Castling Transforms of Prehomogeneous Vector Spaces and Functional Equations

by

Fumihiro Sato and Hiroyuki Ochiai

(Received February 5, 1991)

Introduction

Let \((G, \rho, V)\) and \((\tilde{G}, \tilde{\rho}, \tilde{V})\) be prehomogeneous vector spaces which are the castling transforms of each other. Then it is known that \((G, \rho, V)\) and \((\tilde{G}, \tilde{\rho}, \tilde{V})\) share quite similar properties. In particular there exists a formula relating the functional equation of zeta functions associated with \((G, \rho, V)\) to that of zeta functions associated with \((\tilde{G}, \tilde{\rho}, \tilde{V})\). In a lecture at RIMS of Kyoto University in 1974, Shintani presented such a formula with an outline of its proof in the case where the groups \(G\) and \(\tilde{G}\) are reductive and the singular sets are absolutely irreducible. Although he got no opportunity of publishing his result, the lecture has been reproduced by Muro later in [SM].

The aim of the present paper is to give a complete proof of Shintani’s formula, which simplifies his original argument, in a form extended to multivariable zeta functions over local fields of characteristic 0. As an application, we obtain a relation between the \(b\)-function of \((G, \rho, V)\) and that of \((\tilde{G}, \tilde{\rho}, \tilde{V})\). In Gyoja [Gy], Shintani’s formula has been extended to Gauss sums of prehomogeneous vector spaces defined over a finite field.

We note that this kind of results, which relate properties of a prehomogeneous vector space to the corresponding properties of its castling transform, are quite important, since the castling transform is a standard procedure of constructing new prehomogeneous vector spaces from a given one and plays a crucial role in the classification of prehomogeneous vector spaces (cf. [SK], [Ki2]). In this respect, a recent result of Kajima [Kaj] is of considerable interest, which claims that the functional equation for \((G, \rho, V)\) over a nonarchimedean local field of characteristic 0 implies the functional equation for its castling transform \((\tilde{G}, \tilde{\rho}, \tilde{V})\) (see §2, Theorem 2).

In Section 1 we prove fundamental properties of the castling transform. The main theorem (Theorem 3) is formulated in Section 2 and is proved in Section 4. In Section 3, the relation between the \(b\)-functions of \((G, \rho, V)\) and \((\tilde{G}, \tilde{\rho}, \tilde{V})\) is derived from the main theorem. An application to concrete examples well be given in [S3].
1. Castling transform

1.1. First we recall some basic notions in the theory of prehomogeneous vector spaces (for the detail, see [SK, §2, §4] and [S1, §1, §2]). Let $K$ be a field of characteristic 0 and denote by $\bar{K}$ the algebraic closure of $K$. Let $G$ be a connected linear algebraic group defined over $K$ and $V$ be a finite dimensional $\bar{K}$-vector space with $K$-structure $V_K$. Let $\rho: G \to GL(V)$ be a $K$-rational representation of $G$ on $V$. Then the triple $(G, \rho, V)$ is called a prehomogeneous vector space (abbrev. p.v.) if there exists a proper algebraic subset $S$ of $V$ such that $V_K - S_K$ is a single $\rho(G_K)$-orbit. The algebraic set $S$ is called the singular set of $(G, \rho, V)$. A nonzero rational function $P(v)$ is called a relative invariant of $(G, \rho, V)$ if there exists a rational character $\chi(g)$ of $G$ such that

$$P(\rho(g)v) = \chi(g)P(v) \quad (g \in G, v \in V).$$

Let $S_1, \cdots, S_l$ be the $K$-irreducible hypersurfaces contained in $S$. For each $i (1 \leq i \leq l)$, take a $K$-irreducible polynomial $P_i(v)$ defining $S_i$. Then it is known that any relative invariant $P(v)$ in $K(V)$ of $(G, \rho, V)$ is written uniquely as

$$P(v) = \prod_{i=1}^{l} P_i(v)^{n_i} \quad (c \in K^*, v_1, \cdots, v_l \in \mathbb{Z}).$$

The polynomials $P_1, \cdots, P_l$ are called the basic relative invariants of $(G, \rho, V)$ over $K$. We call $l$ the $K$-rank of $(G, \rho, V)$.

A relative invariant $P(v)$ is called nondegenerate if the Hessian $\det \left( \frac{\partial^2 P}{\partial v_i \partial v_j} \right)$ does not vanish identically. A p.v. $(G, \rho, V)$ is called regular if there exists a nondegenerate relative invariant; and then one can find a nondegenerate relative invariant in $K[V]$. Let $V^*$ be the vector space dual to $V$ and $\rho^*: G \to GL(V^*)$ be the rational representation of $G$ contragredient to $\rho$. The vector space $V^*$ has a $K$-structure canonically defined by the $K$-structure of $V$. Then the representation $\rho^*$ is defined over $K$. We call the triple $(G, \rho^*, V^*)$ the dual of $(G, \rho, V)$. If the p.v. $(G, \rho, V)$ is regular, then its dual $(G, \rho^*, V^*)$ is also a p.v. and is regular. Note that, in general, the dual of a p.v. is not necessarily a p.v.

For a relative invariant $P(v)$, define a rational mapping $\phi_p: V - S \to V^*$ by $\phi_p(v) = \text{grad log } P(v)$. Then we have

$$\phi_p(\rho(g)v) = \rho^*(g)\phi_p(v) \quad (g \in G, v \in V).$$

If $P$ is nondegenerate, then $\phi_p$ gives rise to a birational rational mapping of $V - S$ onto $V^* - S^*$ ($S^*$ = the singular set of $(G, \rho^*, V^*)$). Conversely, if $\phi_p: V - S \to V^*$ is dominant, then some power of $P$ is nondegenerate.

1.2. Let $m$ and $n$ be positive integers with $m > n \geq 1$. We consider a rational representation $\rho_0: H \to GL(m)$ of a connected linear algebraic group $H$. We assume that $H$ and $\rho_0$ are defined over a field $K$ of characteristic 0. Put $G = H \times GL(n)$ and
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$V = M(m, n)$. Also put $\tilde{G} = H \times GL(m - n)$ and $\tilde{V} = M(m, m - n)$. Let $\rho : G \to GL(V)$ and $\tilde{\rho} : \tilde{G} \to GL(\tilde{V})$, respectively, be rational representations of $G$ and $\tilde{G}$ on $V$ and $\tilde{V}$ defined by

$$\rho(h, g_n)v = \rho_0(h)vg_n^{-1} \quad (h, g_n) \in G = H \times GL(n),$$

$$\tilde{\rho}(h, g_{m - n})w = \tilde{\rho}_0(h)^{-1}w'g_{m - n} \quad (h, g_{m - n}) \in \tilde{G} = H \times GL(m - n).$$

The triple $(\tilde{G}, \tilde{\rho}, \tilde{V})$ is called the castling transform of $(G, \rho, V)$ and vice versa. Then we have the following lemma:

**Lemma 1.1** (Sato-Kimura [SK]). The triple $(G, \rho, V)$ is a p.v. if and only if so is its castling transform $(\tilde{G}, \tilde{\rho}, \tilde{V})$.

In the following, we assume that $(G, \rho, V)$ and $(\tilde{G}, \tilde{\rho}, \tilde{V})$ are p.v. with $K$-structure

$$G_K = H_K \times GL(n; K), \quad V_K = M(m, n; K),$$

$$\tilde{G}_K = H_K \times GL(m - n; K), \quad \tilde{V}_K = M(m, m - n; K).$$

1.3. Put $N = \binom{m}{n}$ and let $\Delta_1(v), \ldots, \Delta_N(v)$ (resp. $\tilde{\Delta}_1(w), \ldots, \tilde{\Delta}(w)$) be the minor determinants of $v \in V$ (resp. $w \in \tilde{V}$) of size $n$ (resp. $m - n$).

Let $V_0$ be the vector space of column vectors of $m$ entries and $V_0^*$ the vector space dual to $V_0$. We identify $V$ (resp. $\tilde{V}$) with the direct product of $n$ (resp. $m - n$) copies of $V_0$ (resp. $V_0^*$) in the standard manner. Let $\Delta : V \to \bigwedge^n V_0$ and $\tilde{\Delta} : \tilde{V} \to \bigwedge^{m - n} V_0^*$ be the mappings defined by

$$\Delta(v) = \Delta(v_1, \ldots, v_n) = v_1 \wedge \cdots \wedge v_n \quad (v_1, \ldots, v_n \in V_0)$$

and

$$\tilde{\Delta}(w) = \tilde{\Delta}(w_1, \ldots, w_{m - n}) = w_1 \wedge \cdots \wedge w_{m - n} \quad (w_1, \ldots, w_{m - n} \in V_0^*),$$

respectively. We identify $\bigwedge^n V_0$ with $\bigwedge^{m - n} V_0^*$ via the canonical isomorphism

$$\bigwedge^n V_0 \overset{\cong}{\longrightarrow} (\bigwedge^n V_0)^* \overset{\cong}{\longrightarrow} \bigwedge^{m - n} V_0^*.$$

By taking the standard basis, we may identify $\bigwedge^n V_0$ and $\bigwedge^{m - n} V_0^*$ with $K^n$, so that the mappings $\Delta$ and $\tilde{\Delta}$ are given by

$$\Delta(v) = (\Delta_1(v), \ldots, \Delta_N(v))$$

$$\tilde{\Delta}(w) = (\tilde{\Delta}_1(w), \ldots, \tilde{\Delta}_N(w)).$$

Here the minor determinants are indexed such that

$$\det(v, w) = \sum_{i=1}^N \Delta_i(v)\tilde{\Delta}_i(w).$$
Then it is easy to see that

\[ \Delta(\rho(h, g_n)v) = \det g_n^{-1} \left( \bigwedge^n \rho_0(h) \right) (\Delta(v)) \]  

(1)

and

\[ \tilde{\Delta}(\tilde{\rho}(h, g_{m-n})w) = \left( \frac{\det g_{m-n}}{\det \rho_0(h)} \right) \left( \bigwedge^n \rho_0(h) \right) (\tilde{\Delta}(w)) . \]  

(2)

**Lemma 1.2.**

(i) The $K$-rank of $(G, \rho, V)$ is equal to the $K$-rank of $(\tilde{G}, \tilde{\rho}, \tilde{V})$.

(ii) There exist irreducible homogeneous polynomials $Q_1, \cdots, Q_l \in K[y_1, \cdots, y_N]$ ($l = \text{the } K\text{-rank of } (G, \rho, V)$) such that

\[ P_1(v) = Q_1(\Delta(v)), \cdots, P_l(v) = Q_l(\Delta(v)) \]

are the basic relative invariants of $(G, \rho, V)$ over $K$ and

\[ \tilde{P}_1(\tilde{\omega}) = Q_1(\tilde{\Delta}(\tilde{\omega})), \cdots, \tilde{P}_l(\tilde{\omega}) = Q_l(\tilde{\Delta}(\tilde{\omega})) \]

are the basic relative invariants of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ over $K$.

(iii) Put $d_i = \deg Q_i$. Then there exist $K$-rational characters $\psi_1, \cdots, \psi_l$ of $H$ such that

\[ P_i(\rho(h, g_n)v) = \tilde{\chi}_i(h, g_n) P_i(v), \quad \tilde{\chi}_i(h, g_n) = (\det g_n)^{-d_i} \psi_i(h) \]

\[ \tilde{P}_i(\tilde{\rho}(h, g_{m-n})w) = \tilde{\tilde{\chi}}_i(h, g_{m-n}) \tilde{P}_i(w), \quad \tilde{\tilde{\chi}}_i(h, g_{m-n}) = (\det g_{m-n}/\det \rho_0(h))^{d_i} \psi_i(h) . \]

**Proof.** Since the ring $K[V]^{SL(n)}$ (resp. $K[\tilde{V}]^{SL(m-n)}$) of polynomials in $K[V]$ (resp. $K[\tilde{V}]$) invariant under the action of $SL(n)$ (resp. $SL(m-n)$) is generated by

\[ \Delta_1(v), \cdots, \Delta_N(v) \quad \text{(resp. } \tilde{\Delta}_1(w), \cdots, \tilde{\Delta}_N(w)) \, , \]

the lemma follows immediately from the identities (1) and (2).

1.4. Put

\[ V' = \{ v \in V \mid \text{rank } v = n \} , \]

\[ \tilde{V}' = \{ w \in \tilde{V} \mid \text{rank } w = m-n \} . \]

Then under the identification $\bigwedge^n V_0 = \bigwedge^n V_{\tilde{N}} = K^N$, we have

\[ \Delta(V') = \tilde{\Delta}(\tilde{V}') . \]

and

\[ \Delta(V) = \tilde{\Delta}(\tilde{V}) = \Delta(V') \cup \{ 0 \} = \tilde{\Delta}(\tilde{V}') \cup \{ 0 \} . \]

For simplicity we put

\[ Y = \Delta(V') = \tilde{\Delta}(\tilde{V}') \subset K^N . \]
The group $H \times GL(1)$ acts on $Y$ via

$$ (h, t) \cdot y = t \left( \bigwedge^n \rho_0(h) \right)(y). \quad (3) $$

Let $S$ and $\tilde{S}$ be the singular sets of $(G, \rho, V)$ and $(\tilde{G}, \tilde{\rho}, \tilde{V})$, respectively. Since any fibre of $\Delta$ (resp. $\tilde{\Delta}$) is a principal homogeneous space of $SL(n)$ (resp. $SL(m-n)$), by the identities (1) and (2), there exists an open $H \times GL(1)$-orbit $\Omega$ in $Y$ and we have

$$ V - S = \Delta^{-1}(\Omega) \quad \text{and} \quad \tilde{V} - \tilde{S} = \tilde{\Delta}^{-1}(\Omega). $$

**Lemma 1.3.** The singular set $S$ is a hypersurface in $V$ if and only if the singular set $\tilde{S}$ is a hypersurface in $\tilde{V}$.

**Proof.** If $S$ is a hypersurface, then

$$ V - S = \bigcap_{i=1}^l \{ v \in V \mid P_i(v) = Q_i(\Delta(v)) \neq 0 \}, $$

where $Q_1, \ldots, Q_l$ are the polynomials given in Lemma 1.2. Hence

$$ \Omega = \bigcap_{i=1}^l \{ y \in Y \mid Q_i(y) \neq 0 \}. $$

Therefore

$$ \tilde{V} - \tilde{S} = \tilde{\Delta}^{-1}(\Omega) = \bigcap_{i=1}^l \{ w \in \tilde{V} \mid \tilde{P}_i(w) = Q_i(\tilde{\Delta}(w)) \neq 0 \}. $$

This shows that the singular set $\tilde{S}$ is a hypersurface in $\tilde{V}$ defined by the polynomial $\prod_{i=1}^l \tilde{P}_i$. \hfill \blacksquare

**1.5.** From now on we assume that $K$ is a local field of characteristic 0. Let $H^+$ be an open subgroup of $H_K$ of finite index and put

$$ G^+ = H^+ \times GL(n; K), \quad \tilde{G}^+ = H^+ \times GL(m-n; K). $$

Let $T$ be a subgroup of the multiplicative group $K^*$ of finite index containing the group $\{ \det \rho_0(h) \mid h \in H^+ \}$. For such a $T$, we put

$$ GL(n)_T = \{ g_n \in GL(n; K) \mid \det g_n \in T \}, $$

$$ GL(m-n)_T = \{ g_{m-n} \in GL(m-n; K) \mid \det g_{m-n} \in T \}, $$

$$ G^+_T = H^+ \times GL(n)_T, \quad \tilde{G}^+_T = H^+ \times GL(m-n)_T, \quad H_T^+ = H^+ \times T. $$

Put

$$ V_K = V' \cap V_K, \quad \tilde{V}_K = \tilde{V}' \cap \tilde{V}_K $$

and

$$ Y_K = \Delta(V'_K) = \tilde{\Delta}(\tilde{V}_K' \cap \tilde{V}_K^N). $$
The group $H^+_E$ acts on $Y_K$ via (3). Then we can easily prove the following lemma.

**Lemma 1.4.** There exists a one to one correspondence between the set of open $\rho(G^+_E)$-orbits in $V_K$, the set of open $\tilde{\rho}(\tilde{G}^+_E)$-orbits in $\tilde{V}_K$ and the set of open $H^+_E$-orbits in $Y_K$. For an open $H^+_E$-orbit $\Omega_0$ in $Y_K$, the corresponding orbits in $V_K$ and $\tilde{V}_K$ are given by $\Delta^{-1}(\Omega_0)$ and $\tilde{\Delta}^{-1}(\Omega_0)$, respectively.

**1.6.** Let $(G, \rho^*, V^*)$ and $(\tilde{G}, \tilde{\rho}^*, \tilde{V}^*)$ be the dual of $(G, \rho, V)$ and $(\tilde{G}, \tilde{\rho}, \tilde{V})$, respectively. In the following we identify $V^*$ and $\tilde{V}^*$, respectively, with $V$ and $\tilde{V}$ via the nondegenerate pairing

$$\langle v, v^* \rangle = \text{tr}^* v v^* \quad (v, v^* \in V)$$

and

$$\langle w, w^* \rangle = \text{tr}^* w w^* \quad (w, w^* \in \tilde{V}).$$

Then the representations $\rho^*$ and $\tilde{\rho}^*$ are given by

$$\rho^*(h, g) v^* = \rho_0(h)^{-1} v^* g_n$$

$$\tilde{\rho}^*(h, g_{m-n}) w^* = \rho_0(h) w^* g_{m-n}^{-1}.$$ 

Hence $(G, \rho^*, V)$ and $(\tilde{G}, \tilde{\rho}^*, \tilde{V})$ are the castling transforms of each other.

**Lemma 1.5.** The p.v. $(G, \rho, V)$ is regular if and only if so is $(\tilde{G}, \tilde{\rho}, \tilde{V})$.

**Proof.** It is enough to prove the only if part. Assume that $(G, \rho, V)$ is a regular p.v. Then $(G, \rho^*, V)$ is also a p.v. and, by Lemma 1.1, so is $(\tilde{G}, \tilde{\rho}^*, \tilde{V})$. Denote by $S^*$ and $\tilde{S}^*$ the singular sets of $(G, \rho^*, V)$ and $(\tilde{G}, \tilde{\rho}^*, \tilde{V}^*)$, respectively. Put

$$v_0 = \begin{pmatrix} I_n \\ 0_{(m-n,n)} \end{pmatrix} \in V \quad \text{and} \quad w_0 = \begin{pmatrix} 0_{(m,m-n)} \\ I_{m-n} \end{pmatrix} \in \tilde{V},$$

where $I_r$ is the identity square matrix of size $r$ and $0_{(k,l)}$ is the $k$ by $l$ zero matrix. By a suitable coordinate change, if necessary, we may assume that $v_0 \in V - S$. Then, since $\Lambda(v_0) = \Lambda(w_0)$, we see that $w_0 \in \tilde{V} - \tilde{S}$. Let $P(v) = Q(\Lambda_1(v), \ldots, \Lambda_n(v))$ ($Q \in K[y_1, \ldots, y_n]$) be a nondegenerate relative invariant of $(G, \rho, V)$. Then $\phi_P(v_0) \in V^* - S^*$. For $v = (v_i) \in V$, put

$$A_0(v) = \det \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix}, \quad A_j(v) = \det \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{j1} & \cdots & v_{jj} \\ v_{j+1,1} & \cdots & v_{j+1,n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix}.$$
(n+1 \leq i \leq m, 1 \leq j \leq n). By direct calculation of
\[ \phi_p(v_0)_{ij} = \frac{1}{P(v_0)} \frac{\partial P}{\partial v_{ij}}(v_0), \]
we easily obtain
\[
\phi_p(v_0) = \frac{1}{P(v_0)} \begin{pmatrix}
\frac{\partial Q}{\partial \Delta_0}(\Delta(v_0)) & 0 \\
0 & \ddots & \ddots & \ddots \\
\frac{\partial Q}{\partial \Delta_{n+1,1}}(\Delta(v_0)) & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\frac{\partial Q}{\partial \Delta_{m1}}(\Delta(v_0)) & \ddots & \ddots & \ddots
\end{pmatrix}
\]

Let \( y_0 \) and \( y_{ij} \) be the functions on \( Y \) satisfying
\[
y_0(\Delta(v)) = \Delta_0(v), \quad y_{ij}(\Delta(v)) = \Delta_{ij}(v). \tag{4}
\]
Denote by \( Y^\circ \) the open set of \( Y \) defined by
\[
Y^\circ = \{ y \in Y | y_0 \neq 0 \}.
\]
Then
\[
\{ y_0, y_{ij} | n+1 \leq i \leq m, 1 \leq j \leq n \}
\]
gives a local coordinate system on \( Y^\circ \). For the point \( \Delta(\phi_p(v_0)) \in Y^\circ \), the local coordinates are given by
\[
y_0 = d^n, \quad y_{ij} = (-1)^{n+1} \frac{d^{n-1}}{c} \frac{\partial Q}{\partial \Delta_{ij}}(\Delta(v_0)),
\]
where
\[
P(v_0) = c \quad \text{and} \quad d = \deg Q.
\]
Put \( \tilde{P} = Q \circ \tilde{A} \) and consider the mapping
\[
\phi_{\tilde{P}} = \text{gradlog} \tilde{P} : \tilde{V} - \tilde{S} \rightarrow \tilde{V}.
\]
By similar calculation, we see that the local coordinates of the point \( \tilde{A}(\phi_{\tilde{P}}(w_0)) \in Y^\circ \) are given by
\[ y_0 = d^{m-n}, \quad y_{ij} = (-1)^{n+j} d^{m-n-1} \frac{\delta Q}{\delta \Delta_{ij}} (\tilde{A}(w_0)). \]

Hence
\[ \tilde{A}(\phi_p(w_0)) = d^{m-2n} A(\phi_p(v_0)). \]

Since \( \phi_p(v_0) \in V^* - S^* \), this implies that \( \phi_p(w_0) \in \tilde{V}^* - \tilde{S}^* \). Therefore \( \phi_p(w_0) \in \tilde{V}^* - S^* \). Therefore \( \phi_p(\tilde{V} - \tilde{S}) = \tilde{V}^* - \tilde{S}^* \) and some power of \( \tilde{P} \) is a nondegenerate relative invariant of \((\tilde{G}, \tilde{\rho}, \tilde{V})\). This proves the regularity of \((\tilde{G}, \tilde{\rho}, \tilde{V})\).

\[ \square \]

2. Functional equations

2.1. We keep the notation in §1 and assume that \((G, \rho, V)\) is a regular p.v. defined over a local field \(K\) of characteristic 0. Then by Lemma 1.5, its castling transform \((\tilde{G}, \tilde{\rho}, \tilde{V})\) is also regular.

Let \((G, \rho^*, V)\) and \((\tilde{G}, \tilde{\rho}^*, \tilde{V})\) be the duals of \((G, \rho, V)\) and \((\tilde{G}, \tilde{\rho}, \tilde{V})\), respectively. Here we identify the dual vector spaces \(V^*\) and \(\tilde{V}^*\) with \(V\) and \(\tilde{V}\) as in §1.6. Take a nondegenerate relative invariant \(P(v)\) of \((G, \rho, V)\) in \(K[V]\). Then the mapping \(\phi_p\) induces a \(G^+\)-equivariant homeomorphism of \(V_k - S_k\) onto \(V_k^* - S_k^*\).

Let
\[ V_k - S_k = V_1 \cup \cdots \cup V_v, \quad V_k^* - S_k^* = V_1^* \cup \cdots \cup V_v^*, \]
\[ \tilde{V}_k - \tilde{S}_k = \tilde{V}_1 \cup \cdots \cup \tilde{V}_v, \quad \tilde{V}_k^* - \tilde{S}_k^* = \tilde{V}_1^* \cup \cdots \cup \tilde{V}_v^* \]
be the \(G^+\)-orbit decompositions. As in §1.5, let \(T\) be a subgroup of \(K^*\) of finite index containing the group \(\{\det \rho_0(h) | h \in H^+\}\). We decompose the \(G^+\)-orbits \(V_i, V_i^*, \tilde{V}_i, \text{and } \tilde{V}_i^* \) into \(G_T^+\)-orbits as follows:
\[ V_i = \bigcup_{j=1}^{\mu_i} V_{ij}, \quad V_i^* = \bigcup_{j=1}^{\mu_i} V_{ij}^*, \]
\[ \tilde{V}_i = \bigcup_{j=1}^{\mu_i} \tilde{V}_{ij}, \quad \tilde{V}_i^* = \bigcup_{j=1}^{\mu_i} \tilde{V}_{ij}^*. \]

By Lemma 1.4, we may assume that these orbits are indexed so that
\[ V_{ij} = \phi_p(V_{ij}), \quad \Delta(V_{ij}) = \tilde{A}(-\tilde{V}_{ij}), \quad \Delta(V_{ij}^*) = \tilde{A}(\tilde{V}_{ij}^*), \quad (1 \leq i \leq v, 1 \leq j \leq \mu_i). \]

By Lemma 1.2, we can take polynomials \(Q_1, \cdots, Q_{\ell}\) in \(K[y_1, \cdots, y_N]\) such that
\[ \{P_i(v) = Q_i(\Delta(v)) | 1 \leq i \leq \ell\} \quad \text{and} \quad \{\tilde{P}_i(w) = Q_i(\tilde{A}(w)) | 1 \leq i \leq \ell\} \]
are the basic relative invariants over \(K\) of \((G, \rho, V)\) and \((\tilde{G}, \tilde{\rho}, \tilde{V})\), respectively. Also take polynomials \(Q_1^*, \cdots, Q_{\ell}^*\) in \(K[y_1, \cdots, y_N]\) such that
\[ \{P_i^*(v) = Q_i^*(\Delta(v)) | 1 \leq i \leq \ell\} \quad \text{and} \quad \{\tilde{P}_i^*(w) = Q_i^*(\tilde{A}(w)) | 1 \leq i \leq \ell\} \]
are the basic relative invariants over \( K \) of \((\mathcal{G}, \rho^*, V)\) and \((\mathcal{G}', \tilde{\rho}^*, \tilde{V})\), respectively.

Let \( \chi_i, \tilde{\chi}_i, \chi_i^x, \tilde{\chi}_i^x (1 \leq i \leq l) \) be the \( K \)-rational characters corresponding to \( P_i, \tilde{P}_i, P_i^*, \tilde{P}_i^* \), respectively. By [S1, Lemma 2.4] and Lemma 1.2 (iii), there exists a unimodular matrix \( U=(u_{ij}) \) of size \( l \) satisfying

\[
\chi_i = \prod_{j=1}^{l} \chi_j^{xu_{ij}}, \quad \tilde{\chi}_i = \prod_{j=1}^{l} \tilde{\chi}_j^{xu_{ij}} \quad (1 \leq i \leq l)
\]

(cf. [S1, §3]).

By [S1, Lemma 3.5] and Lemma 1.2 (iii), we can find a \( \lambda=(\lambda_1, \ldots, \lambda_l) \in \left( \frac{1}{2} \mathbb{Z} \right)^l \) satisfying

\[
\det \rho(h, g_n)^2 = \prod_{i=1}^{l} \chi_i(h, g_n)^{2\lambda_i},
\]

\[
\det \tilde{\rho}(h, g_{m-n})^2 = \prod_{i=1}^{l} \tilde{\chi}_i(h, g_{m-n})^{2\lambda_i}.
\]

2.2. Let \( \psi \) be the additive character of \( K \) defined by

\[
\psi(x) = \begin{cases} 
\exp(2\pi \sqrt{-1} \alpha), & \text{if } K = \mathbb{R}, \\
\exp(4\pi \sqrt{-1} \text{Re}x), & \text{if } K = \mathbb{C}, \\
\exp(2\pi \sqrt{-1} \lambda_0(\text{tr}_{K} Q_p(x))), & \text{if } K \text{ is a finite extension of } \mathbb{Q}_p.
\end{cases}
\]

Here we denote by \( \lambda_0 \) the canonical mapping

\[
Q_p \to Q_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}.
\]

A Haar measure \( |dx|_K \) on \( K \) is always normalized such that \( |dx|_K \) is autodual with respect to \( \psi \). We normalize Haar measures \( |dv|_K \) and \( |dw|_K \) on \( V_K=M(m, n; K) \) and \( \tilde{V}_K=M(m, m-n; K) \) by

\[
|dv|_K = \prod_{i=1}^{m} \prod_{j=1}^{n} |dv_{ij}|_K \quad \text{and} \quad |dw|_K = \prod_{i=1}^{m} \prod_{j=1}^{m-n} |dw_{ij}|_K,
\]

respectively.

Denote by \( \mathcal{S}(V_K) \) and \( \mathcal{S}(\tilde{V}_K) \) the Schwartz-Bruhat spaces on \( V_K \) and \( \tilde{V}_K \), respectively. The Fourier transforms of \( f \in \mathcal{S}(V_K) \) and \( \mathcal{S}(\tilde{V}_K) \) are defined by

\[
\mathcal{F}(f)(v) = \int_{V_K} f(v^*) \psi(\text{tr}^r v^w) |dv^*|_K,
\]

\[
\mathcal{F}(\tilde{f})(w) = \int_{\tilde{V}_K} \tilde{f}(w^*) \psi(\text{tr}^r w^w) |dw^*|_K.
\]

Denote by \( \hat{K}^\times \) the group of quasi-characters of the multiplicative group \( K^\times \). A quasi-character \( \omega \in \hat{K}^\times \) can be written as
\[ \omega(\alpha) = \begin{cases} \frac{1}{|\mathfrak{m}|_K} \left( \frac{\alpha}{|\mathfrak{m}|_K} \right)^c & (s \in \mathbb{C}, \varepsilon = 0, 1), \text{ if } K = \mathbb{R}, \\ \frac{1}{|\mathfrak{m}|_K} \left( \frac{\alpha}{|\mathfrak{m}|_K^{1/2}} \right) p & (s \in \mathbb{C}, p \in \mathbb{Z}), \text{ if } K = \mathbb{C}, \\ \left( \frac{\alpha}{|\mathfrak{m}|_K^{\phi}} \right)^{\phi} & (s \in \mathbb{C}, \phi \in \mathfrak{O}_K^\times), \text{ if } K = \text{non-archimedean}, \end{cases} \]

where \(|\cdot|_K\) is the normalized absolute value of \(K\), \(\mathfrak{m}\) is a fixed prime element in \(K\) and \(\mathfrak{O}_K^\times\) is the character group of the unit group of the integer ring \(\mathfrak{O}_K\) of \(K\). Writing \(\omega \in \hat{\mathcal{O}}_K^\times\) as above, we put \(\text{Re}(\omega) = \text{Re}(s)\).

For \(\omega = (\omega_1, \cdots, \omega_l) \in (\hat{\mathcal{O}}_K^\times)^l\), set
\[
\omega(P(v))_{ij} = \begin{cases} \prod_{i=1}^l \omega_i(P_i(v)) & \text{if } v \in V_{ij}, \\ 0 & \text{otherwise}, \end{cases}
\]
\[
\omega(P^*(v))_{ij} = \begin{cases} \prod_{i=1}^l \omega_i(P_i^*(v)) & \text{if } v \in \hat{V}^*_{ij}, \\ 0 & \text{otherwise}, \end{cases}
\]
\[
\omega(\bar{P}(w))_{ij} = \begin{cases} \prod_{i=1}^l \omega_i(\bar{P}_i(w)) & \text{if } w \in \bar{V}_{ij}, \\ 0 & \text{otherwise}, \end{cases}
\]
\[
\omega(\bar{P}^*(w))_{ij} = \begin{cases} \prod_{i=1}^l \omega_i(\bar{P}_i^*(w)) & \text{if } w \in \bar{V}^*_{ij}, \\ 0 & \text{otherwise}. \end{cases}
\]

We define zeta functions attached to \((G, \rho, V), (G, \rho^*, V), (\bar{G}, \bar{\rho}, \bar{V})\) and \((\bar{G}, \bar{\rho}^*, \bar{V})\) as follows:
\[
Z_{ij}(\omega; f) = \int_{V_K} \omega(P(v))_{ij} f(v) \, dv |_{K}, \quad Z^*_{ij}(\omega; f) = \int_{V^*_K} \omega(P^*(v))_{ij} f(v) \, dv |_{K},
\]
\[
\bar{Z}_{ij}(\omega; \bar{f}) = \int_{\bar{V}_K} \omega(\bar{P}(w))_{ij} \bar{f}(w) \, dw |_{K}, \quad \bar{Z}^*_{ij}(\omega; \bar{f}) = \int_{\bar{V}^*_K} \omega(\bar{P}^*(w))_{ij} \bar{f}(w) \, dw |_{K},
\]
\((f \in \mathcal{F}(V_K), \bar{f} \in \mathcal{F}(\bar{V}_K)).\)

The zeta functions converge for \(\text{Re}(\omega_1) > 0, \cdots, \text{Re}(\omega_l) > 0\) and have analytic continuations to meromorphic functions on \((\hat{\mathcal{O}}_K^\times)^l\).

Put
\[
\omega^* = \left( \prod_{i=1}^l (\omega_i |_{K}^{u_{ij}}), \cdots, \prod_{i=1}^l (\omega_i |_{K}^{u_{ij}})^{u_{ij}} \right)
\]
(for the definition of \((u_{ij})\) and \((\lambda_i)\), see (5) and (6)).
The following lemma is easily proved by the uniqueness of relatively invariant distributions on homogenous spaces (cf. [S1, Lemma 5.5], [S2, §2.4]).

**Lemma 2.1.** (i) If the support of \( f \in \mathcal{S}(V_k) \) is contained in \( V_k - S_k^* \), then the zeta functions satisfy the following functional equation:

\[
Z_{ij}(\omega; \mathcal{F}(f)) = \sum_{i'=1}^{v} \sum_{p=1}^{k_i} \Gamma_{ij}^{i'p}(\omega)Z_{i'p}(\omega; f),
\]

where \( \Gamma_{ij}^{i'p}(\omega) \) are meromorphic functions on \((\mathbb{K}^\times)^{l} \) independent of \( f \).

(ii) If the support of \( \tilde{f} \in \mathcal{S}(\tilde{V}_k) \) is contained in \( \tilde{V}_k - S_k^* \), then the zeta functions satisfy the following functional equation:

\[
\tilde{Z}_{ij}(\omega; \mathcal{F}(\tilde{f})) = \sum_{i'=1}^{v} \sum_{p=1}^{k_i} \tilde{\Gamma}_{ij}^{i'p}(\omega)\tilde{Z}_{i'p}(\omega; \tilde{f}),
\]

where \( \tilde{\Gamma}_{ij}^{i'p}(\omega) \) are meromorphic functions on \((\mathbb{K}^\times)^{l} \) independent of \( \tilde{f} \).

The fundamental theorem in the theory of p.v.'s asserts that the functional equations in the lemma above hold without the condition on the the support of test functions.

**Theorem 1** (see [S1, §5, Theorem 1], [S2, §2.4, Theorem k_p]). Assume that \((G, \rho, V)\) satisfies the condition

(S) the singular set \( S \) is a hypersurface, if \( K \) is archimedean,

(F) the singular set \( S \) decomposes into a finite number of \( G \)-orbits. Moreover for any \( G \)-orbit \( S' \) in \( S \), there exists a non-trivial \( K \)-rational character \( \chi \) corresponding to a relative invariant such that \( S' \) is a ker \( \chi \)-orbit, if \( K \) is non-archimedean.

Then the functional equation

\[
Z_{ij}(\omega; \mathcal{F}(f)) = \sum_{i'=1}^{v} \sum_{j'=1}^{v'} \Gamma_{ij}^{i'j'}(\omega)Z_{i'j'}(\omega; f)
\]

holds for any \( f \in \mathcal{S}(V_k) \).

By Lemma 1.3, the condition (S) for \((G, \rho, V)\) implies the same condition of \((\tilde{G}, \tilde{\rho}, \tilde{V})\) and the theorem can also be applied to \((\tilde{G}, \tilde{\rho}, \tilde{V})\), if \( K \) is archimedean. On the other hand, even if \((G, \rho, V)\) satisfies the condition (F), \((\tilde{G}, \tilde{\rho}, \tilde{V})\) does not necessarily satisfy it (cf. [SK, §8]). However we have the following theorem of Kajima:

**Theorem 2** (Kajima [Ka]). Assume that \( K \) is non-archimedean. If the functional equation of the zeta functions \( Z_{ij} \) of \((G, \rho, V)\) in Lemma 2.1 (i) holds for any \( f \in \mathcal{S}(V_k) \), then the functional equation of the zeta functions \( \tilde{Z}_{ij} \) of \((\tilde{G}, \tilde{\rho}, \tilde{V})\) in Lemma 2.1 (ii) holds for any \( \tilde{f} \in \mathcal{S}(\tilde{V}_k) \) and vice versa.

2.3. The problem we now consider is to find a relation between \( \Gamma_{ij}^{i'p}(\omega) \) and
\( P_{\mu}^{ij}(\omega) \). To describe our answer, it is convenient to rewrite the functional equations in Lemma 2.1 (and Theorem 1).

For \( i = 1, \ldots, v \) and \( j = 1, \ldots, \mu_i \), put
\[
G_i^+ = \{(h, g_n) \in G^+ | \rho(h, g_n)V_{ij} = V_{ij}^* \} \\
= \{(h, g_n) \in G^+ | \rho^*(h, g_n)V_{ij}^* = V_{ij}^* \}
\]
and
\[
\tilde{G}_i^+ = \{(h, g_{m-n}) \in \tilde{G}_i^+ | \tilde{\rho}(h, g_{m-n})\tilde{V}_{ij} = \tilde{V}_{ij}^* \} \\
= \{(h, g_{m-n}) \in \tilde{G}_i^+ | \tilde{\rho}^*(h, g_{m-n})\tilde{V}_{ij}^* = \tilde{V}_{ij}^* \}.
\]
The groups \( G_i^+ \) and \( \tilde{G}_i^+ \) are independent of \( j \), since \( G_T^+ \) and \( \tilde{G}_T^+ \) contains the commutator subgroups of \( G^+ \) and \( \tilde{G}^+ \), respectively.

We fix a complete set \( \{(1, g_{ij}) \in G^+ | 1 \leq j \leq \mu_i \} \) (resp. \( \{(1, \tilde{g}_{ij}) \in \tilde{G}_i^+ | 1 \leq j \leq \mu_i \} \) of representatives of the coset space \( G^+/G_i^+ \) (resp. \( \tilde{G}_i^+/\tilde{G}_i^+ \)) satisfying \( g_{i1} = 1 \) (resp. \( \tilde{g}_{i1} = 1 \))
\[
V_{ij} = \rho(1, g_{ij})V_{i1}, \quad V_{ij}^* = \rho(1, g_{ij})V_{i1}^*, \\
\text{(resp.} \quad \tilde{V}_{ij} = \tilde{\rho}(1, \tilde{g}_{ij})\tilde{V}_{i1}, \quad \tilde{V}_{ij}^* = \tilde{\rho}^*(1, \tilde{g}_{ij})\tilde{V}_{i1}^*).\]

Denote by \((K^*/T)^\wedge\) the character group of the finite group \( K^*/T \). For an \( \eta \in (K^*/T)^\wedge \), put
\[
A_\eta = \{i | 1 \leq i \leq v, \eta(\det g_{i}) = 1 \ (\forall (h, g_{i}) \in G_i^+)\} \\
= \{i | 1 \leq i \leq v, \eta(\det g_{m-n}) = 1 \ (\forall (h, g_{m-n}) \in \tilde{G}_i^+)\}
\]
and
\[
A = \{(\eta, i) | \eta \in (K^*/T)^\wedge, i \in A_\eta \}.
\]

For \( (\eta, i) \in A \), define functions \( \omega(P(v))_{\eta,i} \), \( \omega(P^*(v))_{\eta,i} \), \( \omega(\tilde{P}(w))_{\eta,i} \) and \( \omega(\tilde{P}^*(w))_{\eta,i} \) by
\[
\omega(P(v))_{\eta,i} = \sum_{j=1}^{\mu_i} \eta(\det g_{ij})\omega(P(v))_{ij}, \quad \omega(P^*(v))_{\eta,i} = \sum_{j=1}^{\mu_i} \eta(\det g_{ij})\omega(P^*(v))_{ij}, \\
\omega(\tilde{P}(w))_{\eta,i} = \sum_{j=1}^{\mu_i} \eta(\det \tilde{g}_{ij})\omega(\tilde{P}(w))_{ij}, \quad \omega(\tilde{P}^*(w))_{\eta,i} = \sum_{j=1}^{\mu_i} \eta(\det \tilde{g}_{ij})\omega(\tilde{P}^*(w))_{ij}. \tag{7}
\]
These functions are independent of the choice of the representatives \( \{g_{ij}\} \). Conversely, by the orthogonal relation of characters of finite abelian groups, we obtain
\[
\omega(P(v))_{ij} = \frac{1}{\#(K^*/T)^\wedge} \sum_{\eta \in (K^*/T)^\wedge} \eta(\det g_{ij})^{-1} \omega(P(v))_{\eta,i}, \\
\omega(P^*(v))_{ij} = \frac{1}{\#(K^*/T)^\wedge} \sum_{\eta \in (K^*/T)^\wedge} \eta(\det g_{ij})^{-1} \omega(P^*(v))_{\eta,i},
\]
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\[ \omega(\vec{P}(w))_{ij} = \frac{1}{#(K^*/T_i)} \sum_{\eta \in (K^*/T_i)^*} \eta(\det g_{ij})^{-1} \omega(\vec{P}(w))_{n,i} , \]

\[ \omega(\vec{P}^*(w))_{ij} = \frac{1}{#(K^*/T_i)} \sum_{\eta \in (K^*/T_i)^*} \eta(\det g_{ij})^{-1} \omega(\vec{P}^*(w))_{n,i} , \]

where \( T_i = \{ \det(g_{ij}) | (h, g_{n}) \in G_i^+ \} \).

Put

\[ Z_{n,i}(\omega; f) = \int_{V_i} \omega(P(v))_{n,i}f(v)|dv|_K , \quad \tilde{Z}_{n,i}(\omega; f) = \int_{V_i} \omega(P^*(v))_{n,i}f(v)|dv|_K , \]

\[ \hat{Z}_{n,i}(\omega; \vec{f}) = \int_{V_i} \omega(\vec{P}(w))_{n,i}\vec{f}(w)|dw|_K , \quad \hat{Z}_{n,i}(\omega; \vec{f}) = \int_{V_i} \omega(\vec{P}^*(w))_{n,i}\vec{f}(w)|dw|_K \]

\((f \in \mathcal{F}(V_K), \vec{f} \in \mathcal{F}(\vec{V}_K))\). We rewrite the functional equations in terms of \( Z_{n,i}, \tilde{Z}_{n,i}, \hat{Z}_{n,i} \) and \( \hat{Z}_{n,i}^* \).

**Lemma 2.2.** For \( \omega \in (\hat{K}^*)^l \), let \( \omega^* \) be as in Lemma 2.1.

(i) If the support of \( f \in \mathcal{F}(V_K) \) is contained in \( V_K - S_K^0 \), then the zeta functions satisfy the following functional equation:

\[ Z_{n,i}(\omega; \mathcal{F}(f)) = \sum_{\eta \in A_n} \Gamma_{n,i}^{\eta}(\omega)Z_{n,i}^{\eta}(\omega^*; f) \quad ((\eta, i) \in A) , \]

where \( \Gamma_{n,i}^{\eta} \) are meromorphic functions on \((\hat{K}^*)^l\) independent of \( f \).

(ii) If the support of \( \vec{f} \in \mathcal{F}(\vec{V}_K) \) is contained in \( \vec{V}_K - \vec{S}_K^0 \), then the zeta functions satisfy the following functional equation:

\[ \hat{Z}_{n,i}(\omega; \mathcal{F}(\vec{f})) = \sum_{\eta \in A_n} \hat{\Gamma}_{n,i}^{\eta}(\omega)\hat{Z}_{n,i}^{\eta}(\omega^*; \vec{f}) \quad ((\eta, i) \in A) , \]

where \( \hat{\Gamma}_{n,i}^{\eta} \) are meromorphic functions on \((\hat{K}^*)^l\) independent of \( \vec{f} \).

**Proof.** By the identities (7), (8) and Lemma 2.1, we have

\[ Z_{n,i}(\omega; \mathcal{F}(f)) = \sum_{(\eta, i) \in A} \Gamma_{n,i}^{\eta}(\omega)Z_{n,i}^{\eta}(\omega^*; f) \]

for some meromorphic functions \( \Gamma_{n,i}^{\eta}(\omega) \) on \((\hat{K}^*)^l\) independent of \( f \). For a \( g \in GL(n; K) \), put \( f^g(v^*g) = f(v^*g) \). Then it is easy to check the identities

\[ Z_{n,i}(\omega; \mathcal{F}(f^g)) = (\det g)^{-1} \prod_{j=1}^l \omega_j(\chi_j(1, g))Z_{n,i}(\omega; \mathcal{F}(f)) , \]

\[ Z_{n,i}^{\eta}(\omega^*; f^g) = (\det g)^{-1} \prod_{j=1}^l \omega_j(\chi_j(1, g))Z_{n,i}^{\eta}(\omega^*; f) . \]

These identities imply that \( \Gamma_{n,i}^{\eta}(\omega) = 0 \) unless \( \eta = \eta^* \). Putting \( \Gamma_{n,i}^{\eta}(\omega) = \Gamma_{n,i}^{\eta}(\omega) \), we
obtain the first part of the lemma. The second part can be proved similarly.

**Remark.** If \((G, \rho, V)\) (or \((\tilde{G}, \tilde{\rho}, \tilde{V})\)) satisfies the same assumption as in Theorem 1, the functional equations in the lemma above hold for any Schwartz-Bruhat functions.

For a quasi-character \(\omega \in \widehat{K^\times}\), we define the Tate \(\Gamma\)-factor \(\Gamma_\chi(\omega)\) by the functional equation
\[
\mathcal{F}(\omega(\alpha)|| \alpha|_K) = \Gamma_\chi(\omega) \omega(\alpha)^{-1} \quad (\alpha \in K^\times).
\]
For the explicit formulas for \(\Gamma_\chi(\omega)\), we refer to [T]. Put
\[
\Gamma_\chi^{\eta^m}(\omega) = \prod_{i=0}^{n-1} \Gamma_\chi(\omega|_K^i).
\]
Now the following is our main theorem, which will be proved in Section 4.

**Theorem 3.** For \(\omega = (\omega_1, \cdots, \omega_l) \in (\widehat{K^\times})^l\), \(\eta \in (K^\times/T)^\wedge\) and \(i, i^* \in \Lambda_n\), we have
\[
\frac{\Gamma_\chi^{\eta, {i^*}}(\omega)}{\Gamma_\chi^{\eta^m}(\eta|_K \prod_{j=1}^l \omega_j^{d_j})} = \frac{\tilde{\eta}^{i^*}_i(\omega)}{\Gamma_\chi^{\eta^m}(\eta|_K \prod_{j=1}^l \omega_j^{d_j})}
\]
(for the definition of \(d_j = \deg Q_j\), see Lemma 1.2).

3. **b-functions**

3.1. In this section we consider the case \(K = R\). Let \(X_\rho(G)_R\) (resp. \(X_\rho(\tilde{G})_R\)) be the group of \(R\)-rational characters of \(G\) (resp. \(\tilde{G}\)) corresponding to relative invariants of \((G, \rho, V)\) (resp. \((\tilde{G}, \tilde{\rho}, \tilde{V})\)). Then \(X_\rho(G)_R\) (resp. \(X_\rho(\tilde{G})_R\)) is a free abelian group of rank \(l\) generated by \(\chi, \cdots, \chi_l\) (resp. \(\tilde{\chi}, \cdots, \tilde{\chi}_l\)). For \(\chi \in X_\rho(G)_R\) (resp. \(\tilde{\chi} \in X_\rho(\tilde{G})_R\), let \(\delta(\chi) = (\delta(\chi)_1, \cdots, \delta(\chi)_l)\) (resp. \(\delta(\tilde{\chi}) = (\delta(\tilde{\chi})_1, \cdots, \delta(\tilde{\chi})_l)\)) be the element in \(Z^l\) such that
\[
\chi = \prod_{i=1}^l \chi_i^{\delta(\chi)_i}\quad \text{(resp. } \tilde{\chi} = \prod_{i=1}^l \tilde{\chi}_i^{\delta(\tilde{\chi})_i}).
\]
Put \(\delta^*(\chi) = \delta(\chi) U\) and \(\delta^*(\tilde{\chi}) = \delta(\tilde{\chi}) U\) (for the definition of \(U\), see §2.1 (5)). Then the rational functions
\[
P_\chi(v) = \prod_{i=1}^l P_\chi(v)^{\delta(\chi)_i}, \quad P_\chi^*(v) = \prod_{i=1}^l P_\chi^*(v)^{\delta(\chi)_i},
\]
\[
\tilde{P}_\chi(w) = \prod_{i=1}^l \tilde{P}_\chi(w)^{\delta(\tilde{\chi})_i}, \quad \tilde{P}_\chi^*(w) = \prod_{i=1}^l \tilde{P}_\chi^*(w)^{\delta(\tilde{\chi})_i}
\]
satisfy the relative invariance
\[
P_\chi(\rho(h, g_n)v) = \chi(h, g_n) P_\chi(v), \quad P_\chi^*(\rho^*(h, g_n)v) = \chi(h, g_n) P_\chi^*(v),
\]
\[ \tilde{p}^*(\tilde{\varrho}(h, g_{m-n})w) = \tilde{\chi}(h, g_{m-n})\tilde{p}^*(w), \quad \tilde{p}^*\tilde{\varrho}(\tilde{\varrho}^*(h, g_{m-n})w) = \tilde{\chi}(h, g_{m-n})\tilde{p}^*\tilde{\varrho}(w). \]

Assuming that \( \delta^*(\chi)_i \geq 0 \) (resp. \( \delta^*(\tilde{\chi})_i \geq 0 \)) for all \( i \), we define a partial differential operator \( P^*\tilde{p}(\text{grad}_w) \) (resp. \( \tilde{p}^*\tilde{\varrho}(\text{grad}_w) \)) by

\[ P^*\tilde{p}(\text{grad}_w) \exp\langle v, v^* \rangle = P^*\tilde{p}(v^*) \exp\langle v, v^* \rangle, \]

(resp. \( \tilde{p}^*\tilde{\varrho}(\text{grad}_w) \exp\langle w, w^* \rangle = \tilde{p}^*\tilde{\varrho}(w^*) \exp\langle w, w^* \rangle \)).

Then there exist polynomials \( b_x(s) \) and \( \tilde{b}_x(s) \) in \( s_1, \ldots, s_t \) satisfying the identities

\[ P^*\tilde{p}(\text{grad}_w) \left( \prod_{i=1}^l P_i(v)^{s_i} \right) = b_x(s) \left( \prod_{i=1}^l P_i(v)^{s_i + \delta(\chi)} \right), \]

\[ \tilde{p}^*\tilde{\varrho}(\text{grad}_w) \left( \prod_{i=1}^l \tilde{P}_i(w)^{s_i} \right) = \tilde{b}_x(s) \left( \prod_{i=1}^l \tilde{P}_i(w)^{s_i + \delta(\tilde{\chi})} \right), \]

on the universal covering spaces of \( V_C - S_C \) and \( \tilde{V}_C - \tilde{S}_C \), respectively. The functions \( b_x(s) \) and \( \tilde{b}_x(s) \) can be defined for all \( \chi \in X_p(G)_R \) and \( \tilde{\chi} \in X_{\tilde{p}}(\tilde{G})_R \) by the cocycle properties

\[ b_{x\chi}(s) = b_x(s)b_{\chi}(s + \delta(\chi)), \quad \tilde{b}_{\tilde{x}\tilde{\chi}}(s) = \tilde{b}_{\tilde{x}}(s)\tilde{b}_{\tilde{\chi}}(s + \delta(\tilde{\chi})). \]

The functions \( b_x(s) \) and \( \tilde{b}_x(s) \) are called the \textit{b-functions} of \( (G, \rho, V) \) and \( (\tilde{G}, \tilde{\rho}, \tilde{V}) \), respectively (for further details, see [SSM]).

As an application of Theorem 3 over \( K = R \), we can obtain a relation between \( b_x(s) \) and \( \tilde{b}_x(s) \), when \( \delta(\chi) = \delta(\tilde{\chi}) \).

For \( \chi \in X_p(G)_R \) and \( \tilde{\chi} \in X_{\tilde{p}}(\tilde{G})_R \), set

\[ r(\chi) = - \sum_{i=1}^l d_i \delta(\chi)_i \quad \text{and} \quad r(\tilde{\chi}) = - \sum_{i=1}^l d_i \delta(\tilde{\chi})_i. \]

Also set

\[ \beta^e(\chi) = \left\{ \begin{array}{ll} \prod_{i=1}^l \prod_{j=1}^{r(\chi)} (s + i - j) & \text{if } r(\chi) \geq 1, \\ \prod_{i=1}^{n-\frac{r(\chi)}{2}-1} \sum_{j=0}^{r(\chi)-1} (s + i - j) & \text{if } r(\chi) \leq 0. \end{array} \right. \]

**Theorem 4.** If \( \delta(\chi) = \delta(\tilde{\chi}) \) (\( \chi \in X_p(G)_R, \tilde{\chi} \in X_{\tilde{p}}(\tilde{G})_R \)), then

\[ b_x(s) = \beta^e_x(s) \beta_x^{m-n} \left( \sum_{i=1}^l d_i \right) \]

**Remark.** For irreducible regular p.v.'s, the theorem has been obtained by Shintani (see Kimura [Kil, pp. 164–167]).
3.2. Proof of Theorem 4. Let $H^+$ be the identity component of $H_R$ and $T$ the multiplicative group of positive real numbers. Then the group $K^*/T$ is isomorphic to the group $\{ \pm 1 \}$ of order 2 and the character group $(K^*/T)^\wedge$ consists of the trivial character and the sign character $\text{sgn}(\chi) = \alpha/|\chi|_R$. The group $G_T^+$ coincides with the identity component of $G_R$. Hence the sign of any relative invariant with $R$-coefficients does not change on a $G_T^+$-orbit. We put
\[
\varepsilon_{ij}(\chi) = \text{the sign of } P^a(v) \text{ on } V_{ij}, \\
\varepsilon_{ij}^\tau(\chi) = \text{the sign of } P^{a\tau}(v) \text{ on } V_{ij}^\tau, \\
\varepsilon_{ij}(\tilde{\chi}) = \text{the sign of } \tilde{P}(w) \text{ on } \tilde{V}_{ij}, \\
\varepsilon_{ij}^\tau(\tilde{\chi}) = \text{the sign of } \tilde{P}^{a\tau}(w) \text{ on } \tilde{V}_{ij}^\tau.
\]

Then the following lemma gives a relation between the $b$-functions and the functional equations in Lemma 2.2.

**Lemma 3.1.** For $\eta \in (K^*/T)^\wedge$ and $i, i^* \in A_n$, we have
\[
b_{ij}(s) = \varepsilon_{i1}(\chi)\varepsilon_{i1}^*(\chi)(-2\pi \sqrt{-1})^{-nr^2r^2} \frac{\Gamma_{\eta, i^*}^{\eta(\text{sgn})r^2}(\omega_s)}{\Gamma_{\eta, i^*}^{\eta}(\omega_s + \delta(\chi))}, \\
\tilde{b}_{ij}(s) = \varepsilon_{i1}(\tilde{\chi})\varepsilon_{i1}^*(\tilde{\chi})(-2\pi \sqrt{-1})^{-nr^2r^2} \frac{\Gamma_{\eta, i^*}^{\eta(\text{sgn})r^2}(\omega_s)}{\Gamma_{\eta, i^*}^{\eta}(\omega_s + \delta(\chi))},
\]
where
\[
\omega_s = (|\cdot|_{\mathbb{R}}^{1}, \ldots, |\cdot|_{\mathbb{R}}^{1}) \in (\mathbb{R}^\times)^d.
\]

**Proof.** We prove the assertion only for $b_{ij}(s)$. By the identity (7), we have
\[
\Gamma_{\eta, i^*}^{\eta}(\omega) = \frac{1}{\sum_{\mu_i, j, j^* = 1}^{|\mu_i|} \eta(\det g_{ij} \cdot \det g_{ij^*^*}) \Gamma_{\eta, j^*}^{\eta}(\omega)}.
\]
From [S1, (5.8)], it follows that
\[
\Gamma_{\eta, i^*}^{\eta}(\omega_s + \delta(\chi)) = (-2\pi \sqrt{-1})^{-nr^2r^2} b_{ij}(s)^{-1} \\
\times \frac{1}{\sum_{\mu_i, j, j^* = 1}^{|\mu_i|} \varepsilon_{i1}(\chi)\varepsilon_{i1}^*(\chi) \eta(\det g_{ij} \cdot \det g_{ij^*^*}) \Gamma_{\eta, j^*}^{\eta}(\omega_s)}.
\]
Since
\[
\varepsilon_{ij}(\chi) = \varepsilon_{i1}(\chi) \text{ sgn}(\det g_{ij})^{r^2} \quad \text{and} \quad \varepsilon_{ij}^\tau(\chi) = \varepsilon_{i1}^*(\chi) \text{ sgn}(\det g_{ij^*^*})^{r^2},
\]
we get the lemma. 

Now we are in a position to prove Theorem 4. Pur $r = r(\chi) = r(\tilde{\chi})$. Since $\delta(\chi) = \delta(\tilde{\chi})$, we have
\[
\varepsilon_{ij}(\chi) = \varepsilon_{ij}(\tilde{\chi}), \quad \text{and} \quad \varepsilon_{ij}^\tau(\chi) = \varepsilon_{ij}^\tau(\tilde{\chi}).
\]
By Lemma 3.1 and Theorem 3, we have

\[
\begin{align*}
    b_\varepsilon(s) (-2\pi\sqrt{-1})^{-nr} & \cdot \frac{\Gamma^{(n)}(|\cdot|_R^{d_s+m-r}\eta)}{\Gamma^{(n)}(|\cdot|_R^{d_s+m}\eta}(\text{sgn})^r) \\
    &= \tilde{b}_\varepsilon(s) (-2\pi\sqrt{-1})^{-(m-n)r} \cdot \frac{\Gamma^{(m-n)}(|\cdot|_R^{d_s+m-r}\eta)}{\Gamma^{(m-n)}(|\cdot|_R^{d_s+m}\eta}(\text{sgn})^r),
\end{align*}
\]

(9)

where \( d\cdot s = d_1s_1 + \cdots + d_ls_l \). By the formula

\[
\Gamma^{(t)}(|\cdot|_R^{\varepsilon}(\text{sgn})^r) = (\sqrt{-1})^{(1-2z)/2} \cdot \frac{\Gamma((z+\varepsilon)/2)}{\Gamma((1-z+\varepsilon)/2)} \quad (\varepsilon = 0, 1),
\]

we can easily check the identity

\[
\frac{\Gamma^{(t)}(|\cdot|_R^{\varepsilon-r}\eta)}{\Gamma^{(t)}(|\cdot|_R^{\varepsilon}(\text{sgn})^r)} = (-2\pi\sqrt{-1})^r \cdot \prod_{i=1}^{r} (z-i).
\]

Hence

\[
\frac{\Gamma^{(n)}(|\cdot|_R^{d_s+m-r}\eta)}{\Gamma^{(n)}(|\cdot|_R^{d_s+m}\eta}(\text{sgn})^r) = (-2\pi\sqrt{-1})^{nr} \prod_{i=m-n+1}^{m} \prod_{j=1}^{r} (d\cdot s + i - j).
\]

Now the theorem follows immediately from (9) and this identity. 

4. Proof of Theorem 3

4.1. Let \( |dx_n|_K (\xi_n \in SL(n; K)) \) be the Haar measure on \( SL(n; K) \) normalized by

\[
\int_{GL(n; K)} F(x) |dx|_K = \int_{K} \int_{SL(n; K)} F(\xi_n) |dx_n|_K,
\]

where \( |dx|_K = \pi_{1 \leq i \leq n} |dx_i|_K, |d^* t|_K = |dt|_K |t|_K \) and \( t \) is an element in \( GL(n; K) \) such that \( \det t = t \). Then we obtain the following integral formula:

\[
\int_{V_K} F(v) dv = \int_{V_K} |y_0|_K^{m-n(m-n)} |d^* y_0|_K
\]

\[
\times \prod_{n+1 \leq l \leq m} |dy_{ij}|_K \int_{SL(n; K)} F(v_\xi_n) |dx_n|_K
\]

(10)

\( (F \in L^1(V_K, |dv|_K)) \), where \( v_\xi \) is an element in \( V_K \) such that \( A(v_\xi) = y \) and \( \{y_0, y_{ij}\} \) is the coordinate system on \( Y_K \) defined by §1.6 (4). Similarly we obtain
\[ \left( \tilde{V} \in L^1(\tilde{V}_K, |dw|_K) \right), \] where \( w_y \) is an element in \( \tilde{V}_K \) such that \( \tilde{A}(w_y) = y \).

For \( f \in \mathcal{S}(V_K) \) and \( \tilde{f} \in \mathcal{S}(\tilde{V}_K) \), put

\[ M_\Delta(f)(\tilde{A}(w)) = \int_{\mathcal{S}(n;K)} f(v^{\tilde{A}(w)}_{\xi_n}) \, |d\xi_n|_K, \]

\[ M_\tilde{\Delta}(\tilde{f})(\tilde{A}(w)) = \int_{\mathcal{S}(m-n;K)} \tilde{f}(w^{\tilde{A}(w)}_{\xi_{m-n}}) \, |d\xi_{m-n}|_K. \]

Set

\[ Y_1 = \Delta(V_1) \cap Y_1^* = \tilde{A}(\tilde{V}_1) \cap \tilde{V}_1^* \quad \text{and} \quad Y_1^* = \Delta(V_1^*) \cap Y_1^* = \tilde{A}(\tilde{V}_1^*) \cap \tilde{V}_1^* \]

(cf. Lemma 1.4 and §2.1). We may assume that \( \Delta(V_1) \), \( \Delta(V_1^*) \), \( \tilde{A}(\tilde{V}_1) \), \( \tilde{A}(\tilde{V}_1^*) \) contain a point \( y \) such that \( y_0 = 1 \). Then it is easy to prove the following lemma:

**Lemma 4.1.** If \( \text{Re}(\omega_i) > 0 \) (1 \( \leq i \leq l \)), then the following identities hold for \( f \in \mathcal{S}(V_K) \) and \( \tilde{f} \in \mathcal{S}(\tilde{V}_K) \):

\[ Z^*_n,\eta(\omega, f) = \int_{Y_1} \omega(Q^*(y)) |y_0|^{m-n} |y_0|_K | dy_0 | 1 \leq j \leq n \]

\[ Z^*_n,\eta(\omega, \tilde{f}) = \int_{Y_1} \omega(Q^*(y)) |y_0|^{m-n} |y_0|_K | dy_0 | 1 \leq j \leq n \]

where

\[ \omega(Q^*(y)) = \prod_{i=1}^{l} \omega_i(Q^*(y)). \]

**Corollary 4.2.** If \( M_\Delta(f) = M_\tilde{\Delta}(\tilde{f}) \) (\( f \in \mathcal{S}(V_K) \), \( \tilde{f} \in \mathcal{S}(\tilde{V}_K) \)), then

\[ Z^*_n,\eta(\omega, f) = Z^*_n,\eta(\omega, \tilde{f}). \]

**Proof.** If \( \text{Re}(\omega_i) > 0 \) (1 \( \leq i \leq l \)), then Corollary is an immediate consequence of Lemma 4.1. Hence, by analytic continuation, the identities hold for an arbitrary \( \omega \).

Let \( \mathcal{S}(Y_K) \) be space of compactly supported functions on \( Y_K \) which are infinitely differentiable or locally constant, according as \( K \) is archimedean or non-archimedean.
For \( q \in \mathcal{S}(Y_K) \) and \( y \in Y_K \), put
\[
\Phi(\omega, q; y) = \int_{\langle y, y^\ast \rangle \neq 0} \omega(\langle y, y^\ast \rangle)q(y^\ast) |y_0^\ast|_{K} |y_j^\ast|_{K} |y_j^\ast|_{K} \prod_{1 \leq j \leq m} |d\gamma_j^\ast|_{K},
\]
where \( \langle y, y^\ast \rangle = \sum_{i=1}^{N} y_i^\ast y_i \). We also denote by \( \Phi(\omega, q; y) \) the meromorphic function of \( \omega \in \widehat{K}^\times \) defined by the analytic continuation of the integral above.

Denote by \( \mathcal{S}(V_K') \) and \( \mathcal{S}(\hat{V}_K') \) the subspace of \( \mathcal{S}(V_K) \) and \( \mathcal{S}(\hat{V}_K) \) consisting of functions with support contained in \( V_K \) and \( \hat{V}_K \), respectively. The mean value mappings \( M_A \) and \( M_{\hat{A}} \) send \( \mathcal{S}(V_K') \) and \( \mathcal{S}(\hat{V}_K) \), respectively, onto \( \mathcal{S}(Y_K) \).

**Lemma 4.3**
(i) For \( \omega \in \widehat{K}^\times \), \( y \in Y_K \) and \( f \in \mathcal{S}(V_K) \), we have
\[
\int_{\widehat{K}^\times} \omega(t) M_A(\mathcal{F}(f)(ty)) |d^x t|_{K} = \Gamma_k^\ast(\omega) \Phi(\omega^{-1}, M_A(f); y).
\]
(ii) For \( \omega \in \widehat{K}^\times \), \( y \in Y_K \) and \( \hat{f} \in \mathcal{S}(\hat{V}_K) \), we have
\[
\int_{\widehat{K}^\times} \omega(t) M_{\hat{A}}(\mathcal{F}(\hat{f})(ty)) |d^x t|_{K} = \Gamma_k^\ast(\omega) \Phi(\omega^{-1}, M_{\hat{A}}(\hat{f}); y).
\]

**Proof.** We prove only the first part, since the proof of the second part is similar. Take \( v \in \hat{V}_K \) such that \( A(v) = y \) and \( \bar{t} \in GL(n; K) \) such that \( \det \bar{t} = t \). Then we have
\[
\int_{\widehat{K}^\times} \omega(t) M_A(\mathcal{F}(f)(ty)) |d^x t|_{K} = \int_{\widehat{K}^\times} \omega(t) |d^x t|_{K} \int_{SL(n; K)} \mathcal{F}(f)(v \xi_n) |d\xi_n|_{K}
\]
\[
= \int_{GL(n; K)} \omega(\det x) |d\xi|_{K} \mathcal{F}(f)(vx) \frac{|dx|_{K}}{|\det x|_{K}^\ast}.
\]
Recall that, for \( \varphi \in \mathcal{S}(M(n; K)) \),
\[
\int_{GL(n; K)} \omega(\det x) \varphi(x) \frac{|dx|_{K}}{|\det x|_{K}^\ast} = \Gamma_k^\ast(\omega) \int_{GL(n; K)} \omega(\det x)^{-1} \varphi(x^\ast) |dx^\ast|_{K},
\]
where
\[
\varphi(x^\ast) = \int_{M(n; K)} \varphi(x^\ast) \psi(tr^\ast x^\ast) |dx^\ast|_{K}
\]
(cf. [GJ], [S2, Theorem 3.4]). For \( v \), find an \( h \in SL(m; K) \) such that
\[
v = hv_0, \quad v_0 = \begin{pmatrix} I_n \\ 0_{m-n,n} \end{pmatrix}.
\]
Writing
\[
v^\ast = \begin{pmatrix} v_1^\ast \\ v_2^\ast \end{pmatrix} \quad (v_1^\ast \in M(n; K), v_2^\ast \in M(m-n, n; K)),
\]
we put
\[ F^h(v^*_1) = \int_{M(n-n,n;K)} f(lh^{-1}v^*) |dv^*_1|_K. \]

Then we have
\[ \mathcal{F}(f)(v^*_1) = \hat{F}(v^*_1). \]

Hence
\[ \int_{GL(n;K)} \omega(\det x) \mathcal{F}(f)(v^*_1) \frac{|dx|_K}{|\det x|_K^n} = \Gamma_K^{(0)}(\omega) \int_{GL(n;K)} \omega(\det x^*)^{-1} F^h(x^*) |dx_*|_K \]
\[ = \Gamma_K^{(0)}(\omega) \int_{\det(lv^*) \neq 0} \omega(\det(lv^*)) f(v^*) |dv^*|_K. \]

Since
\[ \det(lv^*) = \sum_{i=1}^N A_i(v^*) A_i(v^*), \]
the lemma follows from the integral formula (10).

\[ \text{LEMMA 4.4.} \quad \text{Put } \bar{Y}_l = \{ y \in Y_l \mid y_0 = 1 \} \text{ and } \]
\[ \omega(y)(t) = \eta(t) |t|^m \prod_{j=1}^l \omega_j(t)^{d_j}. \]

Suppose that Re(\omega_j) > 0 (1 \leq j \leq l).

(i) For \( f \in \mathcal{F}(V^*_K) \), we have
\[ Z_{\eta,l}(\omega, \mathcal{F}(f)) = \Gamma_K^{(0)}(\omega) \cdot \int_{\bar{Y}_l} \omega(Q(y)) \Phi(\omega^{-1}_n, M_d(f); \bar{y}) \prod_{n+1 \leq i \leq m, 1 \leq j \leq n} |d\bar{y}_{ij}|_K. \]

(ii) For \( f \in \mathcal{F}(\bar{V}_K) \), we have
\[ Z_{\eta,l}(\omega, \mathcal{F}(f)) = \Gamma_K^{(m-n)}(\omega) \cdot \int_{\bar{Y}_l} \omega(Q(y)) \Phi(\omega^{-1}_n, M_d(f); \bar{y}) \prod_{n+1 \leq i \leq m, 1 \leq j \leq n} |d\bar{y}_{ij}|_K. \]

Proof: Applying Lemma 4.1 to \( Z_{\eta,l} \), we obtain
\[ Z_{\eta,l}(\omega, \mathcal{F}(f)) = \int_{Y_l} \omega(Q(y)) \eta(y_0) |y_0|_K^{m-n} M_d(\mathcal{F}(f)) \bar{y} |d\bar{y}_0|_K \prod_{n+1 \leq i \leq m, 1 \leq j \leq n} |d\bar{y}_{ij}|_K, \]
where
\[ \omega(Q(y)) = \prod_{i=1}^l \omega_i(Q(y)). \]
The mapping $\bar{Y}_i \times K^* \rightarrow Y_i$ defined by $(\bar{y}, t) \mapsto t \bar{y}$ is a homeomorphism and we have

$$Z_{n,i}(\omega, \mathcal{F}(f)) = \int_{\bar{Y}_i} \omega(Q(\bar{y})) \prod_{1 \leq j \leq m} d\bar{y}_j|_K \int_{K^*} \omega_i(t) M_{\mathcal{A}}(\mathcal{F}(f))(t \bar{y})| dt|_K.$$ 

Now the first part follows immediately from Lemma 4.3 (i). By using the integral formula (11) and Lemma 4.3 (ii), the second part can also be proved in the same manner.

**Corollary 4.5.** If $M_{\mathcal{A}}(f) = M_{\mathcal{A}}(\bar{f})$ ($f \in \mathcal{F}(V_k), \bar{f} \in \mathcal{F}(\bar{V}_k)$), then

$$\frac{Z_{n,i}^{*}(\omega, \mathcal{F}(f))}{\Gamma_K^{*}(\eta \cdot \omega_r \prod_{i=1}^{l} \omega_i^{k_i})} = \frac{Z_{n,i}^{*}(\omega, \mathcal{F}(\bar{f}))}{\Gamma_K^{*}(\eta \cdot \omega_r \prod_{i=1}^{l} \omega_i^{k_i})}.$$ 

**Proof.** If $\text{Re}(\omega_i) > 0$ ($1 \leq i \leq l$), then Corollary is an immediate consequence of Lemma 4.4. Hence, by analytic continuation, the identities hold for arbitrary $\omega$.

Now we can complete the proof of Theorem 3. Since

$$Y_K \cong V_k/SL(n; K) \cong \bar{V}_k/SL(m-n; K),$$

the mean value mappings

$$M_{\mathcal{A}} : \mathcal{F}(V_k) \rightarrow \mathcal{F}(Y_K), \quad M_{\mathcal{A}} : \mathcal{F}(\bar{V}_k) \rightarrow \mathcal{F}(Y_K)$$

are surjective. Therefore Theorem 3 follows immediately from Corollaries 4.2 and 4.5.

**References**


Department of Mathematics
Rikkyo University
Nishi-Ikebukuro, Toshima-ku
Tokyo 171, Japan