Cauchy-Kovalevskaja Theorem for
Functional Partial Differential Equations*

by

Takesi YAMANAKA and Hidemitu TAMAKI

(Received March 28, 1980)

§1. Introduction. The purpose of this paper is to extend the Cauchy-
Kovalevskaja-Nagumo theorem [3] to the case where the equation contains
retardations in time and the space variable as well as the unknown function takes its
values in an arbitrary complex Banach space.

First we have to explain our notation and terminology. Throughout this paper
$T_0$ denotes a fixed positive number and we write $I_0=[-T_0, 0]$. If $E$ is a Banach space
we denote by $C(I_0, E)$ the Banach space of all continuous functions of $I_0$ to $E$, the
norm being the usual sup norm. For any subset $W$ of $\mathbb{R} \times E$ ($\mathbb{R}$ — the set of real
numbers) we write

$$(W)_{+} = \{(t, x) \in W \mid t > 0\}, \quad (W)_{0} = \{(t, x) \in W \mid t = 0\}$$

and

$$\overline{W} = \{(t + \theta, x) \mid \theta \in I_0, (t, x) \in W \}$$

and call them the positive part, the horizontal section and the trailed set, respectively, of
$W$. If $\phi$ is a continuous map from the trailed set $\overline{W}$ to a Banach space $F$, we define a
map $\phi$ from $W$ to $C(I_0, F)$ by

$$\phi(t, x)(\theta) = \phi(t + \theta, x) \quad ((t, x) \in W, \theta \in I_0).$$

We call $\phi$ the trail of $\phi$. If $\phi(t, x)$ has a partial derivative with respect to $t$ or $x$, we
denote it by $\partial_t \phi(t, x)$ or $\partial_x \phi(t, x)$, respectively. If $E$ and $F$ are Banach spaces, we
denote by $L(E, F)$ the Banach space of all continuous linear maps of $E$ to $F$.

Now we can state our problem exactly. What we call here a functional partial
differential equation in the unknown function $u$ is an equation of the form

$$\partial_t u(t, x) = f(t, x, \bar{u}(t, x), \partial_x \bar{u}(t, x)), \quad (1.1)$$

where $f$ is a given map from an open set $\Omega$ of $\mathbb{R} \times E \times C(I_0, F) \times C(I_0, L(E, F))$ into
$F$, $E$ and $F$ being Banach spaces. We say that $u$ is a solution of (1.1) in an open set
$W \subset \mathbb{R} \times E$, if $u$ is defined and continuous in the trailed set $\overline{W}$, the quadruplet
$(t, x, \bar{u}(t, x), \partial_x \bar{u}(t, x))$ exists and is in $\Omega$ for all $(t, x) \in W$ and (1.1) is satisfied for all

* This work is partially supported by Grant-in-Aid for Scientific Research 1979 (D), Ministry of
Education, Science and Culture.
Our problem is to seek a solution \( u \) of (1.1) in the positive part \( (W)^+ \) of an open set \( W \subset \mathbb{R} \times E \) with \( (W)_0 \neq \emptyset \) such that the initial condition

\[
(1.2) \quad u(\theta, x) = \psi(\theta, x) \quad ((\theta, x) \in (W)_0),
\]

is satisfied. Here \( \psi \) is a given function from a set \( D = (W)_0 \) to \( F \).

We shall assume that \( f(t, x, u, v) \) is continuous in \( (t, x, u, v) \) and holomorphic in \( (x, u, v) \) and also that \( \psi(\theta, x) \) is continuous in \( (\theta, x) \) and holomorphic in \( x \). Our aim is to show, under the above mentioned conditions, the unique existence of a solution \( u(t, x) \) of the Cauchy problem (1.1)–(1.2) which is holomorphic in \( x \).

For that purpose we shall prepare in \( \S 2 \) an abstract theorem. It is a “retarded type” variant of Nirenberg-Nishida theorem ([4], [5]) on abstract ordinary differential equations in a “scale” of Banach spaces. In \( \S 3 \) we shall make another preparation. We shall state there, for the sake of completeness, the definition of a holomorphic function in a Banach space and some of its important properties which are needed in \( \S 4 \). In \( \S 4 \) we shall state our result exactly and prove it by using the abstract theorem given in \( \S 2 \).

Our result is an extension of that of Nagumo [3]. Recall that Nirenberg [4] stated, “Though we have given no other applications besides the derivation of Nagumo’s result we hope the abstract theorem will find other uses.” An unfinished form of our result was first given in Tamaki [7].

\section{Retarded type Nirenberg-Nishida theorem.}

Let \( B_0 \) be a real or complex vector space and \( \rho_0 \) a positive number. Suppose that for each number \( \rho \) with \( 0 < \rho \leq \rho_0 \) there is a Banach space \( B_\rho \) which is a linear subspace of \( B_0 \). Suppose further that if \( 0 < \sigma < \rho \leq \rho_0 \) and \( x \in B_\sigma \) then \( x \in B_\rho \) and \( \| x \|_\rho \leq \| x \|_\sigma \), where \( \| \cdot \|_\rho \) and \( \| \cdot \|_\sigma \) are the norms of the spaces \( B_\rho \) and \( B_\sigma \), respectively. Such a family \( \{ B_\rho \}_{0 < \rho \leq \rho_0} \) of Banach spaces is called a scale of Banach spaces. Nirenberg [4] considered an ordinary differential equation of the form

\[
(2.1) \quad u'(t) = G(u(t), t)
\]

in the unknown function \( u \) whose values are in a scale \( \{ B_\rho \} \) of Banach spaces. Under certain assumptions on the function \( G(u, t) \) Nirenberg proved the unique existence of a solution of the initial value problem for the differential equation (2.1) and applied his abstract theorem to giving another proof of a theorem of Nagumo [3] which is an extension of the classical Cauchy-Kovalevskaja theorem to the temporally non-analytic case. Later in 1977 Nishida [5] improved the proof of the abstract theorem of Nirenberg’s and succeeded in much simplifying the assumptions on \( G(u, t) \).

Here we want to use the Nirenberg-Nishida method in order to extend Nagumo’s theorem to the case where the differential equation contains retardations in time. For that purpose, of course, it is necessary to modify the Nirenberg-Nishida theorem so that the abstract differential equation (2.1) itself contains retardations in time. We shall give below the necessary modification of the N-N theorem which is in fact an extension of it.

Let \( \{ B_\rho \}_{0 < \rho \leq \rho_0} \) be a scale of (real or complex) Banach spaces and \( T_0 \) the same
fixed positive number as taken in §1. We denote by $D_\rho$ the Banach space of all continuous functions from the interval $[-T_0, 0]$ to the space $B_\rho$, the norm being defined by
\[
\| \phi \|_\rho = \max_{-T_0 \leq t \leq 0} |\phi(t)|_\rho , \quad \phi \in D_\rho.
\]
The family $\{D_\rho\}_{0 < \rho \leq \rho_0}$ naturally becomes a new scale of Banach spaces, each space $D_\rho$ being a linear subspace of the vector space $D_0 = \bigcup_{0 < \rho \leq \rho_0} D_\rho$.

For any $\rho$ with $0 < \rho \leq \rho_0$ and $r > 0$ we denote by $V_{\rho, r}$ the open ball in the space $D_\rho$ centered at the origin and of radius $r$. Let $R$ and $t_0$ be positive numbers. Suppose that $G(\phi, t)$ is a $B_0$-valued function of
\[
(\phi, t) \in \bigcup_{0 < \rho \leq \rho_0} V_{\rho, R} \times [0, t_0]
\]
which satisfies the following conditions (i)--(iii):

(i) If $0 < \sigma < \rho \leq \rho_0$ and $(\phi, t) \in V_{\rho, R} \times [0, t_0]$, then $G(\phi, t) \in B_\sigma$. $G$ is continuous as a function of $V_{\rho, R} \times [0, t_0]$ to $B_\sigma$.

(ii) There is a positive constant $C$ such that, if $0 < \sigma < \rho \leq \rho_0$, $\phi$ and $\psi$ are in $V_{\rho, R}$ and $0 \leq t \leq t_0$, then
\[
|G(\phi, t) - G(\psi, t)|_\sigma \leq C \| \phi - \psi \|_\rho (\rho - \sigma).
\]

(iii) There is a positive constant $K$ such that, if $0 < \rho < \rho_0$ and $0 \leq t \leq t_0$, then
\[
|G(0, t)|_\rho \leq K(\rho_0 - \rho).
\]

With respect to this function $G(\phi, t)$ we consider the following functional differential equation of retarded type:
\[
(2.2) \quad u'(t) = G(tu(t), t),
\]
where $tu(t)$ denotes an element of some space $D_\rho$ obtained from the $B_\rho$-valued unknown continuous function $u$ by the relation
\[
\dot{u}(t)(\theta) = u(t + \theta) \quad (-T_0 \leq \theta \leq 0).
\]
We say that $u$ is a $B_\rho$-valued solution in the interval $(0, t_1)$ of (2.2), if $u$ is a continuous function from the interval $[-T_0, t_1)$ to $B_\rho$, differentiable with respect to any norm $| \cdot |_\sigma$ with $0 < \sigma < \rho$ in $(0, t_1)$ and satisfies (2.2) in $(0, t_1)$.

By similar arguments to Nishida [5] one can prove the following

**Theorem 1.** Under the conditions (i)--(iii) above for the function $G(\phi, t)$ there is a positive number $\tau$ such that for any $\rho$ with $0 < \rho < \rho_0$ there exists a unique $B_\rho$-valued solution of the equation (2.2) in the interval $0 < t < \tau(1 - \rho/\rho_0)$ which satisfies the initial condition
\[
(2.3) \quad u(\theta) = 0 \quad (-T_0 \leq \theta \leq 0).
\]

For a proof see [6].
³. Holomorphic functions on a Banach space. In the next section we shall need the concept of a holomorphic function on a Banach space. Although it is a known concept, we shall state here, for the sake of completeness, its definition and some of its basic properties.

Let $E, F$ be (real or complex) Banach spaces and $k \geq 2$ an integer. We denote by $L^k(E, F)$ the Banach space of all $k$-linear symmetric maps from $E^k$ to $F$, the norm of its element $\phi$ being defined by

$$\| \phi \| = \sup \{ \| \phi(v_1, \ldots, v_k) \| \mid v_i \in E, \| v_i \| \leq 1 \enspace (i = 1, \ldots, k) \}.$$ 

If $\phi \in L^k(E, F)$ and $v \in E$, we write

$$\phi^k = \phi(v^k) = \phi(v, \ldots, v).$$

Also we write $L(E, F) = L^1(E, F)$ and regard $F$ as $L^0(E, F)$.

Let $U$ be an open set of $E$ and $f$ a map from $U$ to $F$. We say that $f$ is analytic at the point $x_0 \in U$, if there is an element $f_k \in L^k(E, F)$ for each integer $k \geq 0$ and there is a positive number $r$ such that the series

$$\sum_{k=0}^{\infty} \| f_k \| s^k$$

converges for all $s$ with $0 < s < r$ and if for any element $x \in E$ with $\| x \| < r$ we have $x \in U$ and

$$f(x) = \sum_{k=0}^{\infty} f_k(x - x_0)^k.$$

If a map $f: U \to F$ is analytic at every point $x_0 \in U$, we say that $f$ is analytic in $U$. When we deal with complex Banach spaces, however, we say holomorphic instead of analytic. We shall state below some basic properties of analytic functions on a Banach space.

PROPOSITION 1. If $f$ is an analytic function of an open set $U$ of a Banach space $E$ to another Banach space $F$, then $f$ is differentiable and its derivative $f'$ is also analytic in $U$.

The proof is obtained, as usual, by termwise differentiating the expansion (3.1). As to the termwise differentiation theorem see, e.g., Lang [2], Chap. V, §9.

PROPOSITION 2. If $f$ is a differentiable function from an open set $U$ of a complex Banach space $E$ to another complex Banach space $F$, then $f$ is holomorphic in $U$.

See Bourbaki [1], n° 3.3.

COROLLARY 1. The composite map of two holomorphic functions is holomorphic.

COROLLARY 2. If $f$ is the same as in Proposition 2, then for any point $x_0 \in U$ there is a neighborhood $V$ of $x_0$ such that $V \subset U$ and $f$ is uniformly continuous in $V$. 
In fact, since the derivative $f'$ is continuous at $x_0$, there is a neighborhood $V \subset U$ of $x_0$ such that $f'$ is bounded in $V$. Thus $f'$ is uniformly continuous in $V$.

**PROPOSITION 3.** If $f$ is the same as in Proposition 2 and $x_0$ is a point of $U$ which is at a finite distance $r$ from the boundary of $U$, then one has

$$\|f'(x_0)\| \leq \frac{1}{r} \sup_{x \in V} \|f(x)\|.$$  

**Proof.** Take an arbitrary unit vector $e$ in the space $E$ and define an $F$-valued function $\phi_e(z)$ of a complex variable $z$ by

$$\phi_e(z) = f(x_0 + ze).$$  

$\phi_e(z)$ is holomorphic for $|z| < r$ and the Cauchy's integral formula

$$f'(x_0)e = \phi'_e(0) = \frac{1}{2\pi i} \int_{|z| = s} \frac{\phi_e(z)}{z^2} \, dz$$

holds for any $s$ with $0 < s < r$. Hence we have

$$\|f'(x_0)e\| \leq \inf_{0 < s < r} \left\| \frac{1}{2\pi i} \int_{|z| = s} \frac{\phi_e(z)}{z^2} \, dz \right\| \leq \frac{1}{r} \sup_{x \in V} \|f(x)\|$$

and

$$\|f'(x_0)\| = \sup_{\|e\| = 1} \|f'(x_0)e\| \leq \frac{1}{r} \sup_{x \in V} \|f(x)\|.$$

q.e.d.

**COLLORARY.** Under the same assumptions as in Proposition 3 one has

$$\|f^{(n)}(x_0)\| \leq \frac{n^n}{r^n} \sup_{x \in V} \|f(x)\| \quad (n = 1, 2, \ldots).$$

**Proof.** Let

$$V_j = \left\{ x \in E \mid \|x - x_0\| < \frac{j}{n} \right\} \quad (j = 1, 2, \ldots, n).$$

By repeated use of Proposition 3 we have

$$\|f^{(n)}(x_0)\| \leq \frac{1}{(r/n)^{j_1}} \sup_{x \in V_1} \|f^{(n-1)}(x)\|$$

$$\leq \frac{1}{(r/n)^2} \sup_{x \in V_2} \|f^{(n-2)}(x)\| \leq \cdots \leq \frac{1}{(r/n)^n} \sup_{x \in V_n} \|f(x)\|.$$

q.e.d.

**PROPOSITION 4.** Let $\{f_n\} (n = 1, 2, \ldots)$ be a sequence of holomorphic functions from an open set $U$ of a complex Banach space $E$ to another complex Banach space $F$. Assume that $\{f_n\}$ converges uniformly to a function $f: U \to F$. Then $f$ is holomorphic.
Proof. Let $x_0$ be a point of $U$ and $r$ be a positive number such that the open ball $V = \{ x \in E \mid \|x - x_0\| < 2r \}$ is contained in $U$. Since $\{ f_n \}$ converges uniformly to $f$ in $V$, it follows from Proposition 3 that the sequence of derivatives $\{ f_n' \}$ converges uniformly in the ball $V_1 = \{ x \mid \|x - x_0\| < r \}$ to a function $g : V_1 \to F$. Hence, by the termwise differentiation theorem, we see that $f$ is differentiable, and therefore holomorphic, in $V_1$ and $f' = g$ there.

COROLLARY. The set of all uniformly continuous holomorphic functions from a bounded open set of a complex Banach space to another complex Banach space is a Banach space by the usual sup norm.

§4. Initial value problem for a functional partial differential equation. Now let us return to the initial value problem (1.1)–(1.2). Note first, however, that we can let, without loss of generality, the initial function $\psi(\theta, x)$ in (1.2) be identically zero, since otherwise we may take

$$w(t, x) = \begin{cases} u(t, x) - \psi(0, x) & \text{if } t \geq 0 \\ u(t, x) - \psi(t, x) & \text{if } -T_0 \leq t \leq 0 \end{cases}$$

as a new unknown function. So we now want to solve the functional differential equation

$$(4.1) \quad \partial_t u(t, x) = f(t, x, \bar{u}(t, x), \partial_x u(t, x))$$

under the initial condition

$$(4.2) \quad u(\theta, x) = 0.$$  

In what follows $E$ and $F$ denote complex Banach spaces. Our purpose in this paper is to prove the following

THEOREM 2. Let $\Omega$ be an open neighborhood of the origin of $\mathbb{R} \times E \times C(I_0, F) \times C(I_0, L(E, F))$ and $f$ a continuous function from $\Omega$ to $F$. Assume that $f(t, x, \phi, \psi)$ is holomorphic in $(x, \phi, \psi)$. Then there is an open neighborhood $W$ of the origin of $\mathbb{R} \times E$ such that there exists a unique solution $u$ of (4.1) in the positive part $(W)_+$ of $W$ satisfying the initial condition (4.2) for $(\theta, x) \in (W)_0$.

To prove the above theorem we first reduce the problem to the case where the differential equation is quasi-linear, namely, of the form

$$(4.3) \quad \partial_t u(t, x) = p(t, x, \bar{u}(t, x)) \partial_x u(t, x) + q(t, x, \bar{u}(t, x)),$$

where the values of the coefficient $p$ are in the space $L(C(I_0, L(E, F)), F)$. Since the reduction procedure is quite standard, we omit it (See Nagumo [3] or Nirenberg [4]) and state the conclusion only. As the conclusion we see that it suffices for our purpose to prove the following

THEOREM 2'. In (4.3) assume that $p$ and $q$ are continuous functions from an open neighborhood $V$ of the origin of the space $\mathbb{R} \times E \times C(I_0, F)$ into $L(C(I_0, L(E, F)), F)$ and
Assume further that \( p(t, x, \phi) \) and \( q(t, x, \phi) \) are holomorphic in \((x, \phi)\). Then there is an open neighborhood \( W \) of the origin of \( \mathbb{R} \times E \) such that there exists a unique solution \( u \) of \((4.3)\) in the positive part \((W)_+ \) of \( W \) satisfying the initial condition \((4.2)\) for \((0, x) \in (W)_0\).

In order to prove the above theorem by the abstract theorem given in \( \S \) 2, however, we need to make some preparation.

For any positive number \( \rho \) we denote by \( X_\rho \) the open ball in the space \( E \) centered at the origin and of radius \( \rho \).

For any positive number \( \rho \) and any Banach space \( G \) we denote by \( B_\rho(G) \) the vector space of all functions from \( X_\rho \) to \( G \) which are uniformly continuous and holomorphic. For each element \( \phi \in B_\rho(G) \) we define its norm \( \| \phi \|_\rho \) by

\[
\| \phi \|_\rho = \sup \{ |\phi(x)| : x \in X_\rho \}
\]

where \( | \cdot | \) is the norm in \( G \). The space \( B_\rho(G) \) with the norm \( \| \cdot \|_\rho \) is a Banach space. If \( \phi \in B_\rho(G) \), \( 0 < \sigma < \rho \) and \( |x| \leq \sigma \), we have by Proposition 3 of \( \S \) 3 and its corollary

\[
|\phi'(x)| \leq |\phi|_\rho / (\rho - \sigma)
\]

and

\[
|\phi''(x)| \leq 4 |\phi|_\rho / (\rho - \sigma)^2.
\]

By \((4.5)\) we see that \( \phi' \) is uniformly continuous in \( X_\sigma \) and belongs to the space \( B_\rho(G) \).

Next we denote by \( D_\rho(G)(\rho > 0) \) the Banach space of all continuous functions from the interval \( I_0 \) to \( B_\rho(G) \), the norm being defined there by

\[
\| \psi \|_\rho = \max_{t \in I_0} |\psi(t)|_\rho.
\]

If \( \psi \in D_\rho(G) \) and \( x \in X_\rho \), it is clear that the map

\[
t \mapsto \psi(t)(x) \quad (t \in I_0)
\]

is an element of the space \( C(I_0, G) \). We denote this element of \( C(I_0, G) \) by \( \psi(x) \). Also we write \( \psi(t, x) = \psi(t)(x) \). So we denote by the same letter \( \psi \) three maps simultaneously, namely, the maps \( I_0 \to B_\rho(G), \quad X_\rho \to C(I_0, G) \) and \( I_0 \times X_\rho \to G \). We denote the map \( I_0 \ni t \mapsto \partial_x \psi(t, x) \in L(E, G) \) by \( \partial_2 \psi(x) \).

**Lemma 1.** If \( \psi \in D_\rho(G) \), then \( \partial_2 \psi(x) \) \( (x \in X_\rho) \) is continuous, i.e., \( \partial_2 \psi(x) \in C(I_0, L(E, G)) \).

In fact, if \( \psi \in D_\rho(G) \), \( |x| < \rho \) and \(-T_0 \leq s \leq t \leq 0\), then we have

\[
|\partial_x \psi(s, x) - \partial_x \psi(t, x)| \leq |\psi(s) - \psi(t)|_\rho / (\rho - |x|),
\]

where \( \partial_x \psi(s, x) \) is the derivative of \( \psi \) with respect to \( x \) evaluated at \( (s, x) \).

---

1) In general, if \( A, B \) and \( C \) are sets and \( f \) is a map from \( A \times B \) to \( C \), there are four maps which are closely connected with \( f \). They are \( f_x : B \ni y \mapsto f(x, y) \in C \) \((x \in A \text{ fixed})\), \( f_y : A \ni x \mapsto f(x, y) \in C \) \((y \in B \text{ fixed})\), \( g : A \ni x \mapsto f_x \in C^g \) \((\text{the set of all maps of } B \text{ to } C)\) and \( h : B \ni y \mapsto f_y \in C^h \). Among these we identify \( g \) and \( h \) with \( f \) and write \( f(x) = g(x) = f_x \) and \( f(y) = h(y) = f_y \). So we consider, in particular, that both \( C(I_0, L(E, F)) \) and \( L(E, C(I_0, F)) \) are subsets of the power set \( F_0^{* \times E} \). Note, however, that these two subsets are not identical.
which shows that \( \partial_2 \psi(x) \) is in \( C(I_0, L(E, G)) \).

From the above proof it is also clear that, for any fixed \( h \in E \), the map \( t \mapsto \partial_x \psi(t, x)h \) is continuous. We denote this map \( I_0 \to G \) (\( \in C(I_0, G) \)) by \( \partial_2 \psi(x)h \).

**Lemma 2.** The map \( E \ni h \mapsto \partial_2 \psi(x)h \) is linear and continuous.

In fact, the linearity is obvious. As for the continuity denote by \( \| \cdot \| \) the norm in the space \( C(I_0, G) \). Then we have

\[
\| \partial_2 \psi(x)h \| = \sup_{t \in I_0} |\partial_x \psi(t, x)h| \leq \sup_{t} |\psi(t)|_\rho|h|/(\rho - |x|)
\]

\[
= \| \psi \|_\rho|h|/(\rho - |x|),
\]

which shows that the linear operator \( h \mapsto \partial_2 \psi(x)h \) is bounded and consequently is continuous.

By Lemmas 1 and 2 we see that \( \partial_2 \psi(x) \) is, in the sense of the footnote of the preceding page, in the intersection \( C(I_0, L(E, G)) \subset L(E, C(I_0, G)) \).

**Lemma 3.** If \( \psi \in D_\rho(G) \), then \( \psi \) is differentiable as a function from \( X_\rho \) to \( C(I_0, F) \) and its derivative \( \psi^\prime : X_\rho \to L(E, C(I_0, F)) \) is given by \( \psi^\prime(x) = \partial_2 \psi(x) \).

In fact, if \( \psi \in D_\rho(G), x \in E, h \in E, t \in I_0 \) and \( \sigma = \max(|x|, |x + h|) < \rho \), then we have

\[
|\psi(t, x+h) - \psi(t, x) - \partial_2 \psi(t, x)h| = \left| \int_0^1 \int_0^1 \xi \partial_x^2 \psi(t, x + \xi \eta)h^2 d\xi d\eta \right|
\]

\[
\leq 4\|\psi\|_\rho|h|^2/(\rho - \sigma)^2,
\]

\[
\|\psi(x+h) - \psi(x) - \partial_2 \psi(x)h\| \leq 4\|\psi\|_\rho|h|^2/(\rho - \sigma)^2
\]

and

\[
\psi^\prime(x) = \partial_2 \psi(x).
\]

**Lemma 4.** If \( \psi \in D_\rho(G) \), then the map \( X_\rho \ni x \mapsto \partial_2 \psi(x) \in C(I_0, L(E, G)) \) is differentiable.

In fact, let \( \psi \in D_\rho(G) \) and \( 0 < \sigma < \rho \). For each fixed \( t \in I_0 \) the map \( X_\rho \ni x \mapsto \partial_2 \psi(t, x) \) is obviously in the space \( B_\rho(L(E, G)) \). Denote this map by \( \phi(t) \). The continuity of the map \( I_0 \ni t \mapsto \phi(t) \) follows again from (4.6). Hence \( \phi \) belongs to the space \( D_\rho(L(E, G)) \). Further, using our notational convention that we regard the element \( \psi \) of \( D_\rho(G) \) as a map from \( X_\rho \) to \( C(I_0, G) \), we can write

\[
\partial_2 \psi(x) = \phi(x) \quad (x \in X_\rho).
\]

Therefore \( \partial_2 \psi(x) \) is differentiable in \( x \in X_\rho \), by Lemma 3.

In what follows we denote the space \( B_\rho(F) \) and \( D_\rho(F) \) simply by \( B_\rho \) and \( D_\rho \), respectively.

Now look at our functional partial differential equation (4.3). First we take, by Corollary 2 of Proposition 2 in §2, a neighborhood \( V' \) of the origin of the space \( \mathbb{R} \times E \times C(I_0, F) \) such that \( V' \subset V \) and \( p, q \) and their partial derivatives with respect to
the third variable are all bounded and uniformly continuous in $V'$. There are positive constants $C_p$, $C_q$, $C_p'$, $C_q'$ such that for all $(t, x, \phi) \in V'$ we have

\begin{equation}
\begin{aligned}
&|p(t, x, \phi)| \leq C_p, \quad |q(t, x, \phi)| \leq C_q, \\
&|\partial_\phi p(t, x, \phi)| \leq C_p', \quad |\partial_\phi q(t, x, \phi)| \leq C_q'.
\end{aligned}
\end{equation}

Next we take a positive number $t_0$ and two open balls $X \subset E$ and $U \subset C(I_0, F)$ centered at the origin and of radius $\rho_0 > 0$ and $R > 0$, respectively, such that $[0, t_0) \times X \times U \subset V'$.

For any $\rho$ with $0 < \rho \leq \rho_0$ and $r > 0$ we denote by $U_{\rho, r}$ the open ball in the space $D_\rho$ centered at the origin and of radius $r$.

If $u \in U_{\rho, R}$, then, by Lemma 1, the expression

\[ v(t, x) = p(t, x, u(x))\partial_2 u(x) + q(t, x, u(x)) \]

is well defined as a point of $F$ for $0 \leq t < t_0$ and $x \in X_p$. It is holomorphic in $x$, because $p(t, x, \phi)$ and $q(t, x, \phi)$ are holomorphic in $(x, \phi)$, $u(x)$ and $\partial_2 u(x)$ are holomorphic in $x$ and composite maps of holomorphic maps are holomorphic. Further, if $0 < \sigma < \rho$, then it is easy to see that $v(t, x)$ is uniformly continuous for $(t, x) \in [0, t_0) \times X_\sigma$. Hence the map $X_\sigma \ni x \mapsto v(t, x)$ is an element of the space $B_\sigma$ for each fixed $t \in [0, t_0)$ and the map $[0, t_0) \ni t \mapsto v(t, \cdot) \in B_\sigma$ is continuous. Therefore, if $0 < \sigma < \rho \leq \rho_0$, we can define a $B_\sigma$-valued continuous function $G(u, t)$ of $(u, t) \in U_{\rho, R} \times [0, t_0)$ by

\begin{equation}
G(u, t)(x) = p(t, x, u(x))\partial_2 u(x) + q(t, x, u(x)) \quad (x \in X_\sigma).
\end{equation}

From the discussion before defining $G(u, t)$ we already know that this function $G$ satisfies the condition (i) in §2. Let us see further that $G$ satisfies the conditions (ii) and (iii) in §2 as well.

From (4.7) we first obtain for $0 < \rho \leq \rho_0$

\[ |G(0, t)|_\rho = \sup \{|q(t, x, 0)| : x \in X_\rho\} \leq C_q,
\]

from which the condition (iii) in §2 trivially follows if we let $K = \rho_0 C_q'$. Further, if $0 < \sigma < \rho \leq \rho_0$, $\phi_i \in U_{\rho, R}$ $(i = 1, 2)$, $x \in X_\sigma$ and $t \in (0, t_0)$, then we have, using (4.4) and (4.7),

\begin{align*}
|G(\phi_1, t)(x) - G(\phi_2, t)(x)| &\leq |p(t, x, \phi_1(x))|_\rho \partial_2 \phi_1(x) - \partial_2 \phi_2(x)| \\
&+ |[p(t, x, \phi_1(x)) - p(t, x, \phi_2(x))]|_\rho \partial_2 \phi_2(x)| \\
&+ |q(t, x, \phi_1(x)) - q(t, x, \phi_2(x))| \\
&\leq C_p \|\phi_1 - \phi_2\|_{\rho/(\rho - \sigma)} + C_p' \|\phi_1 - \phi_2\|_\rho + C_q' \|\phi_1 - \phi_2\|_\sigma \\
&\leq (C_p + C_p' R + C_q' \rho_0) \|\phi_1 - \phi_2\|_{\rho/(\rho - \sigma)},
\end{align*}

which implies the condition (iii) with $C = C_p + C_p' R + C_q' \rho_0$.

By Theorem 1, therefore, we see that there is a positive number $\tau$ such that for any $\rho$ with $0 < \rho < \rho_0$ the initial value problem (2.2)–(2.3) has a unique $B_\rho$-valued solution in the interval $0 < t < \tau(1 - \rho/\rho_0)$. 
For each $\rho$ with $0 < \rho < \rho_0$ we denote by $u_\rho$ the $B_\rho$-valued solution of the initial value problem (2.2)–(2.3) in the interval $0 < t < \tau(1 - \rho/\rho_0)$. Then, if we let
\[ W = \{(t, x) \in \mathbb{R} \times E \mid -T_0 \leq t < \tau, |x| < \rho_0(1 - \max(0, t/\tau))\}, \]
we can define a function $u$ from $W$ to $F$ by
\[ u(t, x) = u_\rho(t)(x), \]
where $\rho$ is any number with $|x| < \rho < \rho_0(1 - \max(0, t/\tau))$. The function $u$ is uniquely defined by (4.9), since $u_\rho(i) = u_\rho(i)$, if $0 < \sigma < \rho$ and $-T_0 \leq t < \tau(1 - \rho/\rho_0)$.

By the relation between (4.3) and (2.2) (through (4.8)) it is clear that the function $u$ defined by (4.9) is a solution of (4.3) in the positive part $(W)_+$ of the open neighborhood of the origin of $\mathbb{R} \times E$ and satisfies (4.2) for $(\theta, x) \in (W)_0$. Conversely it is also clear that a solution of the problem (4.3)–(4.2) gives rise to a solution of the problem (2.2)–(2.3). Thus we have proved Theorem 2′ and accordingly Theorem 2, too.

References


Department of Mathematics
College of Science and Technology
Nihon University
Chiyoda-ku Kanda-Surugadai
Tokyo.