Heegaard Diagrams of Torus Bundles Over $S^1$

by

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1. Introduction

It is well known that every closed connected 3-manifold has a Heegaard splitting. A 3-manifold $M$ is said to be of genus $n$, if $M$ has a Heegaard splitting of genus $n$. Every 3-manifold of genus 1 is either a lens space or $S^2 \times S^1$ in the orientable case and is the twisted $S^2$-bundle over $S^1$ in the non-orientable case. Moreover, 3-manifolds of genus 1 are completely classified in [2], [4] and [5]. In this paper, we shall try to classify a certain class of 3-manifolds of genus 2. Indeed, we shall verify that torus bundles (over $S^1$) of genus 2 are completely classified by a new invariant (Theorem 3). Moreover, since every orientable 3-manifold of genus 2 is a 2-fold branched covering space of $S^3$ branched along a link, by Birman-Hilden-Viro-Takahashi [1], [10], [11], we can verify that every orientable torus bundle of genus 2 is a 2-fold branched covering space of $S^3$ branched along some specified link (Corollary 3.1).

In this paper, we work in the piecewise linear category. $S^n$, $D^n$ denote $n$-sphere and $n$-disk, respectively. Let $X$ be a manifold and $Y$ be a submanifold properly embedded in $X$. Then $N(Y, X)$ denotes a regular neighborhood of $Y$ in $X$. Closure, boundary, interior over one symbol are denoted by $\text{cl}(\cdot)$, $\partial(\cdot)$, $\text{int}(\cdot)$, respectively.

2. Surface-bundles over $S^1$

Let $F$ be a closed connected surface and $\Phi: F \to F$ be a homeomorphism. Moreover let $M$ be the 3-manifold obtained from $F \times I$ by identifying $(x, 0)$ in $F \times 0$ with $(\Phi(x), 1)$ in $F \times 1$. Then $M$ is called a surface-bundle over $S^1$. We denote $M$ also by $M(\Phi)$. It will be noticed that if $F$ is orientable then $M$ is orientable or non-orientable, according as $\Phi$ being orientation-preserving or orientation-reversing. Then by Neuwirth [8], we have;

**PROPOSITION 1.** Let $\Phi_1$ and $\Phi_2$ be self-homeomorphisms of $F$. Then $M(\Phi_1)$ is homeomorphic to $M(\Phi_2)$, if there is a self-homeomorphism $\Psi$ such that $\Psi \Phi_1$ is isotopic to $\Phi_2 \Psi$.

Next we consider the relationship between surface-bundles over $S^1$ and their Heegaard splittings. Let $F$ be a closed connected surface and $g(F)$ be the genus of $F$. 

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That is, if $F$ is orientable (resp. non-orientable), there exist $2 \times g(F)$ (resp. $g(F)$) circles on $F$ such that if we cut $F$ along these circles, the resulting manifold is a 2-disk. We may assume that if $F$ is non-orientable then all of such $g(F)$ circles are one-sided circles. Then we have;

**Theorem 1.** Let $M$ be an $F$-bundle over $S^1$. If $F$ is orientable (resp. non-orientable), $M$ has a Heegaard splitting of genus $2g(F)+1$ (resp. $g(F)+1$).

**Proof.** Let $\Phi$ be a self-homeomorphism of $F$ such that $M=F \times I/\Phi$. We may assume without loss of generality that there exists a point $p$ on $F$ such that $\Phi(p)=p$. Next let $C_1$, $C_2$, $\cdots$, $C_n$ be circles on $F$ satisfying the following conditions;

1. $n=2g(F)$ (resp. $g(F)$), if $F$ is orientable (resp. non-orientable),
2. $C_i \cap C_j = p$, for all $i \neq j$,
3. $F-\bigcup_{k=1}^{n} \text{int}(N(C_k, F))$ is a 2-disk.

Let $C$ be the circle $(p \times I) / \Phi$ in $M$ and $C_k$ be the circle $C_k \times 0$ in $M$ ($k=1, 2, \cdots, n$). Furthermore let $U=\bigcup_{k=1}^{n} C_k \cup C, M$ and $V=M-\text{int}(U)$. We note that $U$ is a non-orientable handle if either $F$ is orientable and $\Phi$ is orientation-reversing or $F$ is non-orientable. (For the definition of non-orientable handles, see [9].) Let $V'$ be $F \times I-\text{int}(N(p \times I, F \times I))$ and $D_i=F \times I-\text{int}(N_i)$, where $i=0, 1$ and $N_0=\bigcup_{k=1}^{n} (C_k \times 0), F \times 0, N_1=\Phi(N_0)$. Then $D_i$ is a 2-disk in $F \times i (i=0, 1)$. Now we may assume that $V$ is obtained from $V'$ by identifying points $x$ in $D_0$ with points $\Phi(x)$ in $D_1$. Since $V'$ is a handle of genus $n$, $V$ is also a handle of genus $n+1$. Thus $M$ has a Heegaard splitting of genus $n+1$. That is, $M=U \cup V$ with $U \cap V=\partial U=\partial V$ and $U$ and $V$ are homeomorphic handles. This completes the proof of the theorem.

From now on, we shall consider surface-bundles over $S^1$ with Heegaard splittings of rather small genus. Let $F$ be a closed surface with positive genus $g(F)$ and $M$ be an $F$-bundle over $S^1$. It is easily verified that $M$ has no Heegaard splittings of genus one. Thus we are interested in the existence of surface-bundles over $S^1$ with Heegaard splittings of genus two. As the first observation, we have;

**Theorem 2.** For an arbitrary positive integer $n$, there exists an orientable $F$-bundle over $S^1$ such that $g(F)=n$ and $M$ has a Heegaard splitting of genus two.

**Proof.** Let $K$ be a torus knot of type $(p, q)$ in $S^3$ with $n=(p-1)(q-1)/2$. Then the knot exterior $E(K)=S^3-\text{int}(N(K, S^3))$ of $K$ is an $F_1$-bundle over $S^1$ such that $\partial F_1 \cong \partial E(K), g(F_1)=n$, and $\partial E(K)=S^1 \times S^1$. Since $K$ is a torus knot, we may assume that $K$ lies on the boundary of an unknotted solid torus $H$ in $S^3$. Let $\alpha$ be a simple arc in $\partial H$ joining distinct points of $K$ with the interior of it disjoint from $K$ such that it is not homotopic on $\partial H$ to any arcs in $K$ joining points $K \cap \alpha$. Then $N(\alpha \cup K, S^3)=V$ is a handle of genus two. Furthermore, $U=S^3-\text{int}(V)$ is also a handle of genus two, since $H-\text{int}(V)$ and $(S^3-\text{int}(H))-\text{int}(V)$ are both solid tori and their intersection is a 2-disk $\partial H-\text{int}(V)$. Let $M$ be a closed 3-manifold obtained by attaching a 2-handle $D^2 \times I$ to $E(K)$ along $\partial F_1$. Then $M$ is an $F$-bundle over $S^1$ such that $F$ is a closed surface with $g(F)=n$ and that $M$ has a Heegaard splitting of genus two. This
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completes the proof of the theorem.

It will be noticed that by Moser [6] all the 3-manifolds given by Theorem 2 are Seifert fibered spaces.

3. Torus-bundles over $S^1$

In this section, we consider only torus-bundles over $S^1$. Let $G$ be the group of $2 \times 2$ matrices over $\mathbb{Z}$ with determinant plus or minus one. Moreover, let $T$ be a torus and $\mathcal{A}(T)$ be the homeotopy group of $T$. Then $\mathcal{A}(T)$ is isomorphic to $G$. Let $\Phi$ be a homeomorphism of $T$ onto itself. Then $\Phi$ is given by a matrix $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ in $G$. Let $M(\Phi)$ be the torus bundle over $S^1$ determined by $\Phi$. A presentation of $\pi_1(M(\Phi))$ is given by

$$\pi_1(M(\Phi)) = \langle x, y, t \mid [x, y] = 1, \quad txt^{-1} = x^ty^a, \quad tyt^{-1} = x^by^a \rangle,$$

where $x, y$ correspond to generators of $\pi_1(T)$.

**Proposition 2.** Let $\Phi_1$ and $\Phi_2$ be self-homeomorphisms of $T$, whose matrices are $A_1$ and $A_2$, respectively. Moreover let $M_1$ and $M_2$ be the torus-bundles over $S^1$ determined by $\Phi_1$ and $\Phi_2$, respectively. Then $M_1$ is homeomorphic to $M_2$ if and only if $A_1$ is a conjugate of $A_2$ or $A_2^{-1}$ in $G$.

**Proof.** One direction comes from Proposition 1. Furthermore, if the Betti number $b(M(\Phi_1)) = 1$, then the converse follows from Theorem 1 in [7]. Suppose that $M(\Phi_1)$ is homeomorphic to $M(\Phi_2)$ and $b(M(\Phi_1)) = 2$ ($i = 1, 2$). Then we have that $H_2(M(\Phi_i), Z) = Z + Z + Z_k$. Let $E$ be the unit matrix and $B_i = A_i - E$ ($i = 1, 2$). It is easily seen that the determinant of $B_i$ is zero. Let $B_i = (\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix})$ ($i = 1, 2$). Then there are integers $u_i$ and $w_i$ such that $(a_i, b_i) = v_i(x_i, \beta_i)$ and $(c_i, d_i) = w_i(x_i, \beta_i)$, where $i = 1, 2$ and $x_i$ and $\beta_i$ are relatively prime integers. Thus there are integers $b_i$ and $d_i$ such that det$(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}) = 1$ ($i = 1, 2$). Then we have that $(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}) = (\begin{smallmatrix} \gamma_i & \delta_i \\ -\gamma_i & -\delta_i \end{smallmatrix}) = (\begin{smallmatrix} a_i + b_i & 0 \\ -c_i & -d_i \end{smallmatrix})$, where $u_i = \delta(\gamma d_i + \delta c_i) - \gamma(\gamma b_i + \delta d_i)$ ($i = 1, 2$). Thus the matrix $A_i$ is conjugate to $(\begin{smallmatrix} a_i + b_i & 0 \\ -c_i & -d_i \end{smallmatrix})$ ($i = 1, 2$). Let $z_i = a_i + b_i + 1$. Since det$(A_i) = \pm 1$, we have that $|z_i| = 1$. Then two cases happen:

**Case (1):** $M(\Phi_1)$ is orientable. In this case, we have that $z_i = 1$. Since $H_1(M(\Phi_i), Z) = Z + Z + Z_k$, we have that $k = |u_i|$. Thus $A_1$ is conjugate to $A_2$, since $(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix})(\begin{smallmatrix} 1 & 0 \\ -1 & -1 \end{smallmatrix}) = (\begin{smallmatrix} 1 & 0 \\ -u & -u \end{smallmatrix})$.

**Case (2):** $M(\Phi_1)$ is non-orientable. In this case, we have that $z_i = -1$. By Hempel [4], $A_1$ is also conjugate to $A_2$, since $(\begin{smallmatrix} -1 & 0 \\ -1 & 0 \end{smallmatrix})(\begin{smallmatrix} -1 & 0 \\ -u & -u \end{smallmatrix}) = E$.

This completes the proof.

By the above argument, if $M$ is a torus-bundle with $H_1(M, Z) = Z + Z + Z_k$, then the corresponding matrix $A$ is conjugate to one of $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$, $(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix})$, $(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix})$.

From now on, we are interested in torus-bundles with Heegaard splittings of genus two. By Theorem 1, every torus-bundle has always a Heegaard splitting of genus three. But some of them have also Heegaard splittings of genus two.
PROPOSITION 3. Let $M(\Phi)$ be a torus-bundle over $S^1$ and $\varepsilon = \pm 1$. If the matrix of $\Phi$ is \((a \ b)\), then $M(\Phi)$ has a Heegaard splitting of genus two.

Proof. By Theorem 1, $M(\Phi)$ has a Heegaard splitting of genus three and the Heegaard splitting $(U, V; F)$ is associated with the presentation of $\pi_1(M(\Phi))$, \([x, y, t] \mid [x, y] = 1, \, txt^{-1} = x^my^q, \, tyt^{-1} = x\). Let $u = u_1 \cup u_2 \cup u_3$ (resp. $v = v_1 \cup v_2 \cup v_3$) be a complete system of meridian-disks properly embedded in $U$ (resp. $V$) such that $\text{cl}(U - N(u, U))$ (resp. $\text{cl}(V - N(v, V))$) is a 3-disk. Let $x, y,$ and $t$ be the canonical generators of the free group $\pi_1(V)$ ($= \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}$). Then we can easily find a homeomorphism $f$ from $\partial U$ onto $\partial V$ such that the induced homomorphism $f_* : \pi_1(\partial U) \to \pi_1(V)$ satisfies $f_*(\partial u_1) = xyx^{-1}y^{-1}$, $f_*(\partial u_2) = x^my^qtx^{-1}t^{-1}$, and $f_*(\partial u_3) = xty^{-1}t^{-1}$.

It will be noticed that $f(\partial u_3)$ bounds a torus with one hole in $V$. We can assume that $f(\partial u_3)$ meets $\partial v_2$ transversely at only one point. Then if $M(\Phi)$ is orientable, by Waldhausen [13] the intersection of $\partial v_2$ and $f(\partial u_1)$ or $f(\partial u_2)$ are eliminated. Next suppose that $M(\Phi)$ is non-orientable. Then we may assume that the generators $x$ and $y$ (resp. $t$) are induced by orientable circles (resp. a non-orientable circle) in $V$. Thus all the circles $f(\partial u_1)$, $f(\partial u_2)$, and $f(\partial u_3)$ are orientable in $\partial V$. Hence the elimination method of the orientable case can also apply to the non-orientable case. Let $u'_1$ and $u'_2$ be the resulting circles on the boundary of $V' = V - \text{int}(N(v_2, V))$. Then $(V'; \partial u_1 \cup \partial u_3, u'_1 \cup u'_2)$ gives a Heegaard diagram of genus two. Thus $M(\Phi)$ has a Heegaard splitting of genus two. This completes the proof.

It will be noticed that if $\varepsilon = -1$ and $m = 2$ (resp. $\varepsilon = +1$ and $m = 3$), $M(\Phi)$ has an orientable (resp. non-orientable) Heegaard diagram of genus two, illustrated in Figure 1.1 (resp. Figure 1.2).

Next we shall verify that the torus-bundles of genus two given by Proposition 3 cover all torus-bundles of genus two.

LEMMA 1. Let $A$ be a matrix in $G$ and $M$ be a torus-bundle determined by $A$. If $\pi_1(M)$ is generated by two generators, then $A$ is conjugate to a matrix \((p \ \frac{q'}{r'})\) with $q' = 1$ or $r' = 1$.

Proof. To avoid complexity, we will verify only the case when $M$ is orientable, and the proof in the case when $M$ is non-orientable is similar. Let $\Pi = \pi_1(M)$ and $A = (p \ \frac{q}{r})$. Suppose that $\Pi = \langle a, b \rangle$, that is, two elements $a$ and $b$ in $\Pi$ generate $\Pi$. By $txt^{-1} = x^py^q$ and $tyt^{-1} = x^r y^s$, we have $t^{-1}xt = x^py^{-q}$ and $t^{-1}yt = x^{-r}y^q$, since $ps - qr = 1$. Thus we have that $tx = x^py^qt$, $ty = x^r y^st$, $t^{-1}x = x^py^{-q}t^{-1}$, and $t^{-1}y = x^{-r}y^qt^{-1}$. Let $z$ be an arbitrary element in $\Pi$. By the above four equations and $xy = yx$, there are three integers $\alpha, \beta, \gamma$, such that $z = x^\alpha y^\beta t^\gamma$. Furthermore such expression of $z$ is unique. For, if $x^\alpha y^\beta t^\gamma = 1$, then the equation $\alpha x + \beta y + \gamma t = 0$ holds in $H_1(M, Z)$. Since $H_1(M, Z) = Z + Z_k$, $x$ and $y$ generate $Z_k$, and $t$ generates $Z$, we have that $\gamma = 0$. Hence $x^\alpha y^\beta = 1$ in $\pi_1(M)$. Here $x, y$ are contained in $\pi_1(T)$. Let $i : \pi_1(T) \to \pi_1(M)$ be the inclusion induced homomorphism. Since $i$ is monic, $x^\alpha y^\beta = 1$ in $\pi_1(M)$.
1 in \( \pi_1(T) \). But \( T \) is a torus, and so \( \alpha = \beta = 0 \).

Now suppose that \( a = x^{\gamma_1 y^{\beta_1} t^{\gamma_1}} \) and \( b = x^{\gamma_2 y^{\beta_2} t^{\gamma_2}} \). We may assume that \( 0 \leq \gamma_1, \gamma_2 \leq \gamma_2 \). Then \( b = x^{\gamma_1 y^{\beta_1} t^{\gamma_1}} x^{\gamma_2 y^{\beta_2} t^{\gamma_2}} = a x^{\gamma_1 y^{\beta_1} t^{\gamma_1}} \) for some integer \( \alpha', \beta' \). Thus we may assume that \( \prod = \langle a, b \rangle \) with \( a = x^{\gamma_1 y^{\beta_1} t^{\gamma_1}} \) and \( b = x^{\gamma_2 y^{\beta_2}} \). Next we can assume without loss of generality that \( \gamma_2 \) and \( \beta_2 \) are relatively prime. Then the element \( b \) can be thought of as a simple loop in \( T \), which is not homotopic in \( T \) to zero. And there is a simple loop \( c \) in \( T \) which meets \( b \) transversely at only one point. Let \( c = c^{a_2 y^{b_2}} \) with \( \det(a_2, b_2) = 1 \). Consequently a new presentation of \( \prod \), \( \langle b, c, t | [b, c] = 1, tbt^{-1} = b^p c^q, tct^{-1} = b^r c^s \rangle \) is obtained and \( \prod = \langle a, b \rangle \) with \( a = b^p c^q t^r \). And so \( \prod = \langle a_1, b \rangle \) with \( a_1 = c^q t^r \). Since \( a_1 \) and \( b \) generate \( t \), we have that \( t = c^{-\beta} a_1 \). Thus \( \prod = \langle a_1, b \rangle \) with \( a_1 = c^{q} t^r \). Since \( t = c^{-\beta} a_1 \), the following presentation of \( \prod \) follows:

\[
\prod = \langle b, c, a_1 | [b, c] = 1, a_1 b a_1^{-1} = b^p c^q, a_1 c a_1^{-1} = b^r c^s \rangle .
\]

Let \( a_1 = g \). For every integer \( m \), we have the following, \( m \) the following,

\[
\begin{align*}
(1) \quad g b^m g^{-1} &= (b^p c^q)^m \\
(2) \quad g^{-1} b g &= (b^\iota c^{-\iota})^m \\
(3) \quad g e^m g^{-1} &= (b^r c^s)^m \\
(4) \quad g^{-1} e g &= (b^{-r} c^{-s})^m
\end{align*}
\]

Since \( \prod = \langle g, b \rangle \), we have that \( c = b^{\iota_1} b^{\iota_2} \cdots b^{\iota_k} \) for some integers \( \iota_1, \iota_2, \ldots, \iota_k \). Then we will verify that \( c \) has an expression \( b^\iota c^\iota g^\iota \) such that \( q_1 \) divides \( \beta \). Since both \( b \) and \( c \) are contained in \( \pi_1(T) \), we may assume without loss of generality that all of the three integers \( \iota_1, \iota_2, \iota_3 \) are non-zero. It is sufficient to verify that an element \( g^\iota b^\iota \), with non-zero integers \( \tau \) and \( \lambda \), in \( \prod \) has an expression \( b^\iota c^\iota g^{\lambda \iota} \) with \( q_1 \) divides \( \beta \). To avoid complexity, we assume that \( \tau \) and \( \lambda \) are both positive. Then by the equations (1) and (2), we have the following,

\[
g^\iota b^\iota = b^{\iota_1 \iota} c^{\iota_2 \iota} \cdots , g^\iota b^\iota = b^{\iota_1 \iota} c^{\iota_2 \iota} \cdots ,
\]

Furthermore, by equation (3) we have that for any integer \( m \), \( g c^m = (b^\iota c^\iota)^m \) \( = b^\iota m c^\iota m \) \( = b^\iota m c^\iota m \) \( = b^\iota m c^\iota m \) \( = b^\iota m c^\iota m \). Thus, at the final step we can obtain the expression of \( g^\iota b^\iota \), \( b^\iota c^\iota g^{\lambda \iota} \), such that \( q_1 \) divides \( \beta \). Consequently, \( c = b^\iota c^\iota g^{\lambda \iota} \) for some integers \( \alpha, \beta, \gamma \) and \( q_1 \) divides \( \beta \). But by the uniqueness of the expression of \( c \), we have that \( \beta = 1 \). Hence \( q = \pm 1 \). Here \( \langle \iota, -\iota \rangle \) is conjugate to \( \langle -\iota, \iota \rangle \). Thus we conclude that \( q = 1 \). This completes the proof of the lemma.

**Lemma 2.** Let \( A = (a \ b) \) be a matrix in \( G \). If \( (q - 1)(r - 1) = 0 \), then \( A \) is conjugate to a matrix \((a \ b)\) in \( G \) with \( \varepsilon = \pm 1 \).

**Proof.** Suppose that \( q = 1 \). In this case, if \( \det(A) = 1 \), then \( A = (a \ b) \). If \( \det(A) = -1 \), then \( A = (a \ b) \). Then the following hold;

\[
\begin{align*}
(p_s \ b \ a) (1 \ b_0) &= (1 \ -b) (p_s \ a) (1 \ b) = (1 \ -b) (p_s \ a) (1 \ b) .
\end{align*}
\]

Thus we set \( m = p + s \). If \( r = 1 \), then the same result is obtained.

Let \( M(m, \varepsilon) \) be a 3-manifold determined by a matrix \((n \ b)\) with \( \varepsilon = \pm 1 \). Then by Lemma 1 and Lemma 2, and Proposition 2, we have;
THEOREM 3. Every torus-bundle over $S^1$ with a Heegaard splitting of genus two is homeomorphic to $M(m, \varepsilon)$ for some integer $m$, and if it is orientable (resp. non-orientable) then $\varepsilon = -1$ (resp. $\varepsilon = 1$). In particular, $M(m, \varepsilon) = M(m', \varepsilon)$ if and only if $m = m'$.

Birman-Hilden-Viro-Takahashi [1], [10], and [11] proved that every orientable

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A Heegaard diagram in the orientable case of $m=2$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{A Heegaard diagram in the non-orientable case of $m=3$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{A link $K(m+4) = K_0 \cup K_4 \cup K_2$.}
\end{figure}
closed 3-manifold with Heegaard splittings of genus two is a 2-fold branched covering space of $S^3$ branched along a link. As illustrated in preceding remark, the manifold $M(2, -1)$ has a Heegaard diagram of genus two given by Figure 1.1. Thus we can determine one type of branched sets of torus-bundles of genus two. Let $K(m+4)$ be the link illustrated in Figure 2. It has two components $K_0$ and $K_1$ (resp. three components $K_0$, $K_1$, and $K_2$) if $m$ is odd (resp. even). We note that the component $K_0$ is unknotted and that $m+4$ is the number of double points in $K_1 \cup K_2$ (resp. $K_0$), when $m$ is even (resp. odd). Then we have;

**Corollary 3.1.** Every orientable torus-bundle of form $M(m, -1)$ is a 2-fold branched covering space of $S^3$ branched along $K(m+4)$.

By the way, there are infinitely many torus-bundles of genus three but not two. It is an interesting problem to decide whether such torus-bundles are 2-fold branched covering spaces of $S^3$ or not. Fox had proved in [3] that $S^1 \times S^1 \times S^1$ is not a 2-fold branched covering space of $S^3$. Thus we will set up the following problem;

**Problem 1.** Which torus-bundles are 2-fold branched covering spaces of $S^3$?

In view of Lemma 1, we raise the following;

**Problem 2.** Are link types of branched sets of every torus-bundle of genus two unique?

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**References**


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