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On the Pullback of a Differential Operator and Its Application to Vector Valued Eisenstein Series

by

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1. Introduction

More than twenty years ago H. Klingen [9] introduced a certain type of Eisenstein series to produce holomorphic Siegel modular forms of degree n from cusp forms of lower degree; these Eisenstein series are very useful analytical tool to reduce questions about Siegel modular forms to questions about cusp forms. The Fourier expansion of these Eisenstein series has been studied intensively over the last ten years from various points of view (rationality properties, integrality properties, explicit formulas, see e.g., [5], [7], [1], [11]).

The results mentioned above cover only the case of scalar-valued Siegel modular forms. However one can also introduce and study such Eisenstein series in the theory of vector-valued Siegel modular forms. We became interested in this problem because of a conjecture formulated in [12] on the denominator of the Fourier coefficients of certain vector-valued Eisenstein series of degree 2 (this conjecture will be settled in section 5).

In this paper, we study the Fourier coefficient of vector valued Klingen-type Eisenstein series of type $\det^k \otimes \mathrm{Sym}^l$. For each $n \geq 1$, let $\rho_{k,l,n}$ be a representation $\det^k \otimes \mathrm{Sym}^l$ of $GL(n, \boldsymbol{C})$ and V its representation space. We denote the space of V-valued Siegel modular forms (resp. cusp forms) of degree n and 'weight' $\rho_{k,l,n}$ with respect to $\Gamma_n = Sp(n, \boldsymbol{Z})$ by $M_{k,l,n}(V)$ (resp. $S_{k,l,n}(V)$). Let Z be a variable on the Siegel upper half plane H_n of degree n. Let $1 \leq r \leq n$ and U be a representation space of $\rho_{k,l,r}$. Assume $U \subset V$ and

$$\rho_{k,l,n}\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} u = \det a^k \rho_{k,l,r}(d) u$$

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for all $a \in GL(n-r, \boldsymbol{C})$, $d \in GL(r, \boldsymbol{C})$ and $u \in U$. For such a pair (U, V), we define the Klingen type Eisenstein series $E(f, V) \in M_{k,l,n}(V)$ attached to $f \in S_{k,l,r}(U)$ by

$$E(f, V)(Z) = \sum_{g \in P_{n,r} \backslash \Gamma_n} ((f \circ \mathrm{pr}_r^n |_{k,l} g)(Z) .$$

Here $P_{n,r}$ is a certain parabolic subgroup of Γ_n (see Sect. 4 for definition) and $\mathrm{pr}_r^n \colon H_n \to H_r$ is a projection defined by

$$\mathrm{pr}_r^n \begin{pmatrix} * & * \\ * & z \end{pmatrix} = z$$

where z is of size r. We compute Fourier coefficients of $E(f, V)$.

For a scalar valued case (i.e. $l=0$), this problem was solved in [1]. One of the main tools is a nice decomposition of Siegel's Eisenstein series shown by Garrett [5]. We generalize his result to the space of vector valued modular forms of weight $\det^k \otimes \mathrm{Sym}^l$. In this case however there is no Siegel Eisenstein series (the construction above does not work for $r=0$, $l>0$; moreover the constant term of any vector valued modular form vanishes, see Weissauer [13, Satz 1]). To avoid this difficulty, we construct, in the section 2, a differential operator whose pullback sends modular forms to modular forms of lower degree. Next, we construct Poincaré series of vector valued modular forms of weight $\det^k \otimes \mathrm{Sym}^l$. These results together with the coset decomposition by Garrett [5, Sect. 2–3] yield the desired pullback formula.

Notation. We put $\Gamma_n = Sp(n, \boldsymbol{Z})$. Let ρ be a representation of $GL(n, \boldsymbol{C})$ with a representation space W. Let H_n be the Siegel upper half plane of degree n and f a W-valued C^∞-function on H_n. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \boldsymbol{R})$, we put

$$(f|_\rho M)(Z) = \rho((cZ+d)^{-1})(f(M\langle Z \rangle))$$

where $Z \in H_n$ and

$$M\langle Z \rangle = (aZ+b)(cZ+d)^{-1} .$$

The W-valued C^∞-modular form f of degree n and of weight ρ is a C^∞-function from H_n to W satisfying

$$f|_\rho M = f$$

for all $Z \in H_n$ and $M \in \Gamma_n$. The space of all such functions is denoted by $M_\rho^\infty(W)$. When ρ is a representation $\det^k \otimes \mathrm{Sym}^l$ of $GL(n, \boldsymbol{C})$, we write $|_\rho$ and $M_\rho^\infty(W)$ as $|_{k,l,n}$ and $M_{k,l,n}^\infty(W)$, respectively. We note $M_{k,l,n}^\infty(W) = \{0\}$ unless $nk \equiv l \bmod 2$. We put

$$M_{k,l,n}(W) = \{f \in M_{k,l,n}^\infty(W) \mid f \text{ is holomorphic on } H_n \text{ (and its cusp)}\}$$

and

$$S_{k,l,n}(W) = \{f \in M_{k,l,n}(W) \mid f \text{ is a cusp form.}\}$$

We omit the subscript ',n' when there is no fear of confusion. For a vector space W, we denote by $W^{(l)}$ its l-th symmetric tensor product. We identify $W^{(0)}$ with $\boldsymbol{C}$. Let $x=(x_1, \cdots, x_n)$ be a row vector consisting of n indeterminates. Through out this paper, we put $V=\boldsymbol{C}x_1 \oplus \cdots \oplus \boldsymbol{C}x_n$. We identify $V^{(l)}$ with $\boldsymbol{C}[x_1, \cdots, x_n]_{(l)}$ where the subscript (l) stands for homogeneous polynomials of degree l. Then $GL(n, \boldsymbol{C})$ acts on $V^{(l)}$ by

$$(gv)(x)=\det g^k v(xg)$$

for $g \in GL(n, \boldsymbol{C})$ and $v \in V^{(l)}$. This is isomorphic to $\det^k \otimes \mathrm{Sym}^l$ and we always use this realization. We also identify $C^\infty(H_n, V^{(l)})$ with $C^\infty(H_n)[x_1, \cdots, x_n]_{(l)}$.

2. Differential operators

Let $Z=(z_{ij})$ be a variable on H_n. For an integer $l \geq 0$ and a function $f \in C^\infty(H_n, V^{(l)})$, we put

$$Df=\left(\frac{1}{2\pi i}\frac{\partial}{\partial Z}f\right)[x],$$

$$Nf=\left(-\frac{1}{4\pi}(\operatorname{Im} Z)^{-1}f\right)[x].$$

and

$$\delta_k f=kNf+Df. \tag{2.1}$$

Here, $\dfrac{\partial}{\partial Z}=\left(\dfrac{1+\delta_{ij}}{2}\dfrac{\partial}{\partial z_{ij}}\right)_{1\leq i,j\leq n}$ and $A[x]=xA^t x$. (Since we are mainly concerned with row vectors, this definition of $A[x]$ is convenient.) Then, Df, Nf and $\delta_k f$ are $V^{(l+2)}$-valued functions. For an integer $l \geq 0$, we put

$$k^{[l]}=\begin{cases} k(k+1)\cdots(k+l-1) & (l>0) \\ 1 & (l=0). \end{cases}$$

Note $A^{[n]}=(-1)^n(-A-n+1)^{[n]}$. We also have

$$(A+B)^{[n]}=\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}A^{[r]}B^{[n-r]}$$

and

$$\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}(A-2r)(-A)^{[r]}(A-2n)^{[n-r]}=\begin{cases} A & (n=0) \\ 0 & (n\neq 0). \end{cases} \tag{2.2}$$

LEMMA 2.1. *The operator δ_{k+l} satisfies*

$$(\delta_{k+l}f)\,|_{k,l+2}M=\delta_{k+l}(f\,|_{k,l}M) \tag{2.3}$$

for $f \in C^\infty(H_n, V^{(l)})$ and $M \in \Gamma_n$. Especially, it maps $M^\infty_{k,l,n}(V^{(l)})$ to $M^\infty_{k,l+2,n}(V^{(l+2)})$. For each integer $l \geq 0$, we have

$$\delta^r_{k+l} = \sum_{i=0}^{r} (k+l+r-i)^{[i]} \binom{r}{i} N^i D^{r-i} . \tag{2.4}$$

Here by δ^r_{k+l} we mean the composition $\delta_{k+l+2r-2} \circ \cdots \circ \delta_{k+l+2} \circ \delta_{k+l}$.

Proof. Let f be a function in $C^\infty(H_n, V^{(l)})$. So $f = f(Z, x)$ is a C^∞-function on H_n as a function in Z and is a homogeneous polynomial of degree l in x. Take a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ_n. Now the operator $|_{k,l}M$ has the following form

$$(f|_{k,l}M)(Z, x) = \det(cZ+d)^{-k} f(M\langle Z\rangle, x(cZ+d)^{-1}) .$$

It is easy to see that

$$\begin{aligned} &D(f(M\langle Z\rangle, x(cZ+d)^{-1})) \\ &= (Df)(M\langle Z\rangle, x(cZ+d)^{-1}) - \frac{l}{2\pi i} f(M\langle Z\rangle, x(cZ+d)^{-1})((cZ+d)^{-1}c)[x] \end{aligned}$$

and

$$D \det(cZ+d)^{-k} = -\frac{k}{2\pi i} \det(cZ+d)^{-k}((cZ+d)^{-1}c)[x] .$$

Therefore we have

$$(Df|_{k,l+2}M)(Z, x) = D(f|_{k,l}M(Z, x)) + \frac{k+l}{2\pi i} f|_{k,l}M(Z, x) \cdot ((cZ+d)^{-1}c)[x] .$$

On the other hand since

$$(\operatorname{Im} M\langle Z\rangle)^{-1} = (\operatorname{Im} Z)^{-1}[cZ+d] - 2i(cZ+d)^t c ,$$

we have

$$(Nf)|_{k,l+2}M(Z, x) = N(f|_{k,l}M(Z, x)) - \frac{1}{2\pi i}(f|_{k,l}M)(Z, x) \cdot ((cZ+d)^{-1}c)[x] .$$

By canceling out unnecessary terms, we obtain

$$((D+(k+l)N)f)|_{k,l+2}M(Z, x) = (D+(k+l)N)(f|_{k,l}M)(Z, x) .$$

This proves the first part.

An easy calculation shows that

$$D(\operatorname{Im} Z)^{-1}[x] = \frac{1}{4\pi}((\operatorname{Im} Z)^{-1}[x])^2 .$$

Since D is a derivation and N is essentially a multiplication,

$$DN^l = -lN^{l+1} + N^l D\,. \tag{2.5}$$

Using induction on r, we have (2.4). $\square$

It is remarkable that the differential operator acting on $M^{\infty}_{k,l,n}(V^{(l)})$ depends only on $k+l$. We note that (2.1) and (2.4) do not explicitly contain n. Let $G_j(t)$ be a formal power series of t defined by

$$G_j(t) = \sum_{l=0}^{\infty} \frac{t^l}{l!\,j^{[l]}} \delta_j^l\,.$$

Following Cohen [4, Sect. 7], we have

$$G_j(t) = e^{tN} \sum_{l=0}^{\infty} \frac{t^l}{l!\,j^{[l]}} D^l\,.$$

In what follows, we put $n=p+q$ where p and q are positive integers. Let $V_1 = \boldsymbol{C}x_1 \oplus \cdots \oplus \boldsymbol{C}x_p$ and $V_2 = \boldsymbol{C}x_{p+1} \oplus \cdots \oplus \boldsymbol{C}x_n$ be two subspaces of V. We note that $V_1^{(l)}$ and $V_2^{(l)}$ are subspaces of $V^{(l)}$ which are stable under the action of $GL(p)\times GL(q)$. Let X be any map $X\colon A \to C^{\infty}(H_n, V^{(l)})$ for any set A. We define two maps $X_{\uparrow}\colon A \to C^{\infty}(H_n, V_1^{(l)})$ and $X_{\downarrow}\colon A \to C^{\infty}(H_n, V_2^{(l)})$ by

$$(X_{\uparrow}(a))(x_1, \cdots, x_p) = (X(a))(x_1, \cdots, x_p, 0, \cdots, 0)$$

and

$$(X_{\downarrow}(a))(x_{p+1}, \cdots, x_n) = (X(a))(0, \cdots, 0, x_{p+1}, \cdots, x_n)\,.$$

Let d^* be the pullback of the diagonal embedding $d\colon H_p \times H_q \to H_n$. Now, for each $l \geq 0$, define an operator

$$L^{(l)}\colon \mathrm{Hol}(H_n, \boldsymbol{C}) \longrightarrow \mathrm{Hol}(H_p \times H_q, V^{(2l)})$$

inductively by

$$d^* \sum_{l=0}^{\infty} \frac{t^l}{l!\,k^{[l]}} D^l = \sum_{l=0}^{\infty} \left(\sum_{\lambda=0}^{\infty} \frac{t^{\lambda}}{\lambda!(k+l)^{[\lambda]}} D_{\uparrow}^{\lambda} \right) \left(\sum_{\lambda=0}^{\infty} \frac{t^{\lambda}}{\lambda!(k+l)^{[\lambda]}} D_{\downarrow}^{\lambda} \right) t^l L^{(l)}\,. \tag{2.6}$$

Lemma 2.2.

$$L^{(l)} = \frac{1}{k^{[l]}} d^* \sum_{0 \leq 2\nu \leq l} \frac{1}{\nu!(l-2\nu)!(2-k-l)^{[\nu]}} (D_{\uparrow} D_{\downarrow})^{\nu} (D - D_{\uparrow} - D_{\downarrow})^{l-2\nu}\,. \tag{2.7}$$

Proof. Since $D_{\uparrow}$ and $D_{\downarrow}$ commute,

$$\left(\sum_{l=0}^{\infty} \frac{1}{l!\,k^{[l]}} D_{\uparrow}^l t^l \right) \left(\sum_{l=0}^{\infty} \frac{1}{l!\,k^{[l]}} D_{\downarrow}^l t^l \right)$$

$$= \sum_{r=0}^{\infty} \frac{1}{k^{[r]}} \left(\sum_{0 \leq \mu \leq r/2} \frac{1}{\mu!(r-2\mu)!\,k^{[\mu]}} (D_{\uparrow} + D_{\downarrow})^{r-2\mu} (D_{\uparrow} D_{\downarrow})^{\mu} \right) t^r\,. \tag{2.8}$$

Since (2.6) uniquely determines $L^{(l)}$, we have only to verify that (2.7) satisfies (2.6). The coefficient of t^l on the right side is

$$\sum_{0\le j\le l/2}\sum_{0\le\rho\le l-2j}\frac{1}{\rho!(l-2j-\rho)!}$$
$$\times\sum_{0\le\mu\le j}\frac{1}{\mu!(j-\mu)!(k+l-\rho-2\mu)^{[\mu]}(-k-l+\rho+2\mu+2)^{[j-\mu]}}$$
$$\times(D_\uparrow+D_\downarrow)^\rho(D_\uparrow D_\downarrow)^j(D-D_\uparrow-D_\downarrow)^{l-2j-\rho}.$$

Using (2.2), we see this is $\dfrac{1}{l!}D^l$. □

Note the direct sum decomposition

$$V^{(l)}=(V_1\oplus V_2)^{(l)}=\bigoplus_{a=0}^{l}V_1^{(a)}\cdot V_2^{(l-a)}$$

where $\cdot$ is a symmetric tensor product. We denote by π_a^l the projection $V^{(l)}\to V_1^{(a)}\cdot V_2^{(l-a)}$. For $f\in\mathrm{Map}(H_n,V^{(l)})$, define π_a^lf by $(\pi_a^lf)(Z)=\pi_a^l(f(Z))$. We have

$$\pi_{a+2}^{l+2}\delta_{k\uparrow}=\delta_{k\uparrow}\pi_a^l, \tag{2.9}$$

$$\pi_a^{l+2}\delta_{k\downarrow}=\delta_{k\downarrow}\pi_a^l. \tag{2.10}$$

Let $f\in C^\infty(H_n)$. For $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in Sp(p)$ and $M'=\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\in Sp(q)$, we put

$$M_\uparrow=\begin{pmatrix}a&0&b&0\\0&1&0&0\\c&0&d&0\\0&0&0&1\end{pmatrix}\in Sp(n)\quad\text{and}\quad M'_\downarrow=\begin{pmatrix}1&0&0&0\\0&a'&0&b'\\0&0&1&0\\0&c'&0&d'\end{pmatrix}\in Sp(n).$$

With this notation, we have

$$\pi_a^ld^*(f|_{k,l,n}M_\uparrow)=(\pi_a^ld^*f)|_{k,a,p}M, \tag{2.11}$$

$$\pi_a^ld^*(f|_{k,l,n}M'_\downarrow)=(\pi_a^ld^*f)|_{k,l-a,q}M'. \tag{2.12}$$

PROPOSITION 2.3. *Let $f\in C^\infty(H_n)$, $M\in Sp(p)$ and $M'\in Sp(q)$. Then*

$$(L^{(l)}f)|_{k,l,p}M|_{k,l,q}M'=L^{(l)}(f|_{k,0,n}M_\uparrow|_{k,0,n}M'_\downarrow). \tag{2.13}$$

Especially if $f\in M_{k,0,n}(C)$, then

$$L^{(l)}f\in M_{k,l,p}(V_1^{(l)})\otimes M_{k,l,q}(V_2^{(l)}). \tag{2.14}$$

Proof. We use induction on l. For $l=0$, this proposition certainly holds because $L^{(0)}=d^*$. Let $l>0$. Multiplying $d^*e^{tN}=e^{t(N_\uparrow+N_\downarrow)}$ on both sides of (2.6), we have

$$d^*\sum_{l=0}^{\infty}\frac{t^l}{l!k^{[l]}}\delta_k^l=\sum_{l=0}^{\infty}\left(\sum_{\lambda=0}^{\infty}\frac{t^\lambda}{\lambda!(k+l)^{[\lambda]}}\delta_{k+l\uparrow}^\lambda\right)\left(\sum_{\lambda=0}^{\infty}\frac{t^\lambda}{\lambda!(k+l)^{[\lambda]}}\delta_{k+l\downarrow}^\lambda\right)t^lL^{(l)}\,.$$

Hence there are constants $c_{l,j,a}\in\boldsymbol{C}$ such that

$$\frac{1}{l!k^{[l]}}d^*\delta_k^lf=L^{(l)}f+\sum_{j=1}^{l}\left(\sum_{a=0}^{j}c_{j,l,a}\delta_{k+l-j\uparrow}^a\delta_{k+l-j\downarrow}^{j-a}\right)L^{(l-j)}f\,. \tag{2.15}$$

By Lemma 2.2, $(L^{(l)}f)(Z)\in V_1^{(l)}\cdot V_2^{(l)}$. Hence

$$\pi_a^{2l}L^{(l)}f=\begin{cases}L^{(l)}f & (a=l)\\ 0 & (a\neq l)\,.\end{cases} \tag{2.16}$$

We apply π_l^{2l} on both sides of (2.15). Then,

$$\begin{aligned}\frac{1}{l!k^{[l]}}\pi_l^{2l}d^*\delta_k^lf&=L^{(l)}f+\sum_{j=1}^{l}\sum_{a=0}^{j}c_{l,j,a}\pi_l^{2l}\delta_{k+l-j\uparrow}^a\delta_{k+l-j\downarrow}^{j-a}L^{(l-j)}f\\&=L^{(l)}f+\sum_{j=1}^{l}\sum_{a=0}^{j}c_{l,j,a}\delta_{k+l-j\uparrow}^a\delta_{k+l-j\downarrow}^{j-a}\pi_{l-2a}^{2l-2j}L^{(l-j)}f\\&=L^{(l)}f+\sum_{1\le j\le l/2}c_{l,2j,j}\delta_{k+l-2j\uparrow}^j\delta_{k+l-2j\downarrow}^jL^{(l-2j)}f\end{aligned}$$

by (2.9), (2.10) and (2.16). Hence (2.13) holds by induction hypotheses and (2.11) and (2.12). Therefore, if $f\in M_{k,0,n}(\boldsymbol{C})$, then

$$L^{(l)}f\in M_{k,l,p}^{\infty}(V_1^{(l)})\otimes M_{k,l,q}^{\infty}(V_2^{(l)})\,.$$

By definition, $L^{(l)}f$ is a holomorphic function on $H_p\times H_q$, which proves (2.14). □

REMARK 2.4. Adding a certain term to D, we obtain a differential operator acting on the space of Jacobi forms. We fix a positive integer m, which is an index. For integers k, l and m, an action $|_{k,l,m}$ of the Jacobi group Γ_n^J of degree n can be defined on $C^\infty(H_n\times\boldsymbol{C}^n, V^{(l)})$. Let $J_{k,l,m,n}^{\infty}(V^{(l)})$ (resp. $J_{k,l,m,n}(V^{(l)})$) be the set of C^∞-Jacobi forms (resp. holomorphic Jacobi forms) of 'weight' $\rho_{k,l,n}$, index m and degree n. These are straightforward generalization of those for scalar valued forms stated in [14]. Let $(\zeta_1,\cdots,\zeta_n)$ be a variable on $\boldsymbol{C}^n$ and put

$$Df=\left(\frac{1}{2\pi i}\frac{\partial}{\partial Z}f-\frac{1}{4m}\left(\frac{1}{2\pi i}\right)^{2t}\left(\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial}{\partial\zeta}\right)f\right)[x]$$

where

$$\frac{\partial}{\partial\zeta}=\left(\frac{\partial}{\partial\zeta_1},\cdots,\frac{\partial}{\partial\zeta_n}\right).$$

We use the same construction for other operators. After much more computation, we can show that (2.3) with $|_{k,l,m}$ instead of $|_{k,l}$ holds for any $f\in C^\infty(H_n\times\boldsymbol{C}^n)$ and

$M\in\Gamma_n^J$ and that (2.5) remains to be valid. So, we obtain the following differential operators:

$$\delta_k: J_{k,l,m,n}^{\infty}(V^{(l)}) \longrightarrow J_{k,l+2,m,n}^{\infty}(V^{(l+2)})$$

and

$$L^{(l)}: J_{k,0,m,n}(\boldsymbol{C}) \longrightarrow J_{k,l,m,p}(V_1^{(l)}) \otimes J_{k,l,m,q}(V_2^{(l)})\,.$$

3. The kernel function

In this section we describe explicitly certain vector valued Poincaré series. For a symmetric positive definite matrix S, we denote by $\sqrt{S}$ the unique symmetric positive definite matrix satisfying $S=\sqrt{S}^2$. As is in the previous section, let $V=\boldsymbol{C}x_1\oplus\cdots\oplus\boldsymbol{C}x_n$. Let $y=(y_1,\cdots,y_n)$ be an another row vector consisting of indeterminates and put $U=\boldsymbol{C}y_1\oplus\cdots\oplus\boldsymbol{C}y_n$. Then inner product

$$\left(\sum_{i=1}^n a_ix_i, \sum_{i=1}^n b_ix_i\right)=\sum_{i=1}^n a_i\overline{b_i}$$

induces an inner product of $V^{(l)}$ defined by

$$(\alpha_1\cdots\alpha_l, \beta_1\cdots\beta_l)=\frac{1}{l!}\sum_{\tau}\prod_{j=1}^{l}(a_{\tau(j)}, \beta_j)$$

where $\alpha_j, \beta_j\in V$ and τ runs over the symmetric group of degree l. It is also denoted by (,). This is invariant under the action of unitary matrices by Sym^l. We extend this inner product $V^{(l)}\times V^{(l)}\to\boldsymbol{C}$ to the map $V^{(l)}\cdot U^{(l)}\times V^{(l)}\to U^{(l)}$ complex linearly by

$$(v_1u, v_2)=(v_1, v_2)u$$

for a monomial u of $y_1,\cdots,y_n$. If $\alpha\in V^{(l)}$ and $\beta\in V^{(l)}\cdot U^{(l)}$, we understand (α,β) to be $\overline{(\beta,\alpha)}$. We fix an isomorphism σ from V to U defined by $\sigma(x_i)=y_i$, which induces an isomorphism (also denoted by σ) from $V^{(l)}$ to $U^{(l)}$. Note

$$(v, (x^ty)^l)=\sigma(v)$$

for any $v\in V^{(l)}$. Put $\rho_{k,l}=\det^k\otimes\mathrm{Sym}^l$. We define the Petersson inner product of $f, g\in M_{k,l,n}^{\infty}(V^{(l)})$ by

$$(f, g)_{k,l}=\int_{\Gamma_n\backslash H_n}(\rho_{k,l}(\sqrt{\mathrm{Im}\,Z})f(Z), \rho_{k,l}(\sqrt{\mathrm{Im}\,Z})g(Z))\det(\mathrm{Im}\,Z)^{-n-1}dZ$$

whenever this integral converges. We again extend it to the map

$$(\ ,\ )_{k,l}: M_{k,l,n}^{\infty}(V^{(l)})\times M_{k,l,n}^{\infty}(V^{(l)})\cdot C^{\infty}(H_n, U^{(l)})\to C^{\infty}(H_n, U^{(l)})\,.$$

Define Poincaré series by

$$P_{k,l,n}(Z, W; V^{(l)}, U^{(l)}) = \sum_{M\in\Gamma_n} \left(\rho_{k,l}(Z-\bar{W})^{-1}(x^ty)^l\right)\Big|_{k,l}M\,,$$

where we regard $(x^ty)^l$ as a $V^{(l)}\cdot U^{(l)}$-valued constant function.

PROPOSITION 3.1. *Let $m = \dim S_{k,l,n}(V^{(l)})$ and $f_1, \cdots, f_m$ be an orthonormal basis of $S_{k,l,n}(V^{(l)})$. Then,*

$$P_{k,l,n}(Z, W; V^{(l)}, U^{(l)}) = C_{k,l,n} \sum_{j=1}^{m} f_j(Z)\cdot\sigma(\overline{f_j(W)}) \tag{3.1}$$

where

$$C_{k,l,n} = 2^{n(n-k+1)-l+1} i^{nk+l} \frac{\pi^{n(n+1)/2}}{k+l-1} \prod_{j=1}^{n-1} \frac{\Gamma(2k-2n+2j-1)(2k-n+j-2)^{[l]}}{(k-n-1+j)\Gamma(2k+j+l-n-1)}\,.$$

Proof. The equation (3.1) is equivalent to

$$(f(Z), P_{k,l,n}(Z, W; V^{(l)}, U^{(l)}))_{k,l} = C_{k,l,n}\sigma(f(W))$$

for all $f \in S_{k,l,n}(V^{(l)})$. Let S_n be the generalized unit circle of degree n:

$$S_n = \{S = {}^tS \in M(n, \boldsymbol{C}) \mid 1_n - S\bar{S} > 0\}\,.$$

A computation similar to that in Klingen [10, Sect. 1] gives

$$\begin{aligned}&(f(Z), P_{k,l,n}(Z, W; V^{(l)}, U^{(l)}))_{k,l}\\ &= 2^{n(n-k+1)-l+1} i^{nk+l} \rho_{k,l}(\sqrt{\operatorname{Im} W}^{-1})\psi_{k-n-1,l,n}\rho_{k,l}(\sqrt{\operatorname{Im} W})\sigma(f(W))\end{aligned}$$

where

$$\psi_{a,l,n} = \int_{S_n} \rho_{a,l}(1_n - S\bar{S})dS\,.$$

Changing the variable S to ${}^tUSU$, we see

$$\psi_{a,l,n} = \rho_{a,l}(U^{-1})\psi_{a,l,n}\rho_{a,l}(U)$$

for any unitary matrix U. Since $\rho_{a,l}$ is an irreducible representation of $U(n, \boldsymbol{C})$, the operator $\psi_{a,l,n}$ is a homothety by Schur's lemma. That is, there exists a constant $c_{a,l,n}$ satisfying $\psi_{a,l,n} = c_{a,l,n}\mathrm{Id}$. Hence the proposition follows from

$$c_{a,l,n} = \frac{\pi^{n(n+1)/2}}{a+n+l} \prod_{j=1}^{n-1} \frac{\Gamma(2a+2j+1)(n+j+2a)^{[l]}}{(a+j)\Gamma(l+n+j+2a+1)}\,. \tag{3.2}$$

We compute $c_{a,l,n}$. Let q_n be a row vector $(1, 0, \cdots, 0)$ of length n.

$$\begin{aligned}c_{a,l,n} &= (\psi_{a,l,n}x_1^l, x_1^l)\\ &= \int_{S_n} \det(1_n - S\bar{S})^a((1_n - S\bar{S})[q_n])^l dS\,.\end{aligned}$$

We set $S=\begin{pmatrix} S_1 & {}^t v \\ v & z \end{pmatrix}$. By Hua [8, Sect. 2.3], especially by Theorem 2.3.2 there,

$$c_{a,l,n}=\frac{\pi}{a+1}\int_{1_{n-1}-S_1\bar{S}_1-{}^t v\bar{v}>0}\frac{\det(1_{n-1}-S_1\bar{S}_1-{}^t v\bar{v})^a((1_{n-1}-S_1\bar{S}_1-{}^t v\bar{v})[q_{n-1}])^l}{(1+\bar{v}(1_{n-1}-S_1\bar{S}_1-{}^t v\bar{v})^{-1}{}^t v)^{a+2}}dvdS_1$$

$$=\frac{\pi}{a+1}\int_{1_{n-1}-S_1\bar{S}_1>0}\det(1_{n-1}-S_1\bar{S}_1)^{a+1}$$

$$\times\int_{1-\bar{u}{}^t u>0}(1-\bar{u}{}^t u)^{2a+2}(\xi_1(1_{n-1}-{}^t u\bar{u}){}^t\overline{\xi_1})^l dudS_1\,,$$

where $\xi_1=\sqrt{1_{n-1}-S_1\bar{S}_1}\,q_{n-1}$. Put

$$\varphi_{a,l,n}=\int_{\substack{1-\bar{u}{}^t u>0\\ u\in\boldsymbol{C}^n}}(1-\bar{u}{}^t u)^a\,\mathrm{Sym}^l(1_n-{}^t u\bar{u})du\,.$$

Using Schur's lemma again, there exists a constant $d_{a,l,n}$ satisfying $\varphi_{a,l,n}=d_{a,l,n}\mathrm{Id}$. Then,

$$\int_{1-\bar{u}{}^t u>0}(1-\bar{u}{}^t u)^{2a+2}(\xi_1(1_{n-1}-{}^t u\bar{u}){}^t\overline{\xi_1})^l du$$

$$=(\varphi_{2a+2,l,n-1}\xi_1^l,\xi_1^l)=d_{2a+2,l,n-1}(\xi_1,\xi_1)^l$$

$$=d_{2a+2,l,n-1}((1_{n-1}-S_1\bar{S}_1)[q_{n-1}])^l\,.$$

Therefore,

$$c_{a,l,n}=\frac{\pi}{a+1}c_{a+1,l,n-1}d_{2a+2,l,n-1}\,. \tag{3.3}$$

The value of $d_{a,l,n}$ is calculated as follows:

$$d_{a,l,n}=(\varphi_{a,l,n}x_1,x_1)$$

$$=\int_{\substack{1-\bar{u}{}^t u>0\\ u\in\boldsymbol{C}^n}}(1-\bar{u}{}^t u)^a((1_n-{}^t u\bar{u})[q_n])^l du$$

$$=\int_{1-\sum_{j=1}^{2n}t_i^2>0}\left(1-\sum_{j=1}^{2n}t_j^2\right)^a(1-t_1^2-t_2^2)^l dt_1\cdots dt_{2n}$$

$$=\pi^n\frac{\Gamma(a+1)}{\Gamma(a+l+n+1)}(n+a)^{[l]}\,. \tag{3.4}$$

By Hua [8, (2.2.6)],

$$c_{a,l,1}=\frac{\pi}{a+l+1}\,. \tag{3.5}$$

Summing up (3.3)–(3.5), we obtain (3.2). $\square$

4. The pullback formula

In this section, we prove a vector valued version of Garrett's Pullback formula. Let p and q be positive integers. To keep notation simple, we put

$$\begin{aligned} x_A&=(x_1, \cdots, x_{p-r}), \\ x_B&=(x_{p-r+1}, \cdots, x_p), \\ x_C&=(x_{p+1}, \cdots, x_{p+q-r}), \\ x_D&=(x_{p+q-r+1}, \cdots, x_{p+q}) \end{aligned}$$

and

$$\begin{aligned} V_{AB}&=\boldsymbol{C}x_1 \oplus \cdots \oplus \boldsymbol{C}x_p, \\ V_B&=\boldsymbol{C}x_{p-r+1} \oplus \cdots \oplus \boldsymbol{C}x_p, \\ V_{CD}&=\boldsymbol{C}x_{p+1} \oplus \cdots \oplus \boldsymbol{C}x_{p+q}, \\ V_D&=\boldsymbol{C}x_{p+q-r+1} \oplus \cdots \oplus \boldsymbol{C}x_{p+q} \end{aligned}$$

for an integer r with $0 \leq r \leq \min(p, q)$. Let $\sigma=\sigma_r$ be an automorphism of $V^{(l)}$ induced from

$$\sigma(x_j)=\begin{cases} x_{j+q} & \text{for } \ p-r<j\leq p, \\ x_{j-q} & \text{for } \ p+q-r<j\leq p+q, \\ x_j & \text{otherwise}. \end{cases}$$

We note that σ exchanges $V_B^{(l)}$ and $V_D^{(l)}$. For $f \in C^\infty(H_r, V_B^{(l)})$ we define $\sigma(f)$ by $(\sigma(f))(z)=\sigma(f(z))$. Let $P_{n,r}$ be the subgroup of Γ_n consisting of all elements whose entries in last $n+r$ rows and first $n-r$ columns vanish. Then Siegel's Eisenstein series $E_k^n(Z)$ of weight k and of degree n is

$$E_k^n(Z)=\sum_{g \in P_{n,0}\backslash \Gamma_n} (1|_{k,0}g)(Z).$$

For $k>n+1$, this converges absolutely and uniformly on any compact set in H_n. We prepare a lemma on a certain finite sum whose proof presented here is due to Prof. D. B. Zagier.

LEMMA 4.1. *For integers $m \geq 0$ and $k \geq 2$,*

$$\sum_{l+2\lambda=m} \frac{(2k-2)^{[l]}}{l!\lambda!(k-1)^{[l]}(k+l)^{[\lambda]}}=\frac{2^m}{m!}.$$

Proof. Denote the expression on the left by A. Then

$$\begin{aligned}\frac{(k+m-1)!}{(k-1)!}A&=\sum_{l+2\lambda=m}\left\{\binom{2k+l-2}{2k-2}+\binom{2k+l-3}{2k-2}\right\}\binom{k+m-1}{\lambda}\\&=\mathrm{Res}_{x=0}\left[\frac{1+x}{(1-x)^{2k-1}}(1+x^2)^{k+m-1}\frac{dx}{x^{m+1}}\right]\\&=\mathrm{Res}_{t=0}\left[(1-2t)^{-k}\frac{dt}{t^{m+1}}\right]\qquad\left(t=\frac{x}{1+x^2}\right)\\&=2^m\binom{k+m-1}{m}.\end{aligned}$$ □

LEMMA 4.2. *Let p and q be positive integers and put $n=p+q$. Let $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_n$ and $Z\in H_n$. Let $k\geq 2$ and $l\geq 0$ be integers. Then*

$$\begin{aligned}&L^{(l)}(\det(cZ+d)^{-k})\\&=d^*\alpha_{k,l}(\det(cZ+d)^{-k})((x_A\ \ x_B\ \ 0\ \ 0)(cZ+d)^{-1}c^t(0\ \ 0\ \ x_C\ \ x_D))^l\end{aligned}\tag{4.1}$$

where

$$\alpha_{k,l}=\left(-\frac{1}{2\pi i}\right)^l\frac{(2k-2)^{[l]}}{l!(k-1)^{[l]}}.$$

Especially, let M be a symmetric matrix of size $0\leq r\leq\min(p,q)$ and put

$$\tilde{M}=\begin{pmatrix}0_{p-r,q-r}&0\\0&M\end{pmatrix},\ \tilde{c}=\begin{pmatrix}0_p&\tilde{M}\\{}^t\tilde{M}&0_q\end{pmatrix}\quad\text{and}$$

$$g_{\tilde{M}}=\begin{pmatrix}1_n&0\\\tilde{c}&1_n\end{pmatrix}.\tag{4.2}$$

(*We understand that $\tilde{c}=0$ for $r=0$.*) *Let $l>0$. Then,*

$$L^{(l)}(1|_{k,0}g_{\tilde{M}})(z,w)=\begin{cases}\alpha_{k,l}\rho_{k,l}(1_r-Mw_3Mz_3)^{-1}(x_B{}^t(x_DM))^l&(r>0),\\0&(r=0),\end{cases}\tag{4.3}$$

where $z=\begin{pmatrix}z_1&{}^tz_2\\z_2&z_3\end{pmatrix}\in H_p$, $w=\begin{pmatrix}w_1&{}^tw_2\\w_2&w_3\end{pmatrix}\in H_q$ and $\rho_{k,l}$ acts on $V_B^{(l)}$.

Proof. For simplicity, we put

$$P=\frac{1}{2\pi i}((cZ+d)^{-1}c)[(x_A\ \ x_B\ \ 0\ \ 0)],$$

$$Q=\frac{1}{2\pi i}(x_A\ \ x_B\ \ 0\ \ 0)((cZ+d)^{-1}c)^t(0\ \ 0\ \ x_C\ \ x_D),$$

$$R=\frac{1}{2\pi i}((cZ+d)^{-1}c)[(0\ \ 0\ \ x_C\ \ \ x_D)]\,,$$

$$S=\frac{1}{2\pi i}((cZ+d)^{-1}c)[x]$$

and

$$\delta=\frac{1}{2\pi i}\det(cZ+d)\,.$$

We actually prove the following equality

$$\sum_{l=0}^{\infty}\frac{t^l}{l!k^{[l]}}D^l\delta^{-k}=\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{(2\pi i)^l\alpha_{k,l}}{(k+l)^{[r]}}W(k,l,r)t^{l+r} \tag{4.4}$$

where

$$W(k,l,r)=\sum_{0\le\mu\le r/2}\frac{1}{\mu!(r-2\mu)!k^{[\mu]}}(D_\uparrow+D_\downarrow)^{r-2\mu}(D_\uparrow D_\downarrow)^\mu(\delta^{-k}Q^l)\,.$$

Then (4.1) follows form it together with (2.6) and (2.8). It is easy to see that

$$D_\uparrow P=-P^2\,,\qquad D_\uparrow Q=-PQ\,,\qquad D_\uparrow R=-Q^2\,,$$
$$D_\downarrow P=-Q^2\,,\qquad D_\downarrow Q=-RQ\,,\qquad D_\downarrow R=-R^2$$

and

$$D\delta=\delta S\,,\qquad DS=-S^2\,.$$

Hence the left hand side of (4.4) is

$$\sum_{l=0}^{\infty}\frac{1}{l!}\delta^{-k}(-tS)^l=\delta^{-k}\exp(-tS)\,. \tag{4.5}$$

We have

$$(D_\uparrow D_\downarrow)^\mu(\delta^{-k}Q^l)=(k+l)^{[\mu]}\sum_{\nu=0}^{\mu}(k+l+\nu)^{[\mu]}\binom{\mu}{\nu}\delta^{-k}Q^{l+2\nu}(PR-Q^2)^{\mu-\nu}$$

and

$$\begin{aligned}&(D_\uparrow+D_\downarrow)^r(\delta^{-k}Q^j(PR-Q^2)^\nu)\\&=\sum_{0\le s\le r/2}(-1)^{r-s}\frac{r!}{(r-2s)!s!}(k+j+\nu)^{[r-s]}\delta^{-k}Q^j(P+R)^{r-2s}(PR-Q^2)^{\nu+s}\end{aligned}$$

by induction on μ and r respectively. Using these formulas, we obtain

$$W(k, l, r) = \sum_{0 \leq j \leq r/2} \sum_{j \leq m \leq r/2} (k+l+j)^{[j]}(1-k-l-r+m-j)^{[r-2m]}$$
$$\times A_{k+l,j}(m)\delta^{-k}Q^{l+2j}(P+R)^{r-2m}(PR)^{m-j}$$

where

$$A_{q,j}(m) = \sum_{0 \leq \mu \leq m-j} \frac{(q+2j)^{[\mu]}(1-q-m-j)^{[m-j-\mu]}}{\mu!(m-j-\mu)!}$$
$$= \begin{cases} 1 & \text{for } m=j, \\ 0 & \text{for } m<j. \end{cases}$$

Hence

$$W(k, l, r) = (-1)^r \sum_{0 \leq j \leq r/2} \frac{(k+l+j)^{[r-j]}}{(r-2j)!j!} \delta^{-k}Q^{l+2j}(P+R)^{r-2j}.$$

Therefore the right hand side of (4.4) is

$$\sum_{l=0}^{\infty} \frac{(2k-2)^{[l]}}{l!(k-1)^{[l]}} \sum_{r=0}^{\infty} \sum_{0 \leq j \leq r/2} \frac{1}{(r-2j)!j!(k+l)^{[j]}} \delta^{-k}Q^{l+2j}(P+R)^{r-2j}(-t)^{l+r}$$
$$= \delta^{-k} \exp(-t(P+R)) \sum_{\mu=0}^{\infty} \left(\sum_{0 \leq j \leq \mu/2} \frac{(2k-2)^{[\mu-2j]}}{(\mu-2j)!j!(k-1)^{[\mu-2j]}(k+\mu-2j)^{[j]}} \right)(-tQ)^{\mu}$$
$$= \delta^{-k} \exp(-t(P+R)) \exp(-2tQ)$$

by Lemma 4.1. This is equal to (4.5) because $P+2Q+R=S$.

Let $g_{\tilde{M}}$ be as in (4.2) and put $J = \tilde{c}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} + 1_n$. We have

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & Mw_2 & Mw_3 \\ 0 & 0 & 1 & 0 \\ Mz_2 & Mz_3 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -Mz_2 & -Mz_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -Mz_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & Mw_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & X \end{pmatrix}$$

where $X = 1 - Mz_3Mw_3$. Therefore $\det J = \det X$. Since J is regular, so is X. Thus,

$$J^{-1}\tilde{c}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -Mw_3X^{-1}M & 0 & {}^t(X^{-1}M) \\ 0 & 0 & 0 & 0 \\ 0 & X^{-1}M & 0 & -X^{-1}Mz_3M \end{pmatrix}.$$

In view of these formulas, (4.3) follows easily from (4.1). □

REMARK 4.3. Lemma 4.2 also holds for $k=1$ if we define $\alpha_{k,l}$ as

$$\alpha_{k,l}=\begin{cases} \left(-\dfrac{1}{2\pi i}\right)^l \dfrac{2(2k-1)^{[l-1]}}{l!k^{[l-1]}} & (l>0) \\ 1 & (l=0). \end{cases}$$

Let $\tilde{S}$ be the symmetric square operator acting on $S_{k,l,n}(V^{(l)})$, which is defined by

$$\tilde{S}f=\sum_M \det M^{-k} \sum_{g\in\Gamma_n\backslash\Gamma_n\begin{pmatrix} M & 0 \\ 0 & M^{-1}\end{pmatrix}\Gamma_n} f|_{k,l}g$$

where M runs over all non-singular integral matrices of size n in elementary form. By Garrett [5, Prop. in Sect. 4], a common eigenfunction of all Hecke operators is an eigenfunction of $\tilde{S}$. Moreover, by [2, (6)], its eigenvalue $\Lambda(f)$ is

$$\zeta(k)^{-1}\prod_{i=1}^n \zeta(2k-2i)^{-1}D_f(k-n)$$

where $\zeta(s)$ is the Riemann zeta function and $D_f(s)$ is the standard L-function of f. For simplicity we put $N_{k,l,n}=\dim S_{k,l,n}(V^{(l)})$ for $n\geq 1$.

PROPOSITION 4.4. *Let p, $q>1$ be integers and $z\in H_p$, $w\in H_q$. Let $k>p+q+1$ and $l\geq 2$ be even integers. For $1\leq r\leq \min(p,q)$, let $\{f_{j,r}\}_{1\leq j\leq N_{k,l,r}}$ be an orthonormal basis of common eigenfunction of $S_{k,l,r}(V_B^{(l)})$. Then,*

$$(L^{(l)}E_k^{p+q})(z,w)=\alpha_{k,l}\sum_{r=1}^{\min(p,q)} C_{k,l,r}\sum_{j=1}^{N_{k,l,r}} \Lambda(f_{j,r})E(f_{j,r},V_{AB}^{(l)})(z)E(\sigma\theta(f_{j,r}),V_{CD}^{(l)})(w) \tag{4.6}$$

where θ is an operator defined by $(\theta f)(z)=\overline{f(-\bar{z})}$.

Proof. Let $g_{\tilde{M}}$ be as in (4.2). By the same computation as in Garrett [6, Sect. 5]

$$\begin{aligned}&\sum_{g_0'\in\Gamma_r} L^{(l)}(1|_{k,0}g_{\tilde{M}})|_{k,l}g_0'(z,w) \\ &=\alpha_{k,l}\det M^{-k}\left(\sum_{g\in\Gamma_r}\rho_{k,l,r}(z_3+w_3)(x_B{}^tx_D)^l|_{k,l,p}g\right)\Big|_{k,l,q}\hat{M}\end{aligned}$$

where $\hat{M}=\begin{pmatrix} M & 0 \\ 0 & M^{-1}\end{pmatrix}$. By Proposition 3.1, this is

$$\alpha_{k,l} \det M^{-k} P_{k,l,r}(z_3, -\bar{w}_3; V_B^{(l)}, V_D^{(l)})|_{k,l,q}\hat{M}$$
$$= \alpha_{k,l} \det M^{-k} C_{k,l,r} \sum_{j=1}^{N_{k,l,r}} f_{j,r}(z_3)(\sigma\theta(f_{j,r})|_{k,l,q}\hat{M})(w_3)\,.$$

Hence, as in [5, Sect. 5], we have

$$(L^{(l)}E_k^{p+q})(z, w)$$
$$= \alpha_{k,l} \sum_{r=1}^{\min(p,q)} C_{k,l,r} \sum_{j=1}^{N_{k,l,r}} \sum_{g_0'' \in P_{p,r}\backslash \Gamma_p} (f_{j,r}\mathrm{pr}_r^p|_{k,l}g_0'')(z)$$
$$\times \sum_{g_1'' \in P_{q,r}\backslash \Gamma_q} ((\tilde{S}\sigma\theta(f_{j,r}))\mathrm{pr}_r^q|_{k,l}g_1'')(w)$$
$$= \alpha_{k,l} \sum_{r=1}^{\min(p,q)} C_{k,l,r} \sum_{j=1}^{N_{k,l,r}} \Lambda(f_{j,r})E(f_{j,r}, V_{AB}^{(l)})(z)E(\sigma\theta(f_{j,r}), V_{CD}^{(l)})(w)\,. \qquad \square$$

5. The Fourier coefficients of vector valued Eisenstein series

In [1] Garrett's pullback formula was used to compute Fourier coefficients of Klingen type (scalar valued) Eisenstein series. In this section we show that our vector valued version (4.7) of the pullback formula allows us to cover the case of vector valued Eisenstein series as well by essentially the same method as in [1].

(5.1) We start from the degree $n=p+q$ Eisenstein sereis $E_k^n(Z)$ with

$$Z = \begin{pmatrix} z & u \\ {}^t u & w \end{pmatrix} \in H_n\,, \qquad z \in H_p\,, \quad u \in \boldsymbol{C}^{(p,q)}\,, \quad w \in H_q$$

and consider its Fourier-Jacobi expansion with respect to $z \in H_p$:

$$E_k^n(Z) = \sum_{R^{(p)} \geq 0} \phi_R(w, u)e(Rz)$$

where $e(*) = \exp(2\pi i \operatorname{Tr}(*))$. Formula (13) of [1, §3] gives an explicit expression for $\phi_R(w, u)$ in the case of positive definite $R = R^{(p)}$. It is convenient to write that formula in the following form as a finite sum, using $\tilde{\phi}_R(Z) := \phi_R(w, u)e(Rz)$ instead of $\phi_R(w, u)$:

$$\tilde{\phi}_R(Z) := \phi_R(w, u)e(Rz) = \sum_{T, \omega_1} a_k^p(T)\tilde{\phi}_{T,\omega_1}(Z)$$

where
- $a_k^p(T)$ is the T-Fourier coefficient of the Eisenstein series E_k^p,
- T runs over all positive definite half integral $p \times p$-matrices,
- ω_1 runs over a set of representatives of $\boldsymbol{Z}^{(p,p)}/GL(p, \boldsymbol{Z})$,
- T and ω_1 satisfy the additional condition $T[\omega_1] = R$; this implies that the sum over T and ω_1 is actually a finite sum.

The function $\tilde{\phi}_{T,\omega_1}(Z)$ has the following expansion:

$$\tilde{\phi}_{T,\omega_1}(Z)=\sum_{M\in P_{q,0}\backslash\Gamma_q}\det j(M,w)^{-k}\sum_{\substack{\omega_3\in\boldsymbol{Z}^{(q,p)}\\ \binom{\omega_1}{\omega_3}\text{ primitive}}} e(TM_{\downarrow}\langle Z\rangle[{}^t\omega_1\ \ {}^t\omega_3]) \tag{5.1}$$

where $j\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, w\right)=cw+d.$

(5.2) Let $p\geq q$. Starting from any $f\in S_{k,l,q}(V_2^{(l)})$, we want to compute the Fourier coefficients of

$$\langle f, (L^{(l)}E_k^{p+q})(-\bar{z}, *)\rangle=\sum_{R^{(p)}\geq 0} d_p(R, f, x)e(Rz). \tag{5.2}$$

Here, we put $x=(x_1, \cdots, x_p)$ and $y=(x_{p+1}, \cdots, x_{p+q})$. For positive definite $R^{(p)}$ the R-Fourier coefficient of (5.2) is given by the formula

$$\begin{aligned} d_p(R, f, x)e(Rz)&=\langle f, (L^{(l)}\tilde{\phi}_R)(-\bar{z}, *)\rangle \\ &=\sum_{T,\omega_1} a_k^p(T)\langle f, (L^{(l)}\tilde{\phi}_{T,\omega_1})(-\bar{z}, *)\rangle\,. \end{aligned} \tag{5.3}$$

We tacitly used that $a_k^p(T)$ is a rational number and that $\tilde{\phi}_{T,\omega_1}$ (and not just $\tilde{\phi}_R$) is a modular form for the embedded group $\Gamma_{\downarrow}$, as can be seen from the expansion (5.1). Using the equivariance properties of the operator $L^{(l)}$, we see that (5.1) implies

$$(L^{(l)}\tilde{\phi}_{T,\omega_1})(z, w)=\sum_{M\in P_{q,0}\backslash\Gamma_q}\rho_{k,l}(j(M,w))^{-1}(L^{(l)}\tilde{\phi}^0_{T,\omega_1})(z, M\langle w\rangle) \tag{5.4}$$

with

$$\tilde{\phi}^0_{T,\omega_1}(Z)=\sum_{\substack{\omega_3\in\boldsymbol{Z}^{(q,p)}\\ \binom{\omega_1}{\omega_3}\text{ primitive}}} e(TZ[{}^t\omega_1\ \ {}^t\omega_3])\,.$$

To compute $d_p(R, f)=d_p(R, f, x)$, $R^{(p)}>0$ from (5.3) and (5.4) we may now apply the standard unfolding procedure to obtain

$$d_p(R, f, x)e(Rz)=\sum_{T,\omega_1} a_k^p(T)$$
$$\times\int_{P_{q,0}\backslash H_q}(\rho_{k,l}(\sqrt{Y})f(w), \rho_{k,l}(\sqrt{Y})(L^{(l)}\tilde{\phi}^0_{T,\omega_1})(-\bar{z}, w))\det(Y)^{-q-1}dXdY$$

with $w=X+iY\in H_q$. Taking into account that

$$P_{q,0}\backslash H_q=\{w=X+iY\in H_q\,|\,X={}^tX\bmod 1,\ Y^{(q)}>0\text{ reduced}\}$$

we may now use the Fourier expansion of f and $L^{(l)}\tilde{\phi}^0_{T,\omega_1}$ to compute (5.5) further. To expalin the Fourier expansion of $L^{(l)}\tilde{\phi}^0_{T,\omega_1}$ we define a polynomial in

$V_1^{(l)}\cdot V_2^{(l)}[t_{ij}]_{1\le i\le j\le n}$ by

$$L^{(l)}(E_\tau)(z,w)=P^{(l)}(T_1,t_2,t_4)e(T_1z)e(t_4w) \tag{5.6}$$

where E_τ denotes the function

$$E_\tau(Z)=e(\tau Z)\,,\qquad Z\in H_n\,,\quad \tau={}^t\tau\in \boldsymbol{C}^{(n,n)}$$

and we use the decomposition

$$\tau=\begin{pmatrix} T_1 & t_2\\ {}^tt_2 & t_4\end{pmatrix},\qquad T_1={}^tT_1\in\boldsymbol{C}^{(p,p)}\,,\quad t_2\in\boldsymbol{C}^{(p,q)}\,,\quad t_4={}^tt_4\in\boldsymbol{C}^{(q,q)}\,.$$

By the formula of Lemma 2.2, we may write down such polynomials quite explicitly:

$$\begin{aligned}P^{(l)}(T_1,t_2,t_4)&=P^{(l)}(T_1,t_2,t_4,\boldsymbol{x},\boldsymbol{y})\\&=\frac{1}{k^{[l]}}\sum_{0\le 2\nu\le l}\frac{1}{\nu!(l-2\nu)!(2-k-l)^{[\nu]}}(T_1[\boldsymbol{x}]t_4[\boldsymbol{y}])^\nu(2\boldsymbol{x}t_2{}^t\boldsymbol{y})^{l-2\nu}\,.\end{aligned} \tag{5.7}$$

Using this notation, we can write down the Fourier expansion of $L^{(l)}\tilde{\phi}^{\,0}_{T,\omega_1}$:

$$(L^{(l)}\tilde{\phi}^{\,0}_{T,\omega_1})(z,w)=\sum_{\substack{\omega_3\in\boldsymbol{Z}^{(q,p)}\\ \binom{\omega_1}{\omega_2}\ \text{primitive}}}P^{(l)}(R,\omega_1T^t\omega_3,T[\omega_3])e(Rz)e(T[\omega_3]w)\,. \tag{5.8}$$

The Fourier expansion of f will be denoted by

$$f(w)=\sum_{S^{(q)}>0}b(S,\boldsymbol{y})e(Sw)\,.$$

Since we integrate over $X={}^tX \bmod 1$, only those $b(S)$ and those $P^{(l)}(R,\omega_1T^t\omega_3,T[\omega_3])$ contribute to (5.5) which satisfy

$$S=T[\omega_3]\,.$$

We obtain

$$\begin{aligned}d_p(R,\boldsymbol{x})=\sum_{T,\omega_1}a_k^p(T)&\int_{\substack{Y^{(q)}>0\\ Y:\,\text{reduced}}}\ \sum_{\substack{\omega_3\in\boldsymbol{Z}^{(q,p)}\\ \binom{\omega_1}{\omega_3}\ \text{primitive}}}\\&(\rho_{k,l}(\sqrt{Y})b(T[\omega_3],\boldsymbol{y}),\ \rho_{k,l}(\sqrt{Y})P^{(l)}(R,\omega_1T^t\omega_3,T[\omega_3],\boldsymbol{x},\boldsymbol{y}))\\&\times\exp(-4\pi\mathrm{Tr}(T[\omega_3]Y)\det(Y)^{-q-1}dY\end{aligned} \tag{5.9}$$

$$=2\sum_{T,\omega_1}a_k^p(T)\sum_{\substack{\omega_3\in GL(q,\boldsymbol{Z})\backslash\boldsymbol{Z}^{(q,p)}\\ \binom{\omega_1}{\omega_3}\text{primitive}}}$$

$$\int_{Y^{(q)}>0} (\rho_{k,l}(\sqrt{Y})b(T[\omega_3], y), \rho_{k,l}(\sqrt{Y})P^{(l)}(R, \omega_1 T^t\omega_3, T[\omega_3], x, y))$$
$$\times \exp(-4\pi \operatorname{Tr}(T[\omega_3]Y)) \det(Y)^{-q-1}dY\,.$$

Here we again use the unfolding procedure, the factor 2 comes in because -1_q acts trivially on Y. One should also notice that in (5.9) only matrices ω_3 of maximal rank actually occur.

To simplify (5.9) further, we introduce an operator of type

$$H_{k,l}^{(q)}(S) = \int_{Y^{(q)}>0} \rho_{k,l}(Y) \exp(-\operatorname{Tr}(SY)) \det(Y)^{-q-1}dY \tag{5.10}$$

where S is a positive definite matrix of size q. For basic properties of such integrals we refer to Godement [6]. Instead of looking at the Fourier coefficients $b(S, y)$ of f, we consider now a kind of modified Fourier coefficients $\hat{b}(S, y)$, defined by

$$\hat{b}(S, y) = H_{k,l}^{(q)}(4\pi S)b(S, y) \tag{5.11}$$

as our basic object of interest. Using the fact that $\rho_{k,l}(\sqrt{Y})$ is a hermitian operator, we may summarize our computations as follows:

PROPOSITION 5.1. *Let k and l be even integers, $k > n+1$ with $n = p+q$, $p \geq q$ and $f \in S_{k,l,q}(V_2^{(l)})$ with Fourier expansion*

$$f(Z) = \sum_S b(S, y)e(SZ)\,.$$

Then, for $R^{(p)} > 0$, the Fourier coefficient $d_p(R, f, x)$ of the function defined by (5.2) *is given by the formula*

$$d_p(R, f, x) = 2 \sum_{T,\omega_1} a_k^p(T) \sum_{\substack{\omega_3 \in GL(q,\mathbf{Z})\backslash \mathbf{Z}^{(q,p)} \\ \binom{\omega_1}{\omega_3} \text{ coprime}}} (\hat{b}(T[\omega_3], y), P^{(l)}(R, \omega_1 T^t\omega_3, T[\omega_3]))\,. \tag{5.12}$$

(5.3) In our formula (5.12) we use modified Fourier coefficients of f to compute unmodified Fourier coefficients $d_p(R, f, x)$. To get rid of this assymmetry we start from

LEMMA 5.2. *There is a non-zero constant $\gamma(k, l, p)$ such that for all positive definite T of size p we have*

$$\det(T)^{k-(p+1)/2} H_{k,l}^{(p)}(4\pi T)P^{(l)}(T, T, T, x, y) = \gamma(k, l, p)(x^t y)^l \tag{5.13}$$

(with $x = (x_1, \cdots, x_p)$, $y = (y_1, \cdots, y_p)$).

Proof. For $Z = \begin{pmatrix} z_1 & {}^t z_2 \\ z_2 & z_4 \end{pmatrix} \in H_{2p}$ with $z_1, z_4 \in H_p$ we start from the well known formula

$$\alpha_k^{(p)}\sum_S \det(z_1+z_2+{}^tz_2+z_4+S)^{-k}=\sum_{T>0}\det(T)^{k-\frac{p+1}{2}}e(T(z_1+z_2+{}^tz_2+z_4)) \quad (5.14)$$

where S runs through all integral symmetric matrices of size p and

$$\alpha_k^{(p)}=(4\pi)^{p(p-1)/4}(2\pi i)^{-pk}\prod_{\nu=0}^{p-1}\Gamma\left(k-\frac{\nu}{2}\right).$$

We apply the operator $L^{(l)}$ on both sides of (5.14). The right hand side yields

$$\sum_{T>0}\det(T)^{k-(p+1)/2}P^{(l)}(T,T,T,x,y)e(T(z_1+z_4)).$$

The left hand side of (5.14) is equal to

$$\alpha_k^{(p)}\alpha_{k,l}\sum_S \rho_{k,l}(z_1+z_4+S)^{-1}(x^ty)^l. \quad (5.15)$$

This follows from (4.3) and the (elementary) formula

$$\det(z_1+z_2+{}^tz_2+z_4)^{-k}=(1|_{k,0,2p}g)|_{k,0,2p}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^{\uparrow}$$

with

$$g=\begin{pmatrix}1_{2p} & 0_{2p}\\ \tilde{c} & 1_{2p}\end{pmatrix}, \qquad \tilde{c}=\begin{pmatrix}0_p & 1_p\\ 1_p & 0_p\end{pmatrix}.$$

By [6, Exp. 10, p. 17] (5.15) equals

$$\alpha_k^{(p)}\alpha_{k,l}c_{k,l,p}\sum_{T>0}H_{k,l}^{(p)}(4\pi T)^{-1}e(T(z_1+z_4))(x^ty)^l.$$

The lemma (and also an explicit expression for the constant $\gamma(k,l,p)$) follows by comparing the Fourier coefficients. □

To compute $\hat{d}_p(R,f,x)$, we start from the observation that

$$P^{(l)}(R,\omega_1T^t\omega_3,T[\omega_3],x,y)=P^{(l)}(R,R,R,x,y\omega_3\omega_1^{-1}).$$

(Note that $y\omega_3\omega_1^{-1}$ has p columns.) Therefore a typical summand of $\hat{d}_p(R,f,x)$ looks as follows:

$$\begin{aligned}&(\hat{b}(T[\omega_3],y),H_{k,l}^{(p)}(4\pi R)P^{(l)}(R,\omega_1T^t\omega_3,T[\omega_3],x,y))\\ &=\gamma(k,l,p)\det(R)^{-k+(p+1)/2}(\hat{b}(T[\omega_3],y),(x^t\omega_1^{-1}{}^t\omega_3{}^ty)^l)\\ &=\gamma(k,l,p)\det(R)^{-k+(p+1)/2}\hat{b}(T[\omega_3],x^t\omega_1^{-1}{}^t\omega_3).\end{aligned}$$

(5.4) Let f be as before, but assume in addition that f is an eigenform for the Hecke algebra associated to $Sp(n,\mathbf{Q})$. It is clear from the pullback formula that the Fourier coefficients of the Eisenstein series

$$E(\sigma f, V_1^{(l)})(z)=\sum c_p(R, f, x)e(Rz)$$

are up to a non-zero constant equal to the numbers $d_p(R, f, x)$. We may now formulate the main result of this section:

THEOREM 5.3. *Let k, l, n, p, q be as in Proposition* 5.1 *and assume that $f \in S_{k,l,q}(V_2^{(l)})$ is an eigen function of all Hecke operators. Then we have for all $R^{(p)}>0$*

$$\hat{c}_p(R, f, x)=2\lambda_{k,l,q}\gamma(k, l, p)\det(R)^{-k+(p+1)/2}$$
$$\times \sum_{T,\omega_1} a_k^p(T) \sum_{\substack{\omega_3 \in GL(q,\mathbf{Z})\backslash \mathbf{Z}^{(q,p)} \\ \binom{\omega_1}{\omega_3}\ \text{coprime}}} \hat{b}(T[\omega_3], x^t\omega_1^{-1}\,{}^t\omega_3)$$

with $\lambda_{k,l,q}=\dfrac{1}{\alpha_{k,l}C_{k,l,q}\Lambda(f)}$.

REMARK 5.4.

(1) Theorem 5.3, in particular the factor $\Lambda(f)$ in the denominator, settles a conjecture made in [12].

(2) By using the concept of "primitive" Fourier coefficients as in [3] one can get rid of the somewhat uncovenient condition "$\begin{pmatrix}\omega_1\\ \omega_3\end{pmatrix}$ primitive". This will be left to the reader.

References

[1] BÖCHERER, S.; Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, *Math. Z.*, **183**, 21–46 (1983).

[2] BÖCHERER, S.; Ein Rationalitätssatz für formulae Heckereihen zur Siegeleschen Modulegruppe, *Abh. Math. Sem. Univ. Hamburg*, **56**, 35–47 (1986).

[3] BÖCHERER, S. and RAGHAVAN, S.; On Fourier coefficients of Siegel modular forms, *J. Reine Angew. Math.*, **384**, 80–101 (1988).

[4] COHEN, H.; Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.*, **217**, 271–285 (1975).

[5] GARRETT, P. B.; Pullbacks of Eisenstein series; applications, in: Automorphic forms of several variables, 114–137, ed. I. Satake and Y. Morita, Birkhäuser, 1984.

[6] GODEMENT, R.; Exposés 5–10, in: Séminaire Henri Cartan 1957/58 Fonctions automorphes, Vol. 1, Paris: 1958.

[7] HARRIS, M.; The rationality of holomorphic Eisenstein series, *Invent. Math.*, **63**, 305–310 (1981).

[8] HUA, L. K.; Harmonic analysis of functions of several complex variables in the classical domains, Translations of mathematical monographs, 6, A.M.S 1963.

[9] KLINGEN, H.; Zum Darstellungssatz für Siegelsche Moduloformen, *Math. Z.*, **102**, 30–43 (1967).

[10] KLINGEN, H.; Über Poincarésche Reihen zur Siegelschen Modulgruppe, *Math. Ann.*, **168**, 157–170 (1967).

[11] MIZUMOTO, S.; On integrality of certain algebraic numbers associated with modular forms, *Math. Ann.*, **265**, 119–135 (1983).

[12] SATOH, T.; On certain vector valued Siegel modular forms of degree two, *Math. Ann.*, **274**, 335–352 (1986).

[13] WEISSAUER, R.; Vektorwertige Siegelsche Modulformen kleinen Gewichtes, *J. Reine Angew. Math.*, **343**, 184–202 (1983).

[14] YAMAZAKI, T.; Jacobi forms and a Maass relation for Eisenstein series, *J. Fac. Sci. Univ. Tokyo*, **33**, 295–310 (1986).

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