On the Meromorphy of Dirichlet Series Corresponding to Siegel Cusp Form of Degree 2 with respect to $\Gamma_0(N)$

by

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§ 0. Introduction

0.1. In [1] and [2], Andrianov constructed the theory of Euler products of Dirichlet series corresponding to holomorphic automorphic forms for the Siegel modular group $Sp_2(\mathbb{Z})$ of degree 2. The method of his theory seems to be applicable even to the case of holomorphic automorphic forms for a certain class of subgroups of $Sp_2(\mathbb{Z})$. In the previous paper [7], we studied the case of congruence subgroups of $Sp_2(\mathbb{Z})$ which is also important from the arithmetic point of view. In this paper, we shall improve the formulation and theorems of [7] to be applicable to a wider class of automorphic forms and show some examples which make the meaning of our main theorems clearer.

0.2. As in [7], we investigate cusp forms of weight $k$ with respect to a congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbb{Z}); \ C \equiv 0 \pmod{N} \right\} \quad (0 < N \in \mathbb{Z})$$

of $Sp_2(\mathbb{Z})$ and a Dirichlet character $\psi$ of modulo $N$. We can define Hecke operators $T(m)\psi \ (m \in \mathbb{N})$ on the space $S_k(\Gamma_0(N), \psi)$ of the forms mentioned above. We take a non-zero automorphic form $F_0 \in S_k(\Gamma_0(N)), \psi$ which is a common eigen form of all Hecke operators $T(m)\psi$ with $(m, N) = 1$:

$$T(m)\psi F_0 = \lambda(m)F_0 \quad \text{for} \quad (m, N) = 1.$$

For the set $\{ \lambda(m); \ (m, N) = 1 \}$ of eigen values, we define a subspace $S_k(\Gamma_0(N), \psi, \lambda)$ of the space $S_k(\Gamma_0(N), \psi)$ by putting

$$S_k(\Gamma_0(N), \psi, \lambda) = \{ F \in S_k(\Gamma_0(N)), \psi); \ F\big|T(m)\psi = \lambda(m)F \text{ for } (m, N) = 1 \}.$$ 

The space $S_k(\Gamma_0(N), \psi, \lambda)$ is invariant under the action of Hecke operators. Define a

\[\text{This paper is a large part of the doctoral dissertation of the author submitted to Tokyo University in 1981.}\]
Dirichlet series by
\[ D_N^\psi(s, \psi) = \sum_{(m, N) = 1} \lambda(m)m^{-s} \quad (\text{Re } s > k) \]
and let \( L(s, \psi^2) \) be the Dirichlet \( L \)-function with character \( \psi^2 \). Set
\[ Z_N^\psi(s, \psi) = L(2(s - k + 2), \psi^2)D_N^\psi(s, \psi). \]
The function \( Z_N^\psi(s, \psi) \) has the following Euler products
\[ Z_N^\psi(s, \psi) = \prod_{p \nmid N} Q_p^\psi(p^{-s})^{-1} \]
in \( \text{Re } s > k \), where \( Q_p^\psi(t) \) is a polynomial of \( t \) with degree 4. Each \( F \in S_k(\Gamma_0(N), \psi) \) has a Fourier expansion of the form:
\[ F(Z) = \sum_{T > 0} a_F(T) \exp(2\pi i \sigma(TZ)), \]
where \( T \) runs through all positive definite half-integral symmetric matrices of size 2. The equivalence class of positive definite primitive half-integral symmetric matrices with given determinant \(-D/4\) corresponds bijectively to the proper \( R \)-ideal class, where \( R \) is an order of \( \mathbb{Q}(\sqrt{D}) \) with discriminant \( D \). Let \( a_1, a_2, \ldots, a_h \) be a complete set of representatives of proper \( R \)-ideal classes and let \( T(a_1), T(a_2), \ldots, T(a_h) \) be the corresponding positive definite primitive half-integral symmetric matrices with determinant \(-D/4\). We take a character \( \chi \) of the group of proper \( R \)-ideal classes and consider the sum
\[ a_F(1; \chi) = \sum_{i=1}^h a_F(T(a_i))\chi(a_i). \]
Let \( \{F_1, F_2, \ldots, F_l\} \) be a basis of the space \( S_k(\Gamma_0(N), \psi, \lambda) \) and put
\[ a(1; \chi) = (a_{F_1}(1; \chi), a_{F_2}(1; \chi), \ldots, a_{F_l}(1; \chi)) \in \mathbb{C}^l. \]
It is easy to show that there exist a certain order \( R \) of an imaginary quadratic field and a certain character \( \chi \) of the group of proper \( R \)-ideal classes with the property \( a(1; \chi) \neq 0 \). We assume that \( N \) is a prime number. We impose the following assumption on the space \( S_k(\Gamma_0(N), \psi, \lambda) \):

(C1) for a suitable choice (see (4.2)') of \( R \) and \( \chi \) which satisfy the condition
\[ a(1; \chi) \neq 0, N \text{ does not divide the discriminant } D(R) \text{ of } R. \]
For each \( F \in S_k(\Gamma_0(N), \psi, \lambda) \), we define a function \( \hat{F} \) by
\[ \hat{F}(Z) = \det(\sqrt{N}Z)^{-h}F(-N^{-1}Z^{-1}), \]
which in fact is an automorphic form of \( S_k(\Gamma_0(N), \psi, \psi^2\lambda) \). We define matrices \( U_\psi \) and \( \hat{U}_\psi \) by putting
\[ (F_1 \mid T(N)_\psi, F_2 \mid T(N)_\psi, \ldots, F_l \mid T(N)_\psi) = (F_1, F_2, \ldots, F_l)U_\psi, \]
\( (F_1 | T(N)_\lambda, F_2 | T(N)_\lambda, \ldots, F_l | T(N)_\lambda) = (\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_l) U_{\lambda} \).

Moreover we put
\[
\hat{a}(1; \chi) = (a_{F_1}(1; \chi), a_{F_2}(1; \chi), \ldots, a_{F_l}(1; \chi)) .
\]

We attach a suitable \( \Gamma \)-factor to the function \( Z^N_\chi(s, \psi) \) putting
\[
\Phi^N(s, \lambda, \psi) = (2\pi)^{-s} \Gamma(s) \Gamma(s-k+2) Z^N_\psi(s, \psi) .
\]

For a complex variable \( s \), we put \( s = 2k - 2 - s \).

Now our main theorems are formulated as follows. In the following theorems we assume that \( N \) is a prime number.

**Theorem 1.** The function \( \Phi^N(s, \lambda, \psi) \) can be continued analytically to a meromorphic function in the whole complex plane and is holomorphic except for a possible simple pole at \( s = k \).

For the functional equation satisfied by the function \( \Phi^N(s, \lambda, \psi) \), we divide two cases.

**Theorem 2.** Let \( \psi \) be a primitive Dirichlet character modulo \( N \) such that \( \psi^2 \) is non-trivial. Suppose that the space \( S_k(\Gamma_0(N), \psi, \lambda) \) satisfies the assumption (C1) above. Then, \( \Phi^N(s, \lambda, \psi) \) satisfies the functional equation:
\[
N^{3s/2} \Phi^N(s, \lambda, \psi) a(1; \psi) (E_i - N^{-s} U_\lambda)^{-1}
\]
\[
= \psi(-1) \omega N^{3s/2} \Phi^N(s, \psi^2 \lambda, \psi) \hat{a}(1; \psi) (E_i - N^{-s} \hat{U}_\lambda)^{-1} ,
\]

where
\[
\omega = N^{-1} \left\{ \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \psi(|x|^2) \exp \left( 2\pi i \frac{x - \bar{x}}{N\sqrt{D}} \right) \right\} .
\]

**Theorem 3.** Let \( \psi \) be the trivial Dirichlet character modulo \( N \). We put \( (\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_l) = (F_1, F_2, \ldots, F_l) U_0 \) with some \( U_0 \in GL_l(\mathbb{C}) \).

Suppose that the space \( S_k(\Gamma_0(N), \psi, \lambda) \) satisfies the condition (C1) above and moreover that
\[
(\psi E_i - N^{-s} U_\lambda)^{-1} (E_i - N^{-s-k+2} U_0)^{-1} \]

We put
\[
\Phi^N(s, \lambda) = \Phi^N(s, \lambda, \psi) (E_i - N^{-s} U_\lambda)^{-1} (E_i - N^{-s-k+2} U_0)^{-1} .
\]

Then we have
\[
a(1; \psi) \Phi^N(s, \lambda) = a(1; \psi) (-1)^k \Phi^N(s, \lambda) .
\]

In [3] Freitag proved the following fact: let \( \Phi \) be the Siegel operator from \( M_k(\Gamma_0(N), \psi) \) to the set of the elliptic modular forms of level \( N \), character \( \psi \). Suppose \( F \) is a common eigen function of all Hecke operators \( T(m)_\psi \) with \( (m, N) = 1 \) and
$F|\Phi \neq 0$. Then $Z_N^\lambda(s, \psi)$ is represented as some products of Dirichlet series corresponding to $F|\Phi$.

The functional equation satisfied by $\Phi^N(s, \psi, \lambda)$ should be independent of the choice of $R$ and coefficient $a(1; \chi)$. Up to now, the problem to obtain the functional equation of $\Phi^N(s, \lambda, \psi)$ in a better form seems to be rather difficult. However in some examples we can derive explicit functional equations from our theorems, where we employ some results of Ibukiyama [5].

Now we show an example. Let $m_1$ and $m_2$ be elements in $\mathbb{Z}^2$ and let $m = (m_1, m_2) \in \mathbb{Z}^4$. For $m = (m_1, m_2)$, put

$$
\theta_m = \theta_m(\tau) = \sum_{n \in \mathbb{Z}^2} \exp(2\pi i((n + m_1/2) \tau (n + m_1/2))^2
+(n + m_1/2)(m_2/2))^2) \quad (\tau \in \mathbb{H}).
$$

$$
X = (\theta_{0000} + \theta_{0010} + \theta_{0001} + \theta_{0011})/4,
$$

$$
Y = (\theta_{0000} \theta_{0010} \theta_{0001} \theta_{0011})^2,
$$

$$
Z = (E^*_4 + 3Y - 4X^2)/12288,
$$

where $E^*_4$ is the normalized Eisenstein series for $Sp_2(Z)$ of weight 4, and

$$
K = (\theta_{0100} \theta_{0110} \theta_{0100} \theta_{0111})^2)/4049.
$$

Then in the detailed version of [5] (preprint), Ibukiyama showed the following fact: let $\phi_0$ be the trivial Dirichlet character of modulo 2 and

$$
F_1 = Y^2Z + XYK - 1024YZ^2 = 5120XYZ,
$$

$$
F_2 = 13XYK - 2X^2YZ - 4608YZ^2 + 5760K^2 - 9728XYZK - 9F_1/4
$$

Then $F_1$ and $F_2$ are cusp forms in $S_2(\Gamma_0(2), \phi_0)$ and

$$
F_1|T(m)_{\phi_0} = \lambda(m)F_1,
$$

$$
F_2|T(m)_{\phi_0} = \lambda(m)F_2.
$$

It can be easily seen that $\{F_1, F_2\}$ is a basis for $S_2(\Gamma_0(2), \phi_0, \lambda)$, and that

$$
U_{\phi_0} = 2^s \begin{pmatrix} -28 & 15 \\ -16 & -28 \end{pmatrix}, \quad U_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Further he proved that the Fourier coefficients of $F_1$ and $F_2$ for $(1/2, 1/2)$ are 6 and $-3/2$ respectively. Therefore we can choose that $R = \mathbb{Z}[(-1 - \sqrt{-3})/2]$ and $\chi$ is the trivial character in the assumption (C1). Then 2, the level of the forms, remains prime in $\mathbb{Q}(\sqrt{D(R)})/\mathbb{Q}$. It is easy to see that $a(1; \chi) = (6, -3/2)$. From Theorem 3, we have the functional equation

$$
\Phi(s) = \Phi(2-s),
$$

for $\Phi(s) = (2\pi)^{-2s} \Gamma(s)^2 2^sZ_2^2(s, \phi_0)(1 + 7 \cdot 2^s - 2^{20-2s})^{-1}$. 

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§ 1. Review on Siegel modular forms and Hecke operators

The Siegel upper half plane of degree \( n \geq 1 \) is the complex manifold of the complex dimension \( n(n+1)/2 \) defined by the following set:
\[
\mathcal{H}_n = \{ Z = X + iY \in M_n(\mathbb{C}); \; X, \; Y \in M_n(\mathbb{R}), \; Z' = Z, \; Y > 0 \},
\]
where \( Z' \) is the transpose of a matrix \( Z \) and \( Y > 0 \) means that \( Y \) is positive definite. Let \( E_n \) be the unit matrix of order \( n \) and \( J_n \) the alternating matrix \((-I_n \, 0 \, E_n)\). Set
\[
GSp_n(\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(\mathbb{R}); \; \begin{pmatrix} A \\ C \end{pmatrix} J_n A = r(\alpha)J_n \begin{pmatrix} A \\ C \end{pmatrix} \right\}. 
\]
It is well-known that \( GSp_n(\mathbb{R}) \) acts transitively on \( \mathcal{H}_n \): if \( \alpha = (\begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in GSp_n(\mathbb{R}), \) then the map
\[
Z \mapsto \langle Z \rangle = (AZ + B)(CZ + D)^{-1} \quad (Z \in \mathcal{H}_n)
\]
is a holomorphic automorphism of \( \mathcal{H}_n \). For a function \( F \) on \( \mathcal{H}_n \), a positive integer \( k \) and \( \alpha = (\begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in GSp_n(\mathbb{R}), \) we define another function \( F\left[ [\alpha]_k \right] \) on \( \mathcal{H}_n \) by the formula
\[
(F\left[ [\alpha]_k \right])(Z) = r(\alpha)^{nk/2} |CZ + D|^{-k} F(\alpha \langle Z \rangle) \quad (Z \in \mathcal{H}_n),
\]
where \( |X| \) is the determinant of a matrix \( X \).

For a positive integer \( N \), we call the set
\[
\Gamma_0^\ast(N) = \{ M = (\begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in Sp_2(\mathbb{Z}); \; C \equiv 0 \pmod{N} \}
\]
a congruence subgroup of \( Sp_2(\mathbb{Z}) \). Let \( \psi \) be a Dirichlet character modulo \( N \) and \( k \) a positive integer. A modular form of degree \( n \) and weight \( k \) with respect to the pair \( (\Gamma_0^\ast(N), \psi) \) is any holomorphic function on \( \mathcal{H}_n \) which satisfies the following condition: for every \( M = (\begin{pmatrix} A & B \\ C & D \end{pmatrix}) \in \Gamma_0^\ast(N) \) and \( Z \in \mathcal{H}_n \), we have the identity
\[
(F\left[ [M]_k \right])(Z) = \psi(\langle A \rangle) F(Z).
\]
In this definition, \( N \) is called the level of the form. All modular forms of degree \( n \) and weight \( k \) with respect to \( (\Gamma_0^\ast(N), \psi) \) form a vector space over \( \mathbb{C} \). We denote the space by \( M_k(\Gamma_0^\ast(N), \psi) \). For \( F \in M_k(\Gamma_0^\ast(N), \psi) \), we can define another form \( F\left| \Phi \right. \) of degree \( n-1 \) by
\[
(F\left| \Phi \right.)(Z_0) = \lim_{\lambda \to \infty} F \left( \begin{pmatrix} Z_0 \\ i\lambda \end{pmatrix} \right) \quad (Z_0 \in \mathcal{H}_{n-1}).
\]
The linear map \( \Phi: M_k(\Gamma_0^\ast(N), \psi) \in F \mapsto F\left| \Phi \right. M_k(\Gamma_0^{n-1}(N), \psi) \) is called the Siegel operator. An element of the set
\[ S_k(\Gamma_0^2(N), \psi) = \{ F \in M_k(\Gamma_0^2(N), \psi); \ (F \mid [M]) \mid \Phi = 0 \text{ for all } M \in Sp_n(\mathbb{Z}) \} \]
is called a cusp form of degree \( n \) and weight \( k \) with respect to \( (\Gamma_0^2(N), \psi) \).

It is easy to see that every modular form \( F \in M_k(\Gamma_0^2(N), \psi) \) has the Fourier expansion
\[
F(Z) = \sum_T a(T) \exp(2\pi i \sigma(TZ)),
\]
where \( T \) runs through all elements of the set
\[
(1.2) \quad P_n = \{ T = (t_{ij}) \in M_n(\Phi); T' = T \geq 0, \ t_{ii}, 2t_{ij} \in \mathbb{Z} \}
\]
and \( \sigma(X) \) is the trace of a matrix \( X \). In the definition (1.2) of \( P_n \), \( T \geq 0 \) means that \( T \) is a positive semidefinite matrix. It is easy to see that
\[
(1.3) \quad a(U'TU) = \psi(|U|) |U|^k a(T),
\]
for all \( U \in GL_n(\mathbb{Z}) \) and all \( T \in P_n \), and that
\[
(1.4) \quad \begin{cases}
    a(T) = O(|T|^k) & |T| \neq 0, \\
    F(Z) = O(|\text{Im } Z|^{-k}),
\end{cases}
\]
where \( O \) depends only on \( F \). Further if \( F \in S_k(\Gamma_0^2(N), \psi) \), then
\[
(1.5) \quad \begin{cases}
    a(T) = 0 \text{ for all } T \in P_n \text{ with } |T| = 0, \\
    a(T) = O(|T|^{k/2}),
\end{cases}
\]
where \( O \) depends only on \( F \).

In the following, we consider Hecke operators on \( M_k(\Gamma_0^2(N), \psi) \). For further details on the facts and definitions cited below, see [2] and [7]. For any positive integer \( m \), we put
\[
\Delta_m(N) = \{ x = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in GSp_2(\mathbb{R}) \cap M_k(\mathbb{Z}); \ (|A|, N) = 1, \ r(x) = m, \ C \equiv 0 \text{ (mod } N) \}. \]

Further we put \( \Delta_0 = \bigcup_{m=1}^{\infty} \Delta_m(N) \) and we write simply \( \Gamma_0 \) or \( \Gamma_0(N) \) for \( \Gamma_0^2(N) \). Since \( \Delta_0 \) form a semi-group, we can define the Hecke ring \( R(\Gamma_0, \Delta_0) \). Especially we put
\[
T(m) = \sum_{x \in \Gamma_0 \backslash \Delta_m(N) / \Gamma_0} \Gamma_0 x \Gamma_0 \in R(\Gamma_0, \Delta_0).
\]
For each double coset \( \Gamma_0 \backslash \Gamma_0 \in R(\Gamma_0, \Delta_0) \) with \( x \in \Delta_m(N) \), a Hecke operator \([\Gamma_0 \backslash \Gamma_0]_{k, \psi}\) on \( M_k(\Gamma_0, \psi) \) is given by the formula
\[
F \mid [\Gamma_0 \backslash \Gamma_0]_{k, \psi} = m^{k-3} \sum_{\gamma \in \Gamma_0 \backslash \Gamma_0 \backslash \Gamma_0} \psi(|A(\gamma)|) (F \mid [\gamma]_{k}) \in M_k(\Gamma_0, \psi),
\]
where \( \gamma = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \) runs through a complete system of representatives \( \Gamma_0 \backslash \Gamma_0 \) modulo \( \Gamma_0 \). Then, the Hecke operator
\[
T(m)_{k, \psi} = \sum_{x \in \Gamma_0 \backslash \Delta_m(N) / \Gamma_0} [\Gamma_0 \backslash \Gamma_0]_{k, \psi}
\]
acts on $M_k(\Gamma_0, \psi)$ by

$$F \mid T(m)_{k, \psi} = \sum_{\alpha \in \Gamma_0 \backslash \Delta \Gamma_0 \Gamma_0} F \mid [\Gamma_0 \varphi \Gamma_0]_{k, \psi}.$$  

(1.6)

We write simply $T(m)_{\psi}$ or $T(m)$ for $T(m)_{k, \psi}$.

Now let $\mathcal{U}$ denote the set of all complex-valued functions $\varphi$ on the set $P_2$ satisfying $\varphi(UTU^*) = \varphi(T)$ for all $U \in SL_2(\mathbb{Z})$ and $T \in P_2$. By (1.3), the Fourier coefficients of any modular form $F \in M_k(\Gamma_0, \psi)$ can be regarded as an element of $\mathcal{U}$. Let $\Gamma^1 = SL_2(\mathbb{Z})$, $g$ an element in $M_2(\mathbb{Z})$ with $|g| > 0$, $(|g|, N) = 1$, and let $\Gamma^1 g \Gamma^1 = \bigcup_j \Gamma^1 g_j$ be a decomposition into disjoint left cosets. For each function $\varphi \in \mathcal{U}$, we set

$$T_{\varphi}(\Gamma^1 g \Gamma^1 \varphi)(T) = \sum_j \varphi(g_j T g_j) \quad (T \in P_2).$$

It is easy to see that $T_{\varphi}$ is independent of the choice of the system of representatives $\{g_j\}$ and that $T_{\varphi}(\Gamma^1 g \Gamma^1 \varphi) \in \mathcal{U}$. Now define operators $\Delta^+(m)$, $\Delta^-(m)$ on $\mathcal{U}$ for $m \in \mathbb{N}$ by

$$\begin{cases}
(\Delta^+(m) \varphi)(T) = \varphi(mT), \\
(\Delta^-(m) \varphi)(T) = \begin{cases} 
\varphi(m^{-1}T) & (m^{-1}T \in P_2), \\
0 & (m^{-1}T \notin P_2).
\end{cases}
\end{cases}$$

(1.7)

Further put, for $m \in \mathbb{N}$ with $(m, N) = 1$,

$$\Pi(m) = T_{\varphi}(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \Gamma^1) \Delta^1(m).$$

(1.8)

Then $\Pi(m)$ is an operator on $\mathcal{U}$.

Let $p$ be a prime and $F \in M_k(\Gamma_0, \psi)$ a common eigen function of all the Hecke operators $T(p^\delta)$ ($\delta = 0, 1, 2, \cdots$):

$$F \mid T(p^\delta) = \lambda(p^\delta) F.$$  

Then, it is easy to show from [Shimura 8] that

$$\sum_{\delta = 0}^\infty \lambda(p^\delta) t^\delta = \frac{P_{\psi}(t)}{Q_{\psi}(t)},$$

(1.9)

where

$$\begin{cases}
P_{\psi}(t) = 1 - p^{2k-4} \psi^2(p) t, \\
Q_{\psi}(t) = 1 - \lambda(p) t + (\lambda(p)^2 - \lambda(p) - p^{2k-4} \psi^2(p)) t^2 - p^{2k-3} \psi^2(p) \lambda t^3 + p^{4k-6} \psi^4(p) t^4
\end{cases} \quad (p \nmid N),$$

(1.10)

Using operators $\Delta^+(m)$, $\Delta^-(m)$ and $\Pi(m)$ [7, Lemma 2.1], we can show the following proposition in the same manner as in [2, Proposition 2.2.1].

**Proposition 1.1.** Let the notation be as above. Suppose that
\[ F(Z) = \sum_{T \in P_2} a(T) \exp(2\pi i \sigma(TZ)) \in M_\delta(G_0, \psi) \]

is a common eigen function of all the Hecke operators \( T(p^\delta) \) for a prime \( p \) and \( \delta \geq 0 \). Then, for any positive definite matrix

\[ T = \begin{pmatrix} a & b \frac{2}{c} \\ b \frac{2}{c} & c \end{pmatrix} \in P_2^+ = \{ T \in P_2^+ ; |T| > 0 \} \]

such that \((a, b, c, p) = 1\), we have the following equalities:

\[
\left\{ \sum_{\delta=0}^{\infty} a(p^\delta T)^\delta \right\} Q^\psi(t) = \begin{cases} 
\begin{aligned}
& a(T) - p^{k-2} \psi(p)(\Pi(p)\alpha)(T)t \\
& + \{p^{2k-4}\psi^2(p)(\Pi(p)^2 - \Pi(p^2) - 1)\alpha\}(T) \\
& + p^{3k-5}\psi^3(p)((\Pi(p)\Delta - (p)\alpha)(T))t^2 
\end{aligned} \\
& \left( \begin{array}{c} p \ \not| \ N \\
\frac{p}{M} \end{array} \right) ,
\end{cases}
\]

§ 2. Fourier coefficients and Euler products of a form

Let \( K = \mathbb{Q}(\sqrt{d_0}) \) be the imaginary quadratic extension of \( \mathbb{Q} \) with discriminant \( d_0 \) \((<0)\) and \( R = R(K) \) the maximal order of \( K \). Put \( \omega = \sqrt{d_0}/2 \) if \( d_0 \equiv 0 \pmod{4} \) and \( \omega = (1 + \sqrt{d_0})/2 \) if \( d_0 \not\equiv 0 \pmod{4} \). Then \( R = \mathbb{Z} + \mathbb{Z}\omega \). The order with conductor \( f \) in \( R \) has the form

\[ R_f(K) = R_f = \mathbb{Z} + \mathbb{Z} f \omega . \]

Clearly \( R = R_1 \), and the discriminant of \( R_f \) is equal to \( D = d_0 f^2 \). For any lattice \( a \) in \( K \), put

\[ R_a = \{ x \in K ; ax = a \} . \]

Then \( R_a \) is an order in \( K \). We call \( R_a \) the order of \( a \) and \( a \) a proper \( R_a \)-ideal. Two lattice \( a_1, a_2 \) in \( K \) are said to be similar if \( a_1 = axa_2 \) for some \( x \not= 0 \) in \( K \) and then we denote \( a_1 \sim a_2 \). For any two lattice \( a_1, a_2 \) in \( K \),

\[
(2.1) \quad a_1 \cdot a_2 = \{ ax \in K ; x \in a_1, \beta \in a_2 \}
\]

is also an lattice in \( K \). If \( R_{a_1} = R_{f_1}, \ R_{a_2} = R_{f_2} \) then

\[
(2.2) \quad R_{a_1 a_2} = R_f ,
\]

where \( f \) is the greatest common divisor of \( f_1 \) and \( f_2 \). Then norm \( N(a) \) of \( a \) is defined by \( N(a) = [R_a : a] \). Then \( N(a_1 a_2) = N(a_1)N(a_2) \). Let \( a \) be a lattice in \( K \). Then \( \bar{a} = \{ x \in K ; \bar{x} \in a \} (\bar{x} \text{ is the conjugate of } x \in K \text{ over } \mathbb{Q}) \) is also a lattice in \( K \), for which we have \( R_{\bar{a}} = R_a \) and

\[
(2.3) \quad a \cdot \bar{a} = N(a)R_a .
\]

Fix an order \( R_0 \) in \( K \). Then it follows from (2.2) and (2.3) that all proper \( R_0 \)-ideals form a commutative group under the multiplication defined by (2.1). The quotient
group of this group by the subgroup of ideals similar to \( R_0 \) is called the class group of the order \( R_0 \). The class group is denoted by \( H(R_0) = H(D) \), where \( D \) is the discriminant of the order \( R_0 \). For any order \( R_0 \) in \( K \), the number \( h(D) \) of elements in the group \( H(D) \) is finite. Suppose that \( f' \mid f \). Then the map

\[
R_f(\mathbb{Q}(\sqrt{d_0})) \ni a \mapsto R_f(\mathbb{Q}(\sqrt{d_0}))a
\]

induces a surjective homomorphism from \( H(d_0 f^2) \) onto \( H(d_0 f'^2) \), which we denote by \( v(f, f') \).

Every positive definite half-integral matrix

\[
T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in P_2^+
\]

(see Proposition 1.1) can be regarded as the matrix of the positive definite integral binary quadratic form

\[
Q(x, y) = ax^2 + bxy + cy^2
\]

and vice versa. When we need to make this correspondence clear, we denote \( T = T_Q \), \( Q = Q_T \) for the matrix (2.4) and the binary quadratic form (2.5). For \( T = (a, b, c) \in P_2^+ \), let \( e(T) = e(Q_T) = (a, b, c) \) be the greatest common divisor of \( a, b \) and \( c \), and put \( D(T) = D(Q_T) = b^2 - 4ac \).

Let \( d_0 \) be a negative integer which is the discriminant of a quadratic field \( K = \mathbb{Q}(\sqrt{d_0}) \). We denote by \( \{\alpha, \beta\} \) the set \( \mathbb{Z}\alpha + \mathbb{Z}\beta \) (\( \alpha, \beta \in K \)). For any proper \( R_f(\mathbb{Q}(\sqrt{d_0})) \)-ideal \( a \) and a \( \mathbb{Z} \)-basis \( \{\alpha, \beta\} \) of \( a \) such that \( \text{Im} (\alpha\beta - \bar{\alpha}\beta) > 0 \), we can define a binary quadratic form \( Q(a) \) and a matrix \( T(a) \in P_2^+ \) by

\[
\begin{align*}
Q(a)(x, y) &= N(a)^{-1}(\alpha x + \beta y)(\bar{\alpha} x + \bar{\beta} y), \\
T(a) &= T_Q(a).
\end{align*}
\]

Then

\[
T(a) = (N(a))^{-1} \begin{pmatrix} |\alpha|^2 & (\alpha\beta + \bar{\alpha}\beta)/2 \\ (\alpha\beta + \bar{\beta}\alpha)/2 & |\beta|^2 \end{pmatrix}.
\]

Note that the class \( \{T(a)\} = \{UT(a)U'; U \in SL_2(\mathbb{Z})\} \) depends only on the ideal class \( \{a\} \) of \( a \). Further, we have \( e(T(a)) = 1 \) and \( D(T(a)) = d_0 f^2 \). Conversely, for a matrix \( T = (a, b, c) \in P_2^+ \) with \( e(T) = 1 \) and \( D(T) = d_0 f^2 \), put \( a(T) = \{a, (b, -\sqrt{D(T)})/2\} \). Then \( T \) is a proper \( R_f(\mathbb{Q}(\sqrt{d_0})) \)-ideal. Further, the ideal class \( \{a(T)\} \) represented by \( a \) depends only on the class \( \{T\} \). It is well-known that this correspondence defines a bijection between the set of all classes \( \{T \in P_2^+, e(T) = 1, D(T) = d_0 f^2\} \) and all proper \( R_f(\mathbb{Q}(\sqrt{d_0})) \)-ideal classes.

Let \( \mathfrak{M} \) be the set of all ideals in all imaginary quadratic extension of \( \mathbb{Q} \) and let \( \hat{\mathfrak{M}} \) be the space of all complex valued function \( \phi \) on the set \( \mathbb{N} \times \mathfrak{M} \) with the property: \( \phi(m; a_1) = \phi(m; a_2) \) if \( a_1 \) and \( a_2 \) are similar. Further, we set \( \mathfrak{M}^* = \{\phi \in \hat{\mathfrak{M}}; \phi(T) = 0 \text{ if } |T| = 0\} \). We say that \( T \in P_2^+ \) is primitive if \( e(T) = 1 \). Then, for any \( T \in P_2^+ \), we have \( T = e(T) T_0 \) with some \( T_0 \) primitive. Hence we can associate \( \phi \in \mathfrak{M}^* \) with a function \( \bar{\phi} \in \hat{\mathfrak{M}} \) by putting \( \phi(T) = \phi(e(T) T_0) = \bar{\phi}(e(T); a(T_0)) \). Since the mapping \( \phi \mapsto \bar{\phi} \) is an
isomorphism from $\mathfrak{U}^*$ to $\hat{\mathfrak{U}}$, we can regard any operator on $\mathfrak{U}^*$ as an operator on $\hat{\mathfrak{U}}$ and conversely. In particular by definitions (1.7) for operators on $\mathfrak{U}^*$, we have

$$(A^+(m)\tilde{\phi})(n; a) = \tilde{\phi}(mn; a),$$

$$(A^-(m)\tilde{\phi})(n; a) = \begin{cases} \tilde{\phi}(nm^{-1}; a) & (m | n), \\ 0 & (m \nmid n), \end{cases}$$

for $m, n \in \mathbb{N}$ and $\tilde{\phi} \in \hat{\mathfrak{U}}$. From [7, i] of Lemma 1–7 we can easily show that $\Pi(m)$ (see (1.8)) with $(m, N) = 1$ has the same properties as $\Pi(m)$ in [2, Theorem 2.3.1, 2.3.2 and Lemma 2.3.2]. Namely we have the following facts.

Let $m \in \mathbb{N}$ and $p$ a prime such that $(p, m) = (p, N) = 1$. Suppose that $a$ be a proper $R_f$-ideal and let $[a]$ denote the class represented by $a$ in the proper $R_f$-ideal class group. Put $e_f = [R_f: R]$ and

$$A = \frac{e_f}{e_f} \sum_{[a_0]} \tilde{\phi}(m; a_0),$$

where the sum extends over all $[a_0]$ such that $[a_0] \in H(d_0(pf)^2)$ and $v(pf, f)a_0 = \{a\}$. Then the following formulae hold:

I) If $(p, f) = 1$ and $p = p\tilde{p}(p \neq \tilde{p})$ in $R_f$, then

$$(\Pi(p^\delta)\tilde{\phi})(m; a) = \tilde{\phi}(m; p^\delta a) + \tilde{\phi}(m; \tilde{p}^\delta a) \quad (\beta \in \mathbb{N}),$$

$$(\Pi(p)\tilde{\phi})(pm; a) = \tilde{\phi}(pm; pa) + \tilde{\phi}(pm; \tilde{p}a) + A.$$

II) If $(p, f) = 1$ and $p = p^\delta$ in $R_f$, then

$$(\Pi(p^\delta)\tilde{\phi})(m; a) = \begin{cases} \tilde{\phi}(m; pa) & (\beta = 1), \\ 0 & (\beta = 2, 3, \cdots), \end{cases}$$

$$(\Pi(p)\tilde{\phi})(pm; a) = \tilde{\phi}(pm; pa) + A.$$

III) If $(p, f) = 1$ and $p$ remains prime in $R_f$, then

$$(\Pi(p^\delta)\tilde{\phi})(m; a) = 0 \quad (\beta \in \mathbb{N}),$$

$$(\Pi(p)\tilde{\phi})(pm; a) = A.$$

IV) If $p | f$, then

$$(\Pi(p)\tilde{\phi})(m; a) = \tilde{\phi}(pm; R_{f/p}a),$$

$$((\Pi(p)^2 - \Pi(p^2) - 1)\tilde{\phi})(m; a) = 0,$$

$$(\Pi(p)\tilde{\phi})(pm; a) = \tilde{\phi}(p^2m; R_{f/p}a) + A.$$

Let $a_1, a_2, \cdots, a_h$ $(h = h(d_0f^2))$ be a complete system of representatives of $H(d_0f^2)$, $\chi$ be a character of $H(d_0f^2)$. Suppose that

$$F(Z) = \sum_{T \in \mathbb{T}_2} a(T) \exp(2\pi i \sigma(TZ)) \in M_h(G_0, \psi)$$
be a common eigen function of the Hecke operators $T(p^\beta)$ for a prime $p$ and $\beta = 0, 1, 2, \cdots$:

$$F \mid T(p^\beta) = \lambda(p^\beta)F.$$ 

Since $a(UTU') = a(T)$ for $U \in SL_2(\mathbb{Z})$ and the class $\{T(a)\} = \{UT(a)U'; U \in SL_2(\mathbb{Z})\}$ depends only on the class $\{a\}$ represented by $a$,

$$(2.8) \quad a(m; \chi) = \sum_{i=1}^{h} a(mT(a_i))\chi(a_i)$$

is well-defined. By the equality (1.11) and properties I), II), III) and IV) of $\Pi(m)$, in some right half plane $\text{Re} \ s > \sigma_0$, we can prove, by the similar way to [2, Theorem 2.4.1],

$$(2.9) \quad \left\{ \sum_{\Delta = 0}^{\infty} a(mp^\Delta; \chi)p^{-\delta s} \right\} Q_p^\phi(p^{-s}) = \begin{cases} 
 a(m; \chi) \prod_{p \mid f} \left( 1 - \frac{\chi(p)\psi(N(p))}{N(p)^{s-k+2}} \right) & \text{for } p \nmid N, p \nmid f, \\
\sum_{i=1}^{h} \chi(a_i) \left( 1 - \frac{\psi(p)\Pi(p)}{p^{s-k+2}} \right) \left( 1 - \frac{\psi^2(p)\Delta^{-}(p)}{p^{s-k+3}} \right) \alpha(mT(a_i)) & \text{for } p \mid N, p \mid f, \\
 a(m; \chi) \prod_{p \mid N} & \text{for } p \mid N
\end{cases}$$

for $(p, m) = 1$, where $p$ runs through all the proper $R_f$-ideals such that $p \mid p$. For an integer $n$, if all prime factors of $n$ divide $N$, then we denote $n \mid N^\infty$. Put

$$\Phi(s, \chi, \psi, n) = \sum_{i=1}^{h} \chi(a_i) \prod_{p \mid f} \left( 1 - \frac{\psi(p)\Pi(p)}{p^{s-k+2}} \right) \left( 1 - \frac{\psi^2(p)\Delta^{-}(p)}{p^{s-k+3}} \right) \alpha(nT(a_i)).$$

Then

$$(2.10) \quad \Phi(s, \chi, \psi, n) = \sum_{i=1}^{h} \chi(a_i) \sum_{\gamma \mid \delta \mid f} \frac{\psi(\delta)\psi^2(\gamma)\mu(\delta)\mu(\gamma)}{\delta^{a-k+2}\gamma^{a-k+3}} \alpha \left( \frac{\delta}{\gamma} nT(R_f, \delta a_i) \right)$$

$(\mu$ is the Möbius function). For $T \in P^+_2$, set

$$(2.11) \quad R_f(T, s) = \sum_{m=1}^{\infty} a(mT)m^{-s}$$

and

$$(2.12) \quad R_f(\chi, s) = \sum_{i=1}^{h} R_f(T(a_i), s)\chi(a_i).$$

Then we obtain
\[ R_F(\chi, s) = \sum_{m=1}^{\infty} a(m; \chi) m^{-s} \]

(see (2.8)). Further we can prove that \( R_F(T, s) \) and \( R_F(\chi, s) \) converge in \( \text{Re} \, s > 2k + 1 \) (resp. \( k + 1 \)) if \( F \in M_k(\Gamma_0, \psi) \) (resp. \( S_k(\Gamma_0, \psi) \)) by the equalities (1.4) and

\[ \lambda(m) (m = 1, 2, \cdots) \] of a modular form \( F \) in \( M_k(\Gamma_0, \psi) \), we can show \( |\lambda(m)| = O(m^c) \), where \( O \) and \( c \) depend only on \( k \). Define a Dirichlet series

\[ D_F^N(s, \psi) = \sum_{(m, N) = 1} \lambda(m) m^{-s}. \]

then, we can see that the Dirichlet series \( D_F^N(s, \psi) \) converges in a right half plane \( \text{Re} \, s > c + 1 \). Let \( L(s, \psi^2) \) be a Dirichlet series \( \sum_{m=1}^{\infty} \psi^2(m) m^{-s} \). Then \( L(s, \psi^2) \) has an Euler products:

\[ L(s, \psi^2) = \prod_p (1 - p^{-s} \psi^2(p))^{-1}. \]

Hence we have, by (1.9) and (1.10),

\[ L(2(s - k + 2), \psi^2) D_F^N(s, \psi) = \prod_{p \nmid N} \frac{1}{Q_p^N(p^{-s})}. \]

Put

\[ Z_F^N(s, \psi) = \prod_{p \nmid N} \frac{1}{Q_p^N(p^{-s})} \]

and

\[ L_d(\chi \psi_f) = \sum_a \chi(\psi_f(a) N(a)^{-s} \quad (D = d_0 f^2) \]

the \( L \)-series of \( R_f \) with the character \( \chi \psi_f(a) = \psi(N(a)) \), where \( a \) runs through all the proper \( R_f \)-ideals whose norms are coprime to \( f \). Then the identity

\[ L(2(s - k + 2), \psi^2) D_F^N(s, \psi) = Z_F^N(s, \psi) \]

holds and by the equality (2.9), we have immediately

**Proposition 2.1.** Suppose that \( F \in M_k(\Gamma_0, \psi) \) satisfy \( F | T(m) = \lambda(m)F \) for \( (m, N) = 1 \). Then

\[ L_d(s - k + 2, \chi \psi_f) = \sum_{n \mid N} \Phi_d(s, \chi, \psi, n) n^{-s} Z_F^N(s, \psi). \]

We also denote \( Z_F^N(s, \psi) \) (resp. \( D_F^N(s, \psi) \)) by

\[ Z_F^N(s, \psi) \quad \text{(resp. } D_F^N(s, \psi)) \]

for \( F \in S_k(\Gamma_0(N), \psi) \) with the condition \( F | T(m) = \lambda(m)F \) (mod \( (m, N) = 1 \)), because \( Z_F^N(s, \psi) \) (resp. \( D_F^N(s, \psi) \)) depend only on the set \( \{ \lambda(m); (m, N) = 1 \} \).
Especially if $F$ is an eigen function of all $T(m)$ ($m = 1, 2, \cdots$), then we get the equality:

$$
\sum_{n \leq N} \Phi_F(s, \chi, \psi, n)n^{-s} = \Phi_F(s, \chi, \psi, 1) \prod_{p \mid N} \frac{1}{Q^\psi_p(p^{-s})}
$$

Therefore if we put

$$
\begin{align*}
Z_F(s, \psi) &= Z^\chi_F(s, \psi) \prod_{p \mid N} Q^\psi_p(p^{-s})^{-1}, \\
D_F(s, \psi) &= \sum_{m=1}^{\infty} \lambda(m)m^{-s} \quad (F \mid T(m) = \lambda(m)F),
\end{align*}
$$

(2.17)

then we can show

$$
L(2(s-k+2), \psi^2)D_F(s, \psi) = Z_F(s, \psi)
$$

and

$$
L_D(s-k+2, \chi\psi_f)R_F(\chi, s) = \Phi_F(s, \chi, \psi, 1)Z_F(s, \psi).
$$

§3. Discontinuous groups and Eisenstein series of Picard type

The set $H = \{u = (z, v) \in \mathbb{C} \times \mathbb{R}; v > 0\}$ is said to be the upper half space of three dimension. It is well-known that $SL_2(\mathbb{C})$ acts on $H$ transitively by the following way: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, $u = (z, v) \in H$, then the map

$$
(3.1) \quad u \mapsto g(u) = \begin{pmatrix} (az+b)(c\bar{z}+\bar{d})+acv^2 & u \\ \bar{A}_d(u) & \bar{A}_d(u) \end{pmatrix}
$$

is an automorphism of $H$, where $A_d(u) = |cz+d|^2 + |c|^2v^2$. An $SL_2(\mathbb{C})$-invariant metric (resp. volume element) of $H$ is given by $(dx^2 + dy^2 + dv^2)^{1/2}/v$ (resp. $dx dy dv/v^3$) ($z = x + iy$).

Let $K = \mathbb{Q}(\sqrt{-d_0})$ and $R_f = R_f(K)$ be as in §2, a a proper $R_f$-ideal such that $(N(a), N) = 1$, and let $\{\alpha_1, \alpha_2\}$ be a $\mathbb{Z}$-basis of a satisfying $\text{Im}(\alpha_1\bar{\alpha}_2 - \alpha_2\bar{\alpha}_1) > 0$. Further we put

$$
J_a = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \bar{\alpha}_1 & \bar{\alpha}_2 \end{pmatrix} \in GL_2(K), \quad \bar{J}_a = \begin{pmatrix} J_a^* & 0 \\ 0 & J_a^{-1} \end{pmatrix} \in Sp_2(K),
$$

$$
g_0 = \begin{pmatrix} D^{-1/4} & 0 \\ 0 & D^{1/4} \end{pmatrix} \in SL_2(\mathbb{C}) \quad (D = d_0f^2)
$$

and

$$
\hat{g} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & \bar{a} & 0 & \bar{b} \\ c & 0 & d & 0 \\ 0 & \bar{c} & 0 & \bar{d} \end{pmatrix} \in Sp_2(\mathbb{C}) \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}).
$$
Then we can define an injective homomorphism $M_a$ of $\text{SL}_2(\mathbb{C})$ into $\text{Sp}_2(\mathbb{R})$ by

$$\text{SL}_2(\mathbb{C}) \ni g \longmapsto M_a(g) = J_a g_0^* g_0^{-1} J_a^{-1} \in \text{Sp}_2(\mathbb{R}).$$

Set $g_0^* = \left( \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{array} \right) \in \text{GL}_2(\mathbb{C})$. Then we can also define a map $Z_a$ from $H$ into $S_2$ by

$$H \ni u = (z, v) \longmapsto Z_a(u) = J_a g_0^* \left( \begin{array}{cc} z & iv \\ -iv & z \end{array} \right) g_0^* J_a \in S_2.$$

Further, we can prove

$$M_a(g) Z_a(u) = Z_a(g(u))$$

for any $g \in \text{SL}_2(\mathbb{C})$ and any $u \in H$. For a proper $R_f$-ideal $\alpha$ such that $(N(\alpha), N) = 1$, put

$$\Gamma(\alpha) = \left\{ g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{K}); a, d \in R_f, c \in N\alpha; b \in \alpha^{-1} \right\}.$$

$$\Gamma_\infty(\alpha) = \left\{ g = \left( \begin{array}{cc} \pm 1 & b \\ 0 & \pm 1 \end{array} \right) \in \Gamma(\alpha) \right\}.$$

Then we can show

$$M_a(\Gamma(\alpha)) \subset \Gamma_0.$$

We denote by $D_a$ (resp. $S_a$) a fundamental domain for $\Gamma(\alpha)$ (resp. $\Gamma_\infty(\alpha)$) on $H$ with respect to the action (3.1). For a cusp form $F \in S_k(\Gamma_0, \psi)$ we put

$$F_a(u) = F(Z_a(u)).$$

Though $Z_a$ depends on the choice of the basis $\{ z_1, z_2 \}$ of $a$, $F_a$ depends only on the ideal $a$. Further if we put $\psi_f(a) = \psi(N(a))$ for a proper $R_f(Q(\sqrt{d_0}))$-ideal $\alpha$, then $F_a$ satisfies

$$F_a(g(u)) = \psi_f((a_g)) A_\alpha(u) F_a(u) \quad ((a_g) = a_g R_f)$$

for any $g = (g^*_0, g^*_1) \in \Gamma(\alpha)$. Moreover we have

$$\begin{cases}
F_a((z, v)) = O(v^{-k}), \\
F_a((z, v)) = O(\exp(-cv)) \quad (v \to \infty), \\
F_a((z, v)) = O(\exp(-c'v^{-1})) \quad (v \to 0, (z, v) \in D_a),
\end{cases}$$

where $c, c' (>0)$ and $O$ depend only on $F$ and $a$. Especially we denote $F_{R_f}$ by $F_0$ and $D_{R_f}$ by $D_0$.

For a cusp form

$$F(Z) = \sum_{T \in P_2^+} a(T) \exp(2\pi i (\sigma(TZ))) \in S_k(\Gamma_0, \psi),$$

we defined $R_f(T(\alpha), s)$ (see (2.7) and (2.11)) and $F_a(u)$. We show a relation between $R_f(T(\alpha), s)$ and $F_a(u)$ in the following
PROPOSITION 3.1. Put $N(a) = N_0$, $D = d_0 f^2$. For any cusp form $F \in S_k(\Gamma_0, \psi)$, it holds that
\[ \int_{D_a} F_a(u) v^{s-1} du = (2\pi)^{-s} D \left| \frac{\Gamma(s)N_0^{-s} + 2sR_f(T(a), s)}{(s-1)/2} \right| \]
(\(du = dxdydv, u = (z, v)\) and \(z = x + iy\) in \(\text{Re } s > k + 1\)).

For the proof, see [7, Lemma 3.1].

Let $E_a(\psi_f, u, s)$ \((u \in H, s \in \mathbb{C})\) be an Eisenstein series of Picard type defined by
\[ E_a(\psi_f, u, s) = \frac{1}{2} N_0^s \psi_f(a) \sum_{d \in \mathbb{Z} \cap \mathbb{Z}(N)} (\psi_f(d)) \left( \frac{v}{|cz + d|^2 + |cv|^2} \right)^s. \]

Then we get

PROPOSITION 3.2. For any cusp form $F \in S_k(\Gamma_0, \psi)$, then equality
\[ (4\pi)^{-s} \Gamma(s) R_f(T(a), s) = \frac{D}{4} \int_{D_0} v^k F_0(u) \]
\[ \times \frac{D}{4} \int_{D_0} \psi_f((du) v^{s-k+2}/v^3) (u = (z, v)) \]
holds in \(\text{Re } s > k + 1\).

Proof. By a result in [6], we see that $E_a(\psi_f, u, s)$ converges absolutely and uniformly in \(\text{Re } s > 2\). Furthermore $R(T(a), s)$ converges \(\text{Re } s > k + 1\). For \(u = (z, v) \in H\), we denote $v(u) = v$. Put
\[ I = \int_{S_a} F_a(u) v(u)^{s-1} du. \]

Since $D_a$ is a fundamental domain for $\Gamma(a)$ on $H$,
\[ S'_a = \bigcup_{\tau \in \Gamma_a(a)/\Gamma(a)} \tau(D_a) \]
becomes a fundamental domain for $\Gamma(a)$ in $H$. Note that $v^k F_0(u)$ is invariant under the transformation of $\Gamma(a)$. Therefore we have
\[ I = \int_{D_a} v(u)^k F_0(u) \sum_{\tau} \psi_f((du))(v(\tau(u)))^{s-k+2} dv/v^3, \]
where $\tau = (\tau_1^*, \tau_2^*)$ runs through a complete system of representatives of $\Gamma(a)/(\Gamma(a))$. By a brief computation, we can see that there exists a matrix $g(a) = (\alpha_1^*, \alpha_2^*) \in SL_2(K)$ such that $\alpha_1, \alpha_2 \in a, \alpha_1 \in Na$ and $\alpha_2^*, \alpha_2^* \in a^{-1}$. It is easy to see that $\mathbf{J}_a \approx g(a) \approx (A_0 B_0, C_0 D_0)$.
then \(|A_0| = N(\alpha^*_a a)\) and
\[ |C_0 Z_{\alpha}(u) + D_0| = N_0^{-1} \Delta_{\alpha^*_a}(u). \]
Change the variable \(u\) to \(g(\alpha)(u)\) in the integrand above and note that \(g(\alpha)(D_0)\) is a fundamental domain for \(\Gamma(\alpha)\) in \(H\), and we have
\[ I = \int_{D_0} (v(u)^k N_0^{-1} F_0(u)) \psi_J(\tau \Delta \alpha^*_a) \sum_{\tau \in \Gamma(\alpha) \setminus \Gamma(\alpha)} \psi_J(\tau(u)(v\tau g(\alpha)(u))) \frac{du}{v^3}. \]
Further put \(\tau g(\alpha) = \tau_0\). Then \(\tau_0\) runs through a complete system of representatives of \(\Gamma(\alpha) \setminus \Gamma(\alpha) g(\alpha)\). Since \(d_i = -c_o \alpha^*_i + d_o \alpha^*_i\), we get \(\psi_J(\tau_0) = \psi_J(\tau(d_i))\) and
\[ I = \int_{D_0} (v(u)^k F_0(u)) \psi_J(\alpha) \sum_{\tau \in \Gamma(\alpha) \setminus \Gamma(\alpha) g(\alpha)} \psi_J(\tau(u)(v\tau g(\alpha))) \frac{du}{v^3}. \]
Moreover we can show that
\[ \Gamma(\alpha) g(\alpha) = \left\{ \begin{array}{l}
\tau_0 = \left( \begin{array}{cc}
a & b \\
c & d \end{array} \right) \in SL_2(K); \ a, b \in a^{-1}, c \in N, \ d \in a \end{array} \right\} \]
and that the inclusion \(\left( \begin{array}{cc}
a & b \\
c & d \end{array} \right) \in \Gamma(\alpha) g(\alpha)\) is equivalent to the conditions \((c, d) = a\) and \(c \equiv 0 \pmod{N}\). Thus we have
\[ I = \frac{1}{2} N_0^{-1} \int_{D_0} (v(u)^k F_0(u)) \psi_J(\alpha) \sum_{\substack{(c, d) = a \\
 c \equiv 0 \pmod{N}} \psi_J(\tau(\alpha)(v\tau g(\alpha))) \frac{v}{|cz + d|^2 + |cv|^2} \frac{du}{v^3}. \]
and hence the equality (3.4) by Proposition 3.1.

It is easy to follow the two propositions.

**Proposition 3.3.** Let \(a\) be a proper \(R_f\)-ideal such that \((N(\alpha), N) = (N(\alpha), f) = 1\). Put \(a_N = \{z \in a; (N(\alpha), N) = 1\} \) and
\[ X_m^N(\alpha) = \{(\gamma^*, \delta^*) \in K \times K; (N(\gamma^*), \delta^*) = a^*, (N(\delta^*), N) = 1\} \]
for \(m \in N\) and an ideal \(\alpha^*\) in \(R_m\) such that \(R_a \supset R_m\). Then
\[ a \times a_N = \bigcup_{(f, f') = 1} F^f \bigcup_{\alpha^*} X^N_{f, f'}(\alpha^*), \]
where \(\alpha^*\) extends over all ideals in \(R_{f, f'}\) such that \(R_a = R_{f, f'}\) and \((N(\alpha^*), N(f') = 1\).

**Proposition 3.4.** In every class of the group \(H(d_0 f^2)\), there exists a proper \(R_f\)-ideal \(a\) such that \((N(\alpha), f) = (N(\alpha), N) = 1\) \((i = 1, 2, \cdots, h)\). Let \(\chi\) be a character of the group \(H(d_0 f^2)\) satisfying the following condition:
(3.5) for every \( f' \in \mathbb{N} \) such that \( f', f'^{-1} > 1 \) and \( (f', N) = 1 \), the character \( \chi \) is non-trivial on the kernel of the surjective homomorphism \( v(f, f'f') : H(d_0f^2) \rightarrow H(d_0(f'f')^2) \).

Put

\[
E^*_a(\psi_f, u, s) = N(a)^{s} \prod_{c, d \in a} \psi_f(d) \left( \frac{v}{|cNz + d|^2 + |cNv|^2} \right)^s
\]

for \( u = (z, v) \in H, s \in \mathbb{C} \) and a proper ideal \( a \) such that \( (N(a), Nf) = 1 \). Let \( L_{d_0f^2}(s, \chi\psi_f) \) be the \( L \)-series of \( R_f \) defined by (2.14) and \( E_a(\psi_f, u, s) \) the Eisenstein series (3.3). Then we have

**Proposition 3.5.** Let \( \chi \) satisfy the condition (3.5). Then, in the domain \( \{ \text{Re} s > 2, u \in H \} \), it holds that

\[
\sum_{i = 1}^{h} \chi(a_i)E^*_a(\psi_f, u, s) = 2L_{d_0f^2}(s, \chi\psi_f) \sum_{i = 1}^{h} \chi(a_i)E_a(\psi_f, u, s).
\]

Further the both sides of the equality (3.7) converge absolutely in the domain \( \text{Re} s > 2 \).

**Proof.** Put a be a proper \( R_f \)-ideal such that \( N(a) = N_0 \) is coprime to \( Nf \). Then via Proposition 3.4, we get

\[
E^*_a(\psi_f, u, s) = \psi_f(a)N_0^s \sum_{(c, d) \in aN} \psi_f(d) \left( \frac{v}{|cNz + d|^2 + |cNv|^2} \right)^s
\]

\[
= \psi_f(a)N_0^s \sum_{f' < f} \psi_2(f')(f')^{-2s} \sum_{a'} \psi_{f/f'}(a')N(a')^{-s} \times \psi_{f/f'}(a')N(a')^s \sum_{(c, d) \in X(f/N(a))} \psi_{f/f'}((d)) \left( \frac{v}{|cNz + d|^2 + |cNv|^2} \right)^s.
\]

where \( a' \) runs through all the proper \( R_{f/f'} \)-ideals in \( R_{f/f} \) satisfying \( (N(a'), f/f') = (N(a'), N) = 1 \). Hence we obtain

\[
E^*_a(\psi_f, u, s) = \psi_f(a)N_0^s \sum_{f' \mid f} \psi_2(f')(f')^{-2s} 2 \sum_{a'} \psi_{f/f'}(a')N(a')^{-s}E_a(\psi_{f/f'}, u, s).
\]

Let \( \{ a_k : 1 \leq k \leq h' \} \) \( h' = h(d_0(f'f')^2) \) be a complete system of representatives of the ideal classes in \( H(d_0(f'f')^2) \) such that \( (N(a_k), N(f'f')) = 1 \) for \( k = 1, 2, \cdots, h' \) and put, for a proper \( R_m \)-ideal \( a'' \),

\[
L_{d_0m}(s, a'', \psi_m) = \sum_{a^*} \psi_m(a^*)N(a^*)^{-s} \quad (N(a^*) = [R_m : a^*], \psi_m(a^*) = \psi(N(a^*)),
\]

where the sum is extended over all the \( R_m \)-proper ideals \( a^* \) satisfying the conditions \( a^* \sim a'' \) and \( (N(a^*), m) = 1 \). Then by the fact that the Eisenstein series \( E^*_a(\psi_{f/f'}, u, s) \) depends only on the class \( \{ a' \} \), we have
\[
\sum_{a} \psi_{j f'}(a')N(a')^{-s} E_a(\psi_{j f'}, u, s) = \sum_{k=1}^{h'} \sum_{a' \sim a_0}^{a \sim a_0} \psi_{j f'}(a')N(a')^{-s} E_a(\psi_{j f'}, u, s)
\]
\[
= \psi_{j f'}(a_0)N(a_0)^{-s} \sum_{k=1}^{h'} L_{d_0}(s, a_0'^{-1}, \psi_{j f'})
\times E_a(\psi_{j f'}, u, s) \quad (a_0 = R_{j f'} a).
\]

From the equalities \(N(a_0') = N(a) = N_0\) and \(\psi_{j f'}(a_0') = \psi_{j f}(a)\), it follows that
\[
E_a^*(\psi_{f'}, u, s) = 2 \sum_{f' \mid f} \psi(f')f'^{-2s} \sum_{k=1}^{h'} L_{d_0}(s, a_0'^{-1}, \psi_{j f'}) E_{a_0}(\psi_{j f'}, u, s).
\]

Hence we have
\[
\sum_{i=1}^{h} \chi(a_i) E_a^*(\psi_{f'}, u, s) = 2 \sum_{f' \mid f} \psi(f')f'^{-2s} \sum_{j,k=1}^{h'} \sum_{1 \leq i \leq h} \chi(a_i)
\times L_{d_0}(s, a_0'^{-1}, \psi_{j f'}) E_{a_0}(\psi_{j f'}, u, s).
\]

If \((f', N) > 1\), then \(\psi(f') = 0\). If \((f', N) = 1\) and \(f' > 1\), then, by the condition (3.5), for any \(j (1 \leq j \leq h')\) we have
\[
\sum_{1 \leq i \leq h} \chi(a_i) = 0.
\]

Namely, we have
\[
\psi(f') \sum_{1 \leq i \leq h} \chi(a_i) = 0 \quad \text{for} \quad 1 < f' \mid f.
\]

Thus we have proved the equality (3.7).

It is not difficult to prove the convergence of left hand side of (3.7).

For a character \(\omega\) of a group, put
\[
\delta^*(\omega) = \begin{cases} 1 & (\text{\(\omega\) is the trivial character}), \\ 0 & \text{(otherwise)}. \end{cases}
\]

For \(x \in \mathbb{Z}\), put
\[
\delta(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}
\]

Further we denote by \(r(N)\) the number of elements of the set \((\mathbb{Z}/N\mathbb{Z})^\times\).

In the same manner as in proving [7, Lemma 3.4], we can show

THEOREM 3.6. Let \(K\), \(R_f\) and \(\psi_f\) be as above. For a proper \(R_f\) ideal \(a\) whose norm \(N_a\) is coprime to \(N_f\), put
$$E_s^*(\psi_f, u, s) = \sum_{c,d \in d} \psi_f((d)) \left( \frac{V}{|cNz + d|^2 |cNv|^2} \right)^s$$

and

$$\Psi_a(\psi_f, u, s) = \pi^{-s} \Gamma(s) \left| \frac{D}{4} \right|^{\frac{s}{2}} E_s^*(\psi_f, u, s).$$

Then the following assertions hold:

i) \( \Psi_a(\psi_f, u, s) \) can be continued to the whole \( s \)-plane as a holomorphic function except possibly for simple poles at \( s = 2 \) and \( s = 0 \). The possible residues are equal to \( r(N) \delta^*(\psi_f) \) and \( -\delta(N-1) \) respectively;

ii) if \( N \nmid d_0 f^2 \) and \( \psi^2 \) is non-trivial or if \( N = 1 \) (and \( \psi \) must be the trivial character), the equation

$$N^{3s/2} \Psi_a(\psi_f, u, s) = (AN^{-1})N^{3(2-s)/2} \Psi_a(\bar{\psi}_f, \tau_N(u), 2-s)$$

holds, where

$$A = A(\psi_f) = \sum_{x \in R_f/NR_f} \psi_f(xR_f) \exp \left( 2\pi i \frac{x - \bar{x}}{N\sqrt{D}} \right)$$

and

$$\tau_N = \begin{pmatrix} 0 & \sqrt{N^{-1}} \\ -\sqrt{N} & 0 \end{pmatrix} \in SL_2(\mathbb{C}).$$

From this Theorem we have immediately

**COROLLARY.** Put

$$\Psi(\chi, \psi_f, u, s) = \sum_{i=1}^{h} \chi(a_i) \Psi_a(\psi_f, u, s) \quad (h = h(D))$$

for a character \( \chi \) of \( H(D) \). Then \( \Psi(\chi, \psi_f, u, s) \) can be continued to the whole \( s \)-plane as a holomorphic function except possibly for simple poles at \( s = 2 \) and \( s = 0 \), and the possible residues are \( \delta^*(\psi_f) \delta^*(\bar{\chi})r(N)h(D) \) and \( -\delta(N-1)\delta^*(\bar{\chi})h(D) \) respectively. Further if \( N \) is a prime such that \( N \nmid d_0 f^2 \) and \( \psi^2 \) is a non-trivial Dirichlet character modulo \( N \), or if \( N = 1 \) (and \( \psi \) must be the trivial character), then we have the functional equation

$$N^{3s/2} \Psi(\chi, \psi_f, u, s) = (AN^{-1})N^{3(s-2)/2} \Psi(\bar{\chi}, \bar{\psi}_f, \tau_N(u), 2-s).$$

Note that if either \( N > 1 \), \( \psi_f \) is non-trivial or \( \chi \) is non-trivial, then \( \Psi(\chi, \psi_f, u, s) \) becomes an entire function of \( s \). For the case that \( N = 1 \) (and \( \psi \) must be the trivial character), Andrianov proved Eq. (3.10) in [2]. In this case, if we put

$$\Psi(\chi, u, s) = \Psi(\chi, \psi_f, u, s),$$

then we have

$$\Phi(\chi, u, s) = \Phi(\bar{\chi}, u, 2-s).$$
From the definitions of $R_F(\chi, s)$ ((2.12)), $E_0(\psi_f, u, s)$ ((3.3)), $E_0(\psi_f, u, s)$ ((3.7)), $\Psi_0(\eta_f, u, s)$ ((3.8)), $\Psi_0(\eta_f, u, s)$ ((3.9)) and from Propositions 3.2 and 3.5, we have

$$
(2\pi)^{-2} \Gamma(s) \Gamma(s-k+2) L_D(s-k+2, \chi \eta_f) R_F(\chi, s)
$$

$$
= e^s \int_{D_0} v(u)^k F_0(u) \Psi(\chi, \psi_f, u, s-k+2) \frac{du}{v^3} \left( c^* = \frac{\pi^{2-k}}{2}, \quad D = \frac{4}{1-k}\right)
$$

for any form $F \in S_k(\Gamma_0, \psi)$ and any character $\chi$ of $H(d_0 f^2)$ satisfying the condition (3.5).

§ 4. Proof of the main theorems

In this section, we assume that $N$ is a prime. Let

$$
F(Z) = \sum_{T \in P_2^+} a(T) \exp(2\pi i Q(TZ)) \in S_k(\Gamma_0, \psi)
$$

be a common eigen function for $T(m)_\psi((m, N) = 1)$:

$$
F \mid T(m)_\psi = \lambda(m) F \quad ((m, N) = 1) \quad (\text{see (1.6))}.
$$

If $F$ is not identically zero, then we can find an integer $D = d_0 f^2$ ($d_0 \in \mathbb{Z}$, $f \in \mathbb{N}$) such that

$$
(4.2) \quad \text{there exists a primitive matrix } T \in P_2^+ \text{ satisfying } D(T) = -4 \mid T \mid = D \quad \text{and the series } R_F(T, s) \neq 0 \quad (\text{see (2.11))};
$$

$$
(4.3) \quad d_0 \quad \text{is the discriminant of the field } K = \mathbb{Q}(\sqrt{d_0}), \quad \text{and for any integer } f' > 1, f' \mid f \text{ and for any matrix } T_0 \in P_2^+ \text{ such that } D(T_0) = d_0 (f/|f'|)^2, \quad \text{the series } R_F(T_0, s) \equiv 0.
$$

Let $a_1, a_2, \ldots, a_h$ ($h = h(D)$) be a complete system of representatives for proper $R_F(K)$-ideal classes such that $(N(a_i), N(f)) = 1 (i = 1, 2, \cdots, h)$ (see Proposition 3.4), $\chi$ a character of $H(D)$. Then, by the condition (4.3), $\Phi_F(s, \chi, \psi, n)$ should be a constant:

$$
(4.4) \quad \Phi_F(s, \chi, \psi, n) = a(n; \chi) \quad (\text{see (2.8) and (2.10)}).
$$

Since $R_F(T, s) = \sum_{m=1}^{\infty} a(mT)m^{-s} \neq 0 \quad (D(T) = d_0 f^2)$, we can choose an integer $v \geq 0$ such that

$$
\sum_{(m, N) = 1} a(mN^v T)m^{-s} \neq 0.
$$

From the properties of Hecke operators, for cusp forms $F \mid T(N)_\psi$, $F \mid T(N^+1)_\psi \in S_k(\Gamma_0, \psi)$, we can show equalities

$$
(4.5) \quad \left\{ (F \mid T(N^v)_\psi) \mid T(m)_\psi = \lambda(m) F \mid T(N^v)_\psi \quad ((m, N) = 1), \right.

$$

$$
\left. (F \mid T(N^v+1)_\psi) \mid T(m)_\psi = \lambda(m) F \mid T(N^v+1)_\psi \quad ((m, N) = 1). \right\}$$
Therefore, by using Proposition 2.1 and the equality (4.4), we get

\[
\begin{align*}
L_D(s-k+2, \chi \psi_f)R_{F|T(N^\nu)\phi}(\chi, s) &= Z^N_F(s, \psi) \sum_{\delta = 0}^{\infty} a(N^\nu + \delta, \chi)N^{-\delta s}, \\
L_D(s-k+2, \chi \psi_f)R_{F|T(N^\nu+1)\phi}(\chi, s) &= Z^N_F(s, \psi) \sum_{\delta = 0}^{\infty} a(N^\nu + \delta + 1, \chi)N^{-\delta s}.
\end{align*}
\]

Since

\begin{equation}
R_{F|T(N^\nu)\phi}(\chi, s) = N^{-s}R_{F|T(N^\nu+1)\phi}(\chi, s) = \sum_{(m, N) = 1} a(mN^\nu; \chi)m^{-s}
\end{equation}

holds, we have

\begin{equation}
L_D(s-k+2, \chi \psi_f) \sum_{(m, N) = 1} a(mN^\nu; \chi)m^{-s} = Z^N_F(s, \psi)a(N^\nu; \chi).
\end{equation}

By the condition (4.5), we can choose a character \( \chi \) of \( H(D) \) satisfying

\begin{equation}
\sum_{(m, N) = 1} a(mN^\nu; \chi)m^{-s} = \left( \sum_{i=1}^{h} \left( \sum_{(m, N) = 1} a(mN^\nu T(a_i))m^{-s} \right) \chi(a_i) \right) \neq 0
\end{equation}

(see (2.8)), and then

\begin{equation}
a(N^\nu; \chi) \neq 0.
\end{equation}

Via property IV) of \( \Pi(p) \) \( (p \mid f) \) and [2, Lemma 3.8.1], we get the following

**Proposition 4.1.** Suppose that \( D = d_0f^2 \) satisfies (4.2) and (4.3), and \( \chi \) is chosen so that (4.9) holds. Then the character \( \chi \) satisfies the condition (3.5).

Let \( \chi \) satisfy (4.9) (so that (3.5)). Then we proved in § 3, for any cusp form \( F \in S_k(\Gamma_0, \psi) \), the equality holds:

\[
(2\pi)^{-2s} \Gamma(s) \Gamma(s-k+2)L_D(s-k+2, \chi \psi_f)R_{F\phi}(\chi, s)
\]

\[
= c^* \int_{D_0} \psi(u)^k F_0(u) \Psi(\chi, \psi_f, u, s-k+2) \frac{du}{v^3} \left( c^* = \left( \frac{n^2-k}{2} \right) \left| \frac{D}{4} \right|^{(k-3)/2} \right)
\]

(see (3.13)). Take \( F \mid T(N^\nu)\phi, F \mid T(N^\nu+1)\phi \) for \( F \). Then

\[
c^* \int_{D_0} \psi(u)^k \{ F \mid T(N^\nu)\phi(\psi(u)) - N^{-s}F \mid T(N^\nu+1)\phi(\psi(u)) \} \Psi(\chi, \psi_f, u, s-k+2) \frac{du}{v^3}
\]

\[
= (2\pi)^{-2s} \Gamma(s) \Gamma(s-k+2)L_D(s-k+2)(R_{F|T(N^\nu)\phi}(\chi, s) - N^{-s}R_{F|T(N^\nu+1)\phi}(\chi, s)).
\]

Hence according to (4.7), we obtain

\[
c^* \int_{D_0} \psi(u)^k \{ F \mid T(N^\nu)\phi(\psi(u)) - N^{-s}F \mid T(N^\nu+1)\phi(\psi(u)) \} \Psi(\chi, \psi_f, u, s-k+2) \frac{du}{v^3}
\]

\[
= (2\pi)^{-2s} \Gamma(s) \Gamma(s-k+2)L_D(s-k+2, \chi \psi_f) \sum_{(m, N) = 1} a(mN^\nu; \chi)m^{-s}.
\]
Therefore we have, by using the equality (4.8),

\[ a(N, \chi)(2\pi)^{-2} \Gamma(s) \Gamma(s - k + 2) Z^N_N(s, \psi) \]

\[ = \epsilon^s \int_{D_0} v(u)^k \left( F \left| T(N)_{\psi} \right|_o(u) - N^{-s} \left( F \left| T(N+1)_{\psi} \right|_o(u) \right) \right) \Psi(\xi, \psi_f, u, s - k + 2) \frac{du}{v^s}. \]

Note that \( a(N, \chi) \neq 0 \) ((4.10)) and \( F \left| T(N)_{\psi} \right|, F \left| T(N+1)_{\psi} \right| \) are elements in \( S_k(\Gamma_0, \psi) \). From the equalities (3.2), the definitions (2.16) of \( Z^N_N(s, \psi) \), (3.9) of \( \Psi(\xi, \psi_f, u, s) \) and Theorem 3.6, we get

**Theorem 4.2.** The function \((2\pi)^{-2} \Gamma(s) \Gamma(s - k + 2) Z^N_N(s, \psi)\) can be continued meromorphically to the whole s-plane and has possibly simple poles at \( s = k \) and \( s = k - 2 \).

If \( F \in S_k(\Gamma_0, \psi) \) has the Fourier expansion of the form (4.1), and we put \( W = \left( -N^2 \right)^{k/2} \in \text{GSP}_2(\mathbb{R}) \), then \( \hat{F} = F \left| [W]_k \right. \) is an element in \( S_k(\Gamma_0, \psi) \). We denote its Fourier expansion by

\[ \hat{F}(Z) = \sum_{T \in \mathbb{P}^2} \hat{a}(T) \exp(2\pi i \sigma(TZ)). \]

Further, for any positive integer \( m \) with the condition \((m, N) = 1\), we have \( \hat{F} \left| \Gamma(m)_{\psi} \right. = \psi^2(m)\lambda(m)\hat{F} \) if \( \hat{F} \left| \Gamma(m)_{\psi} = \lambda(m)F \right. \). We consider the eigen values \( \{ \lambda(m); (m, N) = 1 \} \) of a cusp form in the space \( S_k(\Gamma_0, \psi) \). Put

\[ S_k(\Gamma_0, \psi, \lambda) = \{ F \in S_k(\Gamma_0, \psi); F \left| \Gamma(m)_{\psi} = \lambda(m)F \right. (m, N) = 1 \}. \]

It is easy to see that \( S_k(\Gamma_0, \psi, \lambda) \) is a subspace of \( S_k(\Gamma_0, \psi) \). Let \( \{ F_1, F_2, \cdots, F_l \} \) \((l = \text{dim}_C S_k(\Gamma_0, \psi, \lambda))\) be a basis of \( S_k(\Gamma_0, \psi, \lambda) \) and the Fourier expansion of \( F_j \) be

\[ (F_j(Z)) = \sum_{T \in \mathbb{P}^2} a_j(T) \exp(2\pi i \sigma(TZ)) \quad (1 \leq j \leq l). \]

Further, we put \( F = (F_1, F_2, \cdots, F_l) \) and \( F \left| \Gamma(m)_{\psi} = (F_1 \left| \Gamma(m)_{\psi}, F_2 \left| \Gamma(m)_{\psi}, \cdots, F_l \left| \Gamma(m)_{\psi} \right. (m \in \mathbb{N}) \right. \right. \right) \)

\[ a(m, \xi) = (a_1(m; \chi), a_2(m; \chi), \cdots, a_l(m; \chi)) \]

for a positive integer \( m \) (see (2.8)). Then by the definitions (2.13) of \( Z^N_N(s, \psi) \), (2.16) of \( Z^N_N(s, \psi) \), we get

\[ Z^N_N(s, \psi) = Z^N_N(s, \psi) = \cdots = Z^N_N(s, \psi) = Z^N_N(s, \psi). \]

Since \( \Gamma(m)_{\psi} T(N)_{\psi} = T(N)_{\psi} T(m)_{\psi} \) for \((m, N) = 1\), the space \( S_k(\Gamma_0, \psi, \lambda) \) is invariant by the action of the operator \( T(N)_{\psi} \). Hence there exists a matrix \( U_{\psi} \in M_l(\mathbb{C}) \) such that

\[ F \left| \Gamma(N)_{\psi} = FU_{\psi}. \right. \]

Further, it is easy to show that \( \{ \hat{F}_1, \hat{F}_2, \cdots, \hat{F}_l \} \) is a basis for \( S_k(\Gamma_0, \psi, \psi_2^2\lambda) \). If \( S_k(\Gamma_0, \psi, \lambda) \neq \{0\} \), then we can choose an integer \( D = d_0f^2 \) \((d_0 \text{ is the discriminant of the imaginary quadratic field } \mathbb{Q}(\sqrt{d_0}) \) and \( f \) is a positive integer), a character \( \chi \) of
$H(D)$ and an integer $v \geq 0$ such that

(4.2) there exist a primitive matrix $T \in P_2^+$ satisfying

$$D(T) = D \quad \text{and} \quad R_{F_j}(T, s) \neq 0 \quad \text{for some} \quad j_0 \quad (1 \leq j_0 \leq l);$$

(4.3) $R_{F_j}(T_0, s) = R_{F_j}(T_0, s) \equiv 0$ for all $j = 1, 2, \cdots, l$ and all primitive matrix $T_0 \in P_2^+$ with $D(T_0) = d_0(f/f')^2 (1 < f', f' \mid f)$

(if $R_{F_j}(T_0, s) \neq 0$ for some $j$, then we consider by changing roles between $S_k(T_0, \psi, \lambda)$ and $S_k(T_0, \bar{\psi}, \bar{\psi}^2 \lambda))$, and

$$a(N^v; \chi) \neq 0 \quad \text{(see (4.10))}.$$  

Compare the Fourier coefficients of $F|T(N)_\psi$ and $F U_\psi$. Then from the definition (4.12) of $U_\psi$ and the equality [7, (2.5)], we can see

$$a(N^v; \chi) = a(1; \chi) U_\psi.$$

Therefore, we get

$$a(1; \chi) \neq 0.$$

It follows from the equality (2.10) and the condition (4.3) that equalities

$$\Phi_j(s, \chi, \psi, N^\delta) = a_j(N^\delta; \chi),$$

$$\Phi_j(s, \chi, \psi, N^\delta) = \hat{a}_j(N^\delta; \chi),$$

($j = 1, 2, \cdots, l$ and $\delta = 0, 1, 2, \cdots$) hold. Put

$$F = (F_1, F_2, \cdots, F_l),$$

and

$$R_{F_j}(\chi, s) = (R_{F_j}(\chi, s), R_{F_j}(\chi, s), \cdots, R_{F_j}(\chi, s)),$$

and define a matrix $\hat{U}_\psi \in M_l(C)$ by $\hat{F} | T(N)_\psi F \hat{U}_\psi$. For the Fourier coefficients $\hat{a}_j(T)$ of $\hat{F}_j$ ($j = 1, 2, \cdots, l$), we set

$$\hat{a}_j(m; \chi) = \sum_{i=1}^h \hat{a}_j(T(a_i)) \chi(a_i)$$

(see (2.8)). Further we put $\hat{a}(m; \chi) = (\hat{a}_1(m; \chi), \hat{a}_2(m; \chi), \cdots, \hat{a}_l(m; \chi))$ for a positive integer $m$. Then we have $a(mN; \chi) = a(m; \chi) U_\psi$, $\hat{a}(mN; \chi) = \hat{a}(m; \chi) \hat{U}_\psi$ and hence

$$\left\{ \begin{array}{l}
R_{F|T(N)_\psi}(\chi, s) = R_{F}(\chi, s) U_\psi \\
R_{F|T(N)_\psi}(\chi, s) = R_{F}(\chi, s) \hat{U}_\psi
\end{array} \right.$$

(see [7, (2.5)] and (2.12)).

From Proposition 2.1 and the first equality of (4.13), we can see
\[
\begin{aligned}
&L_p(s-k+2, \chi \psi_j) R_F(\chi, s) = \left\{ \sum_{\delta=1}^{\infty} a_j(N^\delta; \chi) N^{-\delta s} \right\} Z_{\chi}^N(s, \psi), \\
&L_p(s-k+2, \chi \psi_j) R_{F_1|T(N)\psi}(\chi, s) = \left\{ \sum_{\delta=1}^{\infty} a_j(N^{\delta+1}; \chi) N^{-\delta s} \right\} Z_{\chi}^N(s, \psi)
\end{aligned}
\]
for \(j = 1, 2, \ldots, l\). Hence we get
\[
L_p(s-k+2, \chi \psi_j) \{ R_F(\chi, s) - N^{-s} R_{F_1|T(N)\psi}(\chi, s) \} = a(1; \chi) Z_{\chi}^N(s, \psi).
\]
Namely we have
\[(4.14)\quad L_p(s-k+2, \chi \psi_j) \{ R_F(\chi, s)(E_1 - N^{-2} U_\psi) \} = a(1; \chi) Z_{\chi}^N(s, \psi).
\]
Further, by Proposition 2.1 and second equality of (4.13), we can prove
\[(4.15)\quad L_p(s-k+2, \bar{\psi} \psi_j) \{ R_F(\bar{\chi}, \bar{\delta})(E_1 - N^{-s} U_\psi) \} = \bar{a}(1; \bar{\chi}) Z_{\psi_\chi}^N(\bar{\delta}, \bar{\psi})
\]
(\(\bar{\delta} = 2k - 2 - s\)). From (3.13), we have
\[(4.16)\quad (2\pi)^{-2s} \Gamma(s-k+2) L_p(s-k+2, \chi \psi_j) R_F(\chi, s) = c^* \int_{D_0} v(u)^h \mathbf{F}_0(u) \Psi(\chi, \psi_j, u, s-k+2) \frac{du}{v^3},
\]
where \(\mathbf{F}_0 = ((F_1)_0, (F_2)_0, \ldots, (F_l)_0)\), and
\[(4.17)\quad (2\pi)^{-2s} \Gamma(s-k+2) L_p(s-k+2, \bar{\psi} \psi_j) R_F(\bar{\chi}, \bar{\delta}) = c^* \int_{D_0} v(u)^h \mathbf{\bar{F}}_0(u) \Psi(\bar{\chi}, \bar{\psi}_j, u, \bar{\delta}-k+2) \frac{du}{v^3}.
\]
First we consider the case that the conductor of \(\psi\) is \(N\) and \(\psi^2\) is non-trivial. In this case, we have the following

**Theorem 4.3.** Let \(\psi\) be a primitive character modulo \(N\) with \(\psi^2\) non-trivial and put
\[
\Phi^N(s, \lambda, \psi) = (2\pi)^{-2s} \Gamma(s-k+2) Z_{\chi}^N(s, \psi).
\]
Suppose that

(C1) \(N\) does not divide \(d_0 f^2\), the discriminant of \(R_F(\mathbb{Q}(\sqrt{d_0})\), where \(D = d_0 f^2\) satisfies the condition (4.2)' and (4.3)'.

Then we have the following equation:
\[(4.18)\quad a(1; \chi) N^{3s/2} \Phi^N(s, \lambda, \psi)(E_1 - N^{-s} U_\psi)^{-1} = \psi(-1) \omega a(1; \chi) N^{3s/2} \Phi^N(s, \psi^2 \lambda, \psi)(E_1 - N^{-s} U_\psi)^{-1},
\]
where \(\bar{s} = 2k - 2 - s\) and
\[ \omega = N^{-1} \sum_{x \in \mathbb{R}/N\mathbb{R}} \psi(|x|^2) \exp \left( 2\pi i \frac{x - \tilde{x}}{Nf/d_0} \right). \]

(Note that \( a(1; \chi) \neq (0, \cdots, 0) \) (see (4.10').)

**Proof.** If we note that
\[ \nu(\tau_n(u))^k \mathbf{F}_0(\tau_n(u)) = \psi(-1)\nu(\mathbf{F})_0(u) \left( \tau_n = \left( -\sqrt{N} \sqrt{N^{-1}} \right) \right) \]
and that \( \tau_n(D_0) \) is another fundamental domain for \( \Gamma(R_f) \), then by Theorem 3.6, we get
\[ \int_{D_0} \nu(u)^k \mathbf{F}_0(u) \Psi(\chi, \psi_f, u, s - k + 2) \frac{du}{u^3} = \psi(-1)\omega \int_{D_0} \nu(u)^k \mathbf{F}_0(u) \Psi(\tilde{\chi}, \tilde{\psi}_f, u, \tilde{s} - k + 2) \frac{du}{u^3} \]
for \( F = (F_1, F_2, \cdots, F_t) \). Hence from equalities (4.14), (4.15), (4.16) and (4.17), we have the equality (4.18).

Next we consider the case that \( \psi = \phi_0 \) is the trivial Dirichlet character modulo \( N \).

We can easily prove

**Proposition 4.4.** If \( N \) is a prime which remains prime in \( \mathbb{Q}(\sqrt{d_0})/\mathbb{Q} \), then
\[ \Psi(\chi, (\phi_0)_f, u, s) = N^{-s}\{ \Psi(\chi, \tau_n(u), s) - N^{-s}\Psi(\chi, u, s) \}, \]
where \( \Psi(\chi, u, s) \) is defined by (3.11).

If \( F \in S_\chi(\Gamma_0, \phi_0, \lambda) \), then \( \tilde{F} \in S_\chi(\Gamma_0, \phi_0, \lambda) \). Hence we can find a matrix \( U_0 \) such that \( \tilde{F} = F U_0 \) with \( U_0^2 = E_t \). Finally we have

**Theorem 4.5.** Let the notation be as above and put
\[ \Phi^N(s, \lambda) = (2\pi)^{-s}N^s\Gamma(s-k+2)Z_N(s, \phi_0)(E_t - N^{-s}U_0)^{-1}(E_t - N^{-(s-k+2)}U_0)^{-1} \]
Suppose that

\[ C2) \quad N \text{ remains prime in } \mathbb{Q}(\sqrt{D})/\mathbb{Q}, \text{ where } D = d_0f^2 \text{ satisfies} \]
the condition (4.2)' and (4.3)' for some \( f \).

Then the following equation holds:
\[ (4.19) \quad a(1; \chi)\Phi^N(s, \lambda) = (-1)^k a(1; \chi)\Phi^N(s, \lambda) \quad (s = 2k - 2 - \delta). \]

**Proof.** Put
\[ I(s, \chi) = N^{s-k+2} \int_{D_0} \nu(u)^k \mathbf{F}_0(u) \Psi(\chi, (\phi_0)_f, u, s - k + 2) \frac{du}{u^3} \]
Then from (4.14), (4.16) and the definition of $\Phi^N(s, \lambda)$, we have

\begin{equation}
(4.20) \quad c I(s, \chi) = a(1; \chi) N^s \Phi^N(s, \lambda)(E_i - N^{-s} U_{\Theta_0})^{-1}
\end{equation}

where $c = (\pi/N)^{3-k} |D/4|^{(k-3)/2}/2$. Further according to Proposition 4.4, we get

\[ I(s, \chi) = \int_{D_0} v(u)^k F_0(u) \Psi(\chi, \tau_N(u), s-k+2) \frac{du}{u^3} - N^{-(s-k+2)} \int_{D_0} v(u)^k F_0(u) \Psi(\chi, u, s-k+2) \frac{du}{u^3}. \]

Since $v(\tau_N(u)^k F_0(\tau_N(u)) = v(u)^k(\tilde{F}_0(u) = v(u)^k F_0 U_0$, we can easily see

\begin{equation}
(4.21) \quad I(s, \chi) = \int_{D_0} v(u)^k F_0(u) \Psi(\chi, \tau_N(u), s-k+2) \frac{du}{u^3} (E_i - N^{-(s-k+2)} U_0).
\end{equation}

Further we get

\begin{equation}
(4.22) \quad \Psi(\chi, u, s-k+2) = \Psi(\tilde{\chi}, u, \tilde{s}-k+2)
\end{equation}

by (3.12). Therefore we obtain

\begin{equation}
(4.23) \quad I(\tilde{s}, \tilde{\chi}) = \int_{D_0} v^k F_0(u) \Psi(\chi, \tau_N(u), s-k+2) \frac{du}{u^3} (E_i - N^{-\tilde{s}-k+2}) U_0).
\end{equation}

From the equalities (4.20), (4.21), (4.23) and the fact $a(1; \tilde{\chi}) = (-1)^k a(1; \chi)$, finally we get (4.19).

### § 5. Examples

We shall exhibit some examples to make clear the meaning of our theorems.

**Example 1.** Put

\[ \eta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{12}, \quad \theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}, \]

$q = \exp(2\pi i z)$ and $f(z) = \theta(z)^{-1} \sqrt{\eta(2z)}$. Then

\[ f(z) = a_1 q + a_2 q^2 + a_3 q^3 + \cdots \]

is an elliptic cusp form of half integral weight in $S_{11}(\Gamma_0(4))$ (see [4]) and, for Hecke operators $T_{11}(m^2)$ on $S_{11}(\Gamma_0(4))$,

\[ f \mid T_{11}(m^2) = \omega_m f \quad (m = 1, 2, 3, \cdots). \]

For $T = (a/b, b^2/c) \in P^+_2$, set

\[ a(T) = \sum_{m | (a, b, c)} \phi(m) a_{(4ac - b^2)/m^2}. \]

Then, by Kojima [4, Theorem 3] and Ibuki [5],
On the Meromorphy of Dirichlet Series

\[
\psi(f)(Z) = \sum_{T \in \mathcal{P}_2} a(T) \exp(2\pi i \sigma(TZ))
\]
is the cusp form in \( S_6(\Gamma_0(2), \phi_0) \). Note that Ibukiya showed \( \dim S_6(\Gamma_0(2), \phi_0) = 1 \) in [5]. So that we have

\[
\psi(f) \big| T(m)_{\phi_0} = \lambda(m)\psi(f) \quad (m = 1, 2, \ldots), \quad \hat{\psi}(f) = \psi(f).
\]

Further we can show \( a((1/2, 1/2)) = a_3 = -8 \neq 0 \) and 2 remains prime in \( \mathbb{Q}(\sqrt{-3}) \).

Put

\[
\Phi(s) = (2\pi)^{-s} \Gamma(s) \Gamma(s-4) 2^s (1 - 2^{-s})^{-1} (1 - \lambda(2) 2^{-s})^{-1} Z_3^2(s, \phi_0).
\]

Then, by Theorem 4.5, we get

\[
(5.1) \quad \Phi(s) = \Phi(10-s).
\]

N.B. In [4], it was shown that

\[
Z_p(s, \phi_0) = (1 - \lambda(2) 2^{-s})^{-1} Z_3^2(s, \phi_0)
\]

\[
= (1 - 2^{-s}) (1 - 2^{-s})^{-1} \zeta(s-5) \zeta(s-4)
\]

\[
\times \prod_p \left( 1 - \omega_p p^{-s} + \phi_0(p) p^{2s-3} \right)^{-1}
\]

(see (2.17)).

Equation (5.1) can also be proved by using this product.

**Example 2.** Put

\[
S_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 6 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & -2 \\ 2 & 1 & 4 & 0 \\ 1 & -2 & 0 & 4 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 8 & 4 & 3 \\ 1 & 4 & 8 & 7 \\ 0 & 3 & 7 & 8 \end{pmatrix}
\]

and

\[
\tilde{G}(Z) = \sum_{x \in M_{4,1}(Z)} \exp(2\pi i \sigma(x' S_j x Z)) \quad (Z \in \mathfrak{H}_2, j = 1, 2, 3).
\]

Then, in his paper [11], Yoshida showed that

\[
F(Z) = (3 \tilde{G}_1(Z) + \tilde{G}_2(Z) - 2 \tilde{G}_3(Z))/24
\]
is a cusp form in \( S_2(\Gamma_0(11), \phi_0) \) and

\[
(5.2) \quad F \big| T(m)_{\phi_0} = \lambda(m) F
\]

for \( m \in \mathbb{N} \) with \( (m, 2 \cdot 11) = 1 \). It is easy to see that if we put
\[
U_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix},
\]

\[
S_3 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & -1 & 1 \\
1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{pmatrix} \in GL_4(\mathbb{Z}),
\]

then \(11S_j^{-1} = U_jS_jU_j \quad (j=1, 2, 3)\). Further, we can show

\[
\{ x \in M_{4, 2}(\mathbb{Z}); \ x' S_j x = T \} \ \{ y \in M_{4, 2}(\mathbb{Z}); \ y' S_j y = 11T \}
\]

\[
x \quad \longmapsto \quad y = U_jS_jx
\]

\((T \in P_2^+, j=1, 2, 3)\) is one-to-one and onto correspondence. By these facts, we get

\[
\left\{ \begin{array}{c}
\tilde{F} = F \\
F \mid T(11)_{\phi_0} = F.
\end{array} \right.
\]

Put

\[
F(Z) = \sum_{T \in P_2^+} a(T) \exp(2\pi i\sigma(TZ)).
\]

Then \(a(E) = 1 \neq 0\). Hence \(D = D(E) = -4\) and \(\chi_0\) (the trivial character of \(H(-4)\)) satisfy the conditions (4.2) and (4.3). Let

\[
Q(s) = \sum_{j=0}^{4} 2^{-js}a_j T_j,
\]

where \(a_0 = a_2 = 1, a_1 = -1, a_3 = -2, a_4 = 2^2, T_0 = T_4 = T(1), T_1 = T_3 = T(2)\) and \(T_2 = T(2)^2 - T(2^2) = T(1)\), and put

\[
(F \mid Q(s)(Z)) = \sum_{T \in P_2^+} b(T)\exp(2\pi i\sigma(TZ)).
\]

Further set

\[
Z_2^*(s, \phi_0) = \prod_{p \neq 2} Q_2^*(p^{-s})^{-1} \quad (\text{see } (1.10)).
\]

Since \((F \mid T_j) \mid T(m)_{\phi_0} = \lambda(m)F \mid T_j\) for \((m, 2) = 1\) and, \(j=1, 2, 3, 4\), we get, by equations (2.9), (5.2), (5.3) and the definition of \(b_s\),
On the Meromorphy of Dirichlet Series

\[ L_{-4}(s, \phi_0 \text{Norm}) \left\{ \sum_{j=0}^{4} 2^{-js} a_j R_{F | T_j}(X_0, s) \right\} \]
\[ = Z^*_4(s, \phi_0) \left\{ \prod_{\delta=0}^{\infty} b_4(2^\delta E) 2^{-\delta s} \right\} \prod_{p \mid 2} \left( 1 - \frac{1}{N(p)^s} \right)^{-1} \]
\[ = Z^*_4(s, \phi_0) \alpha(E). \]

Hence we have, from (3.13),
\[ (2\pi)^{-2s} \Gamma(s)^2 Z^*_4(s, \phi_0) = \sum_{j=0}^{4} 2^{-js} a_j \int_{D_0} v^2(F \mid T_j T_0(u)) \Psi_0(\chi_0, \phi_0 \text{Norm}, u, s) \frac{du}{v^3}. \]

Thus we have proved that \((2\pi)^{-2s} \Gamma(s)^2 Z^*_4(s, \phi_0)\) can be continued holomorphically to the whole \(s\)-plane except possibly for simple pole at \(s = 2\). The functional equation of \(Z^*_4(s, \phi_0)\) is shown in the following. Put
\[ (F \mid \mathbb{Q}(s)Q(s))(Z) = \sum_{T \in P_4^+} c(T) \exp(2\pi i \sigma(TZ)). \]

Then we get, by using (2.9),
\[ L_{-4}(s, \phi_0 \text{Norm}) \left\{ \sum_{j,k=0}^{4} 2^{-js-ks} a_j a_k R_{F | T_j T_k}(\chi_0, s) \right\} \]
\[ = Z^*_4(s, \phi_0) \left\{ \prod_{\delta=0}^{\infty} c_4(2^\delta E) 2^{-\delta s} \right\} \prod_{p \mid 2} \left( 1 - \frac{1}{N(p)^s} \right)^{-1} \]
\[ = Z^*_4(s, \phi_0) b_4(E). \]

Since the equality (4.16) holds and 11 remains prime in \(\mathbb{Q}(\sqrt{-4})\), we have
\[ (2\pi)^{-2s} \Gamma(s)^2 b_4(E) Z^*_4(s, \phi_0) = \sum_{j,k=0}^{4} 2^{-js-ks} a_j a_k \]
\[ \times \int_{D_0} v^2(F \mid T_j T_k)_0(u) \Psi_1(\chi_0, \tau_{11}(u), s) \frac{du}{v^3} (1 - 11^{-s}). \]

Further, we can prove
\[ b_4(E) = 1 - 2^{-s} + 2^{1-s} - 2^{1-3s} + 2^{2-4s} \]
\[ = (1 + 2^{1-s} + 2^{1-2s})(1 - 2^{-s})(1 - 2^{1-s}). \]

Therefore, if we put
\[
\Phi(s) = (2\pi)^{-s} \Gamma(s)^2 11^s (1 - 11^{-s})^{-1} (1 + 2^{-1-s} + 2^{1-2s})^{-1} \\
(1 - 2^{-s})^{-1} (1 - 2^{1-s})^{-1} Z_2^*(s, \phi_0),
\]
then we get the functional equation

(5.4) \quad \Phi(s) = \Phi(2-s)

by the same way as Theorem 4.5.

N.B. In [10], the \( p \)-factor \((p \neq 11)\) of \( Z_2^*(s, \phi_0) \) is calculated explicitly. By using this result, it is again possible to get (5.4).

References


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