Tsuji points and inner functions

by

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(Received May 4, 1976)

Abstract

Let $D$ be the open unit disk in the complex plane and $C$ the unit circle. Let $f$ be a meromorphic function on $D$. Let $E(f)$ be the set of points $e^{i\phi}$ on $C$, where $f$ can be extended continuously (in the metric of the Riemann sphere) to an open arc of $C$ containing $e^{i\phi}$. Let $T(f)$ be the set of Tsuji points of $f$. In this note the following statement is proved.

**Theorem.** If $f$ is an inner function on $D$, then $T(f) = E(f)$.

1.

Let $D$ be the open unit disk in the complex plane. Let $C$ be the unit circle which is the boundary of $D$. Let $f$ be a meromorphic function in $D$. We denote by $E(f)$ the set of points $e^{i\phi}$ in $C$, where $f$ can be extended continuously (in the metric of the Riemann sphere) to an open arc of $C$ containing $e^{i\phi}$. Let $T(f)$ denote the set of Tsuji points of $f$, that is, the set of points $e^{i\phi}$ in $C$ satisfying the condition

$$\sup_{0 < r < 1, \delta > 0} \int_{\phi - \delta}^{\phi + \delta} f^*(re^{i\theta})d\theta < \infty$$

for some $\delta > 0$, where $f^*(z) = |f'(z)|/1 + |f(z)|^2$. It is clear that if $f$ can be extended analytically to a point $e^{i\phi}$ in $C$, then the point $e^{i\phi}$ is a Tsuji point of $f$. A function $f$ is called a Tsuji function if $T(f) = C$. We say that a function $f$ is normal in $D$ if

$$\sup_{z \in D} (1 - |z|)|f^*(z)| < \infty.$$ 

Clearly, all bounded holomorphic functions on $D$ are normal in $D$.

2.

F. Bagemihl [1] proved that if $\mathcal{L}$ is the Banach space of all holomorphic functions in $D$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\|f\| = \sum_{n=0}^{\infty} |a_n| < \infty$, then the set of all Tsuji functions in $\mathcal{L}$ is a residual

* The author would like to express his thanks to the Princeton University and to the Mathematics Department for their hospitality during his visit at Princeton; and to the Danforth Foundation for the Danforth Fellowship.
subset of $\mathcal{L}$. This means that most functions in $\mathcal{L}$ are not Tsuji functions. Therefore it is important to know how large the Tsuji set $T(f)$ of a function $f$ is. C. Belna and P. Colwell [2] proved the following two statements.

**Lemma A.** If $f$ is a normal meromorphic function in $D$, then $T(f) \subset E(f)$.

**Lemma B.** For each Blaschke product $B(z)$ we have $T(B) = E(B)$.

3.

In this note we try to prove that Lemma B holds for a larger class of normal functions, i.e., the class of inner functions. At first we show that Lemma B holds for all singular functions.

**Lemma 1.** If $S$ is a singular function, then $T(S) = E(S)$.

**Proof.** Since

$$S(z) = \exp \left[ - \int_{-\infty}^{\infty} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right],$$

for $z \in D$, where $\mu$ is a finite positive Borel measure on the unit circle $C$, which is singular with respect to the Lebesgue measure on $C$. By a well-known theorem (see Hoffman [3], p. 68) $S$ is analytic everywhere on the complex plane except at those points of the unit circle which are in the closed support of the measure $\mu$. The function $S$ (or $|S|$) is not continuously extendable from the interior of $D$ to any point in the closed support $\text{supp} (\mu)$ of $\mu$. Therefore $E(f) = C - \text{supp} (\mu) \subset T(f)$. Thus by Lemma A, we have $E(f) = T(f)$. Q.E.D.

We also need the following lemma.

**Lemma 2.** Let $f = BS$, where $B$ is a Blaschke product and $S$ is a singular function. Then $E(B) \cap E(S) = E(f)$.

**Proof.** (a) Since $|f(e^{i\theta})| = 1$ almost everywhere on the unit circle, we have $|f(e^{i\theta})| = 1$ for all $e^{i\theta} \in E(f)$. (b) Suppose that $e^{i\theta} \in E(B) \cap E(S)$ we have either $e^{i\theta} \in E(B)$ or $e^{i\theta} \in E(S)$. If $e^{i\theta} \in E(B)$, then $e^{i\theta}$ is an accumulation point of zeros of $B$; so there exists a sequence $(z_n)_{n=1}^\infty$ in $D$ such that $z_n \to e^{i\theta}$ as $n \to \infty$ and $B(z_n) = 0$ for all $n$. Therefore $e^{i\theta} \in E(f)$; for otherwise, $f(e^{i\theta}) = 0$ which contradicts (a). If $e^{i\theta} \notin E(S)$, then $e^{i\theta}$ is in the closed support of the singular measure $\mu$ which determines $S$. There exists a sequence $(z_n)_{n=1}^\infty$ in $D$ such that $z_n \to e^{i\theta}$ and $S(z_n) \to 0$ as $n \to \infty$; hence, $f(z_n) \to 0$ as $n \to \infty$. It follows that $e^{i\theta} \notin E(f)$, by (a). Therefore we conclude that $E(f) \subset E(B) \cap E(S)$. Since the opposite direction $E(B) \cap E(S) \subset E(f)$ is clear, we have the desired equality. Q.E.D.
4.

Now we are in a position to prove the following result.

THEOREM. *If* $f$ *is an inner function, then* $T(f) = E(f)$.

*Proof.* By Lemma 1, we need only to show that $E(f) \subset T(f)$. Since $f = BS$, where $B$ is a Blaschke product and $S$ is an singular function, by Lemma B, we have $T(B) = E(B)$. Also, by Lemma 1, $T(S) = E(S)$. Hence by Lemma 2, we have $E(f) = E(B) \cap E(S) = T(B) \cap T(S) \subset T(f)$. Q.E.D.

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