}(s)^{\dim(W_{n,0,0,1})}\cdot\delta^{(n)}_{0,1}(s)^{\dim(W_{n,0,1,1})}\,, \tag{77}$$

with $\delta^{(n)}_{0,\varepsilon}(s)$ as in (74) and the spaces $W_{n,0,\varepsilon,1}$ as in (73). Then the total contribution of W_n to the L-factor is

$$\delta^{(n)}(s) \prod_{\substack{r=-n^2 \\ \text{(step 2)}}}^{n^2} L(s+r/2,\ \tau_1)^{\alpha(r,n,n)}\,, \tag{78}$$

where $L(s,\tau_1)$ is as in (67). To see why this is so, we consider for each negative weight $r \in \{-n^2, -n^2+2, \cdots\}$ the space $W_{n,r}$ which contributes to $\delta^{(n)}(s)$. For a suitable basis vector $v \in W_{n,r}$ we just have to see whether it makes a contribution $\delta^{(n)}_{r,0}(s)$ or $\delta^{(n)}_{r,1}(s)$. This depends on whether $\Phi_1(j)$ acts trivially or non-trivially on v. This in turn depends on whether v lies in an irreducible "vertical" G_1–representation whose dimension is $\equiv 1 \mod 4$ or $\equiv 3 \mod 4$. But by definition, for fixed r, there are $\beta(r,j,n)$ "vertical" representations of dimension $n-j+1$.

In case n is even there is an additional contribution to $\delta^{(n)}(s)$ from the spaces $W_{n,0,\varepsilon,1}$, as explained in 2.2. Note that there is no contribution to $\delta^{(n)}(s)$ (but to the factors $L(s,\tau_1)$) from the spaces $W_{n,0,\varepsilon,0}$, on which $\Phi_2(j)$ acts trivially.

It will be proved in the next section that the factors $\delta^{(n)}$ are in fact polynomials (just like the $\delta^{(j)}$ for $j<n$), even though the individual factors $\delta^{(n)}_{r,\varepsilon}$ are not if n is even. Note that, in contrast to the $\delta^{(j)}$ for $j<n$, the factors $\delta^{(n)}$ only depend on n, not on π_1.

Putting everything together we can now state the real counterpart to Proposition 2.2.1.

2.2.2. PROPOSITION. *Let v be a real place of the number field F. Assume that $\pi_{1,v}$ is a discrete series representation of* $\mathrm{PGL}(2,\mathbb{R})$ *and that $\pi_{2,v}$ is a discrete series representation of* $\mathrm{SO}(2n+1,\mathbb{R})$ *which is a constituent of the induced representation considered in Proposition* 1.1.1. *Let Π_v be a representation of* $\mathrm{PGSp}(4n,\mathbb{R})$ *which is a local Langlands lift of the representation $\pi_{1,v}\otimes\pi_{2,v}$ on* $\mathrm{PGL}(2,\mathbb{R})\times\mathrm{SO}(2n+1,\mathbb{R})$ *under the embedding* (16) *of L-groups. Let ρ be the 2^{2n}–dimensional spin representation of* $\mathrm{Spin}(4n+1,\mathbb{C}) = {}^L\mathrm{PGSp}(4n,\mathbb{R})^0$. *Then*

$$L_v(s,\ \Pi_v,\ \rho) = \delta(s) \prod_{j=0}^{n} \prod_{\substack{r=j(j-2n) \\ \text{(step 2)}}}^{j(2n-j)} L(s+r/2,\ \pi_{1,v},\ \mathrm{Sym}^{n-j})^{\beta(r,j,n)}\,. \tag{79}$$

The factor $\delta(s)$ is a polynomial (*depending on l*) *which is given as*

$$\delta(s) = \prod_{j=0}^{n} \delta^{(j)}(s)$$

with polynomials $\delta^{(j)}(s)$ defined in (65), (75) *and* (77). *We have*

$$\delta(s) = \pm\delta(1-s)$$

for some sign $\pm$.

Proof. Since $W = (W_0 \oplus W_{2n}) \oplus \cdots \oplus (W_{n-1}\oplus W_{n+1}) \oplus W_n$, we just have to multiply the contributions (66) and (78). We arranged those in such a way that all the factors $L(s+r/2,\ \tau_1)^{\alpha(r,j,n)}$ combine to give the shifted symmetric power L-functions (coming from the "vertical" representations of $G_1 = \mathrm{SL}(2, \mathbb{C})$). The fact that the functions $\delta^{(j)}(s)$ are polynomials also for $j = n$, and that they have a functional equation, will be proved in the next section. ∎

REMARK. More information on the archimedean Euler factors can be found in [Sch].

2.3. Properties of the δ-factor

To derive the analytic properties of the full spin L-function we need more precise information about the factor $\delta(s)$ appearing in Proposition 2.2.2. In particular we need to compute $\delta(s)/\delta(1-s)$, because this is the contribution of δ to the global ε-factor. Even though $\delta(s)$ can be quite a large quotient of Γ-functions, we shall see that the quotient $\delta(s)/\delta(1-s)$ is just a sign.

The factors $\delta^{(j)}$ for $j < n$ are not mysterious. They are polynomials by definition (see (65)), and they have a functional equation by (64). The sign in the functional equation can be expressed by l, j, n and the numbers $\alpha(r, j, n)$, but we shall refrain from doing so.

In this section we shall prove analogous properties for the factors $\delta^{(n)}$, where they are not obvious.

2.3.1. LEMMA. *Assume $r = -(2m+1)$ for some non-negative integer m. Then*:

i) $\displaystyle \delta^{(n)}_{r,\varepsilon}(s) = (2\pi)^{-(m-1)-\varepsilon} \prod_{i=0}^{m-\varepsilon} \left(s - (m-\varepsilon) - \frac{1}{2} + 2i \right).$

ii) $\displaystyle \frac{\delta^{(n)}_{r,\varepsilon}(s)}{\delta^{(n)}_{r,\varepsilon}(1-s)} = (-1)^{m+1-\varepsilon}.$

Proof. By (69) we have

$$L(s, \tau_2) = 2(2\pi)^{-s-m-1/2}\, \Gamma(s + m + 1/2). \tag{80}$$

The L-factor for the representation (70) is

$$L(s, \tau_3) = \pi^{-(s+\varepsilon)} \Gamma(s') \Gamma(s' + m + 1/2), \quad s' = \frac{s + \varepsilon - m}{2} - \frac{1}{4}, \tag{81}$$

by [Kn2] (3.6). Now (repeated) application of the formulas $\Gamma(s+1) = s\Gamma(s)$ and

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\, \pi^{1/2}\, \Gamma(s) \tag{82}$$

yields

$$L(s, \tau_3) = 2^{1-m} (2\pi)^{-s-\varepsilon+1/2} \Gamma\left(s + \varepsilon + m - \frac{1}{2}\right) \prod_{i=0}^{m-1} (s' + i)^{-1}.$$

It is then easy to compute the quotient $L(s, \tau_2)/L(s, \tau_3)$ and obtain formula i). Formula ii) follows immediately from i). ∎

2.3.2. LEMMA, *Assume $r = -2m$ for some non-negative integer m. Then*:

i) $\displaystyle \delta^{(n)}_{r,\varepsilon}(s) = 2^{1-s-m}\, \pi^{\varepsilon-m} \frac{\Gamma(s+m)}{\Gamma((s+\varepsilon-m)/2)\, \Gamma((s+\varepsilon+m)/2)}$ *if $m > 0$.*

ii) $$\frac{\delta_{r,0}^{(n)}(s)}{\delta_{r,0}^{(n)}(1-s)} = \begin{cases} \tan\left(\dfrac{\pi s}{2}\right) & \textit{for } m \textit{ even}, \\ \tan\left(\dfrac{\pi s}{2}\right)^{-1} & \textit{for } m \textit{ odd}. \end{cases}$$

$$\frac{\delta_{r,1}^{(n)}(s)}{\delta_{r,1}^{(n)}(1-s)} = \begin{cases} \tan\left(\dfrac{\pi s}{2}\right)^{-1} & \textit{for } m \textit{ even}, \\ \tan\left(\dfrac{\pi s}{2}\right) & \textit{for } m \textit{ odd}. \end{cases}$$

Proof. By (69) we have

$$L(s, \tau_2) = 2(2\pi)^{-s-m}\Gamma(s+m)\,. \tag{83}$$

The L-factor for the representation (70) is

$$L(s, \tau_3) = \pi^{-(s+\varepsilon)}\Gamma\left(\frac{s+\varepsilon-m}{2}\right)\Gamma\left(\frac{s+\varepsilon+m}{2}\right). \tag{84}$$

Thus we get i). We shall compute ii) only for $\varepsilon = 0$ and $m > 0$, the other cases being similar. From i) we get

$$\frac{\delta_{r,0}^{(n)}(s)}{\delta_{r,0}^{(n)}(1-s)} = 2^{1-2s}\frac{\Gamma(s+m)\,\Gamma((1-s-m)/2)\,\Gamma((1-s+m)/2)}{\Gamma((s-m)/2)\,\Gamma((s+m)/2)\,\Gamma(1-s-m)}\,. \tag{85}$$

Then, using the formulas

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{86}$$

and (82), it is easy to compute

$$\frac{\Gamma((1-s-m)/2)}{\Gamma((s+m)/2)} = 2^{s+m-1}\pi^{1/2}\frac{1}{\sin(\pi(1-s-m)/2)\,\Gamma(s+m)}\,.$$

Substituting this and the analogous formula with $-m$ instead of m into (85), and using (86) one more time, one obtains

$$\frac{\delta_{r,0}^{(n)}(s)}{\delta_{r,0}^{(n)}(1-s)} = 2^{1-2s}\frac{\sin(\pi(s-m))}{\sin(\pi(1-s-m)/2)\,\sin(\pi(1-s+m)/2)}\,. \tag{87}$$

Now some easy manipulations give the result. ∎

2.3.3. Proposition. *Let $\delta^{(n)}(s)$ be the factor defined in* (75) *and* (77).

i) *Assume n is odd. Then*

$$\frac{\delta^{(n)}(s)}{\delta^{(n)}(1-s)} = \pm 1$$

for some sign ± 1.

ii) *Assume n is even. Then*

$$\frac{\delta^{(n)}(s)}{\delta^{(n)}(1-s)} = 1\,.$$

iii) $\delta^{(n)}(s)$ *is a polynomial with zeros precisely at the points*

$$s \in \Big\{ -\frac{n^2}{2}+1,\ -\frac{n^2}{2}+2,\ \cdots,\ \frac{n^2}{2} \Big\}\,.$$

Proof. i) It is clear by Lemma 2.3.1 and the definition in (75) that $\delta^{(n)}(s)/\delta^{(n)}(1-s)$ is just a sign.

ii) By Lemma 2.3.2 the quotient $\delta^{(n)}(s)/\delta^{(n)}(1-s)$ is a power of $\tan(\pi s/2)$, and what we have to see is that the exponent is actually 0. We shall use notation as before. Let $W' \subset W_n$ be an irreducible subspace under the action of $G_2 = \mathrm{SL}(2,\mathbb{C})$. We shall show that the total contribution of W' to $\delta^{(n)}(s)/\delta^{(n)}(1-s)$ is 1. Let

$$W' = \bigoplus_{\substack{i=-i_0 \\ \text{(step 2)}}}^{i_0} W'_i$$

be the decomposition of W' into weight spaces (i_0 is necessarily even). Let $w'_i \in W'_i$ be a spanning vector. We may and shall assume that each w'_i lies in an irreducible "vertical" representation W''_i of $G_1 = \mathrm{SL}(2,\mathbb{C})$. Since the actions of G_1 and G_2 commute, the dimension of W''_i is independent of i. The contribution to $\delta^{(n)}(s)/\delta^{(n)}(1-s)$ from $\langle w_{-i}, w_i\rangle$ (including $i=0$) is therefore

$$\frac{\delta^{(n)}_{i,\varepsilon}(s)}{\delta^{(n)}_{i,\varepsilon}(1-s)}\,,$$

with ε independent of i.

Case 1: $\dim(W') \in \{1,5,9,\cdots\}$.

In this case the contribution from $\langle w'_{-i_0}, w'_{i_0}\rangle$ cancels that of $\langle w'_{-i_0+2}, w'_{i_0-2}\rangle$, by Lemma 2.3.2 ii). All other contributions from two-dimensional subspaces also cancel in a pairwise manner, so that we are left with the contribution of W'_0. But $\Phi_2(j)$ acts trivially on W'_0 for the dimensions of W' considered, and so W'_0 does not contribute to $\delta^{(n)}(s)$ at all.

Case 2: $\dim(W') \in \{3,7,11,\cdots\}$.

In this case not all the contributions from two-dimensional subspaces cancel; we are left with that of $\langle w'_{-2}, w'_2\rangle$, which is

$$\frac{\delta^{(n)}_{-2,\varepsilon}(s)}{\delta^{(n)}_{-2,\varepsilon}(1-s)} = \tan\Big(\frac{\pi s}{2}\Big)^{2\varepsilon-1}\,.$$

But there is also a contribution from W_0', since $\Phi_2(j)$ acts non-trivially on this space for the dimensions considered. This contribution is

$$\frac{\delta_{0,\varepsilon}^{(n)}(s)}{\delta_{0,\varepsilon}^{(n)}(1-s)} = \tan\left(\frac{\pi s}{2}\right)^{1-2\varepsilon},$$

and thus cancels that of $\langle w_{-2}', w_2' \rangle$.

iii) If n is odd then it is trivially true from Lemma 2.3.1 that $\delta^{(n)}(s)$ is a polynomial. By the same lemma, the zeros of $\delta_{r,\varepsilon}^{(n)}$ are at the points

$$s \in \left\{ \frac{r}{2} + \varepsilon + 1,\ \frac{r}{2} + \varepsilon + 3,\ \cdots,\ -\frac{r}{2} - \varepsilon \right\}.$$

The index r runs through $\{-n^2, -n^2+2, \cdots, -1\}$. Since for $r = -n^2$ and $r = -n^2+2$ we necessarily have $\varepsilon = 0$, the assertion about the zeros of $\delta^{(n)}(s)$ is obvious.

Now assume that n is even. To locate possible poles of $\delta^{(n)}(s)$ note that by Lemma 2.3.2 i) the factor $\delta_{r,\varepsilon}^{(n)}(s)$ has no poles in $\mathrm{Re}(s) > 1$, since the Γ-function has no zeros and poles only at $s \in \{0, -1, -2, \cdots\}$. The same remains true for $\delta_{0,\varepsilon}^{(n)}(s)$ as defined in (74). Thus $\delta^{(n)}(s)$ is holomorphic in $\mathrm{Re}(s) > 1$. By ii), it is also holomorphic in $\mathrm{Re}(s) < 0$. It follows that $\delta^{(n)}(s)$ has only finitely many poles.

Similarly as in the proof of ii) we shall now consider neighboring pairs of factors $\delta_{r,\varepsilon}^{(n)}$. For such a pair we compute, using twice formula (82),

$$\begin{aligned}
&\delta_{r,\varepsilon}^{(n)}(s)\delta_{r+2,\varepsilon}^{(n)}(s)\\
&\quad= \frac{2^{3-2s-2m}\,\pi^{2\varepsilon+1-2m}\,\Gamma(s+m)\,\Gamma(s+m-1)}{\Gamma((s+\varepsilon-m)/2)\,\Gamma((s+\varepsilon+m)/2)\,\Gamma((s+\varepsilon-m+1)/2)\,\Gamma((s+\varepsilon+m-1)/2)}\\
&\quad= (2\pi)^{2(\varepsilon-m)}\frac{\Gamma(s+m-\varepsilon)}{\Gamma(s-m+\varepsilon)}\\
&\quad= (2\pi)^{2(\varepsilon-m)}\prod_{i=0}^{2(m-\varepsilon)-1}(s-m+\varepsilon+i).
\end{aligned}$$

This remains true for $r = -2$, even though the calculation is slightly different. Thus the product of all such pairs is just a polynomial. We have proved that

$$\delta^{(n)}(s) = \text{Polynomial} \cdot \delta_{0,0}^{(n)}(s)^r$$

for some integer r. But $\delta_{0,0}^{(n)}(s)$ has poles at $s \in \{-1, -3, -5, \cdots\}$ and zeros at $s \in \{0, -2, -4, \cdots\}$, and we just saw that $\delta^{(n)}(s)$ has only finitely many poles. It follows that $r = 0$ and $\delta^{(n)}(s)$ is a polynomial.

Since the Γ-function never vanishes, the only zeros of $\delta_{r,\varepsilon}^{(n)}$ in Lemma 2.3.2 i) come from the poles of the denominator. The zeros of $\delta_{r,\varepsilon}^{(n)}$ in $\mathrm{Re}(s) > 0$ are therefore located at

$$s \in \left\{ -\frac{r}{2} - \varepsilon,\ -\frac{r}{2} - \varepsilon - 2,\ \cdots \right\}.$$

Noting that for $r = -n^2$ and $r = -n^2 + 2$ we have $\varepsilon = 0$, it follows that $\delta^{(n)}(s)$ has zeros at $s \in \{n^2/2,\ n^2/2 - 1, \cdots\}$ in $\mathrm{Re}(s) > 0$. By the symmetry expressed in ii) we get the assertion about the zeros of $\delta^{(n)}(s)$. ∎

Amongst the polynomials $\delta^{(j)}$ the most interesting ones are those for $j = n$, which are *universal* (independent of π_1). To compute the orders of the zeros of the polynomials $\delta^{(n)}$ by using a computer, it is convenient to have these orders expressed by the numbers $\beta(r, j, n)$, as in the following lemma. Note that by the functional equation in Proposition 2.3.3 it is enough to know the zeros for $\mathrm{Re}(s) \geq 1/2$. The appendix gives some numerical examples for $n \leq 6$.

2.3.4. LEMMA. *Let $\varepsilon = 0$ if n is even and $\varepsilon = 1$ if n is odd. Let $s = r_0/2$ with some*

$$r_0 \in \{2 - \varepsilon,\ 4 - \varepsilon,\ \cdots,\ n^2\}.$$

Then the polynomial $\delta^{(n)}$ has a zero at s of order

$$\mathrm{ord}_s(\delta^{(n)}) = \sum_{\substack{j=\varepsilon \\ \text{(step 2)}}}^{n} \sum_{\substack{r=r_0+2\varepsilon(j) \\ \text{(step 2)} \\ r \equiv r_0+2\varepsilon(j) \bmod 4}}^{n^2} \beta(r, j, n).$$

Here $\varepsilon(j)$ is as in (76), *namely*

$$\varepsilon(j) = \begin{cases} 0 & \text{if } n - j \equiv 0 \mod 4, \\ 1 & \text{if } n - j \equiv 2 \mod 4. \end{cases}$$

Proof. Assume that n is even; the argument for odd n is similar but even easier. By Proposition 2.2.2 we have

$$\delta^{(n)}(s) = \left(\prod_{\substack{r=-n^2 \\ \text{(step 2)}}}^{-2} \prod_{\substack{j=0 \\ \text{(step 2)}}}^{n} \delta^{(n)}_{r,\varepsilon(j)}(s)^{\beta(r,j,n)} \right) \delta^{(n)}_{0,0}(s)^r$$

for some integer r. As we saw in the previous proof, a factor $\delta^{(n)}_{0,0}$ contributes neither a zero nor a pole in $\mathrm{Re}(s) > 0$, so we need only count the zeros coming from the terms $\delta^{(n)}_{r,\varepsilon(j)}$. As noted before, the zeros of $\delta^{(n)}_{r,\varepsilon(j)}$ in $\mathrm{Re}(s) > 0$ are simple and are located at the points

$$\left\{ -\frac{r}{2} - \varepsilon(j),\ -\frac{r}{2} - \varepsilon(j) - 2,\ \cdots \right\}.$$

Thus $\delta^{(n)}_{r,\varepsilon(j)}$ contributes a zero at $s = r_0/2$ if and only if

$$\frac{r_0}{2} \le -\frac{r}{2} - \varepsilon(j) \quad \text{and} \quad \frac{r_0}{2} \equiv -\frac{r}{2} - \varepsilon(j) \quad \mathrm{mod}\ 2\,.$$

It follows that

$$\begin{aligned}\mathrm{ord}_s(\delta^{(n)}) &= \sum_{\substack{r=-n^2 \\ (\text{step } 2)}}^{-2} \sum_{\substack{j=0 \\ (\text{step } 2) \\ r_0 \le -r-2\varepsilon(j) \\ r_0 \equiv -r-2\varepsilon(j) \bmod 4}}^{n} \beta(r, j, n) \\ &= \sum_{\substack{j=0 \\ (\text{step } 2)}}^{n} \sum_{\substack{r=-n^2 \\ (\text{step } 2) \\ r \equiv -r_0-2\varepsilon(j) \bmod 4}}^{-r_0-2\varepsilon(j)} \beta(r, j, n) = \sum_{\substack{j=0 \\ (\text{step } 2)}}^{n} \sum_{\substack{r=r_0+2\varepsilon(j) \\ (\text{step } 2) \\ r \equiv r_0+2\varepsilon(j) \bmod 4}}^{n^2} \beta(r, j, n)\,,\end{aligned}$$

as claimed. ∎

The lowest-dimensional examples are $n = 1$ (the Saito–Kurokawa case), where

$$\delta^{(1)}(s) = 2\pi(s - 1/2)\,, \tag{88}$$

and $n = 2$, where

$$\delta^{(2)}(s) = (2\pi)^{-4}(s+1)s(s-1)(s-2)\,. \tag{89}$$

For later use we shall also say something about the zeros of $\delta^{(j)}$ for $j < n$. Fix such a j, and consider a negative number $r \in \{j(j-2n), j(j-2n)+2, \cdots\}$ such that $m := l(j-n) - r > 0$. Obviously, from (63), the zeros of $\delta_r^{(j)}(s)$ are located at the points

$$s \in \left\{1 - \frac{m}{2},\ 2 - \frac{m}{2},\ \cdots,\ \frac{m}{2}\right\}.$$

In view of definition (65), the rightmost zero of $\delta^{(j)}$ is therefore located at the point

$$s = \frac{l(j-n) - j(j-2n)}{2} = \frac{1}{2}(-j^2 + (l+2n)j - ln)\,.$$

As a function of j, the polynomial $-j^2+(l+2n)j-ln$ takes its maximum at $j = \frac{1}{2}(l+2n)$, which is greater than $n-1$. Therefore we get the rightmost zero of all the polynomials $\delta^{(j)}$ for $j = n-1$, and it is located at $s = (n^2 - l - 1)/2$. This shows:

$$\text{All the zeros of } \prod_{j=0}^{n-1} \delta^{(j)}(s) \text{ are contained in } [(-n^2+l+3)/2,\ (n^2-l-1)/2]\,. \tag{90}$$

This fact will be used in Corollary 2.4.2 to show that the full L-function $L(s, \Pi, \rho)$ has poles (assuming the expected analytic properties of symmetric power L-functions).

2.4. The spin L-function

We are now ready to state our main result which expresses the (completed) spin L-function of an Ikeda lift in terms of symmetric power L-functions of the original modular form. In the following theorem we shall not use the classical normalization of L-functions but the standard one which makes $\mathrm{Re}(s) = 1/2$ the critical line.

2.4.1. THEOREM. *Let $f \in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$ be a cuspidal eigenform and π its associated automorphic representation of* $\mathrm{PGL}(2,\mathbb{A})$. *Let n be a positive integer with $n \equiv k \mod 2$ and let $F \in S_{k+n}(\mathrm{Sp}(4n,\mathbb{Z}))$ be the Ikeda lift of f. Let Π be the automorphic representation of* $\mathrm{PGSp}(4n,\mathbb{A})$ *associated to F. Then, if $k > n$, the complete spin L-function of Π is given as follows*:

$$L(s,\ \Pi,\ \rho) = \delta(s) \prod_{j=0}^{n} \prod_{\substack{r=j(j-2n) \\ \text{(step 2)}}}^{j(2n-j)} L(s + r/2,\ \pi,\ \mathrm{Sym}^{n-j})^{\beta(r,j,n)}\,. \tag{91}$$

The numbers $\beta(r, j, n)$ are positive integers and are defined in (47). *The factor $\delta(s)$ is a polynomial which depends only on the weight of f. Its zeros are contained in the interval $[1 - n^2/2, n^2/2]$ and it has a functional equation $\delta(s) = \pm\delta(1 - s)$. The L-function $L(s,\ \pi,\ \mathrm{Sym}^0)$ is understood to be the completed Riemann zeta function.*

Proof. This is immediate from Propositions 2.2.1, 2.2.2 and 2.3.3, and from (90). ∎

REMARK. It is most likely that the same formula holds for $k \leq n$. However, one should be careful because it is not immediately clear what the Langlands parameters are for the lowest weight representations corresponding to holomorphic Siegel modular forms of low weight (see the remark after Lemma 1.2.1).

From this theorem one can get the standard analytic properties of the spin L-functions of the lifts to the extend that those of symmetric power L-functions for GL(2) are known. We list some of the properties of $L(s,\ \pi,\ \mathrm{Sym}^m)$ that are expected for any cuspidal automorphic representation π of $\mathrm{GL}(2,\mathbb{A})$ (see [Sh]). Let us assume that $m > 0$.

1) The Euler product defining $L(s,\pi,\mathrm{Sym}^m)$ converges for $\mathrm{Re}(s) > 1$; thus $L(s,\ \pi,\ \mathrm{Sym}^m)$ is holomorphic and non-zero in that region.
2) $L(s,\ \pi,\ \mathrm{Sym}^m)$ can be extended to a meromorphic function on all of $\mathbb{C}$.
3) There is a functional equation of the standard form
$$L(s,\ \pi,\ \mathrm{Sym}^m) = \varepsilon(s,\ \pi,\ \mathrm{Sym}^m) L(1-s,\ \hat{\pi},\ \mathrm{Sym}^m)\,,$$
where $\hat{\pi}$ is the contragredient of π.
4) $L(s,\ \pi,\ \mathrm{Sym}^m)$ never vanishes on the line $\mathrm{Re}(s) = 1$.
5) $L(s,\ \pi,\ \mathrm{Sym}^m)$ is an entire function.

Right now these properties are only known for small values of m. For example, Kim and Shahidi can now prove (2) and (3) up to $m = 9$. However, anything that is known has immediate consequences for the spin L-functions of our lifts because of formula (91). The following corollary gives an example.

2.4.2. COROLLARY. i) *Suppose that properties* 2) *and* 3) *above hold for $m \in \{1, \cdots, n\}$. Then $L(s, \Pi, \rho)$ extends to a meromorphic function on all of $\mathbb{C}$, and there is a sign $\varepsilon \in \{\pm 1\}$ such that the functional equation*

$$L(s, \Pi, \rho) = \varepsilon(s, \Pi, \rho) L(1 - s, \Pi, \rho)$$

holds, where

$$\varepsilon(s, \Pi, \rho) = \varepsilon \prod_{j=0}^{n} \prod_{\substack{r=j(j-2n) \\ \text{(step 2)}}}^{j(2n-j)} \varepsilon(s + r/2, \ \pi, \ \mathrm{Sym}^{n-j})^{\beta(r,j,n)} .$$

ii) *Suppose that moreover properties* 1), 4) *and* 5) *above hold for* $m \in \{1, \cdots, n\}$. *Then* $L(s, \Pi, \rho)$ *has simple poles at* $s = -n^2/2$ *and* $s = n^2/2 + 1$. *The only other possible poles are located at*

$$s \in \left\{ -\frac{n^2}{2} + 1, \ -\frac{n^2}{2} + 2, \ \cdots, \ \frac{n^2}{2} \right\}.$$

Proof. i) Starting from (91) this is a straightforward computation using Proposition 2.3.3. Note that every representation of PGL(2) is self-dual.

ii) Our L-function contains the factor

$$\tilde{Z}(s) := \prod_{\substack{r=-n^2 \\ \text{(step 2)}}}^{n^2} Z(s + r/2)^{\beta(r,n,n)} , \tag{92}$$

where $Z(s)$ is the completed Riemann zeta function. Since $Z(s)$ has simple poles at $s \in \{0, 1\}$ this factor contributes poles at positions

$$s \in \left\{ -\frac{n^2}{2}, \ -\frac{n^2}{2} + 1, \ \cdots, \ \frac{n^2}{2} + 1 \right\}.$$

The poles at $s = -n^2/2, -n^2/2 + 1, n^2/2$ and $n^2/2 + 1$ are simple because $\beta(r, n, n) = 1$ for $r \in \{-n^2, -n^2 + 2, n^2 - 2, n^2\}$; for the other poles we get higher order (an exception is the case $n = 1$, where we have a double pole at $s = 1/2$). By Proposition 2.3.3 iii) and by (90) the poles at $s = -n^2/2$ and $s = n^2/2 + 1$ are not cancelled by $\delta(s)$.

By properties 1) and 4), for $0 \le j < n$, the functions $L(s + r/2, \ \pi, \ \mathrm{Sym}^{n-j})$ only zeros are located in the strip $-r/2 < \mathrm{Re}(s) < -r/2 + 1$. Since r runs from $j(j - 2n)$ to $j(2n - j)$ and $j < n$, the poles at $s = -n^2/2$ and at $s = n^2/2 + 1$ can never be cancelled by those zeros. ∎

We shall make some more remarks on possible poles, assuming 1) through 5) above hold for the symmetric power L-functions. The only poles of $L(s, \ \Pi, \ \rho)$ as in (91) then come from the terms with $j = n$, i.e., from the factor $\tilde{Z}(s)$ defined in (92). For $r \in \{-n^2, \ -n^2 + 2, \ \cdots, \ n^2 + 2\}$ this factor has a pole at $s = r/2$ of order

$$-\mathrm{ord}_{r/2}(\tilde{Z}(s)) = \beta(r, n, n) + \beta(r - 2, n, n) . \tag{93}$$

Note that $\beta(-n^2 - 2, n, n) = \beta(n^2 + 2, n, n) = 0$. Many of these poles will be cancelled by the zeros of the polynomial $\delta^{(n)}$. Using (93) and the formula given in Lemma 2.3.4, it is easy to compute the orders $\mathrm{ord}_{r/2}(\tilde{Z}\delta^{(n)})$ using a computer. The results for $n = 1, \cdots, 6$ are given in the appendix. It turns out that for $n \in \{1, \cdots, 5\}$ there are $n + 1$ poles in $\mathrm{Re}(s) \geq 1/2$, all of them simple, located at

$$s \in \left\{ \frac{n^2}{2} + 1,\ \frac{n^2}{2}, \cdots,\ \frac{n^2}{2} - n + 1 \right\}. \tag{94}$$

For $n \geq 6$ no more than 6 poles appear, all of them simple and located at

$$s \in \left\{ \frac{n^2}{2} + 1,\ \frac{n^2}{2}, \cdots,\ \frac{n^2}{2} - 4 \right\}.$$

This is so because the sequences

$$\alpha(-j(j-2n), j, n),\ \alpha(-j(j-2n)+2, j, n),\ \cdots,$$

and then also the analogous sequences β, become stable in the sense that if we replace n by $n+i$ and j by $j+i$ for $i \geq 0$ then the first terms of the series do not change, and there are the more stable terms the bigger n is (this is not too hard to prove). The orders of the poles are computed from these numbers and thus also become stable. The 6 simple poles at the points (94) are stable for $n \geq 6$, and beyond these poles there are no further ones (see the last columns of the tables "Possible poles" in the appendix). Some of these 6 poles might be cancelled by the polynomials $\delta^{(j)}$ for $j < n$ (but not all of them, as we have seen).

In the case $n = 1$, the Saito–Kurokawa case, our formula (91) says

$$L(s,\ \Pi,\ \rho) = 2\pi(s - 1/2)\, L(s,\ \pi_1)\, Z(s + 1/2)\, Z(s - 1/2)\,,$$

see (53) and (88). It is only in this case that $\tilde{Z}\delta$ happens to have a pole at $s = 1/2$. However, the condition $k \equiv n \mod 2$, where $2k$ is the weight of π_1, forces $L(s,\ \pi_1)$ to vanish at $s = 1/2$. Thus $L(s,\ \Pi,\ \rho)$ is holomorphic at $s = 1/2$, as it should be.

A. Some numerical values

This appendix contains some numerical data obtained with the help of a computer. For $n = 1, \cdots, 6$ we shall list the numbers $\alpha(r, j, n)$ and $\beta(r, j, n)$, defined in section 2.1. Because of $\alpha(-r, j, n) = \alpha(r, j, n)$ and $\alpha(r, 2n - j, n) = \alpha(r, j, n)$ it is enough to restrict to values $r \geq 0$ and $j \leq n$.

We shall also list the orders of $\tilde{Z}$ (defined in (92)) and the polynomial $\delta^{(n)}$ at the points where $\tilde{Z}$ has poles. These orders are given by (93) resp. Lemma 2.3.4.

$\alpha(r, j, 1)$

$r \downarrow,\ j \rightarrow$	0	1
0	1	
1		1

$\beta(r, j, 1)$

$r \downarrow,\ j \rightarrow$	0	1
0	1	
1		1

Possible poles for $n = 1$:

s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(1)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(1)})$
$1/2$	-2	1	-1
$3/2$	-1	0	-1

$\alpha(r, j, 2)$

$r \downarrow,\ j \rightarrow$	0	1	2
0	1		2
1		1	
2			1
3		1	
4			1

$\beta(r, j, 2)$

$r \downarrow,\ j \rightarrow$	0	1	2
0	1		1
1		1	
2			1
3		1	
4			1

Possible poles for $n = 2$:

s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(2)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(2)})$
1	-2	1	-1
2	-2	1	-1
3	-1	0	-1

$\alpha(r, j, 3)$

$r \downarrow,\ j \rightarrow$	0	1	2	3
0	1		3	
1		1		3
2			2	
3		1		3
4			2	
5		1		2
6			1	
7				1
8			1	
9				1

$\beta(r, j, 3)$

$r \downarrow,\ j \rightarrow$	0	1	2	3
0	1		2	
1		1		2
2			2	
3		1		2
4			2	
5		1		1
6			1	
7				1
8			1	
9				1

Possible poles for $n = 3$:

s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(3)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(3)})$
$1/2$	-4	5	1
$3/2$	-4	4	0
$5/2$	-3	2	-1
$7/2$	-2	1	-1
$9/2$	-2	1	-1
$11/2$	-1	0	-1

$\alpha(r, j, 4)$

$r \downarrow,\ j \rightarrow$	0	1	2	3	4
0	1		4		8
1		1		6	
2			3		7
3		1		6	
4			3		7
5		1		5	
6			2		5
7		1		4	
8			2		5
9				3	
10			1		3
11				2	
12			1		2
13				1	
14					1
15				1	
16					1

$\beta(r, j, 4)$

$r \downarrow,\ j \rightarrow$	0	1	2	3	4
0	1		3		4
1		1		5	
2			3		4
3		1		5	
4			3		4
5		1		4	
6			2		3
7		1		3	
8			2		3
9				3	
10			1		2
11				2	
12			1		1
13				1	
14					1
15				1	
16					1

Possible poles for $n = 4$:

s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(4)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(4)})$	s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(4)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(4)})$
1	−8	16	8	6	−3	2	−1
2	−8	12	4	7	−2	1	−1
3	−7	9	2	8	−2	1	−1
4	−6	6	0	9	−1	0	−1
5	−5	4	−1				

$\alpha(r, j, 5)$

$r \downarrow,\ j \rightarrow$	0	1	2	3	4	5
0	1		5		18	
1		1		10		20
2			4		16	
3		1		10		19
4			4		16	
5		1		9		18
6			3		14	
7		1		8		16
8			3		13	
9		1		7		14
10			2		10	
11				5		11
12			2		9	
13				4		9
14			1		6	
15				3		7
16			1		5	
17				2		5
18					3	
19				1		3
20					2	
21				1		2
22					1	
23						1
24					1	
25						1

$\beta(r, j, 5)$

$r \downarrow,\ j \rightarrow$	0	1	2	3	4	5
0	1		4		13	
1		1		9		10
2			4		12	
3		1		9		9
4			4		12	
5		1		8		9
6			3		11	
7		1		7		8
8			3		10	
9		1		6		7
10			2		8	
11				5		6
12			2		7	
13				4		5
14			1		5	
15				3		4
16			1		4	
17				2		3
18					3	
19				1		2
20					2	
21				1		1
22					1	
23						1
24					1	
25						1

Possible poles for $n = 5$:

s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(5)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(5)})$	s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(5)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(5)})$
$1/2$	-20	64	44	$15/2$	-9	10	1
$3/2$	-19	53	34	$17/2$	-7	6	-1
$5/2$	-18	44	26	$19/2$	-5	4	-1
$7/2$	-17	35	18	$21/2$	-3	2	-1
$9/2$	-15	27	12	$23/2$	-2	1	-1
$11/2$	-13	20	7	$25/2$	-2	1	-1
$13/2$	-11	14	3	$27/2$	-1	0	-1

$\alpha(r, j, 6)$

$r \downarrow,\ j \rightarrow$	0	1	2	3	4	5	6
0	1		6		33		58
1		1		15		49	
2			5		31		55
3		1		15		48	
4			5		31		55
5		1		14		46	
6			4		28		51
7		1		13		43	
8			4		27		48
9		1		12		39	
10			3		23		42
11		1		10		35	
12			3		21		39
13				8		30	
14			2		17		32
15				7		26	
16			2		15		28
17				5		21	
18			1		11		22
19				4		17	
20			1		9		18
21				3		13	
22					6		13
23				2		10	
24					5		11
25				1		7	
26					3		7
27				1		5	
28					2		5
29						3	
30					1		3
31						2	
32					1		2
33						1	
34							1
35						1	
36							1

$\beta(r, j, 6)$

$r \downarrow,\ j \rightarrow$	0	1	2	3	4	5	6
0	1		5		27		25
1		1		14		34	
2			5		26		24
3		1		14		33	
4			5		26		24
5		1		13		32	
6			4		24		23
7		1		12		30	
8			4		23		21
9		1		11		27	
10			3		20		19
11		1		9		25	
12			3		18		18
13				8		22	
14			2		15		15
15				7		19	
16			2		13		13
17				5		16	
18			1		10		11
19				4		13	
20			1		8		9
21				3		10	
22					6		7
23				2		8	
24					5		6
25				1		6	
26					3		4
27				1		4	
28					2		3
29						3	
30					1		2
31						2	
32					1		1
33						1	
34							1
35						1	
36							1

Possible poles for $n = 6$:

s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(6)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(6)})$	s	$\mathrm{ord}_s(\tilde{Z})$	$\mathrm{ord}_s(\delta^{(6)})$	$\mathrm{ord}_s(\tilde{Z}\delta^{(6)})$
1	−49	217	168	11	−16	22	6
2	−48	190	142	12	−13	15	2
3	−47	162	115	13	−10	10	0
4	−44	137	93	14	−7	6	−1
5	−40	112	72	15	−5	4	−1
6	−37	92	55	16	−3	2	−1
7	−33	72	39	17	−2	1	−1
8	−28	56	28	18	−2	1	−1
9	−24	42	18	19	−1	0	−1
10	−20	31	11				

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