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On the Quasi-connected Components of a Scheme

by

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Let $S=\operatorname{Spec} R$ be an affine scheme and let $X(S)$ be the space of all connected components of S endowed with the quotient topology. Then it is known that $X(S)$ is homeomorphic to the spectre of the ring of idempotents of R.

In this paper we globalize the above fact to arbitrary quasi-compact scheme S. We show that the spectre of the ring of idempotents of $\Gamma(S, \mathcal{O}_S)$ is homeomorphic to the space of all quasi-connected components of S endowed with the quotient topology.

1. Introduction

Let R be a commutative ring with identity and let $B(R)$ be the set of all idempotents of R. Then it is known that $B(R)$ forms a Boolean algebra if we define $e \wedge f = ef$ and $e \vee f = e+f-ef$ for any e, f in $B(R)$ ([3] p. 5). But, if we define products to be the same as in R and sums as $e \oplus f = e+f-2ef$ (operations in R), it is easy to see that $B(R)$ forms a commutative ring with identity. Indeed, $e+f-2ef=e(1-f)+f(1-e)$ is an idempotent of R and the additive inverse of an idempotent e is e. Let us call this $B(R)$ *the ring of idempotents of* R. The definition of sums given in [2] p. 28 is false, but Proposition II.8 and Proposition II.9 given there without proof are valid by our modified definition of sums. According to [2] Proposition II.9, the set of all connected components of Spec R with the quotient topology is canonically homeomorphic to Spec $B(R)$. We globalize this local result to arbitrary quasi-compact schemes.

2. The ring of idempotents of a commutative ring

We recall here basic properties of the ring of idempotents $B(R)$ of R.

(1) Every open-closed set of Spec R is of the form $V(e)=\{P \in \operatorname{Spec} R \mid P \ni e\}$ for some e in $B(R)$ ([2] Lemma II.2).

(2) Two primes P, Q of R belong to the same connected component of Spec R if and only if they contain the same idempotents ([2] Proposition II.3).

(3) The space of connected components $X(R)$ of Spec R endowed with the quotient topology is a profinite space ([2] Corollary II.4).

(4) A set E of idempotents of R is the set of all idempotents in some prime ideal

of R if and only if E is a maximal Boolean ideal ([2] Proposition II.6).

(5) A set E of idempotents of R is a maximal Boolean ideal of R if and only if E is a prime ideal of the ring of idempotents $B(R)$ of R. Moreover, all prime ideals of $B(R)$ are maximal ([2] Proposition II.8 without proof).

Proof. Recall that a set of idempotents E of R is called a maximal Boolean ideal if;

(a) For every idempotent e of R, either $e \in E$ or $1-e \in E$, but not both;

(b) If e and f are idempotents of R, then $ef \in E$ if and only if $e \in E$ or $f \in E$.

If E is a maximal Boolean ideal, then, by (4), there is a prime ideal P of R such that E is the set of all idempotents in P. If $e, f \in E$, then $e \oplus f = e(1-f) + f(1-e) \in P$. If $e \in E$ and $f \in B(R)$, then $ef \in P$. So E is an ideal of $B(R)$. Since $e \oplus (1-e) = 1$, 1 is not in E by (a). From this and (b), E is a prime ideal. If e is not in E, then $1-e$ is in E and $e^2 = e \equiv 1$ mod. E, which shows that E is maximal. Conversely, if E is a prime ideal of $B(R)$, then E is obviously a maximal Boolean ideal.

(6) Let $i: B(R) \to R$ be the inclusion map (which is not a ring homomorphism) and let P be a prime ideal of R. Then $P \mapsto i^{-1}(P)$ is a surjective continuous map from Spec R to Spec $B(R)$.

Proof. By (4) and (5), $P \mapsto i^{-1}(P)$ is a surjective map from Spec R to Spec $B(R)$. Since the inverse image of a closed set $V^*(I) = \{E \in \text{Spec } B(R) \mid E \supset I\}$ of Spec $B(R)$ is the closed set $V(I) = \{P \in \text{Spec } R \mid P \supset I\}$, where I is an ideal of $B(R)$, this map is continuous.

(7) For any ideals I, J of $B(R)$, following conditions are equivalent.

(i) $V(I) = V(J)$

(ii) $V^*(I) = V^*(J)$

(iii) $I = J$

Proof. (i)$\Leftrightarrow$(ii). This follows from the fact that the canonical map q: Spec $R \to$ Spec $B(R)$ is surjective and $q^{-1}(V^*(I)) = V(I)$. (iii)$\Rightarrow$(i) is trivial. (ii)$\Rightarrow$(iii). This follows from the fact that the radical of I in $B(R)$ is equal to I and the fact that the intersection of all prime ideals in $V^*(I)$ is the radical of I.

(8) The idempotent e of (1) is unique.

Proof. Assume that $V(e) = V(f)$ for some $e, f \in B(R)$. Then $\langle e \rangle = \langle f \rangle$ by (7), where $\langle e \rangle$ is the ideal of $B(R)$ generated by e. So $e = af$ and $f = be$ for some a, b in $B(R)$. Then $ef = af^2 = af = e$. Similarly $ef = f$ and hence $e = f$.

DEFINITION. We say that a family of idempotents $\{f_i\}$ of R is an orthogonal system if $f_i f_j = 0$ for any i, j $(i \neq j)$.

Remark that, if $f_i, f_j \in B(R)$ are orthogonal, then $f_i \oplus f_j = f_i + f_j$.

(9) Let $e_1, \cdots, e_m$ be idempotents of R and let $\langle e_1, \cdots, e_m \rangle$ be the ideal of $B(R)$ generated by $e_1, \cdots, e_m$. Then there exists an orthogonal system of idempotents $f_1, \cdots, f_m$ such that $\langle f_1, \cdots, f_i \rangle = \langle e_1, \cdots, e_i \rangle$ $(i = 1, \cdots, m)$.

Proof. Let $f_1=e_1$ and $f_i=e_i(1-e_1)\ (1-e_2)\cdots(1-e_{i-1})\ (2\leqq i\leqq m)$. Since $e_i(1-e_i)=0$, $f_1, \cdots, f_m$ forms an orthogonal system. Using the induction, we may assume that $e_1, \cdots, e_{i-1}$ are in $\langle f_1, \cdots, f_{i-1}\rangle$. Since $f_i=e_i+a_1e_1+\cdots+a_{i-1}e_{i-1}$ for some $a_1, \cdots, a_{i-1}\in R$, $e_i=s_1f_1+\cdots+s_{i-1}f_{i-1}+f_i$ for some $s_i\in R$. Multiplying $f_k\ (1\leqq k\leqq i-1)$, we have $e_if_k=s_kf_k$. So $e_i=e_if_1+\cdots+e_if_{i-1}+f_i=e_if_1\oplus\cdots\oplus e_if_{i-1}\oplus f_i$ is in $\langle f_1, \cdots, f_i\rangle$.

(10) Let $f_1, \cdots, f_m$ be an orthogonal system of idempotents of R. If $a_1f_1+\cdots+a_mf_m=b_1f_1+\cdots+b_mf_m$ for some $a_1, \cdots, a_m, b_1, \cdots, b_m$ in R, then $a_if_i=b_if_i\ (i=1, \cdots, m)$.

Proof. Multiply f_i to both sides.

(11) Let $g, g_1, \cdots, g_m$ be idempotents of R and let $(g_1, \cdots, g_m)$ be the ideal of R generated by $g_1, \cdots, g_m$. If g is in $(g_1, \cdots, g_m)$, then g is in $\langle g_1, \cdots, g_m\rangle$.

Proof. Let $f_1, \cdots, f_m$ be the orthogonal system of idempotents constructed from $g_1, \cdots, g_m$ by (9). Then, if g is in $(g_1, \cdots, g_m)$, $g=a_1f_1+\cdots+a_mf_m$ for some $a_1, \cdots, a_m\in R$. Since $gf_i=a_if_i$, $g=gf_1+\cdots+gf_m=g(f_1+\cdots+f_m)=g(f_1\oplus\cdots\oplus f_m)$. So g is in $\langle f_1, \cdots, f_m\rangle=\langle g_1, \cdots, g_m\rangle$.

(12) Let $e_1, \cdots, e_m, f_1, \cdots, f_n$ be idempotents of R. Then $(e_1, \cdots, e_m)=(f_1, \cdots, f_n)$ if and only if $\langle e_1, \cdots, e_m\rangle=\langle f_1, \cdots, f_n\rangle$.

Proof. This follows directly from (11).

(13) Spec $B(R)$ is profinite.

Proof. Since $e(1-e)=0$ for any $e\in B(R)$, $D^*(e)=\{E\in \mathrm{Spec}\,B(R)\mid E\not\ni e\}$ is equal to $V^*(1-e)$, which shows that Spec $B(R)$ has a basis consisting of open-closed sets. If P, Q are distinct prime (i.e. maximal) ideals of Spec $B(R)$, then there is an element e such that $e\in P$ and $e\notin Q$. Then open-closed set $V^*(e)$ separates P and Q. Hence Spec $B(R)$ is Hausdorff. Since Spec $B(R)$ is quasi-compact, we know that Spec $B(R)$ is profinite by [2] Proposition I.8.

(14) The space of connected components $X(R)$ of Spec R is canonically homeomorphic to Spec $B(R)$.

Since each quasi-connected component of an affine scheme is a connected component by Corollary II.4, Propostion I.8 and Proposition I.5 in [2], this is a special case of Theorem 7. So we omit the proof.

3. The quasi-connected components of a scheme

Let S be a scheme and let B be the ring of idempotents of $\Gamma(S, \mathcal{O}_S)$.

LEMMA 1. *If $e\in B$, then the presheaf given by $U\mapsto \mathcal{O}_S(U)(e\,|_U)$ is a sheaf, which we denote by $e\mathcal{O}_S$.*

Proof. Let U be an open set of S, let $\{U_i\}$ be an open covering of U, and let s_i be in $(e|_{U_i})\mathcal{O}_S(U_i)$ such that $s_i|_{U_i\cap U_j}=s_j|_{U_i\cap U_j}$ for each i, j. Since $\mathcal{O}_S$ is a sheaf, there is an element $s\in\mathcal{O}_S(U)$ such that $s|_{U_i}=s_i$ for each i. Since $(e|_{U_i})(s|_{U_i})=(e|_{U_i})s_i=s_i$, $(e|_U)s=s$. So s is in $(e|_U)\mathcal{O}_S(U)$.

PROPOSITION 2. *Let e be an idempotent in B and let $V(e)$ be the underlying space of the closed subscheme associated with the sheaf of ideals $e\mathcal{O}_S$. Then $V(e)$ is an open-closed set of S. Moreover, every open-closed set of S is of the form $V(e)$ for some unique $e\in B$.*

Proof. If $e\in B$, then, for each affine open set U of S, $U=V(e|_U)\cap V((1-e)|_U)$ and $V(e|_U)\cap V((1-e)|_U)=\varnothing$. So $V(e)\cup V(1-e)=S$ and $V(e)\cap V(1-e)=\varnothing$, which shows that $V(e)$ is an open-closed set of S. Conversely, let W be an open-closed set of S. If $\{U_i\}$ is an affine open covering, then $U_i\cap W$ is an open-closed set of U_i for each i. By (1), there is an idempotent e_i of $\Gamma(U_i, \mathcal{O}_S)=A_i$ such that $U_i\cap W=V(e_i)$ in Spec $A_i=U_i$. For each i, j, the restrictions of e_i and e_j to $\Gamma(U_i\cap U_j, \mathcal{O}_S)$ coincide, since, for each affine open set V in $U_i\cap U_j$, $e_i|_V=e_j|_V$ by (8). So there is a unique idempotent section $e\in\Gamma(S, \mathcal{O}_S)$ such that $e|_{U_i}=e_i$. Since $V(e)\cap U_i=V(e|_{U_i})=V(e_i)=U_i\cap W$ for each i, $V(e)=W$.

COROLLARY 3. *A scheme S is connected if and only if $\Gamma(S, \mathcal{O}_S)$ has no idempotents except* 0 *and* 1.

DEFINITION. The intersection of all open-closed sets of a topological space which contain a point P is called the quasi-connected component (or for short, the quasi-component) of P. Let $Q(S)$ be the space of all quasi-components of a scheme S, with the strongest topology making the canonical projection $p: S\to Q(S)$ continuous.

PROPOSITION 4. *Let P be a point of a quasi-compact scheme and let $E=q(P)$ be the set of idempotents e of $\Gamma(S, \mathcal{O}_S)$ such that $P\in V(e)$. Then E is a maximal ideal of B.*

Proof. Let $U=$Spec A be an affine neighborhood of P and let j_P be the prime ideal of A corresponding to P. If $e\in E$ and $f\in B$, then $j_P\ni e|_U$. So $(ef)|_U=(e|_U)\,(f|_U)\in j_P$, which means that $P\in V(ef)$, i.e. $ef\in E$. If e, f are in E, then $e|_U\oplus f|_U=e|_U+f|_U-2(e|_U)(f|_U)$ is in j_P. So $P\in V(e\oplus f)$, *i.e.* $e\oplus f\in E$. Since 1 is not in E, $E\neq B$. Assume that E is not maximal. Then there is some $f\in B$ such that $f\notin E$ and $\langle E, f\rangle\neq B$. Since P is not in $V(f)$, P is in $V(1-f)$, i.e. $1-f\in E$. So

$$\bigcap_{e\in E} V(e)\subset V(l-f)\,.$$

Since $S=V(f)\cup V(1-f)$ (disjoint union),

$$\left(\bigcap_{e\in E} V(e)\right)\cap V(f)=\varnothing\,.$$

Let $\{U_i=\text{Spec } A_i\}_{i=1,\cdots,n}$ be a finite affine covering of S. Then

$$\left(\bigcap_{e\in E} V(e\,|_{U_i})\right)\cap V(f\,|_{U_i})=\varnothing\,,$$

that is,

$$V\Big(\sum_{e\in E} A_i(e\,|_{U_i})+A_i(f\,|_{U_i})\Big)=\varnothing\,,$$

i.e.

$$A_i(f\,|_{U_i})+\sum_{e\in E} A_i(e\,|_{U_i})=A_i\,.$$

So there are finite elements $e_1^{(i)}, \cdots, e_{m_i}^{(i)}\in E$ and $a^{(i)}, a_1^{(i)}, \cdots, a_{m_i}^{(i)}\in A_i$ such that $a^{(i)}(f\,|_{U_i})+a_1^{(i)}(e_1^{(i)}\,|_{U_i})+\cdots+a_{m_i}^{(i)}(e_{m_i}^{(i)}\,|_{U_i})=1$. Let $f_1, \cdots, f_N$ be the orthogonal system of idempotents of (9) constructed from all $\{e_j^{(i)}\}$ and f. Since $f_1, \cdots, f_N$ are orthogonal and $\langle e_1^{(1)}, \cdots, e_{m_n}^{(n)}, f\rangle=\langle f_1, \cdots, f_N\rangle, f_1\,|_{U_i}, \cdots, f_N\,|_{U_i}$ are orthogonal and $\langle e_1^{(1)}\,|_{U_i}, \cdots, e_{m_n}^{(n)}\,|_{U_i}, f\,|_{U_i}\rangle=\langle f_1\,|_{U_i}, \cdots, f_N\,|_{U_i}\rangle$. So there are elements $b_1^{(i)}, \cdots, b_N^{(i)}$ of A_i for each i such that $b_1^{(i)}(f_1\,|_{U_i})+\cdots+b_N^{(i)}(f_N\,|_{U_i})=1$. For each i, j and for each affine open set V in $U_i\cap U_j$, $(b_k^{(i)}\,|_V)(f_k\,|_V)=(b_k^{(j)}\,|_V)(f_k\,|_V)$ $(k=1, \cdots, N)$ by (10). So for each k, $\{b_k^{(i)}(f_k\,|_{U_i})\}_{i=1,\cdots,n}$ patch together to give a global section b_k of $\Gamma(S, \mathcal{O}_S)$ such that $b_k\,|_{U_i}=b_k^{(i)}(f_k\,|_{U_i})$. Since $b_1\,|_{U_i}+\cdots+b_N\,|_{U_i}=1$ $(i=1, \cdots, n)$, $b_1+\cdots+b_N=1$. As $(b_kf_k)\,|_{U_i}=b_k\,|_{U_i}$ $(i=1, \cdots, n)$, $b_kf_k=b_k$. So $b_1f_1+\cdots+b_Nf_N=1$. Then, by (12), $\langle E, f\rangle=B$, which is a contradiction. So E is a maximal ideal.

PROPOSITION 5. *Let S be a quasi-compact scheme and let E be a maximal ideal of the ring of idempotents B of $\Gamma(S, \mathcal{O}_S)$. Then there is a point P in S such that the set of idempotents $q(P)$ of Propostion 4 is equal to E.*

Proof. It is sufficient to show that

$$\bigcap_{e\in E} V(e)\neq\varnothing\,,$$

because, if

$$P\in\bigcap_{e\in E} V(e),\quad q(P)=E$$

by Proposition 4. If

$$\bigcap_{e\in E} V(e)=\varnothing\,,$$

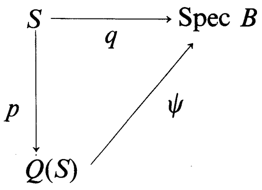
it is easy to see just as in the proof of Proposition 4 that there are elements $f_1, \cdots, f_N$ of E such that $\langle f_1, \cdots, f_N\rangle=B$, so $E=B$, which is a contradiction.

PROPOSITION 6. *Two points P, Q of a quasi-compact scheme S belong to the same quasi-component if and only if $q(P)=q(Q)$.*

Proof. This follows from Proposition 2 and the definition of quasi-components.

THEOREM 7. *The space $Q(S)$ of quasi-components of a quasi-compact scheme S is canonically homeomorphic to Spec B, where B is the ring of idempotents of $\Gamma(S, \mathcal{O}_S)$.*

Proof. By Proposition 4, Proposition 5 and Proposition 6, we have a following commutative diagram;



where p is surjective and continuous, q is surjective and ψ is bijective. For each ideal I of B,

$$q^{-1}(V^*(I))=q^{-1}\left(\bigcap_{e\in I} V^*(e)\right)=\bigcap_{e\in I} q^{-1}V^*(e)=\bigcap_{e\in I} V(e),$$

which shows that q is continuous. From the commutativity of the diagram and the definition of the topology of $Q(S)$, it follows that ψ is continuous. Since $Q(S)$ and Spec B are quasi-compact and Spec B is Hausdorff by (13), ψ is a homeomorphism.

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