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On the Transversal Elements of an Open Ideal

by

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Let I be an open ideal of a semi-local Cohen–Macaulay ring A of Krull dimension 2 and assume that A/P is infinite for each maximal ideal P of A. Then there exist elements x, y of I such that x is a superficial element of I and $(xA+yA)I^n=I^{n+1}$ for some positive integer n. From the consideration of Poincaré series, we know that the multiplicity of I is equal to the length of $A/(xA+yA)$. Furthermore, we show that there exist transversal elements x, y of I such that $(xA+yA)I=I^2$ iff I has a stable superficial element x and I/xA is a stable ideal of one dimensional semi-local Cohen–Macaulay ring A/xA. Then the Poincaré series associated with I has no polynomial part.

1. Introduction

Let I be an open ideal of a semi-local Cohen–Macaulay ring A of Krull dimension 1. Then we say that I is stable if one of the following equivalent conditions is satisfied ([2]);

(i) The length $\lambda(A/I^n)$ is a polynomial in n for all $n>0$.

(ii) There is an element x of I such that $xI=I^2$.

(iii) The Pincaré series associated with I is of the form $a(1-t)^{-1}+b$ for some integers a, b.

In this paper we investigate properties of an open ideal I of a semi-local Chohen–Macaulay ring A of Krull dimension 2. We say that elements x, y of I are transversal elements of I if $(xA+yA)I^n=I^{n+1}$ for some positive integer n. Let x, y be an A-regular sequence and I the integral closure of $xA+yA$, then x, y are transversal elements of I ([2], 2(c)). If x is a superficial element of I and if y is an element of I such that its image in A/xA is I/xA-transversal, then $(xA+yA)I^n=I^{n+1}$ for some positive integer n (Lemma 6). If A/P is infinite for each maximal ideal P of A, then each integrally closed open ideal I is the integral closure of $xA+yA$, where x is any superficial element of I and y is any element of I such that its image in A/xA is I/xA-transversal. We show in Theorem 9 that $(xA+yA)I=I^2$ for some x, y in I iff x is a stable superficial element of I and $yI\equiv I^2$ mod. xA. When I has a stable superficial element x, we show in Theorem 12 that $(xA+yA)I=I^2$ for some y in I iff the Poincaré series associated with I has no polynomial part.

2. The Poincaré series associated with an ideal of a Noetherian ring

Let $A=\bigoplus_{n=0}^{\infty}A_n$ be a Noetherian graded ring, $M=\bigoplus_{n=0}^{\infty}M_n$ a finitely generated graded A-module and λ an additive function with values in $\boldsymbol{Z}$ on the class of all finitely generated A_0-modules. Then the Poincaré series of M with respect to λ is the formal power series $P(M, t)=\sum_{n=0}^{\infty}\lambda(M_n)t^n$. We denote the formal power series $\sum_{n=0}^{\infty}t^n$ by $1/(1-t)$ or $(1-t)^{-1}$.

LEMMA 1. *Let $P(t)=\sum_{n=0}^{\infty}\lambda_n t^n$ be a formal power series in $\boldsymbol{Z}[[t]]$, d an integer $\geqq 1$ and r an integer $\geqq -1$. Then λ_n is a polynomial in n of degree $d-1$ for all $n\geqq r+1$ iff there are integers $a_1, \cdots, a_d$ $(a_d\neq 0)$ and $f(t)\in\boldsymbol{Z}[t]$ with* $\deg f\leqq r$ *such that $P(M, t)=a_d(1-t)^{-d}+a_{d-1}(1-t)^{-d+1}+\cdots+a_1(1-t)^{-1}+f(t)$ (note that* $\deg 0=-\infty$).

Proof. Since $1/(1-t)^d=\sum_{n=0}^{\infty}\binom{n+d-1}{d-1}t^n$, this follows from [4] (20.8).

Remark that, if $r\geqq 0$, then λ_n is a polynomial in n of degree $d-1$ for all $n>r$ iff the Hilbert characteristic function $\chi(n)=\lambda_0+\cdots+\lambda_{n-1}$ is a polynomial in n of degree d for all $n>r$.

THEOREM 2 (Hilbert-Serre). *Let A be a Noetherian ring, I an ideal of A and $G_I(A)=\bigoplus_{n=0}^{\infty}I^n/I^{n+1}$ $(I^0=A)$ the associated graded ring. Then the Poincaré series $P(G_I(A), t)=\sum_{n=0}^{\infty}\lambda(I^n/I^{n+1})t^n$ of $G_I(A)$ is of the form $a_d(1-t)^{-d}+\cdots+a_1(1-t)^{-1}+f(t)$, where $a_1, \cdots, a_d$ are integers and $f(t)\in\boldsymbol{Z}[t]$.*

Proof. Since A is Noetherian, I is finitely generated. Let $x_1, \cdots, x_s$ be generators of I and $\bar{x}_i$ be the image of x_i in I/I^2. Then $G_I(A)=(A/I)[\bar{x}_1, \cdots, \bar{x}_s]$. By [1] Theorem 11.1, $P(G_I(A), t)$ is of the form $g(t)/(1-t)^s$, where $g(t)\in\boldsymbol{Z}[t]$. Expressing $g(t)$ as a polynomial of $1-t$ and cancelling powers of $1-t$, we have the result.

Remark. If A is a Noetherian semi-local ring of Krull dimension $\geqq 1$, I an open ideal of A and λ the length function, then $d=\dim A$ and a_d is a positive integer called the multiplicity of I and denoted by $\mu(I)$. We define the degree of the Poincaré series $P(t)$ as the degree of its polynomial part $f(t)$ and denote it by $r(P)$. If $P(t)=P(G_I(A), t)$, we denote $r(P)$ by $r(I)$.

THEOREM 3. *Let I be an open ideal of a Noetherian semi-local ring A and $I=I_1\cap\cdots\cap I_s$ the primary decomposition of I such that $\sqrt{I_i}\neq\sqrt{I_j}$ $(i\neq j)$. Then $P(G_I(A), t)=\sum_{i=1}^{s}P(G_{I_i}(A), t)$.*

Proof. Since I is an open ideal, each $\sqrt{I_i}$ is a maximal ideal. Thus $\sqrt{I_i}$ and $\sqrt{I_j}$ are coprime for each i, j $(i\neq j)$. Therefore I_i^n and I_j^n are coprime for each $n>0$. Thus $A/I^n\cong A/I_1^n\times\cdots\times A/I_s^n$. So we have $P(G_I(A), t)=\sum_{i=1}^{s}P(G_{I_i}(A), t)$.

PROPOSITION 4. *Let I be an open ideal of a Noetherian semi-local ring A. If $r(I)=r\geqq 0$ (resp. $r(I)=-\infty$), then $r(I^m)=0$ for each $m\geqq r+1$ (resp. $r(I^m)=-\infty$ for each $m\geqq 1$).*

Proof. By Lemma 1, there is a polynomial $g(x)\in \boldsymbol{Q}[x]$ such that $g(n)=\lambda(I^n/I^{n+1})$ for each $n\geqq r+1$ (resp. $n\geqq 0$). So we have

$$\lambda((I^m)^n/(I^m)^{n+1})=\lambda((I^m)^n/I^{mn+1})+\cdots+\lambda(I^{mn+m-1}/I^{mn+m})$$
$$=g(mn)+g(mn+1)+\cdots+g(mn+m-1)$$

for each $n\geqq 1$ and each $m\geqq r+1$ (resp. $n\geqq 0$ and $m\geqq 1$). Now the proposition follows from Lemma 1.

Examples. 1) Let A be an Artinian local ring with the maximal ideal $\mathfrak{m}\neq 0$ and n the maximal integer such that $\mathfrak{m}^n\neq 0$. Then the Poincaré series $P(G_{\mathfrak{m}}(A), t)$ is a polynomial of degree n.

2) Let A be a regular local ring of dimension d and $\mathfrak{m}$ its maximal ideal. Then $P(G_{\mathfrak{m}}(A), t)=1/(1-t)^d$ ([1], Theorem 11.22).

3) Let A be a ring, $a_1, \cdots, a_d\in \mathrm{rad}\,(A)$ and $I=(a_1, \cdots, a_d)$. If $a_1, \cdots, a_d$ is an A-regular sequence, then $P(G_I(A), t)=\lambda(A/I)/(1-t)^d$. This follows from [3] Theorem 27.

4) Let A be a semi-local Cohen–Macaulay ring of Krull dimension 1 and I an open ideal. Then I is stable iff $P(G_I(A), t)$ is of the form $a(1-t)^{-1}+b$ for some a, b in $\boldsymbol{Z}$ ([2], Theorem 1.9). I is a principal ideal iff $P(G_I(A), t)$ is of the form $a(1-t)^{-1}$ ([2], the proof of Theorem 1.9).

3. The Poincaré series associated with an open ideal

Throughout this section we assume that I is a proper open ideal of a semi-local Cohen–Macaulay ring A of Krull dimension 2. Moreover, to guarantee the existence of superficial elements, we assume that A/P is infinite for every maximal ideal P of A. Then, if a regular element x of A is in I, I/xA is an open ideal of the semi-local Cohen–Macaulay ring A/xA of Krull dimension 1. Recall that an element x of an ideal I is called a superficial element of I if there is an integer $r\geqq 0$ such that $(I^n: x)\cap I^r=I^{n-1}$ for each $n>r$.

LEMMA 5 ([2], Lemma 1.8. (ii)). *Let B be a semi-local Cohen–Macaulay ring of Krull dimension 1, J a stable ideal of B and y a J-transversal element (i.e. $yJ^r=J^{r+1}$ for some integer $r>0$). Then $yJ=J^2$.*

Remark that there exists an (I/xA)-transversal element of A/xA for each regular element x in I.

LEMMA 6. (i) *If $(xA+yA)I^n=I^{n+1}$ for some integer $n>0$, then x, y is an A-regular sequence.*

(ii) *If x is a superficial element of I, then there is an element y of I such that $(xA+yA)I^n=I^{n+1}$ for some $n>0$.*

Proof. (i) We have $2=ht(I)=ht(I^{n+1})\leqq ht(xA+yA)\leqq ht(I)=2$. So $ht(xA+yA)=2$. Thus x, y is an A-regular sequence.

(ii) Since altitude $I/xA=1$ by [4] (22.8) and A is a Cohen–Macaulay ring, I/xA is an open ideal of the semi-local Cohen–Macaulay ring A/xA of Krull dimension 1. By the above remark, there exists an element y of I such that $yI^m \equiv I^{m+1}$ mod. xA for some $m>0$. Hence $xA+yI^m \supseteq I^{m+1}$. Let r be a positive integer such that $(I^n\colon x)\cap I^r = I^{n-1}$ for each $n>r$. Muliplying I^r, we have $xI^r + yI^{m+r} \supseteq I^{m+r+1}$. Let z be any element of I^{m+r+1}. Then $z=ax+by$ for some $a\in I^r$ and $b\in I^{m+r}$. So $ax=z-by$ is in I^{m+r+1}. Hence a is in I^{m+r}. Thus $(xA+yA)I^{m+r}=I^{m+r+1}$.

PROPOSITION 7. *Let x be a superficial element of I. Then there is an integer $s>0$ such that $I^n\colon x=I^{n-1}$ for each $n>s$.*

Proof. Let $r>0$ be an integer such that $(I^n\colon x)\cap I^r=I^{n-1}$ for each $n>r$ and y be an element of I such that $(xA+yA)I^m=I^{m+1}$. We will show that $I^n\colon x=I^{n-1}$ for each $n>m+r$. For any element a of $I^n\colon x$, $ax\in I^n=(xA+yA)^r I^{n-r}$. So we have

$$ax=a_0x^r+a_1x^{r-1}y+\cdots+a_{r-1}xy^{r-1}+a_ry^r \quad \text{for some} \quad a_i\in I^{n-r} \qquad (0\leqq i\leqq r)\,.$$

Thus $a_ry^r\equiv 0$ mod. xA. Since y is regular mod. xA, $a_r\equiv 0$ mod. xA, i.e. $a_r=bx$ for some $b\in A$. Dividing by the regular element x, we have

$$a=a_0x^{r-1}+a_1x^{r-2}y+\cdots+a_{r-1}y^{r-1}+by^r\in I^r\,.$$

Hence $a\in I^{n-1}$.

DEFINITION. If x is a superficial element of I, we denote by $r(x)$ the infimum of the integers $s>0$ such that $I^n\colon x=I^{n-1}$ for each $n>s$. If $r(x)=1$, we say that x is a stable superficial element of I.

LEMMA 8. *For all sufficiently large integer $s>0$, I^s has a stable superficial element.*

Proof. If $I^n\colon x=I^{n-1}$ for each $n>s$, then x^s is a stable superficial element of I^s.

THEOREM 9. *Let x, y be elements of I. Then $(xA+yA)I=I^2$ iff x is a stable superficial element of I and $yI\equiv I^2$ mod. xA.*

Proof. Suppose $(xA+yA)I=I^2$. Then it is clear that $yI\equiv I^2$ mod. xA. Let z be any element of $I^n\colon x$ $(n>1)$. Since $(xA+yA)I^{n-1}=I^n$, $xz=ax+by$ for some a, b in I^{n-1}. By Lemma 6, y is a regular element mod. xA. Hence $b\equiv 0$ mod. xA, i.e. $b=cx$ for some $c\in A$. Substituting $b=cx$ and dividing by the regular element x, we have $z=a+cy$. By induction on n, we may assume that c is in I^{n-2}. So z is in I^{n-1}. Thus x is a stable superficial element of I. Conversely, assume that $yI\equiv I^2$ mod. xA and x is a stable superficial element of I. Then $xA+yI\supseteq I^2$, which implies that any element z of I^2 is of the form $ax+by$ with $a\in A$ and $b\in I$. So $ax=z-by\in I^2$. Thus a is in I. Hence $xI+yI=I^2$.

THEOREM 10. *If there exist elements x, y of I such that $(xA+yA)I=I^2$, then the Poincaré series associated with $G_I(A)$ is of the form $a_2(1-t)^{-2}+a_1(1-t)^{-1}$ for some integers a_1, a_2, i.e. $r(I)=-\infty$.*

Proof. By Theorem 2 and the remark after it, $P(G_I(A), t)$ is of the form $a_2(1-t)^{-2}+a_1(1-t)^{-1}+b_0+b_1(1-t)+\cdots+b_r(1-t)^r$. Let $N_n=xA\cap I^n$ and $N=\bigoplus_{n=0}^{\infty}N_n/N_{n+1}$. Then from the exact sequence;

$$0 \longrightarrow xA/N_n \longrightarrow A/I^n \longrightarrow \bar{A}/\bar{I}^n \longrightarrow 0\,,$$

it follows that

$$(*) \qquad P(G_I(A), t)=P(N, t)+P(G_{\bar{I}}(\bar{A}), t)$$

where $\bar{I}=I/xA$ and $\bar{A}=A/xA$. Since

$$\lambda(xA/N_n)=\lambda(xA+I^n/I^n)=\lambda(A/(I^n\colon x))=\lambda(A/I^{n-1})\,,$$

we have

$$\begin{aligned}P(N, t)&=\textstyle\sum_{n=1}^{\infty}\lambda(I^{n-1}/I^n)t^n=tP(G_I(A), t)=P(G_I(A), t)-(1-t)P(G_I(A), t)\\&=a_2(1-t)^{-2}+(a_1-a_2)(1-t)^{-1}\\&\qquad+b_0-a_1+(b_1-b_0)(1-t)+\cdots+(b_r-b_{r-1})(1-t)^r-b_r(1-t)^{r+1}\,.\end{aligned}$$

Since $\bar{I}$ is stable in $\bar{A}$, $P(G_{\bar{I}}(\bar{A}), t)=a(1-t)^{-1}+b$ for some integers a, b. It follows from $(*)$ that $b_r=b_{r-1}=\cdots=b_0=0$, $a=a_2$ and $b=a_1$.

THEOREM 11. *Let x, y be I-transversal elements such that x is a superficial element of I. Then $\mu(I)=\lambda(A/(xA+yA))$.*

Proof. Just as the proof of Theorem 10, we know that $P(N, t)$ is of the form $a_2(1-t)^{-2}+(a_1-a_2)(1-t)^{-1}+f(t)$, where $f(t)$ is a polynomial with $\deg f(t)\leqq r(x)$. From $(*)$, we know that $P(G_{\bar{I}}(\bar{A}), t)$ is of the form $a_2(1-t)^{-1}+g(t)$, for some polynomial $g(t)$. So $\mu(I)=\mu(\bar{I})$. By [2] the proof of Theorem 1.9, $\mu(\bar{I})=\lambda(\bar{A}/\bar{y}\bar{A})$. Thus $\mu(I)=\lambda(\bar{A}/\bar{y}\bar{A})=\lambda(A/(xA+yA))$.

THEOREM 12. *Let x be an element of I, $\bar{A}=A/xA$ and $\bar{I}=I/xA$. Then following conditions are equivalent.*

(i) *$(xA+yA)I=I^2$ for some y in I.*
(ii) *$r(I)=-\infty$ and x is a stable superficial element of I.*
(iii) *$r(I)=-\infty$ and $P(G_{\bar{I}}(\bar{A}), t)=(1-t)P(G_I(A), t)$.*

Proof. (i)⇒(ii). This follows from Theorem 9 and Theorem 10. (ii)⇒(iii). Since x is a stable superficial element, $P(N, t)=tP(G_I(A), t)$. So by $(*)$, we have $P(G_{\bar{I}}(\bar{A}), t)=(1-t)P(G_I(A), t)$. (iii)⇒(i). Let $P(G_I(A), t)$ be of the form $a_2(1-t)^{-2}+a_1(1-t)^{-1}$. Then $P(G_{\bar{I}}(\bar{A}), t)$ is of the form $a_2(1-t)^{-1}+a_1$, which implies that $\bar{I}$ is stable. Moreover, $P(N, t)=tP(G_I(A), t)$ by $(*)$. From the proof of Theorem 10, this implies that, for each $n>1$, $\lambda(A/(I^n\colon x))=\lambda(A/I^{n-1})$, i.e. $I^n\colon x=I^{n-1}$. So x is a stable superficial element of I. Let y be an element of I such that $\bar{y}$ is $\bar{I}$-transversal. Then (i) follows from Lemma 5 and Theorem 9.

LEMMA 13. *Let J be an open ideal of a semi-local Cohen–Macaulay ring B of*

Krull dimension 1 *and* y *a transversal element of* J. *Then* $r(J)=r-1$ *iff* r *is the minimum of the integers* $n>0$ *such that* $yJ^n=J^{n+1}$.

Proof. This follows from [2] Lemma 1.8. (ii) and Theorem 1.9.

THEOREM 14. *Let* x *be a superficial element of* I *with* $r(x)\leqq r$. *If* $r(I)\leqq r-1$, *then* $r(I/xA)\leqq r$ *and* $(xA+yA)I^{r+1}=I^{r+2}$ *for some* y *in* I. *In particular, if* $r(I)=0$ *and* x *is a stable superficial element of* I, *then* $(xA+yA)I^2=I^3$ *for some* y *in* I.

Proof. Let N be as in the proof of Theorem 10. Since $r(x)\leqq r$, we have $\lambda(xA/xA\cap I^n)=\lambda(A/(I^n\colon x))=\lambda(A/I^{n-1})$ for each $n>r$. Thus $(xA/xA\cap I^n)$ is a polynomial in n for each $n>r$. So, by Lemma 1, $r(P(N),t)\leqq r$ and $P(N,t)=tP(G_I(A),t)+f(t)$, where $f(t)$ is a polynomial in $\boldsymbol{Z}[t]$ with $\deg f(t)\leqq r$. It follows from $(*)$ that $r(\bar{I})\leqq r$. By Lemma 13, there exists an element y of I such that $yI^{r+1}\equiv I^{r+2}$ mod. xA. So $xA+yI^{r+1}\supseteq I^{r+2}$. Let $z=ax+by$ $(a\in A,\ b\in I^{r+1})$ by an element of I^{r+2}. Then $ax=z-by\in I^{r+2}$. Hence $a\in I^{r+1}$, which shows that $(xA+yA)I^{r+1}=I^{r+2}$.

THEOREM 15. (i) *If* $r(I)=-\infty$, *then* $(xA+yA)I^m=I^{2m}$ *for some integer* m *and* x, y *in* I^m.

(ii) *If* $r(I)\geqq 0$, *then* $(xA+yA)I^{2m}=I^{3m}$ *for some integer* m *and* x, y *in* I^m.

Proof. By Lemma 8, I^m has a stable superficial element for sufficiently large m. By Proposition 4, for all large m, $r(I^m)=-\infty$ if $r(I)=-\infty$, and $r(I^m)\leqq 0$ if $r(I)\geqq 0$. Now (i) and (ii) follow from Theorem 12 and Theorem 14, respectively.

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