Straight and Strongly Straight Abelian $p$-Groups

by

Makoto Onishi

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§ 0. Introduction and notations

The socles of abelian $p$-groups have been considered by several authors as sources of information concerning the structure of those groups (e.g. summable $p$-groups by K. Honda, and valued vector spaces by L. Fuchs). In 1983, K. Benabdallah and K. Honda introduced the concept of straight basis of a $p$-group, which is closely related to its socle. The concept of straight bases leads quite naturally to the two concepts of straight $p$-groups and strongly straight $p$-groups. In [1], it is proved that the generalized Prüfer group of length $\omega+1$ is straight. In [1] and [11], it is proved independently that any direct sum of cyclic $p$-groups is strongly straight. Moreover, in [1] and [9], it is proved independently that the direct sum of any torsion-complete $p$-group and any divisible $p$-group is strongly straight. In particular, K. Honda [9] discovered the following result: any torsion-complete $p$-group $A$ has the normal straight basis

$$B = \{B(n)\}_{n \geq 0}, \quad B(n) = \{b_{ny}\}_{y \in \Gamma} \cup \left( \bigcup_{m \geq n} \{ p^m - n b_{mi} \}_{i \in I_m} \right),$$

such that

$$pb_{n+1} = b_{ny} + \sum_{i \in I_n} g_{yi} b_{ni} \quad (n \geq 0, y \in \Gamma, i \in I_n, g_{yi} \in N_p),$$

where $N_p = \{0, 1, 2, \ldots, p-1\}$ and $\bigoplus_{n \geq 0}(\bigoplus_{i \in I_n} \langle b_{ni} \rangle)$ is a basic subgroup of $A$. (Every $b_{ny}$ and $b_{ni}$ have the orders $p^n$ and $p^m$, respectively.) Recently, H. K. Wimmer in [13] and [14] showed interesting applications of the concept of straight bases to linear algebra.

The main results of this paper are the next three theorems.

**Theorem 3.5.** $p^{n+1}$-injective $p$-groups are straight.

**Theorem 4.3.** Let $A$ be a reduced strongly straight $p$-group such that $|A| = 2^n$ where $p$ is the cardinality of a basic subgroup of $A$. Then, $A$ is necessarily torsion-complete.

**Theorem 5.3.** There are separable straight $p$-groups which are quasi-complete.
but not strongly straight. In particular, there exist \(2^c\) (c: the cardinality of the continuum) pairwise non-isomorphic quasi-complete, straight \(p\)-groups with the same basic subgroup \(\bigoplus_{n \geq 0} Z(p^{n+1})\) and with the same socle; moreover, all of these are not strongly straight.

Particularly, Theorem 4.3 gives a partial solution to the characterization problem of strongly straight \(p\)-groups, which is a very difficult open problem.

Let \(p\) be an arbitrary but fixed prime. All groups considered in this paper are additively written abelian \(p\)-groups. For all notations and terminologies without explanations, we refer to [6]. However, as was done in [1], we fix a pair of non-zero elements of \(N_p\), \(r\) and \(s\), such that \(r + s = p\).

§ 1. Preliminaries

In [2], K. Benabdallah and K. Honda pointed out the following two important properties. If \(B\) is a straight basis of a \(p\)-group \(A\), then \(B\) is not only a set of generators of \(A\), but any non-zero element of \(A\) can be expressed uniquely as a linear combination of elements of \(B\) with non-negative integer coefficients smaller than \(p\). The other important property is that the relations between the elements of \(B\) give rise to a family of integers called an \(s\)-factor set which determines the group \(A\) up to isomorphism. Further, in [1], [2], [4], [9] and [11], various results have been obtained concerning straight bases, straight \(p\)-groups, and strongly straight \(p\)-groups. The paper [2] is not only the main source of these researches, but is quite indispensable for this paper. Therefore, in this section, we list necessary definitions and theorems in [2].

Throughout this section, let \(A\) be an arbitrary but fixed \(p\)-group.

NOTATIONS 1.1. Let \(B\) be a subset of \(A\) and \(n\) be any non-negative integer. We denote

\[B^n = \{ b \in B \mid b \neq 0 \}, \quad B(n) = \{ b \in B \mid o(b) = p^{n+1} \}\]

\[F_n(B) = \{ \sum ab \mid a \in N_p, b \in B(n) \},\]

where \(o(b)\) denotes the order of the element \(b\).

DEFINITION 1.2. A subset \(B\) of \(A^*\) is said to be a straight basis of \(A\) if \(B(n)\) is a maximal linearly independent subset of \(A(n)\) for every integer \(n \geq 0\).

LEMMA 1.3. For each \(n \geq 0\), the mapping \(\theta_n\) defined by: \(\theta_n(u + A[p^n]) = p^n u (u \in A[p^{n+1}])\), is an isomorphism between \(A[p^{n+1}]/A[p^n]\) and \(S_n = (p^nA)[p]\).

The following proposition is a useful characterization of straight bases.

PROPOSITION 1.4. Let \(B\) be a subset of \(A^*\). Then the following three properties are equivalent.

1°. \(B\) is a straight basis of \(A\).

2°. For all \(n \geq 0\), \(p^n(B(n))\) is a basis of \(S_n = (p^nA)[p]\).
3°. For all \( n \geq 0 \), \( \{ b + A[p^n] \mid b \in B(n) \} \) is a basis of \( A[p^{n+1}] / A[p^n] \).

**Definition 1.5.** For \( n \geq 0 \), let \( C^n \) be a basis of the elementary \( p \)-group \( S_n = (p^nA)[p] \). The family \( C = \{ C^n \}_{n \geq 0} \) is called a sequence of bases of the socle of \( A \). From Proposition 1.4, every straight basis \( B \) induces a sequence of bases of the socle of \( A \), \( C = C(B) \) such that \( C^n = p^n(B(n)) \). The basis \( B \) is then said to be associated with the sequence \( C \).

The next proposition is a significant property of straight bases which is used frequently in this paper.

**Proposition 1.6.** Let \( B \) be a straight basis of \( A \). Then every \( u \in A^* \) can be expressed uniquely as:

\[
u = f_0 + \cdots + f_n \quad (f_i \in F_i(B), i = 0, \ldots, n).
\]

Moreover, \( f_n \neq 0 \) if and only if \( o(u) = p^{n+1} \).

**Corollary 1.7.** Let \( B \) be a straight basis of \( A \). Then every \( u \in A^* \) can be expressed uniquely as a linear combination of elements of \( B \) with coefficients from \( N_p \).

Here we introduce an \( s \)-factor set which determines the group up to isomorphism. The \( s \)-factor set is one of the most important notions in [2], and we often use it in this paper.

Let \( B \) be a straight basis of \( A \) and write

\[
B(n) = \{ b_{n\lambda} \}_{\lambda \in \Lambda_n},
\]

for every \( n \geq 0 \). From Proposition 1.6, every \( n \geq 1 \) and every \( \lambda \in \Lambda_n \), the element \( pb_{n\lambda} \), being of order \( p^n \), is expressible uniquely as:

\[
pb_{n\lambda} = f_0^{n\lambda} + \cdots + f_{n-1}^{n\lambda} \quad (f_i^{n\lambda} \in F_i(B), f_{n-1}^{n\lambda} \neq 0).
\]

Furthermore, every \( f_i^{n\lambda} \) can be written uniquely as:

\[
(I) \quad f_i^{n\lambda} = \sum_{\mu} g_{i\mu}^{n\lambda} b_{\mu} \quad (g_{i\mu}^{n\lambda} \in N_p, \mu \in \Lambda_i).
\]

**Definition 1.8.** The family of elements of \( N_p \)

\[
(1.1) \quad \{ g_{i\mu}^{n\lambda} \mid n \in N, \ 0 \leq i < n, \ \lambda \in \Lambda_n, \ \mu \in \Lambda_i \}
\]

obtained from (I) is called the \( s \)-factor set of \( A \) relative to the straight basis \( B \).

**Theorem 1.9.** Two \( p \)-groups are isomorphic if and only if their \( s \)-factor sets with respect to some straight bases are identical.

**Proposition 1.10.** Let (1.1) be the \( s \)-factor set of \( A \) relative to a straight basis \( B \). Suppose that \( B \) is associated with a sequence of bases \( C = \{ C^n \}_{n \geq 0} \) of the socle of \( A \). Then, \( C^n = \{ c_{n\lambda} \}_{\lambda \in \Lambda_n} \) where \( c_{n\lambda} = p^n b_{n\lambda} \) (\( n \geq 0 \), \( \lambda \in \Lambda_n \)) and
\[ c_{\lambda} = \sum_{\mu} g_{n-1,\lambda}^{n,\mu} c_{n-1,\mu} \quad (n \geq 1). \]

From Proposition 1.10, the set of vectors in \( N^{A_n}_{p^{n-1}} = V_n \)

\[ \{v^{n,\lambda} = (g_{n-1,\lambda}^{n,\mu})_{\mu \in A_n \setminus n-1} ; \lambda \in A_n \} \]

is linearly independent in \( V_n \) considered as a vector space over the field of \( p \) elements. We say for shortness’ sake that the \( v^{n,\lambda} \)'s are independent modulo \( p \).

**Theorem 1.11.** A family of elements of \( N_p \) in the form of (1.1) is the \( s \)-factor set of some \( p \)-group relative to some straight basis if and only if it satisfies the following two conditions:

1°. For any fixed \( n, i, \lambda \in A_n, g_{n-1,\lambda}^{n,\mu} = 0 \) for almost all \( \mu \)'s.

2°. For any fixed \( n \), the set of vectors (II) is independent modulo \( p \).

Let \( B \) be a straight basis of \( A \) and let

\[ \{g_{n,\lambda}^{\mu} \mid n \in N, 0 \leq i < n, \lambda \in A_n, \mu \in A_i \} \]

be the \( s \)-factor set of \( A \) relative to \( B \).

**Definition 1.12.** \( B \) is said to be a normal straight basis if \( g_{n,\lambda}^{\mu} = 0 \) (0 \( \leq i < n - 1 \), \( \lambda \in A_n, \mu \in A_i \)) for every \( n \geq 2 \). In other words,

\[ pb_{n,\lambda} = \sum_{\mu \in A_n \setminus A_{n-1}} g_{n-1,\lambda}^{n,\mu} b_{n-1,\mu} \quad (n \geq 2, \lambda \in A_n). \]

A \( p \)-group \( A \) is called a straight \( p \)-group if there exists a normal straight basis for \( A \). Further, it is said to be a strongly straight \( p \)-group if, for any sequence of bases \( C \) of its socle, there exists a normal straight basis of \( A \) associated with \( C \).

**Proposition 1.13.** The direct sum of a family of straight \( p \)-groups is straight.

**Proposition 1.14.** Any direct summand of a strongly straight \( p \)-group is strongly straight.

**Theorem 1.15.** Let \( S_0 \) be an elementary abelian \( p \)-group and \( \{S_i\}_{i=1}^\infty \) a countable descending sequence of subgroups of \( S_0 \). Then there exists a straight \( p \)-group \( A \) such that \( S_n = (p^nA)[p] \) (\( n \geq 0 \)).

**Definition 1.16.** Let \( A \) and \( A' \) be \( p \)-groups. We say that \( A \) and \( A' \) are \( \omega \)-similar if there exists an isometry between their socles viewed as countably valued vector spaces, in other words, if there exists an isomorphism between \( A[p] \) and \( A'[p] \) which preserves heights. Note that we do not assume preservation of generalized heights.

The next theorem shows the importance of the two notions of straight \( p \)-groups and strongly straight \( p \)-groups.

**Theorem 1.17.** Let \( A \) be a strongly straight \( p \)-group and \( A' \) be a straight
group. If A and A’ are ω-similar, then they are isomorphic.

The following theorem is a remarkable information on the structure of strongly straight p-groups.

**Theorem 1.18.** Any strongly straight p-group A is a direct sum of a separable strongly straight p-group and a divisible p-group.

§ 2. On p^n-extensions of straight and strongly straight p-groups

The main purpose in this section is to prove Proposition 2.3, which is required in Theorem 3.4 of §3.

First we start with the following proposition.

**Proposition 2.1.** If A is a straight p-group, then pA is also straight. Moreover, if A is a strongly straight p-group, then so is pA.

**Proof.** Let B be a normal straight basis of A and write $B(n) = \{b_{n,\lambda}\}_{\lambda \in A_n}$ for every $n \geq 0$. From Definition 1.12, every $pb_{n,\lambda}$ $(n \geq 1, \lambda \in A_n)$ can be written uniquely as:

$$pb_{n,\lambda} = \sum_{\mu \in A_{n-1}} g_{n-1,\mu} b_{n-1,\mu} \quad (g_{n-1,\mu} \in N_p).$$

Hence, we have:

$$p(pb_{n,\lambda}) = \sum_{\mu \in A_{n-1}} g_{n-1,\mu}(pb_{n-1,\mu})$$

for every $n \geq 2$ and every $\lambda \in A_n$. Therefore, by Proposition 1.4 and Definition 1.12, it is straightforward to check that $pB = \{p(B(n))\}_{n \geq 1}$ is a normal straight basis of pA. Thus, if A is straight, then pA is also straight.

Next, suppose that A is strongly straight, and let $C' = (C^{k+1})_{k \geq 0}$ be an arbitrary sequence of bases of the socle of pA. Furthermore, let $C^0$ be any basis of $A[p]$. Clearly, $C^0 \cup C'$ is a sequence of bases of the socle of A. Since A is strongly straight, we have a normal straight basis $B$ of A associated with $C^0 \cup C'$. As we have seen in the first part of this proposition, it follows that $pB$ is a normal straight basis of pA associated with $C'$. Therefore, pA is strongly straight.

By Proposition 2.1 and mathematical induction, we obtain immediately the following corollary.

**Corollary 2.2.** Let n be any natural number. If A is a straight p-group, then $p^nA$ is also straight. Moreover, if A is a strongly straight p-group, then so is $p^nA$.

Let n be any natural number. Then $A'$ is said to be a $p^n$-extension of A, if $A'$ is an essential extension of $A$ such that $A = p^nA'$.

**Proposition 2.3.** Let $A'$ be a $p^n$-extension of a p-group A. If B is a normal straight basis of A, then $A'$ has a normal straight basis $B'$ such that:
\[ B'(0) = B(0), \quad p(B'(n)) = B(n-1) \quad (n \geq 1). \]

Therefore, if \( A \) is straight, then \( A' \) is also straight. Furthermore, if \( A \) is strongly straight, then so is \( A' \).

**Proof.** We put:

\[ C^0 = B(0), \quad C^m + 1 = p^m(B(m)) \quad (m \geq 0). \]

Since \( A' \) is a \( p \)-extension of \( A \), it follows by Proposition 1.4 that \( \{C^k\}_{k \geq 0} \) is a sequence of bases of the socle of \( A' \). Using Theorem 1.15 and its proof, there exist a \( p \)-group \( G \) and a normal straight basis \( K \) of \( G \) such that:

1. \( \Phi^k G = \Phi^k A' \quad (k \geq 0), \)
2. \( K \) is a straight basis associated with \( \{C^k\}_{k \geq 0} \).

In view of the first proof of Proposition 2.1, \( pK \) is a normal straight basis of \( pG \). Therefore, by Proposition 1.10 and (2), it follows that the \( s \)-factor set of \( pG \) relative to \( pK \) and that of \( A \) relative to \( B \) are identical. Hence, from Theorem 1.9 and its proof, there is an isomorphism \( \Phi : pG \rightarrow A \) such that \( \Phi[pG] = 1 \) (identity map). By (1), \( G \) is a \( p \)-extension of \( pG \). Since \( A' \) is a \( p \)-extension of \( A \), there is an isomorphism \( \Phi^* : G \rightarrow A' \) such that \( \Phi^*[G] = 1 \) and \( \Phi^* \) is an extension of \( \Phi \). We put:

\[ B' = \Phi^*(K). \]

Then it is easy to see that \( B' \) is a desired normal straight basis of \( A' \). Thus, if \( A \) is straight, then so is \( A' \).

Next, suppose that \( A \) is strongly straight, and let \( C = \{C^k\}_{k \geq 0} \) be an arbitrary sequence of bases of the socle of \( A' \). Since \( A' \) is a \( p \)-extension of \( A \), \( \{C^{m+1}\}_{m \geq 0} \) is a sequence of bases of the socle of \( A \). Since \( A \) is strongly straight, we have a normal straight basis \( B \) of \( A \) associated with \( \{C^{m+1}\}_{m \geq 0} \). By the first part of this proposition, \( A' \) has a normal straight basis \( B' \) where:

\[ B'(0) = B(0), \quad p(B'(n)) = B(n-1) \quad (n \geq 1). \]

Clearly, \( C^0 \cup \{B'(n)\}_{n \geq 1} \) is a normal straight basis of \( A' \) associated with \( C \). Thus \( A' \) is strongly straight.

By Proposition 2.3 and mathematical induction, we obtain immediately the following corollary.

**Corollary 2.4.** Let \( n \) be any natural number and let \( A' \) be a \( p^n \)-extension of a \( p \)-group \( A \). If \( B \) is a normal straight basis of \( A \), then \( A' \) has a normal straight basis \( B' \) such that:

\[ p^k(B'(k)) = B(0) \quad (0 \leq k < n), \]
\[ p^k(B'(k)) = B(k-n) \quad (k \geq n). \]

Therefore, if \( A \) is straight, then \( A' \) is also straight. Furthermore, if \( A \) is strongly straight, then so is \( A' \).
PROPOSITION 2.5. Let $A$ be a $p$-group and $n$ an integer $\geq 1$. If $p^nA$ is straight, then $A$ is also straight. Moreover, if $p^nA$ is strongly straight, then so is $A$.

Proof. $A$ can be written in the form

$$A = A_0 \oplus A'$$

where $A_0$ is an elementary $p$-group and $A'$ is a $p$-extension of $pA$. If $pA$ is straight, it follows by Proposition 2.3 that $A'$ is straight. Therefore, $A$ is straight by Proposition 1.13.

Next, suppose that $pA$ is strongly straight, and let $C = \{C^n\}_{n \geq 0}$ be an arbitrary sequence of bases of the socle of $A$. Let $C'^0$ be any basis of $A'[p]$. We put:

$$C' = \{C'^0, C^1, C^2, \ldots, C^n, \ldots \}.$$  

Then $C'$ is a sequence of bases of the socle of $A'$. Since $pA$ is strongly straight, it follows by Proposition 2.3 that $A'$ is strongly straight. Hence, $A'$ has a normal straight basis $B'$ associated with $C'$. Clearly, $C'^0 \cup \{B'(n)\}_{n \geq 1}$ is a normal straight basis of $A$ associated with $C$. Thus $A$ is strongly straight.

By induction we obtain the results for $p^nA$.

By Corollary 2.2 and Proposition 2.5, we obtain immediately the following corollary.

COROLLARY 2.6. The direct sum of a strongly straight $p$-group and a bounded $p$-group is strongly straight.

§ 3. The structure of $p^{\alpha+1}$-injective $p$-groups

A $p$-group $A$ is said to be $p^{\alpha+1}$-injective if

$$p^{\alpha+1}\text{Ext}(G, A) = 0 \quad \text{for all } p\text{-groups } G.$$  

As is well known, the $p^{\alpha+1}$-injective $p$-groups $A$ are characterized by $p^{\alpha}A \leq A[p]$ and $A/p^{\alpha}A$ is torsion-complete.

The main objective in this section is to prove that $p^{\alpha+1}$-injective $p$-groups are straight. Moreover, we shall state concretely the structure of $p^{\alpha+1}$-injective $p$-groups. To this end, we require a normal straight basis of the torsion-complete $p$-group, discovered by K. Honda [9].

First of all, we observe the following property on the socles of torsion-complete $p$-groups.

Let $\tilde{V}$ be the torsion-complete $p$-group with the basic subgroup $V = \bigoplus_{n \geq 0} \langle b_n \rangle$, $o(b_n) = p^{n+1}$, $(n \geq 0, i \in I_n)$. Now, let $X$ be the subset of $\tilde{V}[p]$ consisting of all elements of the form
\[
\left( - \sum_{i \in I_0} \lambda_i b_{0i}, - \sum_{i \in I_1} \lambda_i (p b_{1i}), \cdots, - \sum_{i \in I_n} \lambda_i (p^n b_{ni}), \cdots \right) \\
(n \geq 0, i \in I_m, \lambda_i \in \{0, 1\}).
\]

Throughout this paper, we shall use the above subset \( X \) for the torsion-complete \( p \)-group \( \widetilde{V} \).

We start with the following lemma.

**Lemma 3.1.** If \( P \) is a subslice of \( \widetilde{V} \) such that \( \widetilde{V}[p] \neq P \), then we can pick out a subset of \( X \) as the basis of a complement of \( P \) in \( \widetilde{V}[p] \).

**Proof.** Clearly, any element \( z \) of \( \widetilde{V}[p] \) can be written as:

\[
z = z_1 + 2z_2 + \cdots + (p-1)z_{p-1} \quad (z_i \in X, 1 \leq i \leq p-1).
\]

Therefore, \( X \) is a generating set of \( \widetilde{V}[p] \). Hence, the set \( \bar{X} = \{ \bar{x} = x + P \mid x \in X \} \) is a generating set of \( \widetilde{V}[p]/P \), let \( \{ \bar{x}_i \}_{i \in A} \) be a basis of \( \widetilde{V}[p]/P \) contained in \( \bar{X} \). Then \( \{ x_i \}_{i \in A} \) is the desired basis of a complement of \( P \).

Now, we clear up Corollary 1 in p. 452 of [12] which gives the isomorphism theorem on \( p^{o+1} \)-injective \( p \)-groups.

**Lemma 3.2.** Two \( p^{o+1} \)-injective \( p \)-groups are isomorphic if and only if they are \( o \)-similar.

**Proof.** Let \( A_1 \) and \( A_2 \) be \( p^{o+1} \)-injective \( p \)-groups, and suppose that \( A_1 \) and \( A_2 \) are \( o \)-similar. Thus there exists an isometry \( \theta : A_1[p] \rightarrow A_2[p] \). Then, for any basic subgroup \( V_1 \) of \( A_1 \), we can pick out a basic subgroup \( V_2 \) of \( A_2 \) such that \( \theta(V_1[p]) = V_2[p] \). Clearly, there exists an isomorphism \( \theta^* : V_1 \rightarrow V_2 \) such that \( \theta^* | V_1[p] = \theta \).

Obviously, \( \theta^* \) can be extended to an isomorphism \( \overline{\theta^*} : \overline{V_1} \rightarrow \overline{V_2} \). Since \( A_1 \) and \( A_2 \) are \( p^{o+1} \)-injective, there are epimorphisms \( \phi_k : A_k \rightarrow \overline{V}_k \) such that

\[
\ker \phi_k = p^o A_k, \quad \phi_k \mid V_k = 1_{V_k}
\]

for \( k = 1, 2 \) (use Corollary 1 in p. 250 of [5]). Put \( \phi^*_2 = \overline{\theta^*}^{-1} \phi_2 \). Then we have \( \phi^*_2 \theta^* = \phi_1 \) on \( V_1 \) and, moreover,

\[
\phi_1(A_1[p]) = \phi^*_2(A_2[p]).
\]

Therefore, by Lemma 1 in p. 450 of [12], \( \theta^* : V_1 \rightarrow V_2 \) can be extended to an isomorphism \( f : A_1 \rightarrow A_2 \) such that \( \phi_2 f = \phi_1 \).

Our converse assertion is obvious.

Also, Lemma 3.2 can be expressible as:

**Corollary 3.3.** Let \( A_1 \) and \( A_2 \) be \( p^{o+1} \)-injective \( p \)-groups, and let \( V_1 \) and \( V_2 \) be basic subgroups of \( A_1 \) and \( A_2 \), respectively. Moreover, let \( \theta : V_1 \rightarrow V_2 \) be an isomorphism. If there exists an isometry \( \theta^* : A_1[p] \rightarrow A_2[p] \) such that \( \theta^* | V_1[p] = \theta \), then \( \theta \) can be extended to an isomorphism from \( A_1 \) onto \( A_2 \).
THEOREM 3.4. Let $A$ be a $p^{\omega+1}$-injective $p$-group, and let $V = \bigoplus_{n \geq 0} \bigoplus_{i \in I_n} \langle b_{ni} \rangle$ with $\alpha(b_{ni}) = p^{n+1}$ ($n \geq 0$, $i \in I_n$) be any basic subgroup of $A$. Then $A$ has necessarily a normal straight basis $B = \{B(n)\}_{n \geq 0}$,

$$B(n) = \{b_{nk}\}_{\lambda \in A} \cup \{b_{ni}\}_{\mu \in M} \cup \left( \bigcup_{m \geq n} \{ p^m - n b_{mi} \}_{i \in I_m} \right) \quad (n \geq 0),$$

such that:

1°. $p^{\omega}A = \bigoplus_{\lambda \in A} \langle b_{0\lambda} \rangle$.

2°. i) $pb_{1\lambda} = b_{0\lambda} + \sum_{i \in I_1} (rg_{\lambda i})(pb_{1i})$,

ii) $pb_{n+1\lambda} = b_{n\lambda} + \sum_{i \in I_n} (sg_{\lambda i})b_{ni} + \sum_{i \in I_{n+1}} (rg_{\lambda i})(pb_{n+1i})$

$(n \geq 1, \lambda \in A, i \in I_m, g_{\lambda i} \in \{0, 1\})$.

iii) $pb_{n+1\mu} = b_{n\mu} + \sum_{i \in I_n} g_{\mu i}b_{ni}$ $(n \geq 0, \mu \in M, i \in I_m, g_{\mu i} \in N_p)$.

Proof. Since $A$ is $p^{\omega+1}$-injective and $V$ is a basic subgroup of $A$, it is easily seen that there is an epimorphism $\phi : A \to \tilde{V}$, which satisfies the following (1), (2) (use Corollary 1 in p. 250 of [5]).

(1) $\text{Ker } \phi = p^{\omega}A$.

(2) $\phi \mid V = 1_V$.

If $\phi(A[p]) = \tilde{V}[p]$, then $\text{Ker } \phi = 0$ and $A$ is isomorphic to $\tilde{V}$ which is a torsion-complete $p$-group. Hence, as was seen in [9], $A$ has precisely a normal straight basis satisfying our theorem. Now, assume that

$$\phi(A[p]) \neq \tilde{V}[p].$$

Let $\{e_{\lambda i}\}_{\lambda \in \Gamma}$ be a basis of a complement of $V[p]$ in $\tilde{V}[p]$. Since $V[p] \leq \phi(A[p])$ by (2), we may assume that $\{e_{\lambda i}\}_{\lambda \in \Gamma}$ contains a basis of a complement $P$ of $V[p]$ in $\phi(A[p])$.

Choose a proper subset $M$ of $\Gamma$ such that $\{e_{\mu i}\}_{\mu \in M}$ is a basis of $P$, and put $\Lambda = \Gamma \setminus M$.

By Lemma 3.1, we may suppose that $e_{0\lambda} \in X (\lambda \in A)$, since $\phi(A[p]) \neq \tilde{V}[p]$. Therefore, $\tilde{V}$ has a normal straight basis $K = \{K(n)\}_{n \geq 0}$,

$$K(n) = \{e_{n\lambda}\}_{\lambda \in A} \cup \{e_{n\mu}\}_{\mu \in M} \cup \left( \bigcup_{m \geq n} \{ p^m - n b_{mi} \}_{i \in I_m} \right) \quad (n \geq 0),$$

as follows:

(3) i) $pe_{n+1\lambda} = e_{n\lambda} + \sum_{i \in I_n} g_{\lambda i}b_{ni}$ $(n \geq 0, \lambda \in A, i \in I_m, g_{\lambda i} \in \{0, 1\})$,

ii) $pe_{n+1\mu} = e_{n\mu} + \sum_{i \in I_n} g_{\mu i}b_{ni}$ $(n \geq 0, \mu \in M, i \in I_m, g_{\mu i} \in N_p)$.

Set
\[ K'(n) = \{ p e_{n+1} \}_{\lambda \in \Lambda} \cup \{ e_{n\mu} \}_{\mu \in M} \cup \left( \bigcup_{m \geq n} \{ p^{m-n} b_{mij} \}_{i \in I_m} \right) \]

for every \( n \geq 0 \). By Proposition 1.4 and (3), it is straightforward to check that \( K' = \{ K'(n) \}_{n \geq 0} \) is a normal straight basis of \( \bar{V} \). Let \( E \) be a \( p \)-extension of \( \bar{V} \). Then, by Proposition 2.3, \( E \) has a normal straight basis \( T = \{ T(n) \}_{n \geq 0} \) such that

\[ T(0) = K'(0), \quad p(T(n)) = K'(n-1) \quad (n \geq 1). \]

Now, for every \( n \geq 1 \), we can write \( T(n) \) in the form;

\[ T(n) = \{ z_{n\lambda} \}_{\lambda \in \Lambda} \cup \{ u_{n-1\mu} \}_{\mu \in M} \cup \left( \bigcup_{m \geq n-1} \{ p^{m-n+1} u_{mij} \}_{i \in I_m} \right), \]

where

\begin{enumerate}
  \item \( p z_{n\lambda} = p e_{n\lambda} \quad (n \geq 1, \lambda \in \Lambda), \)
  \item \( p u_{n\mu} = e_{n\mu} \quad (n \geq 0, \mu \in M), \)
  \item \( p u_{n i} = b_{n i} \quad (n \geq 0, i \in I_n), \)
  \item \( p z_{n+1\lambda} = z_{n\lambda} + \sum_{i \in I_n} g_{\lambda i} (p u_{n i}) \quad (n \geq 1, \lambda \in \Lambda), \)
  \item \( p u_{n+1\mu} = u_{n\mu} + \sum_{i \in I_n} g_{\mu i} u_{n i} \quad (n \geq 0, \mu \in M). \)
\end{enumerate}

Now, for each \( \lambda \in \Lambda \), we define the elements \( u_{n\lambda} \) as follows:

\begin{enumerate}
  \item \( u_{0\lambda} = p e_{1\lambda} \quad (\lambda \in \Lambda), \)
  \item \( u_{n\lambda} = z_{n\lambda} + \sum_{i \in I_n} (r g_{\lambda i}) u_{n i} \quad (n \geq 1, \lambda \in \Lambda). \)
\end{enumerate}

Then, by i) and ii) of (4), we obtain

\begin{enumerate}
  \item \( p u_{1\lambda} = u_{0\lambda} + \sum_{i \in I_1} (r g_{\lambda i}) p u_{1 i} \quad (\lambda \in \Lambda), \)
  \item \( p u_{n+1\lambda} = u_{n\lambda} + \sum_{i \in I_n} (s g_{\lambda i}) u_{n i} + \sum_{i \in I_{n+1}} (r g_{\lambda i}) p u_{n+1 i} \quad (n \geq 1, \lambda \in \Lambda). \)
\end{enumerate}

(We recall our convention that \( s, r \in N_{p^r} \{ 0 \} \) such that \( s + r = p \). Designate by \( \tilde{x} \) the element \( x + \phi(A[p]) \) of \( E/\phi(A[p]) \) and set for every \( n \geq 0 \),

\[ L(n) = \{ \tilde{u}_{n\lambda} \}_{\lambda \in \Lambda} \cup \{ \tilde{u}_{n\mu} \}_{\mu \in M} \cup \left( \bigcup_{m \geq n} \{ p^{m-n} \tilde{u}_{mij} \}_{i \in I_m} \right). \]

Then, \( L = \{ L(n) \}_{n \geq 0} \) is a normal straight basis of \( E/\phi(A[p]) \).

In fact, assume that
\[
\sum_{\lambda \in \Lambda} \alpha_\lambda (p^n u_{\lambda}) + \sum_{\mu \in M} \alpha_\mu (p^n u_{\mu}) + \sum_{m \geq n} \sum_{i \in I_m} \delta_m (p^m u_{mi}) \in \phi(A[p])
\]
\[
(\lambda \in \Lambda, \mu \in M, i \in I_m, m \geq n, \alpha_\lambda \in N_p, \alpha_\mu \in N_p, \delta_m \in N_p).
\]

Clearly, we have
\[
\sum_{\lambda \in \Lambda} \alpha_\lambda (p^{n+1} u_{\lambda}) + \sum_{\mu \in M} \alpha_\mu (p^{n+1} u_{\mu}) + \sum_{m \geq n} \sum_{i \in I_m} \delta_m (p^{m+1} u_{mi}) = 0.
\]

If \( n = 0 \), then, using i) of (3), ii) of (4), i) of (5) and the independence of \( K(0) \), it is straightforward to check that \( \alpha_\lambda = \alpha_\mu = \delta_m = 0 \) \( (\lambda \in \Lambda, \mu \in M, i \in I_m, m \geq 0) \).

Now, let \( n \geq 1 \). Since \( p^{n+1} z_{\lambda} = 0 \) \( (\lambda \in \Lambda) \), we obtain, by i) of (4), ii) of (5) and the independence of \( p^n(K(n)) \), \( \alpha_\mu = \delta_m = 0 \) \( (\mu \in M, i \in I_m, m \geq n+1) \), and we can write \( \delta_m + \sum_{\lambda \in \Lambda} (\alpha_\lambda z_{\lambda}) = \delta_m p \) \( (i \in I_m, \delta_m \in Z) \). Again, using i) of (4) and ii) of (5), it follows that
\[
\sum_{\lambda \in \Lambda} \alpha_\lambda (p^n e_{\lambda}) + \sum_{i \in I_m} \delta_m (p^m b_{mi}) \in \phi(A[p]).
\]
Moreover, by i) of (3), it holds
\[
\sum_{\lambda \in \Lambda} \alpha_\lambda e_{\lambda} \in \phi(A[p]).
\]
Since \( \{ e_{\lambda} \}_{\lambda \in \Lambda} \) is a basis of a complement of \( \phi(A[p]) \) in \( V[p] \), \( \alpha_\lambda = 0 \) \( (\lambda \in \Lambda) \), and hence \( \delta_m = 0 \) \( (i \in I_m) \). Therefore, for all \( n \geq 0 \), \( p^n(L(n)) \) is independent.

Next, let \( p^n x \in (p^n(E/\phi(A[p]))) \) for any \( n \geq 0 \). Obviously, \( p^{n+2} x = 0 \). Since \( T \) is a straight basis of \( E \), by Proposition 1.6, \( x \) is expressible uniquely as:
\[
x = f_0 + \cdots + f_n + f_{n+1} \quad (f_i \in F_i(T), 0 \leq i \leq n+1).
\]

Then, \( f_{n+1} \) can be written uniquely as:
\[
f_{n+1} = \sum_{\lambda \in \Lambda} \alpha_\lambda z_{\lambda} + \sum_{\mu \in M} \alpha_\mu u_{\mu} + \sum_{m \geq n} \sum_{i \in I_m} \delta_m (p^{m-n} u_{mi})
\]
\[
(\lambda \in \Lambda, \mu \in M, i \in I_m, m \geq n, \alpha_\lambda \in N_p, \alpha_\mu \in N_p, \delta_m \in N_p).
\]

Using i) of (4), it follows by i) of (3) that \( \sum_{\lambda \in \Lambda} \alpha_\lambda e_{\lambda} \in \phi(A[p]) \), since \( p^{n+1} x \in \phi(A[p]) \).

Therefore, \( \alpha_\lambda = 0 \) \( (\lambda \in \Lambda) \). If \( n = 0 \), then it follows by \( T(0) = K(0) \) and i) of (5) that
\[
(L(0)) = (E/\phi(A[p])) \cdot \frac{p^n f_n}{\phi(A[p])}.
\]

Now, let \( n \geq 1 \). Since \( f_n \in F_n(T) \), by i) of (4) and ii) of (5), \( p^n f_n \) can be written uniquely as:
\[
p^n f_n = \sum_{\lambda \in \Lambda} \beta_\lambda (p^n u_{\lambda}) + \sum_{i \in I_n} \left( \sum_{\lambda \in \Lambda} (r_{\lambda} g_{\lambda i}) \right) (p^n u_{\mu}) + x' \quad (\lambda \in \Lambda, \beta_\lambda \in N_p, x' \in \phi(A[p])).
\]

Hence, we have:
\[
p^n x = \sum_{\lambda \in \Lambda} \beta_\lambda (p^n u_{\lambda}) + \sum_{\mu \in M} \alpha_\mu (p^n u_{\mu}) + \sum_{i \in I_n} \left( \delta_m - \sum_{\lambda \in \Lambda} \left( r_{\lambda} g_{\lambda i} \right) \right) (p^n u_{\mu})
\]
\[
+ \sum_{m \geq n+1} \sum_{i \in I_m} \delta_m (p^{m-n} u_{mi}) + x'.
\]

Therefore, \( p^n x \in \langle p^n(L(n)) \rangle \). Namely, \( \langle p^n(L(n)) \rangle = (p^n(E/\phi(A[p]))) \cdot \frac{p^n f_n}{\phi(A[p])} \) \( (n \geq 1) \).

Finally, for all \( n \geq 0 \), \( p^n(L(n)) \) is independent and is a set of generators of \( (p^n(E/\phi(A[p]))) \cdot \frac{p^n f_n}{\phi(A[p])} \). Hence, for every \( n \geq 0 \), \( p^n(L(n)) \) is a basis of \( (p^n(E/\phi(A[p]))) \cdot \frac{p^n f_n}{\phi(A[p])} \).
Thus, by Proposition 1.4 and iii) of (4), (6), \( L \) is a normal straight basis of \( E/\phi(A[p]) \).

Since \( E \) is a \( p \)-extension of \( V \), the mapping
\[
\psi: \bar{e} \mapsto pe, \quad (e \in E)
\]
is an epimorphism of \( E/\phi(A[p]) \) onto \( V \). It follows that:
\[
(7) \quad \psi((E/\phi(A[p]))[p]) = \phi(A[p]).
\]
Since \( T \) is a normal straight basis of \( E \) and since (4) and (6) hold, we have
\[
(8) \quad \text{Ker } \psi = p^{\omega}(E/\phi(A[p])) = \bigoplus_{\lambda \in \Lambda} \langle \bar{u}_{0\lambda} \rangle.
\]

From Corollary 1 in p. 250 of [5] and (8), \( E/\phi(A[p]) \) is \( p^{\omega+1} \)-injective. On the other hand, \( H = \bigoplus_{n \geq 0} (\bigoplus_{i \in I_n} \langle \bar{u}_{ni} \rangle) \) is clearly a basic subgroup of \( E/\phi(A[p]) \). Furthermore, the mapping \( \theta \) such that \( \theta(\bar{u}_{ni}) = b_{ni} \) \( (n \geq 0, \ i \in I_n) \) is an isomorphism of \( H \) onto \( V \). Obviously, we have
\[
r(p^{\omega}A) = r(V[p]/\phi(A[p])) = |A|.
\]

Therefore, using (7), there is an isometry of \( (E/\phi(A[p]))[p] \) onto \( A[p] \) which agrees with \( \theta \) on \( H[p] \). Hence, by Corollary 3.3, there is an isomorphism \( \bar{\theta}: E/\phi(A[p]) \to A \) that extends \( \theta \). Now, we put
\[
\bar{\theta}(\bar{u}_{n\lambda}) = b_{n\lambda} \quad (n \geq 0, \ \lambda \in \Lambda),
\]
\[
\bar{\theta}(\bar{u}_{n\mu}) = b_{n\mu} \quad (n \geq 0, \ \mu \in M),
\]

Moreover, we set
\[
B(n) = \{b_{n\lambda}\}_{\lambda \in \Lambda} \cup \{b_{n\mu}\}_{\mu \in M} \cup \left( \bigcup_{m \geq n} \{p^{m-n}b_{mi}\}_{i \in I_m} \right) \quad (n \geq 0).
\]

Then, clearly, \( B = \{B(n)\}_{n \geq 0} \) is a normal straight basis of \( A \) satisfying the three conditions of our theorem. Thus we reach the desired result.

Hence, we obtain the following theorem which is the main result of this section.

**Theorem 3.5.** Any \( p^{\omega+1} \)-injective \( p \)-group is straight.

**§ 4. Strongly straight \( p \)-groups**

In this section, we give a partial characterization of strongly straight \( p \)-groups (Theorem 4.3). Moreover, from Theorem 4.3, we derive the existence of separable straight \( p \)-groups which are not strongly straight. By Theorem 1.18, reduced strongly straight \( p \)-groups are separable. As is well known, every separable \( p \)-group will be identified with a pure and dense subgroup of a torsion-complete \( p \)-group with the same basic subgroup. Therefore, using a normal straight basis of the torsion-complete \( p \)-group discovered by K. Honda [9], reduced strongly straight \( p \)-groups are
expressible as:

**Lemma 4.1.** Let \( A \) be a reduced strongly straight \( p \)-group. Then, \( A \) has necessarily a normal straight basis \( B = \{ B(n) \}_{n \geq 0} \) where

\[
B(n) = \{ b_{n, \mu} \}_{\mu \in M} \cup \left( \bigcup_{m \geq n} \{ p^{m-n} b_{m, i} \}_{i \in I_m} \right) \quad (n \geq 0),
\]

such that:

1°. \( pb_{n+1, \mu} = b_{n, \mu} + \sum_{i \in I_n} g_{\mu, i} b_{n, i} \quad (n \geq 0, \mu \in M, i \in I_n, g_{\mu, i} \in N_p) \),

2°. \( V = \bigoplus_{n \geq 0} \left( \bigoplus_{i \in I_n} b_{n, i} \right) \) is a basic subgroup of \( A \).

**Proof.** Let \( W = \bigoplus_{n \geq 0} \left( \bigoplus_{i \in I_n} \langle w_{n, i} \rangle \right) \) with \( \omega(w_{n, i}) = p^{n+1} \) \( (n \geq 0, i \in I_n) \) be any basic subgroup of \( A \). Since \( A \) is reduced and strongly straight, by Theorem 1.18, it is separable. Therefore, \( A \) can be identified with a pure and dense subgroup of \( \overline{W} \). Now, let \( \{ w_{0, \gamma} \}_{\gamma \in \Gamma} \) be a basis of a complement of \( W[p] \) in \( \overline{W[p]} \). Since \( W[p] \leq A[p] \leq \overline{W[p]} \), we may assume that \( \{ w_{0, \gamma} \}_{\gamma \in \Gamma} \) contains a basis of a complement \( P \) of \( W[p] \) in \( A[p] \).

Choose a subset \( M \) of \( \Gamma \) such that \( \{ w_{0, \mu} \}_{\mu \in M} \) is a basis of \( P \). On the other hand, \( \overline{W} \) has a normal straight basis \( K = \{ K(n) \}_{n \geq 0} \),

\[
K(n) = \{ w_{n, \gamma} \}_{\gamma \in \Gamma} \cup \left( \bigcup_{m \geq n} \{ p^{m-n} w_{m, i} \}_{i \in I_m} \right) \quad (n \geq 0),
\]

as follows:

\[
p^{n+1} w_{n+1, \gamma} = w_{n, \gamma} + \sum_{i \in I_n} g_{\gamma, i} w_{n, i} \quad (n \geq 0, \gamma \in \Gamma, i \in I_n, g_{\gamma, i} \in N_p).
\]

Put

\[
C^n = \{ p^n w_{n, \mu} \}_{\mu \in M} \cup \left( \bigcup_{m \geq n} \{ p^n w_{m, i} \}_{i \in I_m} \right) \quad (n \geq 0).
\]

Obviously, \( C = \{ C^n \}_{n \geq 0} \) is a sequence of bases of the socle of \( A \). Since \( A \) is strongly straight, we have a normal straight basis of \( A \) associated with \( C \) satisfying the conditions of our lemma.

**Remark.** There exist \( p \)-groups having normal straight bases of the form given above which are not strongly straight.

**Lemma 4.2.** Let \( A \) be a reduced strongly straight \( p \)-group. Then we have either 1° or 2°:

1°. \( A \) is an unbounded torsion-complete \( p \)-group.

2°. For any basic subgroup \( V \) of \( A \), \( 2^{|A|} = 2^{|V|} \).

**Proof.** By Lemma 4.1, \( A \) has a normal straight basis
\[ B = \{ B(n) \}_{n \geq 0}, \quad B(n) = \{ b_{n\mu} \}_{\mu \in M} \cup \left( \bigcup_{m \geq n} \{ p^{m-n}b_{mi} \}_{i \in I_m} \right) \quad (n \geq 0), \]

such that:

1. \[ pb_{n+1} = b_{n} + \sum_{i \in I_n} g_{ni}b_{ni} \quad (n \geq 0, \mu \in M, i \in I_n, g_{ni} \in N_p), \]

2. \[ V = \bigoplus_{n \geq 0} \left( \bigoplus_{i \in I_n} \langle b_{ni} \rangle \right) \quad \text{is a basic subgroup of} \quad A. \]

If \( |M| \leq |V| \), then \( |A| = |V| \). Therefore, 2° holds. Now, let \( |M| > |V| \geq N_0 \). Then it is easily seen that \( |A| = |M| \). By (2), \( V \) is a basic subgroup of \( A \). Hence, we may assume that \( A \) can be identified with a pure and dense subgroup of \( V \). To prove 2°, suppose that 1° does not hold. Since \( A \) is not torsion-complete, it follows that \( X \leq A[p] \) (cf. Proof of Lemma 3.1). Therefore, we can pick out elements \( z_n \) (\( n \geq 1 \)) of order \( p^n \) in \( V \setminus A \) such that

3. \[ p z_{n+1} = z_n + \sum_{i \in I_{n+1}} \alpha_i (p^2b_{n+1}) \quad (n \geq 1, i \in I_{n+1}, \alpha_i \in \{0, 1\}). \]

Further, for every element \( A \) in the power set of \( M \), we put:

\[ B(A)^{(0)} = B(0), \]

\[ B(A)^{(n)} = \left\{ b_{n\lambda} + z_n + \sum_{i \in I_{n+1}} (r_{i\lambda})(p^2b_{n+1}) \right\}_{\lambda \in A} \]

\[ \cup \{ b_{n\mu} \}_{\mu \in M \setminus A} \cup \left( \bigcup_{m \geq n} \{ p^{m-n}b_{mi} \}_{i \in I_m} \right) \quad (n \geq 1). \]

Using (1) and (3), we have the relations:

4. \[ i) \quad p \left[ b_{1\lambda} + z_1 + \sum_{i \in I_2} (r_{i\lambda})(p^2b_{i+1}) \right] = b_{0\lambda} + \sum_{i \in I_0} g_{i\lambda}b_{0i} + \sum_{i \in I_2} (r_{i\lambda})(p^2b_{2i}), \]

\[ ii) \quad p \left[ b_{n+1}\lambda + z_{n+1} + \sum_{i \in I_{n+2}} (r_{i\lambda})(p^2b_{n+2}) \right] = \left[ b_{n\lambda} + z_n + \sum_{i \in I_{n+1}} (r_{i\lambda})(p^2b_{n+1}) \right] + \sum_{i \in I_n} g_{i\lambda}b_{ni} \]

\[ + \sum_{i \in I_{n+1}} (s_{i\lambda})(p^2b_{n+1}) + \sum_{i \in I_{n+2}} (r_{i\lambda})(p^2b_{n+2}) \quad (n \geq 1, \lambda \in A). \]

For every element \( A \) in the power set of \( M \), put \( B(A) = \{ B(A)^{(n)} \}_{n \geq 0} \) and \( A(A) = \langle B(A) \rangle \).

Using (1), (4) and Proposition 1.4, we get

(5) \[ (p^n A(A))[p] = (p^n A)[p] \quad (n \geq 0). \]
(6) \( B^{(A)} \) is a normal straight basis of \( A^{(A)} \).

Now, let \( s \) be the set of all subgroups of \( \vec{V} \) which are isomorphic to \( A \). Then, since \( A \) is strongly straight and since (5) and (6) hold, it follows by Theorem 1.17 that \( A^{(A)} \) is isomorphic to \( A \). Therefore, each \( A^{(A)} \) belongs to \( s \), and hence \(|s| \geq 2^{|M|}\). Moreover, the pure-exact sequence \( 0 \rightarrow V \rightarrow A \rightarrow A/V \rightarrow 0 \) yields the exact sequence \( 0 \rightarrow \text{Hom}(A/V, \vec{V}) \rightarrow \text{Hom}(A, \vec{V}) \rightarrow \text{Hom}(V, \vec{V}) \rightarrow \text{Ext}(A/V, \vec{V}) = 0 \). Hence, \(|\text{Hom}(A, \vec{V})| = |\text{Hom}(V, \vec{V})| \leq |\vec{V}^{|V|} = (|V^{|K_v|} = |V||V| = 2^{|V|} \). Also, since \( s \) is the set of all subgroups of \( \vec{V} \) which are isomorphic to \( A \), it is obvious that \(|s| \leq |\text{Hom}(A, \vec{V})|\). Namely, \(|s| \leq 2^{|V|}\). Here, \( 2^{|M|} \leq |s| \leq 2^{|V|}\). Since \(|A| = |M|\), it follows that \( 2^{|A|} \leq 2^{|V|}\). Conversely, it is straightforward that \( 2^{|A|} \geq 2^{|V|}\). Consequently, we have \( 2^{|A|} = 2^{|V|}\). Since all basic subgroups of \( A \) are isomorphic, \( 2^p \) holds. Thus we reach the desired result.

The following theorem is the main purpose of this section.

**Theorem 4.3.** Let \( A \) be a reduced strongly straight \( p \)-group such that \(|A| = 2^p\) where \( p \) is the cardinality of a basic subgroup of \( A \). Then, \( A \) is necessarily torsion-complete.

**Proof.** Suppose that \( V \) is any basic subgroup of \( A \). If \(|A| = 2^{|V|}\), then \( 2^{|A|} \neq 2^{|V|}\). Therefore, it follows by Lemma 4.2 that \( A \) is torsion-complete.

The next corollary follows immediately from Theorem 4.3.

**Corollary 4.4.** We assume the Continuum Hypothesis. Then any reduced strongly straight \( p \)-group with a countable basic subgroup is either a direct sum of cyclic \( p \)-groups or torsion-complete.

As was seen in Remark 3.10 of [1], by making use of the concept of pure-complete, it can be shown that there exist separable \( p \)-groups which are not strongly straight. On the other hand, let \( \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots \) and \( \beta_1 < \beta_2 < \cdots < \beta_n < \cdots \) be ascending sequences of natural numbers. Then, from Theorem 4.3,

\[
\bigoplus_{n \geq 1} \mathbb{Z}(p^{\alpha_n}) \oplus \bigoplus_{n \geq 1} \mathbb{Z}(p^{\beta_n})
\]

is not strongly straight but straight. Therefore, we obtain the following two results.

**Corollary 4.5.** The direct sum of a direct sum \( H \) of cyclic \( p \)-groups and a torsion-complete \( p \)-group \( T \) is strongly straight if and only if either \( H \) or \( T \) is bounded.

From Corollary 4.5, it follows that the direct sum of two reduced strongly straight \( p \)-groups needs not be strongly straight. Now, in [2], it is stated that the direct sum of a bounded \( p \)-group and a divisible \( p \)-group is strongly straight. However, as was seen in [4], the direct sums of any unbounded direct sum of cyclic \( p \)-groups and
any non-zero divisible \( p \)-group are not strongly straight, but straight. In particular, since a countable \( p \)-group is a direct sum of cyclic groups if and only if it contains no elements \( \neq 0 \) of infinite height, we have the following corollary.

**Corollary 4.6.** Any countable strongly straight \( p \)-group is either a direct sum of cyclic \( p \)-groups or the direct sum of a bounded \( p \)-group and a divisible \( p \)-group.

§ 5. Separable straight \( p \)-groups

From Proposition 2.4 in p. 121 of [1], it is known that strongly straight \( p \)-groups are pure-complete. Also, in [1] and [11], it is proved that any direct sum of cyclic \( p \)-groups is strongly straight. Furthermore, in [1] and [9], it is proved that any torsion-complete \( p \)-group is strongly straight. As is well known, the direct sum of any direct sum of cyclic \( p \)-groups and any torsion-complete \( p \)-group is pure-complete, and any quasi-complete \( p \)-group is pure-complete. Quasi-complete \( p \)-groups have been studied in [7], [10], and others under the name of quasi-closed \( p \)-groups. On the other hand, as was seen in the preceding section, the direct sum of an unbounded direct sum of cyclic \( p \)-groups and an unbounded torsion-complete \( p \)-group is straight but not strongly straight. In this section, we shall show the following two facts: (1) There exist separable straight \( p \)-groups which are not pure-complete. (2) There exist quasi-complete \( p \)-groups which are straight \( p \)-groups but not strongly straight.

Let \( A \) be any \( p \)-group. Then the closure \( 0^+ \) [naturally, in the \( p \)-adic topology of \( A \)] is equal to \( A^1 \). Hence, we claim that quasi-complete \( p \)-groups are separable.

Our starting point is the following result. This result was also obtained in [4] by other methods and for \( p \neq 2 \).

**Proposition 5.1.** Let \( T \) be a torsion-complete \( p \)-group such that \( |T| = 2^n \) and \( p \) is the cardinality of a basic subgroup of \( T \). Put \( n = 2^n \). If \( S \) is a proper dense subsocle of \( T \) such that \( |S| = n \), then \( S \) supports \( 2^n \) pairwise nonisomorphic pure, straight subgroups of \( T \) with the same basic subgroup.

**Proof.** Since \( S \) is a dense subsocle of \( T \), we can choose a basic subgroup \( V \) of \( T \) such that \( S \subset V[p] \). Put

\[
V = \bigoplus_{n \geq 0} \left( \bigoplus_{i \in I_n} \langle v_{ni} \rangle \right), \quad o(v_{ni}) = p^{n+1} \quad (n \geq 0, i \in I_n).
\]

Let \( \{v_{0\gamma}: \gamma \in \Gamma \} \) be a basis of a complement of \( V[p] \) in \( T[p] \). Since \( S \supseteq V[p] \), we may assume that \( \{v_{0\gamma}: \gamma \in \Gamma \} \) contains a basis of a complement \( P \) of \( V[p] \) in \( S \). Choose a proper subset \( M \) of \( \Gamma \) such that \( \{v_{0\gamma}: \gamma \in M \} \) is a basis of \( P \). We claim that \( v_{0\gamma} \in S \) \( (\gamma \in \Gamma \setminus M) \). Since \( T \) is a torsion-complete \( p \)-group, \( T \) has a normal straight basis

\[
B = \{B(n)\}_{n \geq 0}, \quad B(n) = \{v_{n\gamma}: \gamma \in \Gamma \} \cup \left( \bigcup_{m \geq n} \{p^{m-n}v_{m\gamma}: \gamma \in I_m \} \right) \quad (n \geq 0)
\]

as follows:
Straight and Strongly Straight Abelian $p$-Groups

(1) \[
    pv_{n+1} = v_{n} + \sum_{g_{ji}v_{ni}} (n \geq 0, \; \gamma \in \Gamma, \; i \in I_{m}, \; g_{ji} \in N_{p}).
\]

By Lemma 3.1, we may suppose that
\[
g_{ji} \in \{0, 1\} \; (n \geq 0, \; \gamma \in \Gamma \setminus M, \; i \in I_{m}),
\]

since $S \neq T[p]$. Furthermore, it follows that $|M| = n$, since $|S| = n$. Now, let $\gamma$ be an arbitrary but fixed element of $\Gamma \setminus M$, and let $\Psi(M)$ be the power set of $M$. For every $n \geq 0$ and every $\Lambda \in \Psi(M)$, we set
\[
    B^{(A)}(n) = \left\{ v_{n} + p^{2}v_{n+1} \left( \sum_{i \in I_{n} + 1} (rg_{ji})(pv_{n+1}) \right) \right\}_{\lambda \in \Lambda}
\]
\[
    \cup \left\{ v_{n} \right\}_{\mu \in \Lambda \setminus \Lambda} \cup \left( \bigcup_{m \geq n} \left\{ p^{m-n}v_{mi} \right\}_{i \in I_{m}} \right)
\]

Using (1), for every $n \geq 0$ and every $\lambda \in \Lambda$, we have

(2) \[
    p \left[ v_{n+1} + p^{2}v_{n+2} + \sum_{i \in I_{n+1}} (rg_{ji})(pv_{n+1}) \right]
\]
\[
    = \left[ v_{n} + p^{2}v_{n+1} + \sum_{i \in I_{n+1}} (rg_{ji})(pv_{n+1}) \right]
\]
\[
    + \sum_{i \in I_{n}} g_{ji}v_{ni} + \sum_{i \in I_{n+1}} (sg_{ji})(pv_{n+1}) + \sum_{i \in I_{n+2}} (rg_{ji})(p^{2}v_{n+2}).
\]

For each $\Lambda \in \Psi(M)$, we put $B^{(A)} = \{B^{(A)}(n)\}_{n \geq 0}$, and let $A^{(A)} = \langle B^{(A)} \rangle$. Using (1) and (2), we get:

(3) \[
    (p^nA^{(A)})[p] = S \cap p^nT \; (n \geq 0),
\]

(4) \[
    B^{(A)} \text{ is a normal straight basis of } A^{(A)}.
\]

By (4), $A^{(A)}$'s are straight. Also, it is easy to see that $A^{(A)}$'s are pure and dense subgroups of $T$ with the same basic subgroup $V$. If $\Lambda$ and $\Lambda'$ are different elements of $\Psi(M)$, then it holds that $\Lambda \setminus \Lambda' \neq \emptyset$ or $\Lambda' \setminus \Lambda \neq \emptyset$. Assume that $\Lambda \setminus \Lambda' \neq \emptyset$. Then, we can pick out $\lambda \in \Lambda \setminus \Lambda'$. Clearly, $v_{n} + p^{2}v_{n+1} \in A^{(A)}$ for every $n \geq 0$. If $v_{n} + p^{2}v_{n+1}, v_{n} + p^{2}v_{n+1} \in A^{(A)}$ for some $n \geq 1$, then it follows that $p^{2}v_{n+1} \in A^{(A)}$, since $v_{n} \in A^{(A)}$. By (1) and (3), we obtain that $v_{n} + p^{2}v_{n+1} \notin S$. This contradicts $v_{n} \in S$. Therefore, $v_{n} + p^{2}v_{n+1}$ does not belong to $A^{(A)}$. Thus we have $A^{(A)} \neq A^{(A)}$. Since $|\Psi(M)| = 2^n$, it follows that $S$ supports $2^n$ different pure, straight subgroups $A^{(A)}(\Lambda \in \Psi(M))$ of $T$ with the same basic subgroup $V$. Moreover, for any $\Lambda \in \Psi(M)$, it is obvious that $|\text{Hom}(A^{(A)}, T)| = |\text{Hom}(V, T)| = n$. Therefore, the cardinality of the set of all $A^{(A)}$ which are isomorphic to $A^{(A)}$ is at most $n$. Hence the set of nonisomorphic $A^{(A)}$ is of the power $2^n$. Thus we reach the desired result.

Let $T$ be the torsion-complete $p$-group in Proposition 5.1. Then, we can choose
two nonisomorphic pure straight subgroups $A_1$ and $A_2$ of $T$ with the same socle. Let $A$ be the external direct sum of $A_1$ and $A_2$. Put

$$U=\{(a,a)\in A\mid a\in A_1[p]=A_2[p]\}.$$ 

Then, it is easy to see that $U$ does not support a pure subgroup of $A$. Therefore, $A$ is not pure-complete. On the other hand, by Proposition 1.13, $A$ is straight. Hence we have the following corollary.

**Corollary 5.2.** There exist separable straight $p$-groups which are not pure-complete.

Moreover, we have the following theorem from Proposition 5.1.

**Theorem 5.3.** There are separable straight $p$-groups which are quasi-complete but not strongly straight. In particular, there exist $2^\alpha$ pairwise nonisomorphic quasi-complete, straight $p$-groups of the cardinality of the continuum $\mathfrak{c}$ with the same basic subgroup $\bigoplus_{n\geq 0} Z(p^{n+1})$ and with the same socle; moreover, all of these are not strongly straight.

**Proof.** Let $T$ be a torsion-complete $p$-group with a basic subgroup $V\cong \bigoplus_{n\geq 0} Z(p^{n+1})$. Clearly, $|T|=\mathfrak{c}$. Since $T[p]\neq V[p]$, we can choose a countable subsocle $U$ of $T$ such that $V[p]\cap U=0$. By Theorem 7 and its proof in p. 273 of [7], there exists a pure subgroup $A$ of $T$ such that

1. $A[p]\supseteq V[p]$,
2. $|A|=\mathfrak{c}$,
3. $T/A$ is divisible,
4. $A$ is quasi-complete,
5. $A\cap U=0$.

By (3), (5) and the purity of $A$ in $T$, it is obvious that $A$ is not torsion-complete. Further, since the cardinality of a basic subgroup of $A$ is just $\aleph_0$, it follows by Theorem 4.3 and (2) that $A$ is not strongly straight. Namely, $A$ is a separable $p$-group with the cardinality of the continuum $\mathfrak{c}$ which is quasi-complete but not strongly straight. Now, by (2), (3) and (5), $A[p]$ is a proper dense subsocle of $T$ such that $|A[p]|=\mathfrak{c}$. It follows by Proposition 5.1 that $A[p]$ supports $2^\mathfrak{c}$ pairwise nonisomorphic pure, straight subgroups of $T$ with the same basic subgroup $W$. Let $G$ be any member of the set of these subgroups. Since $A[p]=G[p]$, it follows by (2) that $|G|=\mathfrak{c}$. Next, let $S$ be any non-discrete subsocle of $G$. Using Theorem 2 in p. 272 of [7] and (4), $G[p]+S^- = A[p]+S^- = T[p]$, and hence $G$ is quasi-complete. Since $|G|=\mathfrak{c}$ and $|W|=\aleph_0$, $G$ is not strongly straight by Theorem 4.3. Moreover, by $G[p]=A[p]$, (1) and the purity of $G$ in $T$, we get $W\cong V\cong \bigoplus_{n\geq 0} Z(p^{n+1})$. Hence, $G$ is a quasi-complete, straight $p$-group of the cardinality of the continuum $\mathfrak{c}$ with the basic subgroup $W$ and with the socle $A[p]$. Furthermore, $G$ is not strongly straight. Thus we have obtained the desired result.
References


Department of Mathematics
Hachinohe Institute of Technology
Obiraki, Hoachinohe, Aomori 031
Japan