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Saito-Kurokawa Lifting for Odd Weights

by

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0. Introduction

There exists a lifting from modular forms of half integral weights on $\Gamma_0(4)$ into Siegel modular forms of even integral weights of degree two called the Saito-Kurokawa lifting. Many Authors including Kurokawa [Ku], Maass [Ma1], Eichler-Zagier [EZ], Andrianov [An], Kojima [Koj], studied the lifting from various points of views such as the Maass space, the associated L-function, the connection with Jacobi forms and the Weil representation. In particular Maass and Eichler-Zagier ([Ma2], [EZ]) established the whole feature of the theory of Saito-Kurokawa lifting. Moreover in [MRV] Manickam-Ramakrishnan-Vasudevan studied the lifting with square free odd levels.

Recently Duke-Imamoḡlu [DI] reconstructed this lifting with the help of the converse theorem of Imai [Im] and some results of Katok-Sarnak [KS]. This will be one of the rare cases in which the converse theorem of Imai proved to be of use.

On the other hand not so much is known on Saito-Kurokawa lifting in the case of weight k being odd except the misterious work of Maass [Ma3]. Our aim of this paper is to establish a kind of Saito-Kurokawa lifting in odd weight cases by using the method of Duke-Imamoḡlu. It will be natural in this case that we should use Siegel modular foms on the congruence group $\Gamma_0^{(2)}(4)$ with a non-trivial character mod 4. We construct two kinds of liftings from elliptic cusp forms of half-integral weight $k-\frac{1}{2}$ to Siegel modular forms of degree two of weight k on $\Gamma_0^{(2)}(4)$ and study a relationship of these two liftings. We also study these liftings by means of Eichler-Zagier [E-Z] where Jacobi forms are effectively used.

In his article [Ib2] written in Japanese Ibukiyama exhibited a good introductory exposition of the converse theorem and the method of Duke-Imamoḡlu. To complete this work we owe much to his exposition.

1. Saito-Kurokawa lifting for odd weights

1.1. Maass spaces and Saito-Kurokawa lifting. In the sequel we assume that k is an odd positive integer and use the notation $e(z) = \exp(2\pi i z)$. Let $\mathfrak{H}_n$ denote the Siegel upper half space of degree n on which the real symplectic group $Sp_n(\mathbb{R})$ acts in a usual manner; for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_n(\mathbb{R})$ and $Z \in \mathfrak{H}_n$, set

$$MZ = (aZ+b)(cZ+d)^{-1} \in \mathfrak{H}_n \quad \text{and} \quad J(M,Z) = cZ+d\,.$$

If $n=1$, we write $\mathfrak{H}$ for $\mathfrak{H}_1$. Let Γ_n denote the Siegel modular group $Sp_n(\mathbb{Z})$ and $\Gamma_0^{(n)}(4)$ the subgroup of Γ_n consisting of $M \in \Gamma_n$ whose left lower blocks $c(M)$ are congruent to 0 mod 4. In the case of $n=1$ we write $\Gamma_0(4)$ instead of $\Gamma_0^{(1)}(4)$ following the usual notation.

From now on we consider only the cases of $n = 1, 2$. Let χ be a Dirichlet character mod 4. Let $M_k(\Gamma_0^{(2)}(4), \chi)$ denote the space of holomorphic functions F on $\mathfrak{H}_2$ satisfying the condition

$$F(MZ) = \chi(M) \det J(M,Z)^k F(Z) \quad \text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(2)}(4)\,,$$

where we put $\chi(M) = \chi(\det d)$ by abuse of notation. To describe Fourier expansions of modular forms $F \in M_k(\Gamma_0^{(2)}(4), \chi)$ we define $S_2(\mathbb{Z})$ (resp. $S_2^*(\mathbb{Z})$) to be the set of integral (resp. half-integral) symmetric matrices of size two. Each $F \in M_k(\Gamma_0^{(2)}(4), \chi)$ has a Fourier expansion of the form

$$F(Z) = \sum_{T \in S_2^*(\mathbb{Z}),\, T \geq 0} a(T) e(\mathrm{tr}(TZ))\,,$$

where $T \geq 0$ means that T is semi-positive definite.

We now introduce the Maass space $\widetilde{Ma}(k,\chi)$ to be the subspace of $M_k(\Gamma_0^{(2)}(4), \chi)$ consisting of $F \in M_k(\Gamma_0^{(2)}(4), \chi)$ whose Fourier coefficients $a(T)$ satisfy the Maass relation

$$a\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} = \sum_{0<d \mid (m,r,n)} \chi(d) d^{k-1} a \begin{pmatrix} 1 & r/2d \\ r/2d & mn/d^2 \end{pmatrix}$$

for any $T = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \in S_2^*(\mathbb{Z})$, $T \geq 0$, $T \neq 0$. Here, (m,r,n) denotes the greatest common divisor of m, r, n. This type of Maass subspace with a Dirichlet character has been introduced by Kojima [Koj] if k is even. Moreover we define another Maass subspace $Ma(k,\chi)$ by putting

$$Ma(k,\chi) = \widetilde{Ma}(k,\chi) \cap \left\{ F \in M_k(\Gamma_0^{(2)}(4), \chi) \,\middle|\, F\left(Z + \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\right) = F(Z) \right\}.$$

The condition of $F \in Ma(k,\chi)$ amounts to saying that $F \in M_k(\Gamma_0^{(2)}(4), \chi)$ has a Fourier expansion

$$F(Z) = \sum_{T \in S_2(\mathbb{Z}),\, T \geq 0} a(T) e(\mathrm{tr}(TZ))$$

whose Fourier coefficients satisfy the Maass relation

$$a\begin{pmatrix} m & r \\ r & n \end{pmatrix} = \sum_{0<d \mid (m,r,n)} \chi(d) d^{k-1} a \begin{pmatrix} 1 & r/d \\ r/d & mn/d^2 \end{pmatrix} \quad ((m,r,n) \neq (0,0,0))\,.$$

Our aim of this paper is to characterize these Maass subspaces in terms of Saito-Kurokawa lifting.

1.2. Construction of lifting. First we define certain spaces of modular forms of half intrgral weights. For the function $w^{1/2}$ $(w \neq 0)$ we choose the branch with $-\pi < \arg w \leq \pi$. Let $\theta(z) = \sum_{n\in\mathbb{Z}} e(n^2 z)$ $(z \in \mathfrak{H})$ be the usual theta series. This theta series enjoys the the transformation formula

$$\theta(Mz)/\theta(z) = j(M, z) \quad \text{for any } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where $j(M, z)$ is characterized by

$$j(M, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} (cz + d)^{1/2}. \tag{1.1}$$

Here $(\frac{c}{d})$ is Shimura's resideu symbol on whose precise definition we refer the reader to [Sh] and $\varepsilon_d = 1$ (resp. $\varepsilon_d = i$) according to $d \equiv 1 \bmod 4$ (resp. $d \equiv 3 \bmod 4$).

Let $M_{k-1/2}(\Gamma_0(4))$ be the space of elliptic modular forms of weight $k - 1/2$ on $\Gamma_0(4)$. Namely, $M_{k-1/2}(\Gamma_0(4))$ consists of holomorphic functions f on $\mathfrak{H}$ verifying the conditions

(i) $f(Mz) = j(M, z)^{2k-1} f(z)$ for all $M \in \Gamma_0(4)$.

(ii) $f(z)$ is holomorphic at any cusps of $\Gamma_0(4)$.

Let $S_{k-1/2}(\Gamma_0(4))$ denote its subspace consisting of cusp forms. Moreover, $M^+_{k-1/2}(\Gamma_0(4))$ denotes the Kohnen plus space consisting of $f \in M_{k-1/2}(\Gamma_0(4))$ whose Fourier coefficients $a(n)$ at the infinity have to satisfy the condition

$$a(n) = 0 \quad \text{if } (-1)^k n \equiv 1, 2 \bmod 4$$

([Koh]). Finally let $S^+_{k-1/2}(\Gamma_0(4))$ denote the subspce of $M^+_{k-1/2}(\Gamma_0(4))$ consisting of cusp forms. Since we assume k to be odd, the Fourier coefficients $a(n)$ of each $f \in M^+_{k-1/2}(\Gamma_0(4))$ vanish if $n \equiv 2, 3 \bmod 4$.

In this subsection we form liftings from the space $S_{k-1/2}(\Gamma_0(4))$ to the Maass space $\widetilde{Ma}(k, \chi)$ and also to $Ma(k, \chi)$. Let $\varphi \in S_{k-1/2}(\Gamma_0(4))$ and its Fourier expansion be

$$\varphi(\tau) = \sum_{n=1}^{\infty} c(n) e(n\tau).$$

We denote by $S_2(\mathbb{Z})^+$ (resp. $S_2^*(\mathbb{Z})^+$) the subset of $S_2(\mathbb{Z})$ (resp. $S_2^*(\mathbb{Z})$) consisting of positive definite symmetric matrices. Now we define functions $c(T)$ on $S_2(\mathbb{Z})^+$ and $b(T)$ on $S_2^*(\mathbb{Z})^+$ from the Fourier coefficients of φ by putting

$$c\begin{pmatrix} m & r \\ r & n \end{pmatrix} = \sum_{0<d \mid (m,r,n)} \chi(d) d^{k-1} c\left(\frac{\det T}{d^2}\right)$$

and

$$b\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} = \sum_{0<d \mid (m,r,n)} \chi(d) d^{k-1} c\left(\frac{\det 2T}{d^2}\right),$$

respectively. It is not difficult to see that there exists a positive constant M (depending on φ, ψ) such that

$$(1.2)\qquad \left| c\begin{pmatrix} m & r \\ r & n \end{pmatrix} \right| < M(mn)^{k+1/2}\,, \qquad \left| b\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \right| < M(mn)^{k+1/2}\,.$$

We note here that the identities

$$c({}^tUTU) = c(T) \quad \text{and} \quad b({}^tUTU) = b(T)$$

follow from the definition of $c(T)$ and $b(T)$.

Then Saito Kurokawa liftings $\iota(\varphi)$ and $\tilde{\iota}(\varphi)$ in our setting are given by

$$\iota(\varphi)(Z) = \sum_{T \in S_2(\mathbb{Z})^+} c(T)e(\mathrm{tr}(TZ)), \quad \tilde{\iota}(\varphi)(Z) = \sum_{T \in S_2^*(\mathbb{Z})^+} b(T)e(\mathrm{tr}(TZ))\,,$$

where $Z \in \mathfrak{H}_2$. Due to the estimates (1.2) above these $\iota(\varphi)(Z)$ and $\tilde{\iota}(\varphi)(Z)$ define holomorphic functions on $\mathfrak{H}_2$. Then the following theorem holds.

THEOREM 1.1. *If φ is a modular form of half-intrgral weight in $S_{k-1/2}(\Gamma_0(4))$, then $\iota(\varphi) \in Ma(k, \chi), \tilde{\iota}(\varphi) \in \widetilde{Ma}(k, \chi)$.*

The proof of the theorem is reduced to the following Theorem 1.2 which is one of our main results and will be proved later.

Let $\varphi \in S_{k-1/2}(\Gamma_0(4))$ and define a new modular form ψ by putting

$$(1.3)\qquad \psi(\tau) = \sqrt{2}(-1)^{\frac{k-1}{2}} 4^{\frac{1}{2}-k} \left(\frac{\tau}{i}\right)^{\frac{1}{2}-k} \varphi\left(-\frac{1}{4\tau}\right)\,.$$

Then it is known and easy to see that $\psi \in S_{k-1/2}(\Gamma_0(4))$. Here we set

$$(1.4)\qquad F(Z) = \iota(\varphi)(Z) \quad \text{and} \quad G(Z) = \tilde{\iota}(\psi)(Z)\,.$$

THEOREM 1.2. *Under the notation above the transformation formula*

$$(1.5)\qquad F(-(4Z)^{-1}) = \det\left(\frac{2Z}{i}\right)^k G(Z)$$

holds and moreover if $\psi \in S^+_{k-1/2}(\Gamma_0(4))$, then

$$F(-(4Z)^{-1}) = \det\left(\frac{2Z}{i}\right)^k F(Z)\,, \quad \textit{namely}, \quad F = G\,.$$

Here we give a proof of Theorem 1.1 with the help of Theorem 1.2.

Proof of Theorem 1.1. Our task is to prove the modularity of F:

$$(1.6)\qquad F(MZ) = \chi(M)\det J(M, Z)^k F(Z)$$

for any $M \in \Gamma_0^{(2)}(4)$. Since the congruence subgroup $\Gamma_0^{(2)}(4)$ is generated by the following three kinds of elements

$$t(s') := \begin{pmatrix} 1_2 & s' \\ 0 & 1_2 \end{pmatrix} \quad (s' \in S_2(\mathbb{Z})), \quad \begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix} \quad (U \in GL_2(\mathbb{Z})),$$

$$v(4s) := \begin{pmatrix} 1_2 & 0 \\ 4s & 1_2 \end{pmatrix} \quad (s \in S_2(\mathbb{Z}))$$

(see [Ib1], Lemma 2.1), we have only to prove for these generators. The transformation formulas (1.6) for $M = t(s')$, $\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}$ are valid from the definition (1.4) of F.

Let $M = v(4s)$ $(s \in S_2(\mathbb{Z}))$. Then

$$F(v(4s)Z) = F\big(-(-4(sZ + (1/4)1_2)Z^{-1})^{-1}\big).$$

Making use of (1.5) in Theorem 1.2 we have

$$\begin{aligned} F(v(4s)Z) &= \det\left(\frac{-2(sZ + (1/4)1_2)Z^{-1}}{i}\right)^k G(-(sZ + (1/4)1_2)Z^{-1}) \\ &= \det(4sZ + 1_2)^k \det\left(\frac{2Z}{i}\right)^{-k} G(-(4Z)^{-1} - s). \end{aligned}$$

Here we have, by the definition (1.4) of G,

$$G(-(4Z)^{-1} - s) = G(-(4Z)^{-1}).$$

Using again (1.5) in Theorem 1.2 in an opposite direction, we get

$$G(-(4Z)^{-1}) = \det\left(\frac{2Z}{i}\right)^k F(Z).$$

Therefore,

$$F(v(4s)Z) = \det(4sZ + 1_2)^k F(Z),$$

which proves (1.6) for $M = v(4s)$. The Maass relation for F immediately follows from the definition of $F = \iota(\varphi)$. Hence, $F \in Ma(k, \chi)$. Moreover Theorem 1.2 implies that $G = \tilde{\iota}(\varphi) \in M_k(\Gamma_0(4), \chi)$ and hence that $G \in \widetilde{Ma}(k, \chi)$. $\square$

We prove Theorem 1.2 by using the method of Duke-Imamoḡlu, but the proof will be postponed until the next subsection. We have to make some preparatory arguments.

First we give a definition of Maass wave forms and explain how we regard them so-called Grössen characters on $\mathcal{P}_2$, the symmetric space of positive definite real symmetric matrices of size two. A function $v : \mathfrak{H} \to \mathbb{C}$ is called a Maass wave form of weight 0, if v satisfies the following three conditions

(i) $v(Mz) = v(z)$ $(\forall M \in SL_2(\mathbb{Z}))$

(ii) v is a C^∞-function on $\mathfrak{H}$ with respect to x, y which verifies a differential equation $\Delta v = -\lambda v$ with some $\lambda \in \mathbb{C}$, where $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is a $SL_2(\mathbb{R})$-invariant differential operator on $\mathfrak{H}$.

(iii) There exists a certain $\alpha > 0$ with the growth condition $v(x + iy) = O(y^\alpha)$ $(y \to \infty)$.

A Maass wave form v is called a common eigen form, if it is an eigen form of all Hecke operators T_p, whose definition we refer to [KS], p. 199.

Let $\mathcal{P}S_2$ denote the subset of $\mathcal{P}_2$ consisting of $Y \in \mathcal{P}_2$ whose determinants coincide with 1. Then each element $g \in SL_2(\mathbb{R})$ acts on $\mathcal{P}_2$ and also on $\mathcal{P}S_2$ via $Y \to {}^t g^{-1} Y g^{-1}$ and, as is well-known there exists a natural diffeomorphism from the upper half plane $\mathfrak{H}$ onto $\mathcal{P}S_2$ which is compatible with the action of $SL_2(\mathbb{R})$. Namely if we put, for $z = x+iy \in \mathfrak{H}$,

$$Y(z) = \begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & y^{-1}(x^2+y^2) \end{pmatrix},$$

then $Y(z) \in \mathcal{P}S_2$ and moreover $Y(gz) = {}^t g^{-1} Y(z) g^{-1}$. Via this diffeomorphism each Maass wave form v can be identified with a function on $\mathcal{P}S_2$ and with the one on $\mathcal{P}_2$ by a natural extension. We may call such function on $\mathcal{P}_2$ a Grössen character of Maass. Namely, a function $u : \mathcal{P}_2 \to \mathbb{C}$ is called a Grössen character of Maass, if it satisfies the following two conditions

(i) $u(cY) = u(Y)$ for any $c > 0$ and $Y \in \mathcal{P}_2$.

(ii) for some Maass wave form v, $u(Y) = v(z)$ $(Y \in \mathcal{P}S_2, z \in \mathfrak{H})$, where Y is corresponding to z by $Y = Y(z)$.

A Maass wave form v of weight 0 (and accordingly the corresponding u) is called *even*, if it satisfies

$$v(-\bar{z}) = v(z) \quad (\text{resp. } u(\widetilde{Y}) = u(Y)),$$

where $\widetilde{Y} = {}^t I_0 Y I_0$ with $I_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. On the contrary a Maass wave form v of weight 0 is called odd, if $v(-\bar{z}) = -v(z)$. In this case a Grössen character u corresponding to v is also called odd. For a general theory of Maass' Grössen characters we refer to [Ma1].

Moreover we have to introduce Maass wave forms of weight 1/2 to describe the Maass wave form version of the Shimura correspondence. For $r \in \mathbb{C}$ let T_r^+ denote the $\mathbb{C}$-linear space consisting of functions $g : \mathfrak{H} \to \mathbb{C}$ satisfying the following three conditions:

(i) Each g is a C^∞-function of x and y verifying the transformation formula

$$g(Mz) = g(z) j(M, z) |cz + d|^{-1/2}$$

for all $M \in \Gamma_0(4)$ and it has the growth condition at any cusps of $\Gamma_0(4)$; namely there exists $\alpha > 0$ such that for all $M \in SL_2(\mathbb{Z})$

$$|g(Mz)| = O(y^\alpha) \quad (y \to \infty).$$

(ii) g has a Fourier expansion of the form

$$g(z) = \sum_{n \in \mathbb{Z}} B(n, y) e(nx), \tag{1.7}$$

where the Fourier coefficients $B(n, y)$ for $n \neq 0$ are given by

$$B(n, y) = b(n) W_{\operatorname{sign} n/4, ir/2}(4\pi y |n|).$$

Here $W_{\alpha,\beta}$ is the usual Whittaker function.

(iii) If $n \equiv 2, 3 \bmod 4$, then necessarily $B(n, y) = 0$.

The next theorem describes the Shimura correspondence from the space of Maass wave forms of weight 0 to the space of Maass wave forms of weight 1/2. This has been proved by Katok-Sarnak [KS], Theorem in the case of v being cusp forms and by Duke-Imamoḡlu [DI] in the case of v being a constant function or Eisenstein series.

For $Y \in \mathcal{P}_2$ we denote by z_Y the point z in $\mathfrak{H}$ determined by $Y(z) = \frac{1}{\sqrt{\det Y}} \cdot Y$.

THEOREM 1.3 (Katok-Sarnak, Duke-Imamoḡlu). *Let v be an even common eigen Maass wave form of weight* 0 *and assume that $\Delta v = -(\frac{1}{4} + r^2)v$ with some $r \in \mathbb{C}$. Then there exists $g \in T_r^+$ such that concerning the Fourier coefficints $b(-n)$ of g for positive n we have*

$$b(-n) = n^{-3/4} \sum_{T \in S_2^*(\mathbb{Z})^+/SL_2(\mathbb{Z}), \det 2T = n} v(z_T)|AutT|^{-1} \quad (n \in \mathbb{Z}_{>0}),$$

where T runs through all the $SL_2(\mathbb{Z})$-equivalence classes of elements of $S_2^*(\mathbb{Z})^+$ with $\det 2T = n$ *and* $AutT := \{U \in SL_2(\mathbb{Z}) \mid {}^tUTU = T\}$ *is the unit group of T.*

1.3. Proof of Theorem 1.2. Now we give a proof of our Theorem 1.2.

Let $\varphi(z) = \sum_{n=1}^{\infty} c(n)e(nz)$ be a Fourier expansion of $\varphi \in S_{k-1/2}(\Gamma_0(4))$. Then ψ is given as before by (1.3) and has a Fourier expansion of the form: $\psi(z) = \sum_{n=1}^{\infty} a(n)e(nz)$. Recalling the the definition of $F(Z)$, $G(Z)$, we have

$$F(Z) = \iota(\varphi)(Z) = \sum_{T \in S_2(\mathbb{Z})^+} c(T)e(\mathrm{tr}(TZ))$$

$$G(Z) = \tilde{\iota}(\psi)(Z) = \sum_{T \in S_2^*(\mathbb{Z})^+} a(T)e(\mathrm{tr}(TZ)),$$

where $c(T)$ and $a(T)$ are given by

$$(1.8) \qquad c\begin{pmatrix} m & r \\ r & n \end{pmatrix} = \sum_{0<d \mid (m,r,n)} \chi(d)d^{k-1}c\left(\frac{\det T}{d^2}\right)$$

and

$$(1.9) \qquad a\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} = \sum_{0<d \mid (m,r,n)} \chi(d)d^{k-1}a\left(\frac{\det 2T}{d^2}\right),$$

respectively.

Now take any even Grössen character u on $\mathcal{P}_2$ and let v be a Maass wave form corresponding to u with $\Delta v = -(\frac{1}{4} + r^2)$. Then there exists a Maass wave form g of weight 1/2 belonging to the space T_r^+ which corresponds to v by the Shimura correspondence (Theorem 1.3). We compute Mellin transforms of F and G. Set

$$(1.10) \qquad \xi_2(G, u; s) := \int_{\mathcal{R}} G\left(\frac{iY}{2}\right)(\det Y)^s u(Y)dv(Y),$$

where $\mathcal{R}$ denotes a fundamental domain of $SL_2(\mathbb{Z})$ in $\mathcal{P}_2$ and

$$dv(Y) := (\det Y)^{-3/2}dy_{11}dy_{12}dy_{22} \quad (Y = (y_{ij}))$$

is an invariant volume form on $\mathcal{P}_2$. Making use of some results in Maass [Ma1], § 10, we have

$$\xi_2(G, u; s) = 2\pi^{1/2}\pi^{-2s}\Gamma\left(s - \frac{1}{4} + \frac{ir}{2}\right)\Gamma\left(s - \frac{1}{4} - \frac{ir}{2}\right)D_2(G, u, s),$$

where $D_2(G, u, s)$ denotes the Koecher-Maass series attached to G with u:

$$D_2(G, u, s) = \sum_{T \in S_2^*(\mathbb{Z})^+/SL_2(\mathbb{Z})} \frac{a(T)u(T)}{|Aut(T)|(\det T)^s}.$$

Moreover we set

$$\xi_2(F, u; s) := \int_{\mathcal{R}} F\left(\frac{iY}{2}\right)(\det Y)^s u(Y) dv(Y) \tag{1.11}$$

and

$$D_2(F, u, s) = \sum_{T \in S_2(\mathbb{Z})^+/SL_2(\mathbb{Z})} \frac{c(T)u(T)}{|Aut(T)|(\det T)^s}.$$

Then similarly,

$$\xi_2(F, u; s) = 2\pi^{1/2}\pi^{-2s}\Gamma\left(s - \frac{1}{4} + \frac{ir}{2}\right)\Gamma\left(s - \frac{1}{4} - \frac{ir}{2}\right)D_2(F, u, s).$$

We see easily from the estimate (1.2) that $D_2(G, u, s)$ and $D_2(F, u, s)$ are absolutely convergent if $\mathrm{Re}(s)$ is sufficiently large and accodingly that $\xi_2(G, u; s)$ and $\xi_2(F, u; s)$ are also absolutely convergent and indicate holomorphic functions in the same region of s.

Since $a(I_0 T I_0) = a(T)$, it follows that, if u is an odd Grössen character, $D_2(G, u, s)$ and $D_2(F, u, s)$ are identically zero. So we have only to consider the case of u being even (and accordingly v even).

Let $Prim_2^*(\mathbb{Z})^+$ denote the set of positive definite primitive half-integral symmetric matrices of size 2. Using the expression (1.9) of $a(T)$, we have

$$\begin{aligned}
D_2(G, u, s) &= \sum_{T_0 \in Prim_2^*(\mathbb{Z})^+/SL_2(\mathbb{Z})} \sum_{e=1}^{\infty} \frac{a(eT_0)u(eT_0)}{|Aut(eT_0)|(\det eT_0)^s} \\
&= \sum_{T_0 \in Prim_2^*(\mathbb{Z})^+/SL_2(\mathbb{Z})} \sum_{e=1}^{\infty} \sum_{0<d\,|\,e} \chi(d) d^{k-1} \frac{a(\det(2eT_0)/d^2)u(T_0)}{|Aut(T_0)|e^{2s}(\det T_0)^s} \\
&= \sum_{d=1}^{\infty}\sum_{m=1}^{\infty} \sum_{T_0 \in Prim_2^*(\mathbb{Z})^+/SL_2(\mathbb{Z})} \chi(d) d^{-2s+k-1} \frac{a(\det(2mT_0))u(mT_0)}{|Aut(mT_0)|(\det mT_0)^s} \\
&= L(2s-k+1, \chi) \sum_{T \in S_2^*(\mathbb{Z})^+/SL_2(\mathbb{Z})} \frac{a(\det 2T)u(T)}{|Aut(T)|(\det T)^s}.
\end{aligned}$$

By using Theorem 1.3 we easily have

$$\begin{aligned} D_2(G,u,s) &= 4^s L(2s-k+1,\chi) \sum_{T\in S_2^*(\mathbb{Z})^+/SL_2(\mathbb{Z})} \frac{a(\det 2T)v(z_T)}{|Aut(T)|(\det 2T)^s} \\ &= 4^s L(2s-k+1,\chi)\sum_{n=1}^{\infty}\frac{a(n)b(-n)}{n^{s-3/4}}\,. \end{aligned}$$

Consequently,

$$\begin{aligned} \xi_2(G,u;s) &= 2\pi^{1/2}\pi^{-2s}4^s\Gamma\left(s-\frac{1}{4}+\frac{ir}{2}\right)\Gamma\left(s-\frac{1}{4}-\frac{ir}{2}\right) \\ &\quad\times L(2s-k+1,\chi)\sum_{n=1}^{\infty}\frac{a(n)b(-n)}{n^{s-3/4}}\,. \end{aligned} \tag{1.12}$$

Similarly we have

$$D_2(F,u,s) = 2^{3/2}L(2s-k+1,\chi)\sum_{n=1}^{\infty}\frac{a(n)b(-4n)}{n^{s-3/4}}$$

and

$$\begin{aligned} \xi_2(F,u;s) &= 2\pi^{1/2}(\pi^{-2s}2^{3/2})\Gamma\left(s-\frac{1}{4}+\frac{ir}{2}\right)\Gamma\left(s-\frac{1}{4}-\frac{ir}{2}\right) \\ &\quad\times L(2s-k+1,\chi)\sum_{n=1}^{\infty}\frac{a(n)b(-4n)}{n^{s-3/4}}\,. \end{aligned} \tag{1.13}$$

To express the right hand sides of the above identities (1.12), (1.13) as a kind of Rankin-Selberg convolution we recall real analytic Eisenstein series on $\Gamma_0(4)$. Let Γ_∞ denote the subgroup of $\Gamma_0(4)$ consisting of matrices $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ $(n\in\mathbb{Z})$. If we set

$$E_\infty(z,s) = \sum_{M\in\Gamma_\infty\backslash\Gamma_0(4)} \chi(d)\left(\frac{cz+d}{|cz+d|}\right)^k (\operatorname{Im} Mz)^s\,,$$

then this series converges absolutely for $\mathrm{Re}(s) > 1$ and becomes the Eisenstein series on $\Gamma_0(4)$ with χ at cusp ∞. Moreover we define the Eisenstein series at cusp 0 by putting

$$E_0(z,s) = \left(\frac{z}{|z|}\right)^k E_\infty\left(-\frac{1}{4z},s\right)\,.$$

Multiplying the gamma factor we set

$$\begin{aligned} \widetilde{E}_\infty(z,s) &= 2^{3s}\pi^{-s}\Gamma(s+k/2)L(2s,\chi)E_\infty(z,s)\,, \\ \widetilde{E}_0(z,s) &= 2^{3s}\pi^{-s}\Gamma(s+k/2)L(2s,\chi)E_0(z,s)\,. \end{aligned}$$

It is known and easy to see that these Eisenstein series can be analytically continued to meromorphic functions of s which satisfy the functional equation

$$\widetilde{E}_\infty(z,s) = (-i)\widetilde{E}_0(z,1-s)\,. \tag{1.14}$$

For $g \in T_r^+$ we define a new modular form g_0 by

$$g_0(z) = \sqrt{2}\left(\frac{z}{i}\right)^{-1/2} |z|^{1/2} g(-1/4z)\,.$$

Then it has a Fourier expansion of the form ([DI], [Ib2])

$$g_0(z) = \sum_{m \in \mathbb{Z}} B(4m, y/4) e^{2\pi i m x}\,,$$

where $B(n, y)$'s are Fourier coefficients of g given in (1.7).

We set

$$(1.15)\quad \begin{aligned} \Lambda_\infty(\psi, g, s) &= \int_{\Gamma_0(4)\backslash\mathfrak{H}} y^{\frac{k}{2}-\frac{1}{4}} \psi(z) g(z) \widetilde{E}_\infty(z, s) \frac{dxdy}{y^2} \\ \Lambda_\infty(\varphi, g_0, s) &= \int_{\Gamma_0(4)\backslash\mathfrak{H}} y^{\frac{k}{2}-\frac{1}{4}} \varphi(z) g_0(z) \widetilde{E}_\infty(z, s) \frac{dxdy}{y^2}\,. \end{aligned}$$

Unfolding the integral (1.15) faithfully, we have

$$\begin{aligned} \Lambda_\infty(\psi, g, s) &= 2^{3s}\pi^{-s}\Gamma(s + k/2)L(2s, \chi) \times \int_0^\infty \int_0^1 y^{s+\frac{k}{2}-\frac{1}{4}} \psi(z) g(z) \frac{dxdy}{y^2} \\ &= 2^{3s}\pi^{-s}\Gamma(s + k/2)L(2s, \chi)(4\pi)^{-s-\frac{k}{2}+\frac{5}{4}} \left(\sum_{n=1}^\infty \frac{a(n)b(-n)}{n^{s+k/2-5/4}}\right) I(k, r; s)\,, \end{aligned}$$

where we put

$$I(k, r; s) = \int_0^\infty y^{s+\frac{k}{2}-\frac{5}{4}} e^{-\frac{y}{2}} W_{-1/4, ir/2}(y) \frac{dy}{y}\,.$$

It is known and easy to compute from a certain integral expression for $W_{-1/4,ir/2}(y)$ that

$$I(k, r; s) = \frac{\Gamma\left(s + \frac{k}{2} + \frac{ir}{2} - \frac{3}{4}\right)\Gamma\left(s + \frac{k}{2} - \frac{ir}{2} - \frac{3}{4}\right)}{\Gamma\left(s + \frac{k}{2}\right)}$$

(see [Ib2], Appendix). Therefore,

(1.16)

$$\begin{aligned} \Lambda_\infty(\psi, g, s) = {}& 2^{3s}\pi^{-s}(4\pi)^{-s-\frac{k}{2}+\frac{5}{4}}\Gamma\left(s + \frac{k}{2} + \frac{ir}{2} - \frac{3}{4}\right)\Gamma\left(s + \frac{k}{2} - \frac{ir}{2} - \frac{3}{4}\right) \\ & \times L(2s, \chi)\left(\sum_{n=1}^\infty \frac{a(n)b(-n)}{n^{s+k/2-5/4}}\right). \end{aligned}$$

We see from this expression that $\Lambda_\infty(\psi, g, s)$ is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large. Define a positive constant $c(k)$ by

$$c(k) = 2^{3k/2-2}\pi^{1/4-k/2}\,.$$

Comparing the right hand sides of the both identities (1.12), (1.16), we have

$$(1.17)\quad \xi_2(G, u, s) = c(k)2^s \Lambda_\infty\left(\psi, g, s - \frac{k-1}{2}\right).$$

In a manner similar to the above argument we get

$$\begin{aligned}\Lambda_\infty(\varphi, g_0, s) =& 2^{3s}\pi^{-s}(4\pi)^{-s-\frac{k}{2}+\frac{5}{4}}\Gamma\left(s+\frac{k}{2}+\frac{ir}{2}-\frac{3}{4}\right)\Gamma\left(s+\frac{k}{2}-\frac{ir}{2}-\frac{3}{4}\right) \\ &\times L(2s,\chi)\left(\sum_{n=1}^{\infty}\frac{a(n)b(-4n)}{n^{s+k/2-5/4}}\right)\end{aligned}$$

and accordingly

$$\xi_2(F,u,s) = c(k)2^{\frac{3}{2}-s}\Lambda_\infty\left(\varphi, g_0, s-\frac{k-1}{2}\right). \tag{1.18}$$

By the property of $E_\infty(z,s)$ and by a usual procedure of treating with Rankin-Selberg convolution we see that the functions $\Lambda_\infty(\psi, g, s)$ and $\Lambda_\infty(\varphi, g_0, s)$ can be continued to meromorphic functions in the whole s plane which are holomorphic in $\mathrm{Re}(s) \geq 1/2$. Moreover these integrals are absolutely convergent where the Eisenstein series $E_\infty(z,s)$ has no poles. Set $\sigma_2 = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}$. A direct computation shows that

$$\mathrm{Im}(\sigma_2 z)^{\frac{k}{2}-\frac{1}{4}}\varphi(\sigma_2 z)g_0(\sigma_2 z)\widetilde{E}_\infty(\sigma_2 z, s) = (-i)2^{k-\frac{3}{2}}y^{\frac{k}{2}-\frac{1}{4}}\psi(z)g(z)\widetilde{E}_0(z,s).$$

Replacing z with $\sigma_2 z = -1/4z$, one has

$$\Lambda_\infty(\varphi, g_0, s) = (-i)2^{k-\frac{3}{2}}\int_{\Gamma_0(4)\backslash\mathfrak{H}} y^{\frac{k}{2}-\frac{1}{4}}\psi(z)g(z)\widetilde{E}_0(z,s)\frac{dxdy}{y^2}.$$

Thus with the help of (1.14) we have the functional equation

$$\Lambda_\infty(\varphi, g_0, s) = 2^{k-3/2}\Lambda_\infty(\psi, g, 1-s). \tag{1.19}$$

Thanks to this functional equation $\Lambda_\infty(\psi, g, s)$ and $\Lambda_\infty(\varphi, g_0, s)$ can be entire functions of s. Furthermore with the help of a standard argument using Phragmén-Lindelöf theorem we observe that these functions are bounded in any vertical strip.

PROPOSITION 1.4. *The functions $\xi_2(F,u;s)$ and $\xi_2(G,u;s)$ can be analytically continued to entire functions of s and are bounded in any vertical strip. Moreover they satisfy the functional equation*

$$\xi_2(F,u;s) = \xi_2(G,u;k-s).$$

Proof. The analytic continuation and the boundedness of $\xi_2(F,u;s)$ and $\xi_2(G,u;s)$ are easily derived from those of $\Lambda_\infty(\psi, g, s)$ and $\Lambda_\infty(\varphi, g_0, s)$ with the use of the identities (1.17) and (1.18). The desired functional equation also follows from (1.17), (1.18) and (1.19). □

The final step is to employ Imai's theorem [Im] to prove the transformation formula

$$F(-(4Z)^{-1}) = \det\left(\frac{2Z}{i}\right)^k G(Z). \tag{1.20}$$

The converse theorem [Im] of Imai (Kaori Ohta) in our situation is reformulated as follows.

THEOREM 1.5. *Assume k is a positive odd integer. Let $a(T)$ and $c(T)$ be functions on $S_2^*(\mathbb{Z})^+$ and $S_2(\mathbb{Z})^+$, respectively satisfying the following conditions*

(i) $a({}^tUTU) = a(T)$, $c({}^tUTU) = c(T)$ *for any* $U \in GL_2(\mathbb{Z})$.

(ii) *There exist some positive constants M, c such that*

$$\left| c\begin{pmatrix} m & r \\ r & n \end{pmatrix} \right| < M(mn)^c, \quad \left| a\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \right| < M(mn)^c.$$

Define holomorphic functions $F(Z)$, $G(Z)$ by putting

$$F(Z) == \sum_{T \in S_2(\mathbb{Z})^+} c(T)e(\mathrm{tr}(TZ))$$

$$G(Z) == \sum_{T \in S_2^*(\mathbb{Z})^+} a(T)e(\mathrm{tr}(TZ)).$$

Associate the integrals $\xi_2(G, u, s)$, $\xi_2(F, u, s)$ for any even Grössen character u by (1.10) *and* (1.11), *respectively.*

Assume moreover that $\xi_2(F, u, s)$ and $\xi_2(G, u, s)$ can be analytically continued to entire functions of s which are bounded in any vertical strip and that the functional equation

$$\xi_2(F, u; s) = \xi_2(G, u; k - s)$$

holds. Then we have the transformation formula

$$F(iY^{-1}/2) = \det(Y)^k G(iY/2) \quad (Y \in \mathcal{P}_2)$$

which in turn is nothing but the identity

$$F(-(4Z)^{-1}) = \det\left(\frac{2Z}{i}\right)^k G(Z).$$

REMARK. As is pointed out in [Ib2], the assumptions of the theorem are a little different from those of Imai [Im]. We follow the reformulation of [DI], Theorem 2. But no proof is given there. Ibukiyama [Ib2] gave a proof to reformulate the converse theorem in the manner given in [DI].

We continue the proof of Theorm 1.2. In our stuation $F = \iota(\varphi)$ and $G = \tilde{\iota}(\psi)$. For these F and G we have proved that all the assumptions in Imai's theorem hold. Hence we obtain the desired transformation formula (1.20).

The rest we have to do is to prove $F = G$ if $\psi \in S_{k-1/2}^+(\Gamma_0(4))$. Let $\psi = \sum_{n=1}^\infty a(n)e(nz) \in S_{k-1/2}^+(\Gamma_0(4))$, where $a(n) = 0$ if $n \equiv 2, 3 \bmod 4$. In this case φ determined by (1.3) has the following Fourier expansion ([Koh])

$$\varphi(z) = \sum_{n=1}^\infty a(4n)e(nz),$$

which implies that $c(n) = a(4n)$. Hence we see from (1.8) that

$$c\begin{pmatrix} m & r \\ r & n \end{pmatrix} = \sum_{0<d \mid (m,r,n)} \chi(d)d^{k-1}a\left(\frac{\det 2T}{d^2}\right) = a\begin{pmatrix} m & r \\ r & n \end{pmatrix}.$$

For $T = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \in S_2^*(\mathbb{Z})^+$, $\det(2T) = 4mn - r^2 \equiv 0, 3 \bmod 4$. Therefore $a(T) = 0$ if r is odd. Consequently $F = G$ follows. Thus we have completed the proof of Theorem 1.2.

REMARK. It is likely that $F = \iota(\varphi)$ and $G = \tilde{\iota}(\psi)$ are cusp foms, but we cannot prove at present.

2. Eichler Zagier's method

In this section we study our liftings by means of Jacobi forms due to Eichler-Zagier [EZ]. We continue the assumption that k is a positive odd integer.

2.1. Jacobi forms. First we define the $\mathbb{C}$-linear space $J_{k,m}(\Gamma_0(4), \chi)$ of Jacobi forms of weight k and index m on $\Gamma_0(4)$ with χ a non-trivial character mod 4 following [E-Z]; namely any holomorphic function $\phi(\tau, z)$ on $\mathfrak{H} \times \mathbb{C}$ is in the space $J_{k,m}(\Gamma_0(4), \chi)$, if and only if ϕ satisfies the following three conditions

(J-i) $\phi(\tau, z + \lambda\tau + \mu) = e(-\lambda^2 m\tau - 2\lambda m z)\phi(\tau, z) \quad (\forall\lambda, \mu \in \mathbb{Z})$

(J-ii) $\phi(M(\tau, z)) = \chi(d)(c\tau + d)^k e\left(\frac{cmz^2}{c\tau+d}\right)\phi(\tau, z) \quad \left(\forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)\right)$,

where $M(\tau, z) = (M\tau, z(c\tau + d)^{-1})$.

(iii) ϕ is holomorphic at any cusps of $\Gamma_0(4)$, namely, for each $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\phi \mid M$ has a Fourier expansion of the form

$$(\phi \mid M)(\tau, z) = \sum_{n,r\in\mathbb{Z}, 4n \geq Nr^2} c(n, r) e\left(\frac{n}{N}\tau + rz\right),$$

where $(\phi \mid M)(\tau, z) = (c\tau + d)^{-k} e\left(-\frac{cmz^2}{c\tau+d}\right)\phi(M(\tau, z))$ and N, a natural number is suitably chosen for each M.

In this section we consider only the case of $m = 1$. Each $\phi \in J_{k,1}(\Gamma_0(4), \chi)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n,r\in\mathbb{Z}, 4n\geq r^2} c(n, r) e(n\tau + rz). \tag{2.1}$$

If we define theta series $\theta_j(\tau, z)$ $(j = 0, 1)$ by putting

$$\theta_j(\tau, z) = \sum_{n\in\mathbb{Z}} e((n + j/2)^2\tau + 2(n + j/2)z),$$

then as usual any $\phi \in J_{k,1}(\Gamma_0(4), \chi)$ can be written in a linear combination of the theta series:

$$\phi(\tau, z) = h_0(\tau)\theta_0(\tau, z) + h_1(\tau)\theta_1(\tau, z). \tag{2.2}$$

The theta transformation formula for the theta series is well-known; for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$\begin{pmatrix} \theta_0(M(\tau, z)) \\ \theta_1(M(\tau, z)) \end{pmatrix} = e\left(-\frac{cz^2}{c\tau + d}\right)(c\tau + d)^{1/2} U(M) \begin{pmatrix} \theta_0(\tau, z) \\ \theta_1(\tau, z) \end{pmatrix}, \tag{2.3}$$

where the branch of $(c\tau + d)^{1/2}$ is chosen in the same manner as in (1.1) and $U(M)$ is a certain unitary matrix of size two depending on M. We set as in [AB]

$$\omega(M) = \det U(M),$$

which is independent of the choice of the branch above. In particular if $M \in \Gamma_0(4)$, then it is known that

$$(c\tau + d)^{1/2} U(M) = \begin{pmatrix} j(M, \tau) & 0 \\ 0 & \mu(M, \tau) \end{pmatrix}. \tag{2.4}$$

Here $j(M, \tau)$ is Shimura's factor of automorphy given by (1.1) and $\mu(M, \tau)$ is also a factor of automorphy on $\Gamma_0(4)$ with weight $1/2$. We know by [AB] that if we write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, then

$$\omega(M) = \chi(d) e(b/4). \tag{2.5}$$

It is immediate to see from (2.2), (2.3) and (2.4) that for any $M \in \Gamma_0(4)$

$$h_0(M\tau) = j(M, \tau)^{2k-1} h_0(\tau) \quad \text{and} \quad h_1(M\tau)\mu(M, \tau) = \chi(M)(c\tau + d)^k h_1(\tau). \tag{2.6}$$

We have, by (2.4) and (2.5),

$$\chi(M)(c\tau + d)^k \mu(M, \tau)^{-1} = e(-b/4) j(M, \tau)^{2k-1}.$$

So let $M^*_{k-1/2}(\Gamma_0(4))$ denote the space consisting of holomorphic functions f satisfying the conditions

(i) $f(M\tau) = j(M, \tau)^{2k-1} e(-b/4) f(\tau) \quad (\forall M \in \Gamma_0(4))$.

(ii) f is holomorphic at any cusp of $\Gamma_0(4)$.

According to (2.2), (2.6) and the definitions of $M_{k-1/2}(\Gamma_0(4))$, $M^*_{k-1/2}(\Gamma_0(4))$, the direct sum decomposition of the space $J_{k,1}(\Gamma_0(4), \chi)$ holds true:

PROPOSITION 2.1. *The space $J_{k,1}(\Gamma_0(4), \chi)$ is isomorphic to the direct sum $M_{k-1/2}(\Gamma_0(4)) \bigoplus M^*_{k-1/2}(\Gamma_0(4))$ via the linear map $\sigma_k : \phi \mapsto (h_0(\tau), h_1(\tau))$.*

2.2. Lifting. For each $\phi \in J_{k,1}(\Gamma_0(4), \chi)$ and for a natural number m the operator V_m called Eichler-Zagier's operator is defined in a manner similar to [EZ] by

$$(\phi|_{k,1} V_m)(\tau, z) = m^{k-1} \sum_{\substack{M \in \Gamma_0(4) \backslash M_2^* \\ \det M = m}} \chi(a)(c\tau + d)^{-k} e\left(-\frac{cmz^2}{c\tau + d}\right) \phi\left(M\tau, \frac{mz}{c\tau + d}\right),$$

where

$$M_2^* = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; \det M \neq 0,\ c \equiv 0 \bmod 4,\ (a, 2) = 1 \right\}.$$

The well-definedness of the operator is easily verified and the transformation formula (J-ii) for $\phi|_{k,1}V_m$ above can be proven without difficulty. Moreover we may rewrite

$$(2.7) \qquad (\phi|_{k,1}V_m)(\tau, z) = m^{k-1} \sum_{\substack{0<a|m \\ (a,2)=1}} \sum_{b \bmod d} \chi(a) d^{-k} \phi\left(\frac{a\tau + b}{d}, \frac{mz}{d}\right),$$

where we put $d = m/a$. Then the transformation formula (J-i) for $\phi|_{k,1}V_m$ can be easily derived from this expression. Hence, $\phi|_{k,1}V_m$ is a Jacobi form of $J_{k,m}(\Gamma_0(4), \chi)$.

Let $\phi(\tau, z)$ have the Fourier expansion of the form (2.1). Then as is easily seen from (2.7), $\phi|_{k,1}V_m$ has the Fourier expansion

$$(2.8) \qquad (\phi|_{k,1}V_m)(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn-r^2\geq 0}} \left(\sum_{0<d|(m,r,n)} \chi(d) d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right) \right) e(n\tau + rz).$$

The following theorem is fundamental concerning the Maass space $\widetilde{Ma}(k, \chi)$.

THEOREM 2.2. *The space $J_{k,1}(\Gamma_0(4), \chi)$ of Jacobi forms corresponds bijectively to the Maass subspace $\widetilde{Ma}(k, \chi)$ via the linear map $l : J_{k,1}(\Gamma_0(4), \chi) \to \widetilde{Ma}(k, \chi)$ given by*

$$l(\phi)\begin{pmatrix} \zeta & z \\ z & \tau \end{pmatrix} = \phi_0(\tau, z) + \sum_{m=1}^{\infty} (\phi|_{k,1}V_m)(\tau, z) e(m\zeta) \quad (\phi \in J_{k,1}(\Gamma_0(4), \chi)).$$

Here we have

$$\phi_0(\tau, z) = \left(\frac{(2/i)^{k-1}\Gamma(k)L(k, \chi)}{\pi^k} + \sum_{n=1}^{\infty} \left(\sum_{0<d|n} \chi(d) d^{k-1} \right) e(n\tau) \right) c(0, 0),$$

where $c(0, 0)$ is the first Fourier coefficient of ϕ.

Proof. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we write $M_\perp$ (resp. $M^\perp$) instead of the matrix

$$\begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix} \in \Gamma_0^{(2)}(4) \quad \left(\text{resp.} \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix} \right).$$

Set, for simplicity, $Z = \begin{pmatrix} \zeta & z \\ z & \tau \end{pmatrix} \in \mathfrak{H}_2$. Then by a direct computation,

$$\begin{aligned} l(\phi)(M_\perp Z) &= \phi_0(M(\tau, z)) + \sum_{m=1}^{\infty} (\phi|_{k,1}V_m)(M(\tau, z)) e\left(m\zeta - \frac{cmz^2}{c\tau + d}\right) \\ &= \phi_0(M(\tau, z)) + \chi(d)(c\tau + d)^k \sum_{m=1}^{\infty} (\phi|_{k,1}V_m)(\tau, z) e(m\zeta). \end{aligned}$$

We note here that $\phi_0(\tau, z)$ coincides with the Eisenstein series

$$E_k(\tau, \chi) = \sum_{M \in \Gamma_\infty \backslash \Gamma_0(4)} \frac{\chi(d)}{(c\tau + d)^k}$$

up to some constant factor. Namely,

$$\phi_0(\tau, z) = \frac{(2/i)^{k-1}\Gamma(k)L(k, \chi)}{\pi^k} E_k(\tau, \chi) \cdot c(0, 0) .$$

Therefore,

$$\begin{aligned} l(\phi)(M_\perp Z) &= \chi(d)(c\tau + d)^k l(\phi)(Z) \\ &= \chi(M_\perp) \det(J(M_\perp, Z))^k l(\phi)(Z) . \end{aligned}$$

On the other hand, in view of the Fourier expansion (2.8) of $\phi|_{k,1} V_m$ we may write

(2.9)

$$l(\phi)(Z) = a(0) + \sum_{\substack{m,n,r \in \mathbb{Z}, 4mn - r^2 \geq 0 \\ (m,n,r) \neq (0,0,0)}} \left(\sum_{0 < d | (m,r,n)} \chi(d) d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right) \right) e(m\zeta + rz + n\tau),$$

where $a(0)$ is given by

$$a(0) = \frac{(2/i)^{k-1}\Gamma(k)L(k, \chi)}{\pi^k} c(0, 0) .$$

Next take

$$V^\sharp = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \quad \text{with} \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Set, for simplicity, $F(Z) = l(\phi)(Z)$. By the symmetry of (2.9) with respect to ζ and τ, we have

$$F(V^\sharp Z) = F(VZV) = F(Z) .$$

Therefore for any $M \in \Gamma_0(4)$,

$$\begin{aligned} F(M^\perp Z) &= F(V^\sharp M_\perp V^\sharp Z) \\ &= F(M_\perp V^\sharp Z) \\ &= \chi(M_\perp) \det J(M_\perp, V^\sharp Z)^k F(V^\sharp Z) \\ &= \chi(M_\perp)(c\zeta + d)^k F(Z) , \end{aligned}$$

from which we get the desired transformation formula

$$F(M^\perp Z) = \chi(M^\perp) \det J(M^\perp, Z)^k F(Z) .$$

Since $\Gamma_0^{(2)}(4)$ is generated by the following elements

$$M_\perp , \quad M^\perp \ (M \in \Gamma_0(4)) , \quad t(s') \ (s' \in S_2(\mathbb{Z})) , \quad \begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix} \ (U \in GL_2(\mathbb{Z})) ,$$

we may conclude that $l(\phi) = F \in M_k(\Gamma_0^{(2)}(4), \chi)$. It is easy to see from (2.9) that the Maass relation for $l(\phi)$ holds and hence that $l(\phi) \in \widetilde{Ma}(k, \chi)$.

Conversely take any $F(Z) = \sum_{T \in S_2^*(\mathbb{Z}),\, T \geq 0} a(T)e(\mathrm{tr}(TZ)) \in \widetilde{M}a(k, \chi)$. If we put

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4n - r^2 \geq 0}} a\begin{pmatrix} 1 & r/2 \\ r/2 & n \end{pmatrix} e(n\tau + rz),$$

then, $\phi \in J_{k,1}(\Gamma_0(4), \chi)$ and the Maass relation for F implies that $F = l(\phi)$. □

Finally we consider the restriction of l to $\sigma_k^{-1}(M_{k-1/2}(\Gamma_0(4)))$ via the isomorphism

$$\sigma_k : J_{k,1}(\Gamma_0(4), \chi) \cong M_{k-1/2}(\Gamma_0(4)) \oplus M^*_{k-1/2}(\Gamma_0(4)).$$

Take any $\phi \in \sigma_k^{-1}(M_{k-1/2}(\Gamma_0(4)))$ and write $\phi = \varphi(\tau)\theta_0(\tau, z)$ with $\varphi \in M_{k-1/2}(\Gamma_0(4))$. We define the map $l^\sharp : M_{k-1/2}(\Gamma_0(4)) \to \widetilde{M}a(k, \chi)$ by putting

$$l^\sharp(\varphi) = l(\phi).$$

Then the following proposition is immediate to see.

PROPOSITION 2.3. *The map $l^\sharp$ induces an isomorphism from $M_{k-1/2}(\Gamma_0(4))$ onto the Maass subspace $Ma(k, \chi)$. Moreover $l^\sharp$ restricted to $S_{k-1/2}(\Gamma_0(4))$ coincides with the previous lifting ι. Namely, if $\varphi \in S_{k-1/2}(\Gamma_0(4))$, then $l^\sharp(\varphi) = \iota(\varphi)$.*

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