The Torsion Product of Totally Projective $p$-Groups

by

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In this paper we consider the structure of the torsion product, $\text{Tor}(A, B)$, of abelian $p$-groups $A$ and $B$. Our emphasis is on the class of totally projective $p$-groups. We are able to show that the Ulm lengths $\lambda(A)$, $\lambda(B)$ play a crucial role in the structure of $\text{Tor}(A, B)$. In particular we show that for $A$ and $B$ totally projective $\text{Tor}(A, B)$ is totally projective if and only if $\min\{\lambda(A), \lambda(B)\} \leq \Omega$, the first uncountable ordinal. Additionally we show for arbitrary $p$-groups $A, B$ having unequal Ulm lengths greater than $\Omega$ that $\text{Tor}(A, B)$ is not totally projective.

In what follows all groups are reduced abelian $p$-groups. Let $H_\alpha$ denote the generalized Pruefer group of length $\alpha$, $f_\delta(A)$ the $\delta$th Ulm-Kaplansky invariant of $A$, and $C(p^n)$ a cyclic group of order $p^n$. In addition we use d.s.c. for direct sum of countables and $\perp$ to denote a direct summand.

Before proceeding to the first of our results we need the following definition and theorems:

**Definition.** A subgroup $B$ of the group $A$ is isotype if $B \cap p^nA = p^nB$ for all ordinals $\alpha$.

**Theorem 1** (Griffith [2] p. 78). If $A$ is a d.s.c. group and $B$ an isotype subgroup of $A$ such that $\lambda(B) < \Omega$ then $B$ is a d.s.c. group.

**Theorem 2** (Nunke [5]). If $\lambda(A) > \lambda(B) \geq \omega$, then $\text{Tor}(A, B)$ is a d.s.c. group $\Leftrightarrow$

(i) $B$ is a d.s.c. group

(ii) If $\beta \geq \omega$ is such that $f_\beta(B) \neq 0$, then every $p^\beta$-high subgroup of $A$ is a d.s.c. group.

We are now prepared to prove the following theorem.

**Theorem 3.** For $\alpha$ an arbitrary ordinal and $\beta \leq \Omega$, $\text{Tor}(H_\alpha, H_\beta)$ is a d.s.c. group.

**Proof.** If $\alpha < \Omega$ and $\beta < \Omega$ then $H_\alpha$ and $H_\beta$ are countable so $\text{Tor}(H_\alpha, H_\beta)$ is countable.

If $\alpha = \Omega$ and $\beta \leq \Omega$ then since

$$H_\Omega = \bigoplus_{\delta < \Omega} H_\delta$$

we have

$$\text{Tor}(H_\alpha, H_\beta) = \bigoplus_{\delta < \Omega} \text{Tor}(H_\alpha, H_\beta)$$
a d.s.c. group.

If $\beta \leq \omega$ then since $H_\alpha = C(p^\alpha)$ and

$$H_\alpha = \bigoplus_{n < \omega} C(p^n)$$

it follows that Tor($H_\alpha$, $H_\beta$) is a direct sum of cyclics so it's a d.s.c. group.

Thus we may assume that $\beta$ is a fixed ordinal, $\omega < \beta \leq \Omega$. Let $\alpha > \Omega$ be an arbitrary but fixed ordinal. Our proof will be by induction on $\alpha$.

For $\tau < \Omega$ Tor($H_\tau$, $H_\beta$) is a d.s.c. group by our above comments. Assume for all $\tau < \alpha$ that Tor($H_\tau$, $H_\beta$) is a d.s.c. group.

**Case 1.** If $\alpha$ is a limit ordinal then

$$\text{Tor}(H_\alpha, H_\beta) = \bigoplus_{\tau < \alpha} \text{Tor}(H_\tau, H_\beta)$$

which, by our induction hypothesis, is a d.s.c. group.

**Case 2.** If $\alpha$ is not a limit ordinal then let $\alpha = \gamma + 1$. Consider the homomorphism

$$\eta : H_{\gamma + 1} \longrightarrow \frac{H_{\gamma + 1}}{p^\gamma H_{\gamma + 1}} \cong H_\gamma,$$

and for $\delta$ any ordinal such that $f_\delta(H_\beta) \neq 0$ let $H$ be a $p^\delta H_{\gamma + 1}$ high subgroup of $H_{\gamma + 1}$. Since $\delta < \gamma$, $\eta$ is height preserving on $H$ which implies that $H$ can be imbedded isomorphically in a $p^\delta H_{\gamma}$ high subgroup $\tilde{H}$ of $H_\gamma$. In addition since $H$, as a $p^\delta H_{\gamma + 1}$ high subgroup, is isotype in $H_{\gamma + 1}$, the height preserving of $\eta$ implies that $H$ is isotype in $\tilde{H}$. Now by our induction hypothesis Tor($H_\gamma$, $H_\beta$) is a d.s.c. group, so by Theorem 2 (ii) $\tilde{H}$ is a d.s.c. group. Since $H$ is an isotype subgroup of $\tilde{H}$ Theorem 1 implies that $H$ is a d.s.c. group. By our choice of $\delta$, and since $H$ and $H_\beta$ are d.s.c. groups, Theorem 2 implies that Tor($H_\alpha$, $H_\beta$) is a d.s.c. group.

**Corollary 4.** If $A$ and $B$ are totally projective groups such that $\min\{\lambda(A), \lambda(B)\} \leq \Omega$ then Tor($A$, $B$) is a d.s.c. group.

**Proof.** Let $\lambda(A) = \gamma$, $\lambda(B) = \delta \leq \Omega$. $A$ and $B$ totally projective implies

$$A \perp \bigoplus_{\alpha \leq \gamma} H_\alpha$$

and

$$B \perp \bigoplus_{\beta \leq \delta} H_\beta$$

which implies

$$\text{Tor}(A, B) \perp \text{Tor}(\bigoplus_{\alpha \leq \gamma} H_\alpha, \bigoplus_{\beta \leq \delta} H_\beta) = \bigoplus_{\alpha \leq \gamma} \bigoplus_{\beta \leq \delta} \text{Tor}(H_\alpha, H_\beta).$$

By Theorem 3 each Tor($H_\alpha$, $H_\beta$) is a d.s.c. group so Tor($A$, $B$) as a direct summand of a d.s.c. group is itself a d.s.c. group.

We note that since the Ulm invariants of Tor($A$, $B$) can be computed in terms of
the Ulm invariants of $A$ and $B$ (Nunke [4]), we have a complete characterization of Tor$(A, B)$ for $A$ and $B$ as in Corollary 4.

Before continuing we need the following short exact sequences. For $\lambda(A) > \delta \geq \omega$, and $M$ a $p^\delta A$-high subgroup of $A$

(I) $M \rightarrow A \rightarrow A/M$ is $p^{\delta+1}$ pure and $A/M$ is divisible. For $B$ arbitrary the above sequence induces

(II) Tor$(M, B) \rightarrow$ Tor$(A, B) \rightarrow \bigoplus B$ also $p^{\delta+1}$ pure (Nunke, [5]). Additionally we will make use of:

**Theorem 5** (Nunke, [6]). If $C \rightarrow E \rightarrow D$ is $p^\gamma$ pure with $\gamma$ a limit ordinal not cofinal with $\omega$, $E$ reduced, $D$ not zero and divisible, then $C$ is not $p^{\gamma+n}$ projective for $n < \omega$.

We are now prepared to prove the following lemma.

**Lemma 6.** If $\gamma$ is a limit ordinal not cofinal with $\omega$, $A$ a group such that $\lambda(A) > \gamma + n$ for some $n$, $M$ a $p^{\gamma+n} A$-high subgroup of $A$, then $M$ is not totally projective.

**Proof.** The sequence (I) $M \rightarrow A \rightarrow A/M$ is $p^{\gamma+n+1}$ pure so it is $p^{\gamma}$-pure. $A/M \neq 0$ as $\lambda(A) > \gamma + n$ so, by Theorem 5, $M$ is not $p^{\gamma+n}$ projective. Since $\lambda(M) < \gamma + n$, if $M$ were totally projective it would be $p^{\gamma+n}$-projective which it is not. Thus $M$ is not totally projective.

We have one more lemma, a generalization of results by Nunke [5], relating total projectivity of Tor$(A, B)$ to conditions on the groups $A$ and $B$, to consider. First, two well known facts concerning totally projective groups (Fuchs [1]).

Fact (1) For $B$ totally projective, if $f_\beta(B) \neq 0$ then there exists $B' \perp B$ such that $\lambda(B') = \beta + 1$.

Fact (2) For $B$ totally projective, if $\lambda(B) > \alpha$, $\alpha$ a limit ordinal, then there exists $n$ such that $f_\alpha+n (B) \neq 0$.

**Lemma 7.** Assume Tor$(A, B)$ is totally projective.

(i) If $\lambda(A) > \lambda(B)$ then $B$ is totally projective.

(ii) If $\lambda(A) \geq \lambda(B) = \gamma + 1$, $B$ totally projective, and $M$ a $p^\gamma A$-high subgroup of $A$ then $M$ is totally projective.

(iii) If $\lambda(A) \geq \lambda(B)$, $B$ is totally projective, and for some $\gamma$, $f_\gamma(B) \neq 0$ then every $p^\gamma A$-high subgroup of $A$ is totally projective.

**Proof.** To prove (i) let $\lambda(B) = \gamma$. Let $M$ be a $p^\gamma A$-high subgroup of $A$ then the sequence (I) $M \rightarrow A \rightarrow A/M \neq 0$ is $p^{\gamma+1}$-pure and the sequence (II) Tor$(M, B) \rightarrow$ Tor$(A, B) \rightarrow \bigoplus B$ is also $p^{\gamma+1}$-pure. Since $\lambda(\text{Tor}(A, B)) = \lambda(B) = \gamma$ and Tor$(A, B)$ is assumed totally projective it is $p^\gamma$-projective. Thus the sequence splits by Nunke [4] which implies $B \perp \text{Tor}(A, B)$ and thus $B$ as a direct summand of a totally projective is totally projective.

For (ii) choose $M$ as above and notice that $B$ totally projective and $\lambda(B) = \gamma + 1$ implies that $B$ is $p^{\gamma+1}$-projective, thus by Nunke [6] the sequence (II) splits. Thus Tor$(M, B) \perp \text{Tor}(A, B)$ so Tor$(M, B)$ is totally projective. Since $\lambda(M) < \lambda(B)$ (i) implies $M$ is totally projective.
To prove (iii) we assume \( f_\gamma(B') \neq 0 \) for some \( \gamma \). By Fact (1) there exists \( B' \perp B \) such that \( \lambda(B') = \gamma + 1 \). Let \( B = B' \oplus C \). Then \( \text{Tor}(A, B) = \text{Tor}(A, B') \oplus \text{Tor}(A, C) \) which implies that \( \text{Tor}(A, B') \) is totally projective. Now choose \( M \) a \( p^nA \)-high subgroup of \( A \). Since \( \lambda(A) \geq \lambda(B') = \gamma + 1 \) and \( B' \) is totally projective (ii) implies that \( M \) is totally projective.

We are now in a position to prove our next theorem.

**Theorem 8.** If \( A \) and \( B \) are totally projective, \( \lambda(A) > \Omega, \lambda(B) > \Omega \), then \( \text{Tor}(A, B) \) is not totally projective.

**Proof.** We assume w.o.l.o.g. that \( \lambda(A) \geq \lambda(B) \) and that \( \text{Tor}(A, B) \) is totally projective. Since \( \lambda(B) > \Omega \) by our Fact (2) there exists an \( n \) such that \( f_\Omega + n(B) \neq 0 \). Choose \( M \) a \( p^{\Omega + n}A \)-high subgroup of \( A \). By Lemma 7 (iii) \( M \) is totally projective. However, since \( f_\Omega + n(B) \neq 0 \) implies \( \lambda(B) > \Omega + n \) which gives \( \lambda(A) > \Omega + n \) and since \( \Omega \) is a limit ordinal not cofinal with \( \omega \) Lemma 6 implies that \( M \) is not totally projective, a contradiction. Therefore \( \text{Tor}(A, B) \) is not totally projective.

Combining Corollary 4 and Theorem 8 we can now give our main theorem.

**Theorem 9.** For \( A \) and \( B \) totally projective, \( \text{Tor}(A, B) \) is totally projective if and only if \( \min(\lambda(A), \lambda(B)) \leq \Omega \).

**Proof.** If \( \min(\lambda(A), \lambda(B)) \leq \Omega \) then by Corollary 4, \( \text{Tor}(A, B) \) is a d.s.c. so it is totally projective. On the other hand assume \( \text{Tor}(A, B) \) is totally projective. If \( \min(\lambda(A), \lambda(B)) > \Omega \) then by Theorem 8, \( \text{Tor}(A, B) \) is not totally projective, a contradiction. Therefore \( \min(\lambda(A), \lambda(B)) \leq \Omega \).

Since totally projective groups of length less than or equal \( \Omega \) are d.s.c. we get an immediate corollary.

**Corollary 10.** For \( A \) and \( B \) totally projective, \( \text{Tor}(A, B) \) is totally projective if and only if it is a d.s.c. group.

We make the observation that in the statement of Theorem 8, \( A \) may be taken to be an arbitrary reduced \( p \)-group and the proof still holds. This raises the question as to whether both groups, \( A \) and \( B \), may be taken to be arbitrary. A partial answer is given in the following theorem.

**Theorem 11.** If \( A \) and \( B \) are groups such that \( \lambda(A) > \lambda(B) > \Omega \) then \( \text{Tor}(A, B) \) is not totally projective.

**Proof.** Suppose \( \text{Tor}(A, B) \) is totally projective, \( \lambda(A) > \lambda(B) \) implies by Lemma 7 (i) that \( B \) is totally projective. The rest of the proof follows as in Theorem 8.

The case \( \lambda(A) = \lambda(B) > \Omega \) is an open problem.

We conclude this paper by giving two applications of our results. The first characterizes \( p^nA \)-high subgroups of the totally projective group \( A \) for ordinals \( \alpha \leq \Omega \). In particular we show that such subgroups are themselves totally projective and, in addition, are isomorphic.

**Theorem 12.** If \( A \) is totally projective then for arbitrary \( \alpha \leq \Omega \) all \( p^nA \)-high
subgroups of $A$ are totally projective and isomorphic.

Proof. Let $M$ be $p^nA$-high in $A$. Then, as before, the $p^{n+1}$ pure sequence $M \rightarrow A \rightarrow \bigoplus Z(p^n)$ gives $\text{Tor}(A, H_{q}) \cong \text{Tor}(M, H_{q}) \oplus (\bigoplus H_{q})$. Since $\text{Tor}(A, H_{q})$ is totally projective by Corollary 4 and $\text{Tor}(M, H_{q}) \perp \text{Tor}(A, H_{q})$ we have $\text{Tor}(M, H_{q})$ is totally projective. Since $\lambda(M) < \lambda(H_{q}) = \alpha$, Lemma 7 implies that $M$ is totally projective. If $M'$ is another $p^nA$-high subgroup of $A$ then by Irwin and Walker [3] $f_{q}(M) = f_{q}(M') \forall \delta$ which, since $M$ and $M'$ are totally projective, implies that $M \cong M'$.

Our second application gives a rather interesting example based on Corollary 4. In particular we describe certain groups $A$ which have the property that $\text{Tor}(A, A) \cong A$.

Example. Let $A$ be totally projective, $\lambda(A) \leq \Omega$, and $f_{q}(A) = r_{q+1}(A) \geq \chi_{0}$ for all $\alpha < \lambda(A)$. From our assumption and from Nunke [4] we have

$$f_{q} \left( \text{Tor}(A, A) \right) = f_{q}(A) f_{q}(A) + f_{q}(A) r_{q+1}(A) + r_{q+1}(A) f_{q}(A) = f_{q}(A).$$

Since $A$ is totally projective and, by Corollary 4, $\text{Tor}(A, A)$ is also totally projective, it follows that $\text{Tor}(A, A) \cong A$. For a specific choice satisfying our requirements consider $H_{\alpha^2}$ and notice that $\text{Tor}(H_{\alpha^2}, H_{\alpha^2}) \cong H_{\alpha^2}$.

References


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