Abelian Groups Related to Mitchell’s Problem

by

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All groups considered in this paper are abelian. For the general notation, we refer to Fuchs [3].

The general purification problem is to ascertain precisely which subgroups of a subgroup $A$ of a group $G$ are the intersections of $A$ with pure subgroups of $G$.

Mitchell showed in [5] that if $G$ is a $p$-group and $K$ is a neat subgroup of $G^1$, then there exists a pure subgroup $P$ of $G$ such that $P \cap G^1 = K$.

He then raised the problem whether the converse holds, i.e. “if $P$ is pure in $G$, then is $P \cap G^1$ neat in $G^1$?"

F. Richman and C. P. Walker solved the purification problem when $G$ is a $p$-group and $A = G^1$ in [6]. By [6], it is immediate that the answer to Mitchell’s problem is negative in general.

But it is an interesting problem to characterize the group $G$ in which the intersection of each pure subgroup and $G^1$ is neat.

In this paper, we shall characterize the groups mentioned above for the arbitrary case as well as the case of $p$-groups. Namely, we shall prove the next theorem which determines such groups.

Before we formulate the theorem, we introduce a condition and a definition on a group $G$ for ready reference.

Condition (M). If $P$ is a pure subgroup of $G$, then $P \cap G^1$ is a neat subgroup of $G^1$.

Definition. Let $G$ be a $p$-group. If $G$ is the direct sum of a bounded subgroup and a divisible subgroup, then $G$ is called a B.D.-group.

Theorem. A group $G$ satisfies the condition (M) if and only if either of the following two holds:

(i) $G^1$ is torsion and for each prime $p$, either $G_p$ is a B.D.-group or $(G_p)^1$ is elementary.

(ii) $G^1$ is not torsion and for each prime $p$, $G_p$ is a B.D.-group.

§ 1. Preliminaries

We first quote some results which will be frequently used afterwards.
(1.1) (J. M. Irwin and E. A. Ealker [4]). If \( G \) is a group and \( S \) is a pure subgroup of \( G \), then we have \( p^{n}S = S \cap p^{n}G \) for each prime \( p \).

(1.2) (F. Richman and C. P. Walker [5]). Let \( G \) be a group, \( K \) a subgroup of \( G^{1} \) and \( \alpha \) the final rank of a high subgroup of \( G \). Then there exists a pure subgroup \( P \) of \( G \) such that \( P \cap G^{1} = K \) if and only if \( K \) is \( \alpha \)-quasi-neat in \( G^{1} \).

(If \( \alpha \) is a cardinal number, we say that \( K \) is \( \alpha \)-quasi-neat in \( G \) if \( |(pG \cap K)/pK| \leq \alpha \). We add that the final rank of a high subgroup of \( G \) is uniquely determined by \( G \).)

(1.3) (K. Benabdallah and J. M. Irwin [2]). Let \( G = B \oplus D \) be a \( p \)-group, where \( B \) is bounded and \( D \) is divisible. Then every pure subgroup \( K \) of \( G \) is the direct sum of a bounded subgroup and the divisible subgroup \( K \cap D \).

(1.4) (J. M. Irwin and E. A. Walker [4]). Let \( G \) be a group and \( K \) be a subgroup of \( G^{1} \). Then all \( K \)-high subgroups of \( G \) are pure in \( G \).

(1.5) (J. Bečvár [1]). Let \( G \) be a group and \( G_{p} \) be a B.D.-group for each prime \( p \). Then each pure subgroup of \( G \) is an isotype subgroup of \( G \).

§ 2. \( p \)-Groups

In this section, we shall determine all \( p \)-group \( G \)'s with the condition (M).

**Proposition 2.1.** A \( p \)-group \( G \) satisfies the condition (M) if and only if either \( G \) is a B.D.-group or \( G^{1} \) is elementary.

**Proof.** Suppose that either \( G \) is a B.D.-group or \( G^{1} \) is elementary. When \( G^{1} \) is elementary, each subgroup of \( G^{1} \) is neat in \( G^{1} \). So we may assume that \( G^{1} \) is a B.D.-group. Let \( S \) be a pure subgroup of \( G \). Then, by (1.3), \( S \) is a B.D.-group and the divisible part of \( S \) is equal to \( S \cap G^{1} \). Thus \( S \cap G^{1} \) is neat in \( G^{1} \).

Conversely, suppose that \( G \) satisfies the condition (M) and \( pG^{1}[p] \neq 0 \). Let \( H \) be a high subgroup of \( G \). We shall prove that \( H \) is bounded.

Put \( r(H) = \alpha \). Suppose that \( \alpha \) is finite. Since \( H \) is reduced, we have \( \text{Fin} r(H) = 0 \) and hence \( H \) is bounded. Thus we may assume that \( \alpha \) is infinite. In this case, \( \langle a \rangle \) is \( \alpha \)-quasi-neat for a nonzero element \( a \) of \( pG^{1}[p] \). By (1.2), there exists a pure subgroup \( P \) of \( G \) such that \( P \cap G^{1} = \langle a \rangle \). Then \( p\langle a \rangle = 0 \). On the other hand, we have \( p(P \cap G^{1}) = (P \cap G^{1}) \cap pG^{1} = P \cap pG^{1} = \langle a \rangle \neq 0 \). This is a contradiction. Hence \( H \) is bounded.

Furthermore, since \( H \) is pure in \( G \) by (1.4), \( H \) is a direct summand of \( G \). Then it follows that \( G = H \oplus G_{0} \) for some subgroup \( G_{0} \) of \( G \). As \( G^{1} = (G_{0})^{1} \), we have \( G_{0} \subseteq G^{1} \). On the other hand, since \( H \oplus G^{1} \) is essential in \( G \), we have \( G_{0}[p] \subseteq G[p] \subseteq H \oplus G^{1} \). Let \( x \in G_{0}[p] \) and \( x = h + g \), \( h \in H \), \( g \in G^{1} \). We have \( x - g = h \in H \cap G_{0} = 0 \) and hence \( x = g \in G^{1} \). As \( G_{0}[p] \subseteq G^{1} = (G_{0})^{1} \), \( G_{0} \) is divisible. Hence \( G \) is a B.D.-group. This proposition has been thus proved.
§ 3. Arbitrary groups

First we show a few lemmas which will be used in the proof of our Theorem.

LEMMA 3.1. Let $G$ be a group. Then we have $(G_i)^1 = (G^1)$, and $(G_p)^1 = (G^1)_p$ for each prime $p$.

Proof. By (1.1), we have

$$(G_p)^1 = \bigcap_q q^\infty G_p = \bigcap_q (G_p \cap q^\infty G) = G_p \cap \left( \bigcap_q q^\infty G \right) = G_p \cap G^1 = (G^1)_p$$

where $q$ ranges over all primes. Similarly, we have

$$(G_i)^1 = \bigcap_q q^\infty G_i = \bigcap_q (G_i \cap q^\infty G) = G_i \cap \left( \bigcap_q q^\infty G \right) = G_i \cap G^1 = (G^1)_i.$$

LEMMA 3.2. Let $G$ be a torsion group and put $G = \bigoplus q G_q$. Then we have $G^1 = \bigoplus q^\infty G_q$.

Proof. Since

$$p^\infty G = \left( \bigoplus q^\infty G_q \right) \oplus p^\infty G_p,$$

we have

$$G^1 = \bigcap_p p^\infty G = \bigoplus p^\infty G_p.$$

Writing $q$ in place of $p$, we have the desired equality.

LEMMA 3.3. A torsion group $G$ satisfies the condition (M) if and only if, for each prime $p$, $G_p$ satisfies the condition (M).

Proof. Suppose that $G_p$ satisfies the condition (M) for each prime $p$. Let $S$ be a pure subgroup of $G$ and put $S = \bigoplus q S_q$. By (3.2), we have

$$(S \cap G^1) \cap pG^1 = S \cap \left( \left( \bigoplus q^\infty G_q \right) \oplus p^\infty G_p \right).$$

Let $g$ be an element of $S \cap pG^1$. Then it follows that $g = s_0 + s_1 + \cdots + s_n$ where $s_0 \in S_p \cap p^\infty G_p$ and $s_i \in S_{q_i} \cap q_i^\infty G_{q_i}$ for each $i = 1, 2, \cdots, n$.

Now we have

$$p(S_{q_i} \cap q_i^\infty G_{q_i}) = S_{q_i} \cap q_i^\infty G_{q_i},$$

and by hypothesis, $S_p \cap p^\infty G_p = (S_p \cap p^\infty G_p) \cap p^\infty G_p = p(S_p \cap p^\infty G_p)$. Hence it follows that, for each $i = 1, 2, \cdots, n$, $s_i = px_i$ where $x_i \in S_{q_i} \cap q_i^\infty G_{q_i}$ and $s_0 = px_0$ for some $x_0 \in S_p \cap p^\infty G_p$. Thus $g = p(x_0 + x_1 + \cdots + x_n) \in p(S \cap G^1)$.
Lemma 3.4. Let $G$ be a group. If $G$ satisfies the condition (M), then $G$, satisfies the condition (M).

Proof. Let $S$ be a pure subgroup of $G$, and we have $S \cap G^1 = S \cap pG^1$ for each prime $p$. Let $s$ be an element of $S \cap p(G_i)$. Then $s \in S \cap pG^1 = p(S \cap G^1)$. So it follows that $s = px$ for some $x \in S \cap G^1$. Using $x \in G_i$ and (3.1), we have $x \in (G_i)^1$ and so $x \in S \cap (G_i)^1 = p(S \cap (G_i)^1)$.

Lemma 3.5. Let $G$ be a group satisfying the condition (M) and $p$ be a prime. If $G_p$ is unbounded and $(G_p)^1$ is elementary, then $G^1$ is torsion.

Proof. First we note that $G_p \cap pG^1 = 0$. In fact, by hypothesis, we have $G_p \cap pG^1 = (G_p \cap G^1) \cap pG^1 = p(G_p \cap G^1) = p(G_p)^1 = 0$. Suppose that $G^1$ is not torsion. Then there exists an element $g$ of $pG^1$ with $o(g) = \infty$ and elements $g_1, g_2, \cdots$ of $G$ such that $p^{-1}g_1 = g$ for every $i = 1, 2, \cdots$. Let $B$ be a high subgroup of $G_p$. Then $B$ is unbounded as $(G_p)^1$ is elementary. So there exists a linearly independent set $\{b_1, b_2, \cdots\}$ in $B$ such that $o(b_i) = p^i$ for each $i = 1, 2, \cdots$.

Put $X = \langle (G_p)^1, pg, g_1+b_1, g_2+b_2, \cdots \rangle$. We show that $g \notin X$. Suppose that $g \in X$, i.e.

$$g = x + z_0pg + z_1(g_1+b_1) + \cdots + z_k(g_k+b_k),$$

where $z_0, z_1, \cdots, z_k$ are integers and $x \in (G_p)^1$. Then

$$(*) \quad -(x + z_1b_1 + \cdots + z_kb_k) = z_0pg - g + z_1g_1 + \cdots + z_kg_k.$$  

From $(*)$ it follows that, in case of $k \geq 2$,

$$-p^{k-1}z_kb_k = p^{k-1}(z_0pg - g + z_1g_1 + \cdots + z_kg_k) \in G_p \cap pG^1 = 0,$$

and $p \mid z_k$. From $(*)$ it follows that, in case of $k \geq 3$,

$$-p^{k-2}z_{k-1}b_{k-1} - p^{k-2}z_kb_k = p^{k-2}(z_0pg - g + z_1g_1 + \cdots + z_kg_k) \in G_p \cap pG^1 = 0,$$

and hence $p \mid z_{k-1}$ and $p^2 \mid z_k$. Finally we have $p^{k-1} \mid z_{k-1}p^{k-2} \mid z_{k-1}$, $\cdots$, $p \mid z_2$. Now from $(*)$ it follows that

$$x + z_1b_1 + \cdots + z_kb_k \in G_p \cap pG^1 = 0,$$

and hence $x = 0$ and $p^i \mid z_i$ for each $i = 1, \cdots, k$. Write $z_i = p^{i-1}z_i'$ for each $i = 2, \cdots, k$; from $(*)$ it follows that $(z_0p - 1 + z_1 + \cdots + z_k) = 0$ — a contradiction, since $p \mid z_1$, $p \mid z_2$, $\cdots$, $p \mid z_k$.

Let $H$ be a subgroup of $G$ maximal with respect to the properties $X \subseteq H$, $g \notin H$. Then $H$ is pure in $G$ in analogy with the proof of Theorem 1 in [1].

If $pg \in p^{a+1}H$, then it follows that $pg = ph$ for some $h \in p^a H$. Since $p(g-h) = 0$, we have $g-h \in (G_p)^1 \subseteq H$ and so $g \in H$, a contradiction. Hence $pg \notin p^{a+1}H$ and thus $pg \notin p(H \cap G^1)$. On the other hand, since $pg \in H \cap pG^1$, we have $p(H \cap G^1) \subseteq H \cap pG^1$.

Proof of Theorem. Suppose that $G$ satisfies the condition (M). By (3.3), (3.4),
and (2.1), we have, for each prime $p$, either $G_p$ is a B.D.-group or $(G_p)^1$ is elementary.

If $G^1$ is not torsion, then, by (3.5), $G_p$ is a B.D.-group for each prime $p$.

Conversely, let $S$ be a pure subgroup of $G$. Then we have

$$S \cap G^1 = S \cap \left( \bigcap_p p^o G \right) = \bigcap_p (S \cap p^o G) = \bigcap_p p^o S = S^1$$

by (1.1).

First suppose that (i). As $G^1$ is torsion, $S^1$ is torsion. By (3.1), it follows that

$$p(S \cap G^1) = pS^1 = p(S_i)^1 = p(S_i \cap (G_i)^1) = S_i \cap p(G_i)^1 = S \cap p(G^1_i) = S \cap pG^1.$$

Next suppose that (ii). For each prime $p$, since $G_p$ is a B.D.-group, $S_p$ is a B.D.-group by (1.3). Then it follows that $S = B_p \oplus D_p \oplus K$, where $B_p$ is a bounded part of $S_p$, $D_p$ is the divisible part of $S_p$, and $K \leq S$.

Now put $H = D_p \oplus K$. Then we have $p^o S = p^o H$ and so $S^1 = H^1$. First we will prove that $p^o K$ is $p$-divisible. Let $x \in p^o K$, then there exist elements $k_n$ such that $x = p^n k_n$ for $n = 1, 2, \cdots$. Since $x = p^n k_n = p^{n+1} k_{n+1}$, it follows that $p^n (k_n - p k_{n+1}) = 0$ and hence $k_n - p k_{n+1} \in K_p = 0$. Hence $x = pk_1$ and $k_1 = pk_2 = p^2 k_3 = \cdots = p^n k_{n+1} = \cdots$, and so $p^o K$ is $p$-divisible. Thus $p^o H$ is $p$-divisible.

To finish the proof, it is sufficient to prove that $pH^1 = H^1 \cap pG^1$. Suppose that $h = pg \in H^1 \cap pG^1$, $g \in G^1$. Then we have $h \in H^1 \subseteq p^o H$. Since $p^o H$ is $p$-divisible, it follows that $h = ph'$ for some $h' \in p^o H$. As $pg = ph'$, we have $g - h' \in p^o G[p] = (G^1)_p[p] \subseteq G^1$. Hence $h' \in G^1 \cap H = H^1$. Thus the proof of Theorem is completed.

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References


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