A Simple Proof of a Theorem of Stratton

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Notation in this note follows that of [3], and [4].
A major step towards classifying the nil rank two torsion free groups is the following theorem of Stratton [4, Theorem 3.3]:

**THEOREM.** Let $G$ be a rank two torsion free group. If $G$ is not nil then $T(G)$ possesses a unique minimal element, and $\# T(G) \leq 3$.

As indicated in the introduction to [3], an examination of the proof of Freedman's theorem [1] reveals that she proved more than she stated. Namely, "Let $G$ be a rank two torsion free group. If $G$ is not nil, and $T(G)$ possesses a unique minimal element, then $\# T(G) \leq 3$.”

In lieu of this fact, it suffices to prove the following:

**LEMMA.** Let $G$ be a rank two torsion free group. If $G$ is not nil then $T(G)$ possesses a unique minimal element.

Stratton proved the above Lemma [4, Corollary 3.2]. However his proof relies on rather heavy machinery (the nucleus of a group, the splitting of certain exact sequences of modules over a principal ideal ring, quasi-idempotents, etc.). The following is a short and simple proof of the above Lemma, and hence of the above Theorem.

**Proof of Lemma.** Let $R = (G, \cdot)$ be a ring satisfying $R^2 \neq 0$. Suppose there exist $x, y \in R$ with $t(x) \neq t(y)$, and $x \cdot y \neq 0$. Clearly either $t(x \cdot y) > t(x)$, or $t(x \cdot y) > t(y)$. Suppose that $t(x \cdot y) > t(x)$. Then every element of $G$ is a linear combination of $x$, and $xy$ over the rationals, and so $t(x)$ is the unique minimal element in $T(G)$. It may therefore be assumed that:

(A) $x \cdot y = 0$ for all $x, y \in R$ with $t(x) \neq t(y)$.

Suppose that $\# T(G) \geq 3$. Let $x \in G$. There exist $0 \neq y, 0 \neq z \in G$ such that $t(x)$, $t(y)$, $t(z)$ are distinct types. $x = ay + bz$, with $a, b$ rational numbers. Hence $x^2 = ax \cdot y + bx \cdot z = 0$ by (A). Let $0 \neq x, 0 \neq y \in R$, with $t(x) = t(y)$. If $x$ and $y$ are independent, then every element in $G$ is a linear combination of $x$ and $y$ over the rationals, and so $t(x)$ is the unique minimal element in $T(G)$. Therefore it may be assumed that there exist nonzero integers $n, m$ such that $nx = my$. This yields that $nm(x \cdot y) = n^2 x^2 = 0$. Since $G$ is torsion free $x \cdot y = 0$, or
(B) \[ x \cdot y = 0 \quad \text{for all} \quad x, y \in G \quad \text{with} \quad t(x) = t(y). \]

Clearly (A) and (B) yield that \( R^2 = 0 \), a contradiction.

Therefore (A) implies that \( \#T(G) \leq 2 \). Let \( T(G) = \{ \tau_1, \tau_2 \} \). Choose \( x_i \in G \) such that \( t(x_i) = \tau_i \), \( i = 1, 2 \), and put \( x = x_1 + x_2 \). If \( t(x) = \tau_1 \), then the fact that \( x_2 = x - x_1 \) yields that \( \tau_2 \geq \tau_1 \). Similarly \( t(x) = \tau_2 \) implies that \( \tau_1 \geq \tau_2 \). In either case \( T(G) \) possesses a unique minimal element.

The above theorem of Stratton's reduces the problem of classifying nil torsion free groups \( G \) of rank two, to the case \( \#T(G) \leq 3 \). In what follows, the case \( \#T(G) = 3 \) will be considered.

Notation. \( \cong \), and \( = \) will signify quasi-isomorphism, and quasi-equality respectively. \( N(G) \), \( E(G) \), and \( \tilde{E}(G) \) will respectively denote the nucleus of a group \( G \), [4, p. 200], the ring of endomorphisms of \( G \), and the ring of quasi-endomorphisms of \( G \).

**PROPOSITION.** Let \( G \) and \( H \) be torsion free groups with \( G \cong H \). Then \( G \) is nil if
and only if \( H \) is nil.

**Proof.** It may be assumed that \( G \) and \( H \) are subgroups of a group \( A \), and that \( G \cong H \). Suppose that \( H \) is not nil. Let \( R = (H, \cdot) \) be a ring, with \( R^2 \neq 0 \). There exist positive integers \( n, m \) such that \( nG \leq H \), and \( mH \leq G \). For \( g_1, g_2 \in G \), define \( g_1 \cdot g_2 = (ng_1) \cdot (ng_2) \). It is readily seen that \( S = (G, \ast) \) is a ring. Let \( h_1, h_2 \in R \) such that \( h_1 \cdot h_2 \neq 0 \). Then \( (mh_1) \ast (mh_2) = n^2 m^2 (h_1 \cdot h_2) \neq 0 \). Hence \( S^2 \neq 0 \), and \( G \) is not nil. An identical argument shows that if \( G \) is not nil, then neither is \( H \).

**OBSERVATION.** Let \( G \) be a rank two, torsion free group with \( \#T(G) = 3 \). Then \( G \) is not nil if and only if \( G \cong G_1 \oplus G_2 \), with \( G_i \) a rank one torsion free group, \( i = 1, 2 \), and there exist \( i, j, k \in \{1, 2\} \) such that \( t(G_i) \cdot t(G_j) \leq t(G_k) \).

**Proof.** Suppose that \( G \) is quasi-decomposable. Then \( G \cong G_1 \oplus G_2 \), with \( G_i \) a rank one torsion free group, \( i = 1, 2 \). By the above Proposition, \( G \) is nil if and only if \( G_1 \oplus G_2 \) is nil. It is well known that \( G_1 \oplus G_2 \) is nil if and only if for all \( i, j, k \in \{1, 2\} \), \( t(G_i) \cdot t(G_j) \leq t(G_k) \).

Suppose that \( G \) is strongly indecomposable. Then \( E(G) = Q \) (the ring of rational numbers), [1, Theorem 8.4]. Hence \( E(G) = \{ r \in Q \mid r \in G, \text{ for all } x \in G \} = N(G) \). By [4, Lemma 2.2], \( G \) is nil.

**References**

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