On the Mordell-Weil Lattices

Dedicated to Professor K. Kodaira for his 75th birthday

by

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0. Introduction

In [S2, I], we have announced the basic results on the Mordell-Weil lattices, the key idea of which is to view the Mordell-Weil group of an elliptic curve over a function field (or of an elliptic surface) as a lattice with respect to a suitable pairing.

In this paper, we shall prove these results and thus give the foundation of the theory of Mordell-Weil lattices.

The paper consists of two parts. In the part I, we reconsider the problems treated in our old paper [S1, §1] and reorganize the results there, with some improvement, in a way appropriate for introducing the notion of the Mordell-Weil lattices. We are concerned with the basic finiteness theorems for an elliptic surface, i.e. that of the Mordell-Weil group and the Néron-Severi group, and the relationship between these two. We give a detailed exposition on the subject from the viewpoint to be explained in §1.

In the part II, we introduce the notion of Mordell-Weil lattices by defining the height pairing on the Mordell-Weil group via the intersection theory on an elliptic surface. This viewpoint fits remarkably well in the standard theory of algebraic surfaces, and we can establish the basic results on the Mordell-Weil lattices, based on the theory of elliptic surfaces due to Kodaira [K] and to Bombieri and Mumford [BM] for its characteristic $p$ version. As the first application of this theory, we consider the case of rational elliptic surfaces, and obtain the structure theorem and the generator theorem of the Mordell-Weil group of such a surface. The case with higher rank, announced in [S2, II], will be treated here. The general case will be treated in [OS], where a complete classification of the Mordell-Weil lattice for such a surface is given.

This theory has a wide variety of applications. The simplest case of rational elliptic surfaces alone has already provided many interesting results on such subjects as the following:

1. Theory of algebraic equations of type $E_r$ ($r=6, 7, 8$)
2. Construction of elliptic curves over $\mathbb{Q}(t)$ with rank up to 8, together with explicit generators
3. Construction of Galois representations of type $E_r$
(4) Hasse zeta functions and Artin L-functions
(5) Exceptional curves on del Pezzo surfaces
(6) Deformation of singularities
cf. [S2, II, III], [S4], [S5], [S6], [S7], and we expect more to come.

Beyond the case of rational elliptic surfaces, we have studied so far only some simplest case of unirational elliptic surfaces in characteristic $p$, but their Mordell-Weil lattices turn out to be lattices of some interest from the point of view of sphere packings (cf. [S3]). This last topic has been independently and more thoroughly investigated by N. Elkies [E].

Finally it should be noted that the height pairing defined here is essentially the same as the canonical height on an abelian variety, due to Néron [N2], Tate [T2] or Manin [M1] (cf. [L2], [Se]), in the case of an elliptic curve over a function field, and also as the pairing defined by Cox and Zucker [CZ] in the complex case. But, with the basic idea to view the Mordell-Weil group as a lattice, we are naturally led to a simple algebraic definition of the height pairing by means of the intersection theory on an elliptic surface. Thus the situation is quite analogous to the case of arithmetic surfaces, due to Faltings [F] and Hriljac [H]. It would be interesting to pursue further analogy between the arithmetic and geometric cases. For instance, we may ask: Can one define some class of elliptic curves defined over $\mathbb{Q}$ which forms an arithmetic analogue of the class of rational elliptic surfaces?

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Part I. The Mordell-Weil group and the Néron-Severi group of an elliptic surface

1. The basic theorems

The ground field $k$ is an algebraically closed field of arbitrary characteristic. Let $C$ be a smooth projective curve over $k$, and let $S$ be an elliptic surface over $C$. By this we always mean the following: $S$ is a smooth projective surface with a relatively
minimal elliptic fibration

\[ f: S \to C, \]

that is, \( f \) is a surjective morphism such that 1) almost all fibres are elliptic curves and 2) no fibres contain an exceptional curve of the first kind (i.e. a smooth rational curve with self-intersection number \(-1\)). Throughout the paper, we assume:

(1.1) \begin{enumerate}
\item[(i)] \( f \) has a global section \( O \), and
\item[(ii)] \( f \) is not smooth, i.e., there is at least one singular fibre.
\end{enumerate}

Let \( E \) denote the generic fibre of \( f: S \to C \). Then \( E \) is an elliptic curve defined over the function field of \( C \), \( K = k(C) \), given with a \( K \)-rational point \( O \). Let \( E(K) \) denote the group of \( K \)-rational points of \( E \), with the origin \( O \).

Conversely, if we start from an elliptic curve \( E \) over \( K \), we can naturally consider the associated elliptic surface \( f: S \to C \) (the Kodaira-Néron model of \( E/K \)), whose generic fibre is \( E \). The existence and the uniqueness, up to an isomorphism, of the Kodaira-Néron model is well-known ([K], [N1], [T1]). Roughly speaking, \( S \) is obtained from \( E/K \) as follows: writing down the equation of \( E \) with coefficients in \( K \), say in the Weierstrass form, we can easily construct a smooth quasi-projective surface \( S' \) with a smooth elliptic fibration \( f': S' \to C' \) having the generic fibre \( E/K \) and a section \( O': C' \to S' \), where \( C' \) is the curve \( C \) minus a finite number of points. Then, at each point \( v \) of \( C - C' \), we fill in a suitable fibre \( F_v \) to \( S' \) which is a reducible divisor in general, to obtain a smooth projective surface \( S \) with the required properties.

It is indispensable in studying an elliptic surface to know the singular fibres, and we shall freely use the results of [K], [N1], [T1]. Given an elliptic surface \( f: S \to C \), let \( F_v = f^{-1}(v) \) denote the fibre over \( v \in C \), and let

\[
\text{Sing}(f) = \{ v \in C \mid F_v \text{ is singular} \}
\]

\[
R = \text{Red}(f) = \{ v \in C \mid F_v \text{ is reducible} \}.
\]

For each \( v \in R \), let

\[
F_v = f^{-1}(v) = \Theta_{v,0} + \sum_{i=1}^{m_v-1} \mu_{v,i} \Theta_{v,i} \quad (\mu_{v,0} = 1)
\]

where \( \Theta_{v,i} \) (0 \( \leq \) \( i \) \( \leq \) \( m_v - 1 \)) are the irreducible components of \( F_v \), \( m_v \) being their number, such that \( \Theta_{v,0} \) is the unique component of \( F_v \) meeting the zero section.

Now the global sections of \( f: S \to C \) are in a natural one-to-one correspondence with the \( K \)-rational points of \( E \). Namely, given a section \( s: C \to S \) of \( f \) (\( f \circ s = \text{id}_C \)), let \( P \) be its restriction to the generic fibre \( E \), which is a \( K \)-rational point of \( E \). Conversely, given a \( P \in E(K) \), the \( k \)-locus of \( P \) in \( S \), say \( \Gamma \), is a curve in \( S \) and the restriction of \( f \) induces an isomorphism of \( \Gamma \) onto \( C \), since \( C \) is smooth and hence normal. Thus there is a unique section \( s: C \to S \) with \( \text{Im}(s) = \Gamma \). Therefore we identify the sections of \( f \) with the \( K \)-rational points of \( E \), and use the same notation \( E(K) \) to denote the group of sections of \( f \). For \( P \in E(K) \), \( (P) \) denotes the prime divisor of
S which is the image of the section \( P : C \to S \) (i.e. \( (P) = \Gamma \) as above); \( (P) \) will also be called a section. In particular, \( (O) \) denotes the zero section as a curve in \( S \).

**Theorem 1.1.** Under the assumption (1.1), \( E(K) \) is a finitely generated abelian group.

This is a special case of the Mordell-Weil theorem (cf. [L2, Ch. 6], [Se, Ch. 4]), valid more generally for an abelian variety \( A/K \), \( K \) being a "global" field in suitable sense. Thus the group \( E(K) \) is called the **Mordell-Weil group** of the elliptic curve \( E/K \), or of the elliptic surface \( f : S \to C \). This group is closely related to another important group \( NS(S) \), the **Néron-Severi group** of the surface \( S \), which is defined as the group of divisors on \( S \) modulo algebraic equivalence.

**Theorem 1.2.** Under the same assumption, \( NS(S) \) is finitely generated and torsion-free.

The finiteness is again a special case of the theorem of the base (cf. [L2, Ch. 5]), valid for more general projective varieties. The general proof for this is to consider the Jacobian variety of some auxiliary curve on a given variety, and to apply the Mordell-Weil theorem. The proof of the latter usually consists of 2 steps: 1) the weak Mordell-Weil theorem that \( A(K)/mA(K) \) is finite for some \( m > 1 \) and 2) the "infinite descent" analysing the points of bounded height. The argument for 1) is of number-theoretic nature, which shows that certain unramified extensions of \( K \) are finite.

In the case of an elliptic surface, we can (and we shall) prove these fundamental theorems by more geometric method based on the intersection theory on an algebraic surface. Moreover the relation of the two theorems is explicitly given by Theorem 1.3 below. The arguments involving the height will be sublimed to the theory of Mordell-Weil lattices, which will be treated in the part II.

**Theorem 1.3.** Let \( T \) denote the subgroup of \( NS(S) \) generated by the zero section \( (O) \) and all the irreducible components of fibres. Then, under (1.1), there is a natural isomorphism

\[
E(K) \cong NS(S)/T,
\]

which maps \( P \in E(K) \) to \( (P) \mod T \).

Since \( T \) is finitely generated (see below), Theorems 1.1 and 1.2 are equivalent to each other under Theorem 1.3. We shall prove Theorems 1.2 and 1.3 below, which implies Theorem 1.1.

2. Intersection theory on an elliptic surface

First let \( S \) be arbitrary smooth projective surface over \( k \), and let us consider the group of divisors on \( S \) and its subgroups:
(2.1) \[ \mathcal{D}(S) = \{ D = \sum_j n_j \Gamma_j \mid n_j \in \mathbb{Z}, \Gamma_j \text{ irreducible curves on } S \} \]
\[ \cup \]
\[ \mathcal{D}_a(S) = \{ D \sim 0 \text{ (algebraically equivalent to zero)} \} \]
\[ \cup \]
\[ \mathcal{D}(S) = \{ D \sim 0 \text{ (linearly equivalent to zero)} \} \].

The components \( \Gamma_j \) of a divisor \( D \) on \( S \) are assumed to be defined over the ground field \( k \), unless otherwise mentioned. The quotient groups

(2.2) \[ NS(S) = \mathcal{D}(S)/\mathcal{D}_a(S) \]

and

(2.3) \[ Pic^0(S) = \mathcal{D}_a(S)/\mathcal{D}(S) \]

are respectively called the Néron-Severi group and the Picard variety of \( S \); the latter has the structure of an abelian variety over \( k \) (cf. [L1]).

Now the intersection number \( (D \cdot D') \) of two divisors \( D \) and \( D' \) on \( S \) is a well-defined integer, which depends only on the algebraic equivalence classes of \( D \) and \( D' \) (cf. [Mu], [W1]). It defines a symmetric bilinear pairing on \( NS(S) \):

(2.4) \[ NS(S) \times NS(S) \rightarrow \mathbb{Z}. \]

We take for granted the following axiom on the cycle map:

**Theorem 2.1.** There is a finite-dimensional vector space \( H^2(S) \) over a field \( \Omega \) of characteristic 0, equipped with a non-degenerate pairing \( H^2(S) \times H^2(S) \rightarrow \Omega \), and a homomorphism (cycle map)

(2.5) \[ \gamma : NS(S) \rightarrow H^2(S) \]

preserving the pairing: i.e., for any \( D, D' \in \mathcal{D}(S) \), we have

(2.6) \[ (D \cdot D') = (\gamma(D) \cdot \gamma(D')). \]

As is well-known, we can take as \( H^2(S) \) any one of the following: if \( k = \mathbb{C} \), \( H^2(S^{an}, \mathcal{O}) \) (\( S^{an} \): the compact complex surface defined by \( S \), \( \mathcal{O} = \mathcal{O} \); for \( k \) general, \( H^2_{et}(S, \mathcal{O}_l) \) (the \( l \)-adic etale cohomology, \( l \neq \text{char}(k) \)), \( \mathcal{O} = \mathcal{O}_l \), or the crystalline cohomology if \( \text{char}(k) > 0 \). Note that the kernel of the cycle map consists of those classes of \( D \) which are numerically equivalent to 0, i.e.

(2.7) \[ (D \cdot \Gamma) = 0 \quad \text{for all curves } \Gamma \text{ on } S, \]

and hence it is independent of the choice of \( \gamma \).

As an immediate consequence, we have

**Corollary 2.2.** The Néron-Severi group \( NS(S) \) modulo numerical equivalence is finitely generated and torsion-free.
Now we go back to the case of an elliptic surface. In order to prove Theorem 1.2, we have to show that numerical equivalence coincides with algebraic equivalence on such a surface. This will be done in the next section, after we make some preparation here, reviewing the necessary facts on an elliptic surface.

Let us call an irreducible curve \(\Gamma\) on \(S\) vertical (or horizontal) if it is (or is not) contained in some fibre \(F_v\). We denote by

\[
\mathcal{D}_{\text{ver}} \quad \text{or} \quad \mathcal{D}_{\text{hor}}
\]

the subgroup of \(\mathcal{D}(S)\) generated by vertical (or horizontal) curves. By definition, we have

\[
T = \text{the image of } \mathcal{D}_{\text{ver}} + \mathcal{Z}(\mathcal{O}) \quad \text{in} \quad NS(S).
\]

**Proposition 2.3.** \(T\) is a torsion-free group generated by the divisor classes of the following:

\[
(\mathcal{O}), \ F \quad \text{and} \quad \Theta_{v,i} \ (v \in R, 1 \leq i \leq m_v - 1),
\]

where \(F\) stands for any fibre. In particular, we have

\[
\text{rk } T = 2 + \sum_{v \in R} (m_v - 1).
\]

**Proof.** Consider the intersection matrix of the curves in (2.10). Then the assertion will follow from Lemmas 2.4, 2.5, 2.6 below.

**Lemma 2.4.** Any two fibres of \(f, F_v\) and \(F_{v'}\) \((v, v' \in C)\), are algebraically equivalent to each other. In particular, we have

\[
F \simeq \Theta_{v,0} + \sum_{v \in R} (m_v - 1) \Theta_{v,i} \quad (v \in R).
\]

**Proof.** This is clear from the definition of algebraic equivalence (any two members of an algebraic family of divisors on \(S\) are algebraically equivalent) and (1.4).

**Lemma 2.5.** For each \(v \in R\), the intersection matrix

\[
A_v = ((\Theta_{v,i} \cdot \Theta_{v,j}))_{1 \leq i, j \leq m_v - 1}
\]

is negative-definite.

For a simple (classification-free) proof, see [BM, §1]. A case by case proof using the classification of singular fibres is just an exercise in linear algebra, but it gives more useful information which will be used later (cf. Lemma 7.2).

**Lemma 2.6.** For any section \(P \in E(K)\) and any fibre \(F\), we have

\[
(PF) = (OF) = 1, \quad (F^2) = (F_v \cdot F_{v'}) = 0.
\]

**Proof.** Obvious.
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Here and in what follows, let us simplify the notation by writing
\[(PD) = ((P) \cdot D), \quad (PQ) = ((P) \cdot (Q)), \quad (P^2) = (PP), \quad \text{etc.}\]
for \(P, Q \in E(K)\) and \(D \in \mathcal{D}(S)\).

**Lemma 2.7.** For any section \(P \in E(K)\), we have
\[(P^2) = (O^2) = -\chi < 0 \quad (\chi = \text{arithmetic genus of } S).\]

This is a consequence of the following two important results.

**Theorem 2.8.** For an elliptic surface \(f : S \to C\), the canonical bundle \(K_S\) is given by the formula:
\[(2.17) \quad K_S = f^*(\mathfrak{l} - f),\]
where \(\mathfrak{l}\) is the canonical bundle of the base curve \(C\) and \(f\) is a line bundle of degree \(-\chi\) on \(C\), isomorphic to the normal bundle to the zero section \((O)\) in \(S\), such that \(\mathcal{O}(f) = R^1 f_* (\mathcal{O}_S)\). Thus we have
\[(2.18) \quad K_S \approx (2g - 2 + \chi) F, \quad (K_S^2) = 0 \quad (g: \text{genus of } C)\]
\[(2.19) \quad \deg(f) = (O^2) = -\chi.\]

For the proof, we refer to [BM, §1] or [K, §12]. (Note that, under the assumption (1.1), there are no multiple fibres.)

**Theorem 2.9.** Under the assumption (1.1), the arithmetic genus \(\chi\) of \(S\) is positive:
\[(2.20) \quad \chi = \chi(\mathcal{O}_S) > 0.\]

For a nice proof of this “well-known” result, see the recent paper of Oguiso [O].

3. Algebraic and numerical equivalence

**Theorem 3.1.** On an elliptic surface satisfying (1.1), the following conditions on a divisor \(D\) are equivalent to each other:
(a) \(D\) is algebraically equivalent to 0.
(b) \(nD\) is algebraically equivalent to 0 for some \(n > 0\).
(c) \(D\) is numerically equivalent to 0, i.e.,
\[(3.1) \quad (D \cdot \Gamma) = 0 \quad \text{for any curve } \Gamma \text{ on } S.\]

**Proof.** The implication (a) \(\Rightarrow\) (b) and (b) \(\Rightarrow\) (c) being obvious, we have only to show that (c) implies (a). We give a proof, following an idea of M. Inoue [I].

Take a divisor \(D\) on \(S\), and assume that (3.1) holds. Then, by the Riemann-Roch theorem and (2.20), we have
\[(3.2) \quad h^0(S, \mathcal{O}(D)) - h^1(S, \mathcal{O}(D)) + h^2(S, \mathcal{O}(D)) = \frac{1}{2} ((D^2) - (D \cdot K_S)) + \chi(\mathcal{O}_S) = \chi > 0.\]

Hence we have either (i) \( h^0(S, \mathcal{O}(D)) > 0 \) or (ii) \( h^2(S, \mathcal{O}(D)) > 0 \). In case (i), \( D \) will be linearly equivalent to an effective divisor satisfying (3.1), hence \( D \sim 0 \) and we are done. Let us assume (ii). Then we have, by the Serre duality,

\[(3.3) \quad h^0(S, \mathcal{O}(K - D)) = h^2(S, \mathcal{O}(D)) > 0,\]

where \( K = K_S \) is the canonical bundle of \( S \). It follows that \( K - D \) is equivalent to some effective divisor, say \( D' \). For any fibre or its component \( \Theta \) of \( f \), we have

\[(3.4) \quad (D' \cdot \Theta) = (K \cdot \Theta) - (D \cdot \Theta) = 0,\]

by (2.18) and (3.1). Therefore, all the irreducible components of \( D' \) are contained in some fibres, and we can write

\[ D' = \sum_j a_j F_j + \sum_{v \in \mathbb{N}} \sum_{i \geq 1} b_{v,i} \Theta_{v,i} \quad (a_j, b_{v,i} \in \mathbb{Z}). \]

By (3.4) and Lemma 2.5, we see easily that \( b_{v,i} = 0 \) for all \( v, i \). It follows that \( D' \) is algebraically equivalent to some multiple of a fibre \( F \). By (2.18) again, \( D \) is also algebraically equivalent to such: \( D \approx mF \). Taking the intersection number with the zero section \( (O) \), we see \( m = 0 \) from (3.1). Hence \( D \approx 0 \). q.e.d.

**Proof of Theorem 1.2.** This is an immediate consequence of Corollary 2.2 and Theorem 3.1.

4. The Picard variety of an elliptic surface

Next we consider the Picard variety \( \text{Pic}^0(s) \) of an elliptic surface \( S \). Recall the situation at the beginning of §2.

The group of divisors on the base curve \( C \) has the filtration similar to (2.1):

\[(4.1) \quad \mathcal{D}(C) \supseteq \mathcal{D}_0(C) \supseteq \mathcal{D}_0(C),\]

with the quotient groups:

\[(4.2) \quad \mathcal{D}(C)/\mathcal{D}_0(C) \simeq \mathbb{Z}\]

and

\[(4.3) \quad \text{Pic}^0(C) = \mathcal{D}_0(C)/\mathcal{D}_0(C) = \text{Jac}(C).\]

Here \( \mathcal{D}_0(C) \) is the group of divisors of degree 0 on \( C \), and \( \text{Jac}(C) \) is the Jacobian variety of \( C \).
THEOREM 4.1. For an elliptic surface $f : S \to C$ satisfying (1.1), the Picard variety is naturally isomorphic to the Jacobian variety of the base curve $C$ via the induced map $f^*$:

$$f^*: \text{Jac}(C) = \text{Pic}^0(C) \simeq \text{Pic}^0(S).$$

Proof. Since $f$ admits the section $O$, we see that $f^*$ is an injection. We must show that it is also surjective. In other words, given a divisor $D$ algebraically equivalent to $0$ on $S$, we have to find some divisor $b$ of degree $0$ on $C$ such that $f^*(b) \sim D$.

Before continuing the proof, we insert a remark.

It is convenient to view the generic fibre $E$ of $f : S \to C$ as a curve lying on $S$. For this purpose, take a generic point $u$ of $C$ over $k$ in the sense of Weil [W1], and identify $k(u)$ with $K = k(C)$ and the fibre $f^{-1}(u)$ with $E$; thus $E$ is an elliptic curve on $S$ defined over $K$. For any $D \in \mathcal{D}(S)$, the intersection product $D \cdot E$ of $D$ and $E$ on $S$ is a divisor on $E$ (a 0-cycle), defined as follows: write $D$ as a sum of a horizontal divisor $D'$ and a vertical divisor $D''$. Then $D'$ and $E$ intersect properly on $S$ so that the intersection product $D' \cdot E$ is a well-defined 0-cycle, which is $K$-rational and of degree $(D' \cdot E) = (D \cdot E)$ (cf. [W1]). We define

$$D \cdot E = D' \cdot E \in \mathcal{D}(E)_{K} = \{ \text{K-rational divisors on } E \},$$

We sometimes call it the restriction of $D$ to $E$ and denote it by $D|_E$. A horizontal irreducible curve $\Gamma$ on $S$ is uniquely recovered from $\Gamma \cdot E$ as the $k$-locus of any intersection point of $\Gamma$ and $E$.

LEMMA 4.2. If $D \cdot E$ is linearly equivalent to $0$ on $E$, then $D$ is linearly equivalent to some vertical divisor $D' \in \mathcal{D}_{\text{vert}}$ on $S$.

Proof. It is obvious that, if $D \cdot E = 0$, then $D$ is vertical. Suppose that $D \cdot E \neq 0$ on $E$. Then, by definition, there exists a function $h \in K(E)$, $h \neq 0$, such that $D \cdot E = (h)$. Since the function field $K(E)$ of $E$ over $K$ is equal to the function field $k(S)$ of $S$ over $k$, we have a function $g \in k(S)$ corresponding to $h$, which has the property that $(g)|_E = (h)$. Then $D' = D - (g)$ satisfies $D' \cdot E = 0$, hence it is vertical, and $D$ is linearly equivalent to $D'$.

LEMMA 4.3 (Abel's theorem for an elliptic curve $E$ over $K$). There is a natural isomorphism $\text{Pic}^0(E) \simeq E$. In particular, we have

$$\alpha : \mathcal{D}_a(E)_K/\mathcal{D}_a(E)_K \simeq E(K)$$

in which $P \in E(K)$ corresponds to the equivalence class of $P - O$.

Proof. This is well-known, and indeed this is how one defines the group law on an elliptic curve.

Now we go back to the proof of Theorem 4.1. Suppose that $D \neq 0$. Then we have $(D \cdot E) = (D \cdot F) = 0$. Thus $D \cdot E$ is an element of $\mathcal{D}_a(E)_K$ so that it defines some $P \in E(K)$ by (4.6); by definition, we have

$$D \cdot E \sim (P)|_E - (O)|_E.$$
Letting $D' = D - (P) + (O)$, we have $D' \sim E \sim 0$. By Lemma 4.2, $D' \sim D''$ for some $D'' \in \mathcal{D}_{\text{ver}}$. Hence we can write

$$(4.7) \quad D \sim (P) - (O) + \sum_j a_j F_{v_j} + \sum_{v \in R} \sum_{i \geq 1} b_{v,i} \Theta_{v,i}$$

for some $a_j, b_{v,i} \in \mathbb{Z}$. First assume that $P$ belongs to the subgroup of $E(K)$ defined by

$$(4.8) \quad E(K)^0 = \{ P \mid (P) \text{ intersects } \Theta_{v,0} \text{ for all } v \in R \}.$$ 

This is of finite index in $E(K)$ (cf. §8 below). Then, considering the intersection number of each $\Theta_{v,i}$ with both sides of (4.7), we see that $b_{v,i} = 0$ for all $v, i$, by the same argument as in the proof of Lemma 3.1. Then (4.7) implies

$$(4.9) \quad D \sim (P) - (O) + f^*(\mathfrak{b}), \quad \mathfrak{b} = \sum_j a_j \mu_j.$$ 

Taking the intersection number with $(P) - (O)$, we have

$$0 = (P^2) + (O^2) - 2(PO)$$

cf. (2.15) for the notation. In view of Lemma 2.7, this is possible only when $P = O$ (since otherwise $(PO) > 0$). Then $D \sim f^*(\mathfrak{b})$ by (4.9), and $\deg(\mathfrak{b}) = ((O) \cdot D) = 0$. This proves the assertion.

Next, to treat the general case, we take a positive integer $m$ such that $mE(K)$ is contained in $E(K)^0$. By Abel's theorem, we see that the difference of $(mP) - (O)$ and $m((P) - (O))$ is a vertical divisor up to linear equivalence. Hence the relation (4.7), multiplied by $m$, gives a similar relation in which $D$ and $P$ are replaced by $mD$ and $mP$. By the above case, we then have $mD \sim f^*(c)$ for some $c$. This implies that

$$m \text{Pic}^0(S) \subset \text{Pic}^0(C) = \text{Jac}(C).$$

Since multiplication by $m$ is an isogeny of an abelian variety (cf. [L1], [W2]), we conclude that the Picard variety of $S$ has the same dimension as the Jacobian variety of $C$. Hence $f^*$ gives an isomorphism of $\text{Pic}^0(S)$ to $\text{Jac}(C)$, with the inverse map $O^*$ induced by the morphism $O : C \to S$. This completes the proof of Theorem 4.1.

### 5. Proof of Theorem 1.3

We keep the notation of the previous section. By (4.5) and (4.6), we can define a homomorphism

$$(5.1) \quad \psi : \mathcal{D}(S) \to E(K)$$

by letting

$$(5.2) \quad \psi(D) = a(D \cdot E - (D \cdot E)O).$$

By definition, $\psi(D) = P \in E(K)$ is characterized by the relation:
(5.3) \[ D \cdot E - (D \cdot E)O \sim P - O \quad \text{(linear equivalence on } E) \].

**Lemma 5.1.** For any divisor \( D \) on \( S \), let \( P = \psi(D) \in E(K) \). Then \( D \) is uniquely written in the form:

\[
D \cong (P) + (d-1)(O) + nF + \sum_{v \in R} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i}
\]

where \( d, n \) and \( b_{v,i} \) are integers defined as follows:

\[
d = (D \cdot F), \quad n = (d-1)\chi + (OD) - (PO).
\]

\[
\begin{pmatrix}
  b_{v,1} \\
  \vdots \\
  b_{v, m_v - 1}
\end{pmatrix}
= A_v^{-1}
\begin{pmatrix}
  (D \cdot \Theta_{v,1}) - (P \Theta_{v,1}) \\
  (D \cdot \Theta_{v,m_v-1}) - (P \Theta_{v,m_v-1})
\end{pmatrix}
\]

where \( A_v \) is the matrix defined by (2.13).

**Proof.** Given a divisor \( D \), let \( D' = D - (P) - (d-1)(O), d = (D \cdot F) \). Then, by (5.3), \( D' \cdot E \sim 0 \) and hence \( D' \) is linearly equivalent to some vertical divisor by Lemma 4.2. Therefore we can write

\[ D' \approx nF + \sum_{v \in R, i \geq 1} b_{v,i} \Theta_{v,i} \]

for some \( n, b_{v,i} \in \mathbb{Z} \). Taking the intersection number with \( (O) \) gives:

\[ n = (D' \cdot (O)) = (OD) - (PO) - (d-1)(O^2). \]

Similarly, intersection with \( \Theta_{v,j} \) \((j \geq 1)\) gives the matrix relation:

\[ ((\Theta_{v,j} \cdot \Theta_{v,i})(b_{v,i}) = ((D' \cdot \Theta_{v,j})) = ((D \cdot \Theta_{v,j}) - (P \Theta_{v,j})), \]

which has a unique solution, by Lemma 2.5, as given in (5.6). q.e.d.

**Lemma 5.2.** The map \( \psi \) is a surjective homomorphism such that

\[
\ker(\psi) = \mathcal{D}_a(S) + \mathbb{Z}(O) + \mathcal{D} \overline{\text{er}}.
\]

Therefore it induces an isomorphism

\[
\overline{\psi} : NS(S)/T \cong E(K).
\]

**Proof.** The surjectivity is clear: for any \( P \in E(K) \), we have \( \psi((P)) = P \). Let us denote the right hand side of (5.7) by \( W \). Then the above lemma implies that \( \ker(\psi) \subset W \). Conversely, take any \( D \in W \). Then we have by Theorem 4.1

\[ D \sim f^*(d) + d(O) + D' \]

for some divisor \( d \) on \( C \) and \( d \in \mathbb{Z}, D' \in \mathcal{D} \overline{\text{er}} \). Hence \( D \cdot E \sim dO \) and \( \psi(D) = O \). Thus \( W \subset \ker(\psi) \), and we have proven (5.7).

The second assertion (5.8) then follows immediately by the definition (2.2) and (2.9). q.e.d.
Proof of Theorem 1.3. Let us define a map

\[ \tilde{\varphi} : E(K) \to NS(S)/T \]

by \( \tilde{\varphi}(P) = (P) \mod T \). By what we have seen above, it is now obvious that \( \tilde{\varphi} \) is the inverse map of \( \tilde{\psi} \), defined in (5.8). This completes the proof of Theorem 1.3.

Thus we have proven Theorems 1.1, 1.2, 1.3 stated in §1.

Corollary 5.3. The Mordell-Weil rank, \( r = \text{rk} E(K) \), is given by the formula:

\[ r = \text{rk} \ NS(S) - \text{rk} \ T 
\]

\[ = \text{rk} \ NS(S) - 2 \sum_{v \in \mathbb{R}} (m_v - 1). \]

As for the torsion subgroup of \( E(K) \), we have

\[ E(K)_{\text{tor}} \cong T'/T, \quad T' = T \otimes \mathbb{Q} \cap NS(S). \]

Part II. The Mordell-Weil lattice of an elliptic surface

6. Preliminaries on lattices

We fix the terminology concerning lattices used in this paper, recalling some basic facts; for details, we refer to [CS].

By a lattice we mean a free \( \mathbb{Z} \)-module of finite rank, say \( L \), given with a symmetric non-degenerate bilinear pairing

\[ \langle , \rangle : L \times L \to \mathbb{Q}. \]

An integral lattice is a lattice with a \( \mathbb{Z} \)-valued pairing.

The determinant of \( L \), \( \text{det} \ L \), is defined as the absolute value of the determinant of the Gram matrix \( I = (\langle x_i, x_j \rangle) \) where \( \{x_1, \cdots, x_r\} \) is a \( \mathbb{Z} \)-basis of \( L \) (\( r = \text{rk} \ L = \) the rank of \( L \)):

\[ \text{det} \ L = |\text{det}(\langle x_i, x_j \rangle)|. \]

We define the positive- (or negative-) definiteness or the signature of a lattice by that of the matrix \( I \), noting that these properties are independent of the choice of a basis. Let us denote by \( L^- \) the opposite lattice of \( L \), i.e.,

\[ L^- = \text{the module } L \text{ with the pairing } - \langle , \rangle. \]

An integral lattice \( L \) is called even if \( \langle x, x \rangle \in 2\mathbb{Z} \) for all \( x \) in \( L \), and unimodular if \( \text{det} \ L = 1 \).

The dual lattice \( L^* \) of an integral lattice \( L \) is defined by
(6.3) \[ L^* = \{ x \in L \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in L \} . \]

We have

(6.4) \[ \det L^* = 1/(\det L) \quad \text{and} \quad [L^*:L] = \det L . \]

A sublattice, say \( T \), of \( L \) is a submodule of \( L \) such that the restriction of \( \langle \cdot, \cdot \rangle \) to \( T \times T \) is non-degenerate. The orthogonal complement of \( T \), denoted by \( T^\perp \), is defined by

(6.5) \[ T^\perp = \{ x \in L \mid \langle x, y \rangle = 0 \text{ for all } y \in T \} . \]

A sublattice \( T \) of \( L \) is called primitive if the quotient \( L/T \) is torsion-free. The primitive closure of a sublattice \( T \) of \( L \) is:

(6.6) \[ T^\ast = \{ x \in L \mid nx \in T \text{ for some positive integer } n \} . \]

If \( L' \) is a sublattice of finite index in \( L \), then

(6.7) \[ \det L' = \det L \cdot [L:L]^2 . \]

For a sublattice \( T \) of \( L \), \( T + T^\perp \) is a direct sum and is of finite index in \( L \), and we have

(6.8) \[ \det T \cdot \det T^\perp = \det L \cdot [L: T + T^\perp]^2 . \]

Moreover, if \( T \) is primitive in a unimodular lattice \( L \), then

(6.9) \[ \det T = \det T^\perp = [L: T \oplus T^\perp] . \]

The root lattices \( A_n, D_n, E_6, E_7 \) and \( E_8 \) (cf. [B, Ch. 6], [CS, Ch. 4]).

These are positive-definite even integral lattices of rank indicated by the subscript. We list the determinant, the minimal norm \( \mu \) and the number of minimal vectors \( \tau \) for each lattice and its dual lattice:

(6.10)

<table>
<thead>
<tr>
<th></th>
<th>( E_8 )</th>
<th>( E_7 )</th>
<th>( E_6 )</th>
<th>( D_6 )</th>
<th>( D_6^* )</th>
<th>( A_6 )</th>
<th>( A_6^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \det )</td>
<td>1</td>
<td>2</td>
<td>1/2</td>
<td>3</td>
<td>1/3</td>
<td>4</td>
<td>1/4</td>
</tr>
<tr>
<td>( \mu )</td>
<td>2</td>
<td>2</td>
<td>3/2</td>
<td>2</td>
<td>4/3</td>
<td>2</td>
<td>1 (( \geq 4 ))</td>
</tr>
<tr>
<td>( \tau )</td>
<td>240</td>
<td>126</td>
<td>56</td>
<td>72</td>
<td>54</td>
<td>2n(( n-1 ))</td>
<td>2n(( n \geq 5 ))</td>
</tr>
</tbody>
</table>

In particular, \( E_8 \) is the unique positive-definite even unimodular lattice of rank 8. For any sublattice \( A_1 \) of \( E_8 \) (i.e. \( A_1 = \mathbb{Z}x \) for some \( x \in E_8 \) with norm 2; such an \( x \) is called a root), its orthogonal complement \( A_1^\perp \) is isomorphic to \( E_7 \). Similarly, for any sublattice \( A_2 \) (or \( A_1^\perp ) \) of \( E_8 \), its orthogonal complement is isomorphic to \( E_6 \) (or \( D_6 \)).

(6.11) \[ A_1^\perp \simeq E_7, \quad A_2 \simeq E_6, \quad (A_1^\perp)^\perp \simeq D_6 . \]
7. The Néron-Severi lattice

Now we reconsider the results in the part I from the viewpoint of lattices. Let $S$ be an elliptic surface. By Theorems 1.3 and 3.1, the Néron-Severi group $NS(S)$ becomes an integral lattice with respect to the intersection pairing $(\cdot, \cdot)$, which will be called the Néron-Severi lattice of $S$. Its rank $\rho$ (the Picard number of $S$) is given by

$$\rho = \text{rk } NS(S)$$

$$= r + 2 + \sum_{v \in R} (m_v - 1) \geq 2 \quad \text{(cf. (5.10))}.$$  

**Theorem 7.1** (Hodge index theorem). The Néron-Severi lattice of an elliptic surface is an indefinite lattice of signature $(1, \rho - 1)$.

For the proof, see [Mu].

Consider the subgroup $T$ of $NS(S)$ generated by the zero section $(O)$ and all the irreducible components of fibres of $f : S \to C$. By Proposition 2.3, $T$ is a sublattice of $NS(S)$ such that

$$T = U \oplus \bigoplus_{v \in R} T_v$$  

where we set

$$U = \langle (O), F \rangle$$

$$T_v = \langle \Theta_{v,i} | 1 \leq i \leq m_v - 1 \rangle \quad (v \in R).$$

By (2.14) and (2.16), $U$ is a unimodular indefinite lattice with the intersection matrix

$$
\begin{pmatrix}
-\chi & 1 \\
1 & 0
\end{pmatrix}.
$$

By Lemma 2.5, each $T_v$ ($v \in R$) is negative-definite. More precisely, a case by case checking using the classification of singular fibres (cf. [K], [N1] or [T1]) proves the following:

**Lemma 7.2.** The opposite lattice $T^-_v$ is a root lattice of rank $m_v - 1$, determined by the type of the singular fibre $F_v$ as follows:

\[
\begin{array}{cccccccc}
\text{Type of } F_v & I_m & I^*_m & II^* & III^* & IV^* & IV & III \\
T^-_v & A_{m-1} & D_{m+4} & E_8 & E_7 & E_6 & A_2 & A_1 \\
\end{array}
\]
The determinant of $T_v$ is equal to the number $m_v^{(1)}$ of the simple components of $F_v$ (cf. (1.4) and (6.10)):

$$\det T_v = m_v^{(1)} = \#\{i | 0 \leq i \leq m_v - 1, \mu_{v,i} = 1\}.$$  

**Definition 7.3.** We call $T$ the *trivial sublattice* of $NS(S)$. Its rank and determinant are given by

$$\text{rk } T = 2 + \sum_{v \in \mathcal{R}} (m_v - 1)$$  

$$\det T = \prod_{v \in \mathcal{R}} m_v^{(1)}.$$  

We call the orthogonal complement $T^\perp$ of $T$ in $NS(S)$ the *essential sublattice* of $NS(S)$.

**Theorem 7.4.** The essential sublattice of the Néron-Severi group $NS(S)$, $L = T^\perp$, is a negative-definite even integral lattice. Its rank is equal to the Mordell-Weil rank of $E(K)$:

$$r = \rho - 2 - \sum_{v \in \mathcal{R}} (m_v - 1),$$

and its determinant is given by

$$\det L = \det NS(S) \cdot [NS(S) : L + T]^2 / \det T.$$  

**Proof.** Since the signature of the lattice $NS(S)$ is $(1, \rho - 1)$ and that of $T$ is $(1, \text{rk } T - 1)$, $L$ is negative-definite. Next, for any element of $L$, take a representative divisor $D$ and write

$$D = \sum_j n_j \Gamma_j \quad (\Gamma_j: \text{irreducible curves}).$$

By the adjunction formula, we have

$$(K \cdot \Gamma_j) + (\Gamma_j^2) = 2g(\Gamma_j) - 2$$

for each $j$, where $g(\Gamma_j)$ denotes the virtual genus. It follows that

$$(\Gamma_j^2) \equiv (K \cdot \Gamma_j) \equiv \chi(F \cdot \Gamma_j) \mod 2$$

by the canonical bundle formula (2.18). Therefore we have

$$(D^2) = \sum_j n_j^2 (\Gamma_j^2) + 2 \sum_{j < k} n_j n_k (\Gamma_j \cdot \Gamma_k)$$

$$= \sum_j n_j \chi(F \cdot \Gamma_j) \mod 2$$

$$= \chi(F \cdot D) = 0,$$

because $D \perp T$ and $F \in T$. This shows that $L$ is an even lattice.
The rank of $L = T^\perp$ is equal to $\text{rk} \, NS(S) - \text{rk} \, T$, hence (7.10) is a restatement of (7.1). (7.11) follows from (6.8).

The same proof gives the following result, which is useful in the later application:

**Theorem 7.5.** Let $W$ denote the orthogonal complement of the rank 2 sublattice $U = \langle (O), F \rangle$ in $NS(S)$:

$$W = U^\perp = \langle (O), F \rangle^\perp.$$  

Then $W$ is a negative-definite even integral lattice of rank $\rho - 2$, with $\det = \det NS(S)$. The essential sublattice $L$ of $NS(S)$ is equal to the orthogonal complement of $\bigoplus_{\nu \in \mathbb{R}} T_\nu$ in $W$.

We call $W^-$ (a positive-definite even lattice) the **frame** of an elliptic surface $S$.

**8. The Mordell-Weil lattices**

By Theorem 1.3, we know that the Mordell-Weil group $E(K)$ of an elliptic surface $S$ is canonically isomorphic to the quotient group $NS(S)/T$. In order to define a good pairing on $E(K)$, the first thing to try would be to split this isomorphism and embed $E(K)$ into $NS(S)$. We shall show that this can be done with a slight modification. Note that there will be no harm in passing from $NS(S)$ to $NS(S)_\mathbb{Q} = NS(S) \otimes \mathbb{Q}$ since it is torsion-free (Theorem 1.2).

**Lemma 8.1.** For any $P \in E(K)$, there exists a unique element of $NS(S)_{\mathbb{Q}}$, say $\varphi(P)$, satisfying the condition:

(i) $\varphi(P) \equiv (P) \mod T_{\mathbb{Q}}$, and

(ii) $\varphi(P) \perp T$.

Explicitly, $\varphi(P)$ is the class of the divisor $D_P$ with $\mathbb{Q}$-coefficients:

$$D_P = (P) - (O) - ((PO) + \chi)F + \sum_{\nu \in \mathbb{R}} (\Theta_{\nu,1}, \ldots, \Theta_{\nu,m_\nu - 1}) (-A_v^{-1}) \binom{(P\Theta_{\nu,1})}{(P\Theta_{\nu,m_\nu - 1})}.$$

**Proof.** Suppose that $\varphi(P)$ has a representative $D \in \mathcal{D}(S) \otimes \mathbb{Q}$. Then the first condition of (8.1) says that

$$D \cong (P) + d(O) + nF + \sum_{\nu \in \mathbb{R}, i \geq 1} b_{\nu,i} \Theta_{\nu,i} \quad (d, n, b_{\nu,i} \in \mathbb{Q}).$$

By the second condition of (8.1), the intersection number of $D$ with $F$, $(O)$ and $\Theta_{\nu,j}$ should be 0, which gives $d = -1$, $n = -((PO) + \chi)$ and a unique solution of $b_{\nu,i}$ so that $D$ is algebraically equivalent to $D_P$ given by (8.2). Conversely, the class of $D_P$ satisfies (8.1).

q.e.d.
**Lemma 8.2.** The map

\[(8.3)\]

\[\varphi : E(K) \to NS(S)_Q\]

is a group homomorphism such that

\[(8.4)\]

\[\text{Ker}(\varphi) = E(K)_{\text{tor}}.\]

**Proof.** By Abel's theorem on \(E\) (Lemma 4.3), we see that

\[\varphi(P) + \varphi(Q) \equiv (P + Q) \mod T_Q \quad (P, Q \in E(K)).\]

By the uniqueness in Lemma 8.1, we have then \(\varphi(P + Q) = \varphi(P) + \varphi(Q)\). Hence \(\varphi\) is a group homomorphism. For \(P \in E(K)\), we have \(\varphi(P) = 0\) if and only if some multiple of \((P - 0)\) is algebraically equivalent to an integral linear combination of \(F\) and \(\Theta_{\nu,1}\). By Theorem 4.1, the latter holds if and only if that multiple is linearly equivalent to a vertical divisor. Restricting to \(E\) and using Abel's theorem, we see that this is the case if and only if \(P\) is a torsion point. \(\text{q.e.d.}\)

**Lemma 8.3.** Let \(L = T^1\) be the essential sublattice of \(NS(S)\), and let \(m\) be the least common multiple of \(m^{(1)}_v\) defined by (7.7):

\[(8.5)\]

\[m = L.C.M.\{m^{(1)}_v | v \in R\}.\]

Then we have

\[(8.6)\]

\[\text{Im}(\varphi) \subseteq \frac{1}{m} L\]

and \(\varphi\) induces an injection:

\[(8.7)\]

\[\varphi' : E(K)/E(K)_{\text{tor}} \subseteq \frac{1}{m} L \subseteq L_Q = L \otimes Q.\]

**Proof.** By (7.7), we have

\[\det(-A_v) = \det T_v = m^{(1)}_v.\]

Hence, by (8.2), \(mD_p\) is an integral divisor for any \(P \in E(K)\), which implies that \(m\varphi(P) \in L\). This proves (8.6), and (8.7) follows from Lemma 8.2. \(\text{q.e.d.}\)

**Theorem 8.4.** For any \(P, Q \in E(K)\), let

\[(8.8)\]

\[\langle P, Q \rangle = -(\varphi(P) \cdot \varphi(Q)).\]

Then it defines a symmetric bilinear pairing on \(E(K)\), which induces the structure of a positive-definite lattice on \(E(K)/E(K)_{\text{tor}}\).

**Proof.** The pairing (8.8) is obviously symmetric, and it is bilinear because \(\varphi\) is a homomorphism. Now the lattice \(L\), with the intersection pairing \((\cdot)\), is negative-definite (Theorem 7.4), and hence we have
\[
\langle P, P \rangle = -(\varphi(P) \cdot \varphi(P)) \geq 0 \quad \text{for any } P \in E(K)
\]
and, noting (8.4), we have
\[
\langle P, P \rangle = 0 \iff \varphi(P) = 0 \iff P \in E(K)_{\text{tor}}. \quad \text{q.e.d.}
\]

**Definition 8.5.** The pairing (8.8) on the Mordell-Weil group \( E(K) \) will be called the *height pairing*, and the lattice
\[
(E(K))/E(K)_{\text{tor}} \langle \cdot, \cdot \rangle
\]
will be called the *Mordell-Weil lattice* of the elliptic curve \( E/K \) or of the elliptic surface \( f : S \rightarrow C \).

**Theorem 8.6 (Explicit formula for the height pairing).** For any \( P, Q \in E(K) \), we have
\[
\langle P, Q \rangle = \chi + (PO) + (QO) - (PQ) - \sum_{v \in R} \text{contr}_v(P, Q)
\]
\[
\langle P, P \rangle = 2\chi + 2(PO) - \sum_{v \in R} \text{contr}_v(P).
\]

**Notation.** Here \( \chi \) is the arithmetic genus of \( S \), and \( (PO) \) is the intersection number of the sections \( (P) \) and \( (O) \), and similarly for \( (QO), (PQ) \). The term \( \text{contr}_v(P, Q) \) stands for the local contribution at \( v \in R \), which is defined as follows: suppose that \( (P) \) intersects \( \Theta_{v,i} \) and \( (Q) \) intersects \( \Theta_{v,j} \). Then we let
\[
\text{contr}_v(P, Q) = \begin{cases} (-A_v^{-1})_{i,j} & \text{if } i \geq 1, j \geq 1 \\ 0 & \text{otherwise} \end{cases}
\]
where the first one means the \((i,j)\)-entry of the matrix \((-A_v^{-1})\). Further we set
\[
\text{contr}_v(P) = \text{contr}_v(P, P).
\]

**Proof.** By the formula (8.2), we have
\[
\langle P, Q \rangle = -(D_P \cdot D_Q) = -(D_P \cdot (Q))
\]
because \( D_Q \equiv (Q) \mod T_0 \) and \( D_P \perp T \). Hence
\[
\langle P, Q \rangle = -(PO) - (QO) - ((PO) + \chi) \\
- \sum_{v \in R} ((Q\Theta_{v,1}), \cdots, (Q\Theta_{v,m_v-1}))(A_v^{-1})(P\Theta_{v,1}) \\
(P\Theta_{v,m_v-1}).
\]
This is equivalent to (8.11) in view of the definition of \( \text{contr}_v \). By taking \( Q = P \), (8.11) reduces to (8.12).

q.e.d.

It is a simple matter to write down the explicit values of (8.13); cf. [CZ]. Arrange \( \Theta_i = \Theta_{v,i} \) \((i = 0, 1, \cdots, m_v - 1)\) so that the simple components are numbered as in
the figure below.

For the other types of reducible fibres, the numbering is irrelevant. Assume that \((P)\) intersects \(\Theta_{v,i}\) and \((Q)\) intersects \(\Theta_{v,j}\) with \(i \geq 1, j \geq 1\). Then we have the following table: the second row gives the value of \(\text{contr}_v(P, Q)\) for the case \(i = j\) as well as \(\text{contr}_v(P)\), and the third row is for the case \(i < j\) (interchange \(P, Q\) if necessary).

<table>
<thead>
<tr>
<th>(T_v)</th>
<th>(A_1)</th>
<th>(E_7)</th>
<th>(A_2)</th>
<th>(E_6)</th>
<th>(A_{b-1})</th>
<th>(D_{b+4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>type of (F_v)</td>
<td>(\text{III} )</td>
<td>(\text{III}^*)</td>
<td>(\text{IV})</td>
<td>(\text{IV}^*)</td>
<td>(I_b (b \geq 2))</td>
<td>(I_b^* (b \geq 0))</td>
</tr>
<tr>
<td>(\text{contr}_v(P))</td>
<td>1/2</td>
<td>3/2</td>
<td>2/3</td>
<td>4/3</td>
<td>(i(b - i)/b)</td>
<td>(\begin{cases} 1 &amp; (i = 1) \ 1 + b/4 &amp; (i &gt; 1) \end{cases})</td>
</tr>
<tr>
<td>(\text{contr}_v(P, Q))</td>
<td>—</td>
<td>—</td>
<td>1/3</td>
<td>2/3</td>
<td>(i(b - j)/b)</td>
<td>(\begin{cases} 1/2 &amp; (i = 1) \ (2 + b)/4 &amp; (i &gt; 1) \end{cases})</td>
</tr>
</tbody>
</table>

Now we turn our attention to the subgroup \(E(K)^0\) of \(E(K)\), appearing in the proof of Theorem 4.1. Recall its definition (4.8):

\[
E(K)^0 = \{ P \in E(K) \mid (P) \text{ meets } \Theta_{v,0} \text{ for all } v \in R \}.
\]

It is of finite index in \(E(K)\); in fact, \(mE(K) \subset E(K)^0\) for \(m\) in (8.5). By (8.11) and (8.12), we have:

\[
\langle P, Q \rangle = \chi -(PO) + (QO) - (PO) \quad \text{if } P \text{ or } Q \in E(K)^0.
\]

\[
\langle P, P \rangle = 2\chi + 2(PO) \quad \text{if } P \in E(K)^0.
\]

**Theorem 8.7.** \(E(K)^0\) is a torsion-free subgroup of \(E(K)\). Viewed as a lattice with respect to the height pairing, it is a positive-definite even integral lattice, with minimal norm \(\geq 2\chi\). We have

\[
\text{det } E(K)^0 = \text{det } NS(S) \cdot [E(K) : E(K)^0]^2 / \text{det } T.
\]

**Proof.** For any \(P \in E(K)\), \(P \neq O\), we have \((PO) \geq 0\) because the sections \((P)\) and \((O)\) are distinct irreducible curves on \(S\). Hence, if \(P \in E(K)^0\), then we have by (2.20) and (8.19)

\[
\langle P, P \rangle \geq 2\chi > 0.
\]
Thus $P$ is not torsion by Lemma 8.2. Further (8.19) shows that the lattice $E(K)^0$ is even and positive-definite, with minimal norm $\geq 2\chi$. The formula (8.20) follows from (7.11).

**Definition 8.8.** The lattice $(E(K)^0, \langle , \rangle)$ will be called the narrow Mordell-Weil lattice of the elliptic curve $E/K$ or of the elliptic surface $f : S \to C$.

**Theorem 8.9.** Via the map $\varphi : E(K) \to NSW(S)_Q$, the narrow Mordell-Weil lattice $E(K)^0$ is isomorphic to the essential sublattice $L$ of $NS(S)$ up to sign. In other words, $E(K)^0$ is isomorphic to the opposite lattice $L^-$ of $L$. Furthermore, the Mordell-Weil lattice $E(K)/E(K)_{tor}$ is embedded via $\varphi$ into the dual lattice of $L^-$, and we have a commutative diagram of lattices:

\[
\begin{array}{c}
E(K)/E(K)_{tor} \\
\cup \\
E(K)^0 \\
\cong \\
L^-
\end{array}
\]

**Proof.** If $P \in E(K)^0$, then $(P\Theta_{v_i}) = 0$ for all $v \in R$, all $i \geq 1$. By (8.2), $\varphi(P)$ is the class of the (integral) divisor

\[
D_p = (P) - (O) - ((PO) + \chi)F.
\]

Hence $\varphi(P) \in T^\perp = L$. Conversely, take any element of $L$ and represent it by a divisor $D$. Letting $\psi$ be as in (5.1), we let $P = \psi(D) \in E(K)$. Then, by Lemma 5.1, we have $D \equiv (P) \mod T$, and $D \perp T$. By the uniqueness in Lemma 8.1, we have $D \cong D_p$. This shows that $D_p$ is an integral divisor, which is the case only if $(P\Theta_{v_i}) = 0$ for all $i \geq 1$, i.e., $P \in E(K)^0$. It is easy to see that the maps $\varphi|_{E(K)^0}$ and $\psi|_L$ are inverse to each other. This proves the first assertion in view of the definition of the height pairing.

Next observe that (8.18) implies:

$$\langle P, Q \rangle \in \mathbb{Z} \quad \text{for any } P \in E(K)^0 \text{ and } Q \in E(K).$$

Hence the Mordell-Weil lattice is contained in the dual lattice of the narrow Mordell-Weil lattice, and we have the diagram (8.22).

**Corollary 8.10.** Assume that $f : S \to C$ is an elliptic surface with no reducible fibres. Then $E(K) = E(K)^0$ is a positive-definite even integral lattice such that

\[
det E(K) = \det NS(S).
\]

**Proof.** This is immediate from Theorem 8.7.

**Remark 8.11.** A natural question: Can one replace the inclusion $E(K)/E(K)_{tor} \subset L^-$ in (8.22) by equality? The answer is no, in general. Some examples are given in [S3] or [S2, §4]. On the other hand, this is the case if the Néron-Severi lattice is unimodular. This will be discussed in the next section.

Finally, we note some functorial properties of the height pairing.
Proposition 8.12. Let $K'$ be a finite extension of $K$. For an elliptic curve $E$ over $K$, let $E' = E \times_k K'$, an elliptic curve over $K'$. For any $P, Q \in E(K)$, let $P', Q' \in E'(K')$ be induced by $P, Q$. Then
\begin{equation}
\langle P', Q' \rangle_{E'(K')} = [K': K] \cdot \langle P, Q \rangle_{E(K)}.
\end{equation}

Proof. It suffices to consider the case $P = Q \in E(K)^0$, because both sides of (8.25) are symmetric and bilinear. Let $f : S \to C$ (or $f' : S' \to C'$) be the elliptic surface associated to $E/K$ (or $E'/K'$). Then $S'$ is birational to the fibre product $S \times_C C'$, and there is a natural rational map $g : S' \to S$. Denoting by $S_1$ the open set of $S$ which is the complement of the union of $\Theta_{v,i}$ ($v \in R$, $i \geq 1$), we can find a suitable open set $S_1'$ containing the curves $(P')$ and $(O')$ such that the restriction of $g$ to $S_1'$ is a morphism $g_1 : S_1' \to S_1$.

Now, by the definition of the height pairing, we have
\begin{equation}
\langle P, P \rangle = -(D_P \cdot D_P),
\end{equation}
where $D_P$ is the divisor on $S$ defined by (8.23). Observe that
\begin{equation}
g_1^*(P) = (P'), \quad g_1^*(O) = (O'), \quad g_1^*(F) = dF' \quad (d = [K': K]).
\end{equation}
Letting $D' = g_1^*(D_P)$, we have
\begin{equation}
D' = (P') - (O') - d((PO) + \chi)F'.
\end{equation}
By the standard property of intersection number, we have
\begin{equation}
(D' \cdot (O')) = (g_1^*(D_P) \cdot g_1^*(O)) = d(D_P \cdot (O)) = 0.
\end{equation}
Then (8.26) and (8.27) imply
\begin{equation}
(P'O') + \chi' - d((PO) + \chi) = 0.
\end{equation}
This proves (8.25), by the formula (8.19) for $E$ and for $E'$. q.e.d.

Proposition 8.13. Suppose that $k$ is the algebraic closure of a perfect field $k_0$. Let $C_0$ be an absolutely irreducible curve defined over $k_0$ and let $E_0$ be an elliptic curve defined over $k_0(C_0)$. Further let $E$ be the elliptic curve over $K = k(C_0)$ obtained from $E_0$ by base extension. Then, for any automorphism $\sigma \in Gal(k/k_0)$, we have
\begin{equation}
\langle P^\sigma, Q^\sigma \rangle = \langle P, Q \rangle \quad \text{for any} \quad P, Q \in E(K).
\end{equation}
Namely the height pairing is stable under the action of $Gal(k/k_0)$.

Proof. Under the assumption, the elliptic surface $f : S \to C$ is defined over $k_0$. Letting $\phi$ be as in Lemma 8.1, we claim
\begin{equation}
\phi(P)^\sigma = \phi(P)^\sigma \quad \text{for any} \quad P \in E(K).
\end{equation}
Recall that $\phi(P)$ is uniquely determined as the divisor class of $D$ such that
\begin{equation}
D \equiv (P) \mod T_0 \quad \text{and} \quad (D \cdot \Theta) = 0 \quad \text{for any} \quad \Theta \in T.
\end{equation}
Now the trivial lattice $T$ is stable under any $\sigma \in \text{Gal}(k/k_0)$, because $\sigma$ fixes the classes of $(O)$ and $F$ and interchanges the irreducible components $\Theta_{v,l}$ of fibres. Hence (8.30) implies

$$D^\sigma \equiv (P^\sigma) \mod T_q \quad \text{and} \quad (D^\sigma \cdot \Theta) = 0 \quad \text{for any} \quad \Theta \in T$$

by the invariance of the intersection number under $\sigma$. By the uniqueness in Lemma 8.1, this proves (8.29).

Then (8.28) immediately follows from the definition (8.8):

$$\langle P^\sigma, Q^\sigma \rangle = -\langle \varphi(P^\sigma) \cdot \varphi(Q^\sigma) \rangle = -\langle \varphi(P)^\sigma \cdot \varphi(Q)^\sigma \rangle = \langle P, Q \rangle$$

again by the invariance of the intersection number under $\sigma$. q.e.d..

9. The unimodular case

**Theorem 9.1.** Let $S$ be an elliptic surface such that $\text{NS}(S)$ is a unimodular lattice, i.e.,

(9.1) \hspace{1cm} \det \text{NS}(S) = 1.

Then the Mordell-Weil lattice $E(K)/E(K)_{\text{tor}}$ is the dual lattice of the narrow Mordell-Weil lattice $E(K)^0$. In other words, with the notation of Theorem 8.9, we have

(9.2) \hspace{1cm} E(K)/E(K)_{\text{tor}} \approx L^{-\ast}.

**Proof.** By Theorem 8.9, it suffices to prove

(9.3) \hspace{1cm} [E(K)/E(K)_{\text{tor}} : E(K)^0] = [L^\ast : L].

Let $T'$ denote the primitive closure (6.6) of the trivial lattice $T$ in $\text{NS}(S)$, i.e., $T' = T \otimes \mathbb{Q} \cap \text{NS}(S)$. Then we have

(9.4) \hspace{1cm} T'^\perp = T^\perp = L.

Applying (6.9) to $T' + L \subset \text{NS}(S)$, we have

(9.5) \hspace{1cm} \det L = \det T' = [\text{NS}(S) : L + T'].

By (6.4), this gives

(9.6) \hspace{1cm} [L^\ast : L] = [\text{NS}(S)/T' : L].

On the other hand, Theorem 1.3 implies

(9.7) \hspace{1cm} E(K) \approx \text{NS}(S)/T \quad \text{and} \quad E(K)_{\text{tor}} \approx T'/T.

Therefore we have

(9.8) \hspace{1cm} E(K)/E(K)_{\text{tor}} \approx \text{NS}(S)/T'.

Now (9.6) and (9.8) prove the desired equality (9.3). q.e.d.
It will be convenient to reformulate Theorem 9.1 in the following way, by using Theorem 7.5. Let $W$ denote the “frame” of $S$ (which was denoted by $W^-$ in Theorem 7.5); by definition, we have

\[(9.9)\quad NS(S) = U \oplus W^- , \quad U = \langle (O), F \rangle , \quad W^- = U^\perp .\]

By (9.1), $W$ is a positive-definite even unimodular lattice. Let

\[(9.10)\quad V = \bigoplus_{v \in R} T_v^- \subset W\]

be the root lattice associated to the reducible fibres of $f : S \to C$. By Lemma 7.2, this is a direct sum of simple root lattices of type $A, D, E$. By (7.2), the trivial lattice $T$ is equal to the direct sum

\[(9.11)\quad T = U \oplus V^- .\]

Letting $T'$ (or $V'$) be the primitive closure of $T$ (or $V$), we have

\[(9.12)\quad T' = U \oplus V'^- .\]

Then (9.7) implies

\[(9.13)\quad E(K) \simeq W/V , \quad E(K)_{\text{tor}} \simeq V'/V .\]

Let $M = V^\perp$ be the orthogonal complement of $V$ in $W$. Obviously we have

\[(9.14)\quad M = V^\perp = L^- \quad (L = T^\perp \text{ in } NS(S)) ,\]

and the narrow Mordell-Weil lattice $E(K)^0 \simeq M$ via the map $\varphi$.

We can summarize the above as follows:

**Theorem 9.2.** Let $S$ be an elliptic surface such that $NS(S)$ is a unimodular lattice. Then the frame $W$ of $S$ is a positive-definite even unimodular lattice. Let $V = \bigoplus_{v \in R} T_v^-$ be the root lattice associated to the reducible singular fibres of $f : S \to C$.

Then the narrow Mordell-Weil lattice $E(K)^0$ is isomorphic to the lattice $M = V^\perp$, the orthogonal complement of $V$ in $W$, and the Mordell-Weil lattice $E(K)/E(K)_{\text{tor}}$ is isomorphic to the dual lattice $M^*$ of $M$. Thus we have

\[(9.15)\quad E(K)/E(K)_{\text{tor}} \simeq M^* , \quad E(K)^0 \simeq M .\]

The index of $M$ in $M^*$ as well as the determinant of $M$ is given by

\[(9.16)\quad [M^* : M] = \det M = \det V/|E(K)_{\text{tor}}|^2\]

where

\[(9.17)\quad \det V = \prod_{v \in R} m_v^{(1)} , \quad |E(K)_{\text{tor}}| = [V' : V] .\]
As for the Mordell-Weil group $E(K)$, it is isomorphic to the quotient group $W/V$, and furthermore, there is an isomorphism:

$$(9.18) \quad E(K) \simeq M^* \oplus (V'/V),$$

which preserves the height pairing.

Proof. We have already proven everything in the above, except for the formula (9.16), which follows from (9.5), by using (6.4), (6.7) and (9.13). q.e.d.

10. Rational elliptic surfaces

In this section, we apply the preceding results to the case where $S$ is a rational elliptic surface with a section. Namely we suppose that $S$ is a (smooth projective) rational surface over $k$ with a relatively minimal elliptic fibration $f : S \to C$, having a global section $O : C \to S$. Then we have $C = P^1$ (the projective line) and $K = k(P^1) = k(t)$ is a rational function field. We note that the assumption (ii) of (1.1) that $f$ be non-smooth is automatically satisfied in this case. As usual, $E/K$ is the generic fibre.

We recall the well-known results on such a surface $S$. For the convenience of the reader, we outline the proof assuming the general theory of algebraic surfaces.

**Lemma 10.1.** (i) The arithmetic genus $\chi$ of $S$ is equal to 1. (ii) The Néron-Severi lattice $NS(S)$ is unimodular and of rank $\rho = 10$.

Proof. We have $\chi = p_g - q + 1$ with $p_g$ the geometric genus, $q$ the irregularity, both of which are 0 for a rational surface. Hence $\chi = 1$. Then, by Noether's formula, $c_2 + (K^2) = 12\chi$ where $c_2$ is the Euler number of $S$, which reduces now to $c_2 = 12b_2$ by (2.18). On the other hand, $c_2 = b_2 + 2 - 2b_1$ (the alternating sum of Betti numbers) gives $b_2 = 10$, since $b_1 = 0$ as the Picard variety of $S$ is trivial by (4.5). For a rational surface, $\rho = b_2$ holds, hence $\rho = 10$. Finally $det NS(S)$ is a birational invariant (blowing up a point adds to $NS$ an exceptional curve $\Gamma$ with $(\Gamma^2) = -1$) and $NS(P^2)$ is unimodular. Hence $det NS(S) = 1$. q.e.d.

**Lemma 10.2.** The frame $W$ of $S$ is the root lattice $E_8$.

Proof. As we remarked in Theorem 9.2, $W$ is a positive-definite even unimodular lattice, and in the present case, of rank $\rho - 2 = 8$. Hence we have $W \cong E_8$ by the uniqueness mentioned in §6. q.e.d.

By Theorem 9.2, we have a complete description for the Mordell-Weil lattice as well as the Mordell-Weil group of a rational elliptic surface, in terms of the reducible singular fibres:

**Theorem 10.3.** Given a rational elliptic surface $f : S \to P^1$ with a section, let $R$ be the set of reducible fibres (1.3), and for $v \in R$, let $m_v$ (or $m_v^{(1)}$) be the number of (simple) irreducible components $\Theta_v$ of $f^{-1}(v)$. Let
be the root lattice associated with the reducible fibres, which is a sublattice of \( W = E_8 \). Let \( M = V^\perp \) be the orthogonal complement of \( V \) in \( W \), and let \( V' \) be the primitive closure of \( V \) in \( W \). Then we have

\[
(10.2) \quad r = \text{rk } E(K) = 8 - \sum_{v \in R} (m_v - 1)
\]

\[
(10.3) \quad E(K)_{\text{tor}} \simeq V' / V, \quad n = |E(K)_{\text{tor}}| = [V' : V]
\]

\[
(10.4) \quad E(K)^0 \simeq M, \quad \det M = \prod_{v \in R} m_v(1) / n^2
\]

\[
(10.5) \quad E(K) / E(K)_{\text{tor}} \simeq M^*, \quad \det M^* = 1 / \det M
\]

\[
(10.6) \quad E(K) \simeq M^* \oplus (V' / V).
\]

In particular, \( E(K) \) is torsion-free if \( \det V = \prod_{v \in R} m_v(1) \) is square-free.

As a consequence, the determination of the Mordell-Weil groups (or lattices) of rational elliptic surfaces reduces to the study of root sublattices in \( E_8 \). Let us see below how this works for the case of higher rank: \( r = 8, 7 \) or \( 6 \). The general case will be treated in [OS], where we make use of Dynkin’s result [D] on the classification of root lattices contained in \( E_8 \).

First of all, note that \( r \leq 8 \) by (10.2). Further we have

(i) \( r = 8 \iff f \) has no reducible fibres \( (R = \emptyset) \).

(ii) \( r = 7 \iff \) there is only one reducible fibre \( f^{-1}(v) \) and \( m_v = 2 \) (type \( I_2 \) or \( III \)).

(iii) \( r = 6 \iff \)

\( a \) there is only one reducible fibre \( f^{-1}(v) \) and \( m_v = 3 \) (type \( I_3 \) or \( IV \)), or

\( b \) there are exactly two reducible fibres \( R = \{v, v'\} \) with \( m_v = m_{v'} = 2 \).

**Theorem 10.4** (Structure theorem). Assume that \( r \geq 6 \). Then the Mordell-Weil group \( E(K) \) of a rational elliptic surface is torsion-free, and the structure of the Mordell-Weil lattice is the dual lattice of the root lattice of type \( E_8, E_7, E_6 \) or \( D_6 \) according to the above cases (i), \( \cdots \), (iii)). Namely we have

(i) \( r = 8 \).

\[
(10.7) \quad E(K) = E(K)^0 \simeq E_8 \quad \text{det} = 1, \quad \mu = 2, \quad \tau = 240
\]

(ii) \( r = 7 \).

\[
(10.8) \quad \bigcup_\cup \cup \cup \cup \cup = 2 \quad E(K) \simeq E_7 \quad \text{det} = 1 / 2, \quad \mu = 3 / 2, \quad \tau = 56
\]

\[
\bigcup_\cup \cup \cup \cup \cup = 2 \quad E(K)^0 \simeq E_7 \quad \text{det} = 2, \quad \mu = 2, \quad \tau = 126
\]
(iiiia) $r = 6$. 

$$E(K) \cong E_6^\ast \quad \text{det} = 1/3, \quad \mu = 4/3, \quad \tau = 54$$  \hspace{1cm} (10.9)  

$$E(K)^0 \cong E_6 \quad \text{det} = 3, \quad \mu = 2, \quad \tau = 72$$

(iiib) $r = 6$. 

$$E(K) \cong D_6^\ast \quad \text{det} = 1/4, \quad \mu = 1, \quad \tau = 12$$  \hspace{1cm} (10.10)  

$$E(K)^0 \cong D_6 \quad \text{det} = 4, \quad \mu = 2, \quad \tau = 60$$

Here $v$ is the index of a lattice in its dual lattice, $\mu$ is the minimal norm and $\tau$ is the number of minimal vectors in each lattice.

Proof. In each case, we show that $E(K)^0$ is the root lattice as above and that $E(K)$ has no torsion. Then Theorem 10.3 determines $E(K)$. The values for $v$, det, $\mu$ or $\tau$ for these lattices are well-known (cf. [B, Ch. 6], [CS, Ch. 4] or [M2, Ch. 4]) and summarized in (6.10).

With the notation of Theorem 10.3, we have by (7.6)

$$V = \{0\}, \ A_1, \ A_2 \text{ or } A_1^{\oplus 2}$$

according to the cases (i), \hdots, (iii). Then, by (6.11), we have

$$M = V^\perp \cong E_8, \ E_7, \ E_6 \text{ or } D_6.$$  

On the other hand, we have det $V = 1$, 2, 3 or 4 by (6.10). Letting $n = [V' : V] = |E(K)_{tor}|$, we know that det $V' = \text{det} V/n^2$ is an integer. Hence, in the first 3 cases, it follows that $n = 1$, i.e., $E(K)$ has no torsion. In the case (iii), $n = 1$ or 2. If $n = 2$, then $V'$ is not primitive and its primitive closure $V'$ would be an even unimodular lattice of rank 2, a contradiction. Thus $E(K)$ is torsion-free. q.e.d.

Now, for any $P \in E(K), \ P \neq O$, the norm $\langle P, P \rangle$ is computed by the formula (8.12) and (8.16):

$$\langle P, P \rangle = 2 + 2(PO)$$  \hspace{1cm} (10.11)  

$$\begin{cases} 
0 & \text{for } P \in E(K)^0 \\
1/2 & \text{(ii)} \\
2/3 & \text{for } P \notin E(K)^0 \\
1 \text{ or } 1/2 & \text{(iii)}
\end{cases}$$

In particular, this implies that $\langle P, P \rangle > 0$, which gives another proof that $E(K)$ is torsion-free in these cases.

Let us call $P \in E(K)$ a minimal rational point or a minimal section if $\langle P, P \rangle$ has the smallest positive value in the Mordell-Weil lattice.

By comparing (10.11) with Theorem 10.4, we have:

**Lemma 10.5.** For the cases (i), \hdots, (iii), $P \in E(K)$ is a minimal section if and
only if the following condition is satisfied:
case (i). \( (PO) = 0 \), i.e., \( P \) is disjoint from \( O \).
cases (ii) and (iii). \( (PO) = 0 \) and \( (P \Theta_{v_0,0}) = 0 \); the latter means that \( P \) passes through
the non-identity component \( \Theta_{v_i} \) \((i \geq 1)\).
case (iii). \( (PO) = 0 \) and \( (P \Theta_{v_0,1}) = (P \Theta_{v_i,1}) = 1 \).

**Theorem 10.6.** The number of the minimal sections (with minimal norm \( \mu \)) in
the Mordell-Weil lattice \( E(K) \) is 240 \((\mu = 2)\), 56 \((\mu = 3/2)\), 54 \((\mu = 4/3)\) or 12 \((\mu = 1)\),
according to the cases (i), \cdots or (iii). They contain generators of \( E(K) \) in case (i),
(ii) or (iii), while the next minimal sections of norm 3/2 are also needed in case (iii).

**Proof.** The corresponding fact to the second assertion for the lattices \( E_8, 
E_7^3, E_6^8 \) or \( D_6^8 \) can be easily verified.
q.e.d.

**Lemma 10.7.** For any rational elliptic surface, the number of the sections \( P 
with \( (PO) = 0 \) is finite and at most 240. Every such \( P \) has norm \( \langle P, P \rangle \leq 2 \).

**Proof.** For any \( P \) with \( (PO) = 0 \), the divisor
\[
D_P = (P) - (O) - F
\]
defines an element \( \xi_P \) of \( \langle O, F \rangle \subset NS(S) \) such that \( (\xi_P^2) = -2 \). In other words,
\( \xi_P \) gives a "root" of the frame \( W \cong E_8 \). Moreover the correspondence \( P \mapsto \xi_P \)
is one-to-one by Theorem 4.1 and Lemma 4.3. Hence the number of \( P \) with \( (PO) = 0 \)
is bounded by \( \tau(E_8) = 240 \). Since \( \langle P, P \rangle \leq 2 + 2(PO) \) in general, the last assertion is
clear.
q.e.d.

**Theorem 10.8.** The Mordell-Weil group of a rational elliptic surface is generated
by the sections \( P \) with \( (PO) = 0 \), hence by those \( P \) with norm \( \langle P, P \rangle \leq 2 \).

**Proof.** This is shown in Theorem 10.6 in case \( r \geq 6 \). The general case follows
from [OS].

We can make the above results more explicit by writing down the equation of
the elliptic curve \( E \) over \( K = k(t) \) in the Weierstrass form. Suppose that the minimal
Weierstrass equation of \( E \) over \( K \) is given by
\[
y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t), \quad a_4(t) \in k[t].
\]
The the associated elliptic surface \( f : S \rightarrow \mathbf{P}^1 \) is a rational surface if and only if (with
the notation (1.2))
\[
\text{deg } a_4(t) \leq i \text{ (all } i) \text{ and } \text{Sing}(f) \neq \emptyset.
\]
This is a consequence of the canonical bundle formula (Theorem 2.8) and
Castelnuovo's criterion of rationality (see [AS, (2.5)]). In case \( char(k) \neq 2, 3 \), we can
take the equation in a more familiar form:
\[
y^2 = x^3 + p(t)x + q(t), \quad p(t), q(t) \in k[t].
\]
Then the condition (10.14) is equivalent to
\begin{equation}
(10.14)' \quad \deg p(t) \leq 4, \quad \deg q(t) \leq 6 \quad \text{and} \quad \Delta = 4p(t)^3 + 27q(t)^2 \quad \text{is not a constant, i.e. } \Delta \neq k.
\end{equation}

**Lemma 10.9.** Let $P = (x, y) \in E(K)$, $P \neq O$. Then the section $(P)$ is disjoint from the zero section $(O)$ if and only if $x$ and $y$ are polynomials in $t$ of degree $\leq 2$ or 3, i.e., of the form:
\begin{equation}
(10.15) \quad x = gt^2 + at + b, \quad y = ht^3 + ct^2 + dt + e,
\end{equation}
with $a, b, \ldots, g, h \in k$.

**Proof.** If $(P)$ and $(O)$ intersect at some point lying over $v \in P^1$, $v \neq \infty$, then the $x$-coordinate of $P$ must have a pole at $v$. Hence if $(P)$ and $(O)$ are disjoint over $P^1 - \{\infty\}$, then $x$ is a polynomial in $t$. At $v = \infty$, rewrite (10.9) in terms of $s = 1/t$, $X = x/t^2$, $Y = y/t^3$, and apply the same argument to see that $x$ is of degree $\leq 2$ in $t$. q.e.d.

By Lemma 10.7 and Theorem 10.8, we have:

**Theorem 10.10 (Generator theorem).** For any elliptic curve $E$ over $K = k(t)$ defined by the Weierstrass equation (10.13) with (10.14), there are at most 240 $K$-rational points $P = (x, y)$ of the form
\begin{equation}
(10.15) \quad x = gt^2 + at + b, \quad y = ht^3 + ct^2 + dt + e,
\end{equation}
and they generate the Mordell-Weil group $E(K)$.

More precise results can be obtained for the generators of the Mordell-Weil group $E(K)$, when $E$ is more specified. For instance, for the Weierstrass version of Theorem 10.6, see [S2, Th. 3.2] and [S6]. (We take this opportunity to make a correction to [S2]: In the statement of Theorem 3.2 of [S2], make an additional assumption that the reducible singular fibres are of additive type. Without this condition, the conclusion of that theorem must be slightly modified.)

Finally we go back to a coordinate-free situation, and give a proof for the following result of Manin [M1, Th. 6], which is, we hope, considerably simpler than Manin’s elaborate proof.

**Theorem 10.11.** Consider a pencil of cubic curves in $P^2$ such that (i) every member of the pencil is irreducible and (ii) the 9 base points are distinct. If we denote by $S$ the blowing up of $P^2$ at the 9 base points, the pencil defines an elliptic fibration $f : S \to P^1$ such that the 9 exceptional curves arising from the blowing up give 9 sections $P_0, \ldots, P_8$ of $f$. Choose $P_0$ as the zero section $O$, and let $E$ be the generic fibre of $f$. Then $P_1, \ldots, P_8$ generate a subgroup of index 3 in $E(K)$, and there is a unique point $Q \in E(K)$ such that $\sum_i P_i = 3Q$ which, together with $P_1, \ldots, P_7$, generate the full group $E(K)$. 
Proof. Since $f : S \to \mathbb{P}^1$ has no reducible fibres by (i), we have $r = 8$ and $E(K) \cong E_8$ by Theorem 10.4. Since the 9 exceptional curves $(P_i)$ are disjoint to each other, we have by (8.19)

\begin{equation}
\langle P_i, P_j \rangle = \chi + (P_i P_0) + (P_j P_0) -(P_i P_j)
\end{equation}

\[\begin{cases}
1 & i \neq j \\
2 & i = j \quad (i, j \geq 1)
\end{cases}
\]

and hence

$$\det(\langle P_i, P_j \rangle) = 9 = 3^2.$$ Then $P_1, \ldots, P_8$ generate a subgroup, say $H$, of index 3 by (6.7). Take $Q \in E(K)$ which is not in $H$. Since $3Q \in H$, we can write

\begin{equation}
3Q = \sum_{i=1}^{8} n_i P_i \quad (n_i \in \mathbb{Z}).
\end{equation}

We may assume (by adjusting $Q$ by some element of $H$) that

\begin{equation}
-1 \leq n_i \leq 1 \quad \text{(all $i$)}.
\end{equation}

Since $E(K)$ is an integral lattice, $\langle Q, P_i \rangle$ is an integer. Now

$$3\langle Q, P_i \rangle = \langle 3Q, P_i \rangle = 2n_i + \sum_{j \neq i} n_j \quad \text{(by (10.16))}$$

$$= n_i + N, \quad N = \sum_j n_j.$$

Thus $n_i \equiv -N \pmod{3}$ for all $i$, and (10.18) implies that $n_i = n_1$ for all $i$. Hence we see that $3Q$ is either $O$ or $\pm \sum_i P_i$. Since $E(K)$ is torsion-free, the latter holds. Then it is obvious that $Q, P_1, \ldots, P_7$ generate the full Mordell-Weil group $E(K)$.

q.e.d.

References


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