Modular forms obtained from $L$-functions with Grössen-characters of $\mathbb{Q}(\sqrt{-3})$

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(Received January 20, 1966)

1. Let

$$L_{1k}^{(1)}(s) = \sum_{(\mu) \in \mathbb{Q}(\sqrt{-3})} \mu^{2k-1} \chi_{j}(\mu) N(\mu)^{-s}$$

be the $L$-functions with Grössen-characters $\mu^{2k-1} \chi_j(\mu)$ for ideals $(\mu)$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$ over the rational number field $\mathbb{Q}$, where $\chi_j(\mu)$ denotes a Klassen-character modulo $2\sqrt{-3}$ if $k \equiv 0, 1 \pmod{3}$, $\chi_j(\mu)$ denotes a Klassen-character modulo $\sqrt{-3}$ if $k \equiv 2 \pmod{3}$, $N(\mu)$ is the norm of $\mu$, and the summation is taken over all integral ideals $(\mu)$ of $\mathbb{Q}(\sqrt{-3})$. By a simple computation we have

$$L_{1k}^{(1)}(s) = \sum_{\mu \equiv 1 \pmod{2 \sqrt{-3}}} \mu^{2k-1} N(\mu)^{-s}, \quad (N(\mu) \equiv 1 \pmod{6}),$$

and

$$L_{1k}^{(2)}(s) = \sum_{\mu \equiv 1 \pmod{\sqrt{-3}}} \mu^{2k-1} N(\mu)^{-s}, \quad (N(\mu) \equiv 1 \pmod{3}).$$

From the functional equations of $L_{1k}^{(j)}(s) (j=1, 2)$, we can prove that the Dirichlet series $L_{1k}^{(1)}(s)$ and $L_{1k}^{(2)}(s)$ have signature\(^1\)

$$(6, 2k, \pm 1) \quad \text{and} \quad (3, 2k, \pm 1),$$

respectively (cf. [3], Beispiel 3).

Let $F_{1k}^{(j)}(\tau)$ be the function corresponding to $L_{1k}^{(j)}(s)$ under Mellin's transformation $(j=1, 2)$. Then we have

$$F_{1k}^{(1)}(\tau) = \sum_{\mu \equiv 1 \pmod{2 \sqrt{-3}}} \mu^{2k-1} q^{N(\mu)}, \quad N(\mu) \equiv 1 \pmod{6}, \quad \text{where} \quad q = e^{\frac{2\pi i}{6}},$$

(2) $F_{1k}^{(2)}(\tau) = \sum_{\mu \equiv 1 \pmod{\sqrt{-3}}} \mu^{2k-1} q^{N(\mu)}, \quad N(\mu) \equiv 1 \pmod{3}, \quad \text{where} \quad q = e^{\frac{2\pi i}{3}}.$

The functions $F_{1k}^{(1)}(\tau)$ and $F_{1k}^{(2)}(\tau)$ are holomorphic in the upper half-plane of the complex variable $\tau$ and further, as a function of $q$ each is holomorphic in the domain $0 < |q| < 1$ and take the value 0 at $q=0$. Moreover they satisfy the following transformation laws:

$$F_{1k}^{(1)}(\tau + 6) = F_{1k}^{(1)}(\tau),$$

$$F_{1k}^{(2)}(\tau + 3) = F_{1k}^{(2)}(\tau),$$

and

\(^1\) For the definition of 'signature', see Hecke ([2], p. 665).
\[ F_{3k}^{(1)} \left( \frac{-1}{\tau} \right) = \pm (-1)^{k \tau_{3k}} F_{3k}^{(1)}(\tau), \]

(3)

\[ F_{2k}^{(1)} \left( \frac{-1}{\tau} \right) = \pm (-1)^{k \tau_{2k}} F_{2k}^{(1)}(\tau). \]

2. Let \( \Gamma(1) \) be the inhomogeneous modular group, and \( \Gamma(1)^c \) be the commutator subgroup of \( \Gamma(1) \). Then it is well known that \( \Gamma(1)^c \) is of index 6 in \( \Gamma(1) \) and a free group of rank 2 generated freely by \( S^{-1}T^{-1}ST \) and \( TS^{-1}T^{-1}S \), where \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \)

We shall denote by \( \Gamma_1 \) a maximal subgroup of \( \Gamma(1) \) such that

\[ F_2^{(1)} \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^k F_2^{(1)}(\tau) \]

for all \((a \ b) \in \Gamma_1\). It is easy to show that \( \Gamma_1 \supset \Gamma(1)^c \). By (1), we obtain the following transformation law for \( F_2^{(1)}(\tau) \):

\[ F_2^{(1)}(\tau + 1) = e^{2 \pi i} F_2^{(1)}(\tau); \]

and hence \( \Gamma_1 \) is of index \( \geq 6 \) in \( \Gamma(1) \). Therefore we have

\[ \Gamma_1 = \Gamma(1)^c. \]

We shall denote by \( \Gamma_2 \) a maximal subgroup of \( \Gamma(1) \) such that

\[ F_3^{(1)} \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^k F_3^{(1)}(\tau), \]

and

\[ F_{2k}^{(1)} \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^k F_{2k}^{(1)}(\tau) \]

for all \((a \ b) \in \Gamma_2\) and \(k = 1, 2, 3, \ldots\). \( \Gamma_2 \supset \Gamma_1 \) is obvious. By (1) and (2), we have the following transformation laws

\[ F_3^{(1)}(\tau + 1) = e^{\frac{2 \pi i}{6}} F_3^{(1)}(\tau), \]

and

\[ F_{2k}^{(1)}(\tau + 1) = e^{\frac{2 \pi i}{2k}} F_{2k}^{(1)}(\tau) \]

for all positive integers \(k\), respectively. By the above transformation laws and (3) we can prove that the generators \( S^{-1}T^{-1}ST \) and \( TS^{-1}T^{-1}S \) of \( \Gamma(1)^c \) belong to \( \Gamma_2 \). Therefore \( \Gamma_2 \supset \Gamma(1)^c \), and hence we have

\[ \Gamma_2 = \Gamma(1)^c. \]
Consequently all the functions $F_{2k}^{(1)}(\tau)$ and $F_{2k}^{\tau_0}(\tau)$ are cusp forms of dimension $-2k$ for $\Gamma(1)^{\tau_0}$.

It is known that a fundamental domain of the group $\Gamma(1)^{\tau}$ is given by

$$D = \left\{ \tau = x + iy; \ y > 0, \ \left| x - \frac{1}{2} \right| < 3, \ \left| \tau - k \right| > 1 \ (k = -3, -2, \ldots, 3, 4) \right\},$$

and $D$ has only one fixed point, i.e., the point $i\infty$. Therefore the compactified Riemann surface associated with $\Gamma(1)^{\tau}$ is of genus 1.

It is known that the function $F_{2k}^{(1)}(\tau)$ is of the form

$$F_{2k}^{(1)}(\tau) = \Delta^{k/2}(\tau),$$

where $\Delta(\tau)$ is the Ramanujan function. This relation has been given by Schoeneberg [5].

We shall denote by $S_{2k}(\Gamma(1)^{\tau})$ the linear space of all cusp forms of dimension $-2k$ with respect to the group $\Gamma(1)^{\tau}$.

The result obtained above is summarized as follows:

**Theorem 1.** The dimension of the linear space $S_{2k}(\Gamma(1)^{\tau})$ is:

$$\dim S_{2k}(\Gamma(1)^{\tau}) = \begin{cases} 1, & \text{if } k = 1, \\ k-1, & \text{if } k > 1. \end{cases}$$

Moreover, \{\Delta^{k/2}(\tau), \Delta^{k-n/2}(\tau)G_{2n}(\tau)\}_{n=1}^{k-1} gives a basis for the above space, where $\Delta(\tau)$ is the Ramanujan function, and $G_{2n}(\tau)$ is the normalized Eisenstein series of dimension $-2n$ for $\Gamma(1)$.

3. We have the following linear relations from the results in §2:

(i) if $k = 3n + 3$,

$$F_{6n+3}^{(1)}(\tau) = \sum_{m=0}^{[n/2]} a^{(1)}(m) \Delta^{\frac{6m+1}{2}} G_{2(3n-6m+1)}(\tau),$$

where $a^{(1)}(0) = 1$.

(ii) if $k = 3n + 1$,

$$F_{6n+1}^{(1)}(\tau) = \sum_{m=0}^{[n/2]} b^{(1)}(m) \Delta^{\frac{6m+1}{2}} G_{2(n-2m)}(\tau),$$

where $b^{(1)}(0) = 1$.

(iii) if $k = 3n + 2$,

Moreover, the functions $F_{sk}^{(1)}(\tau)$ are cusp forms of dimension $-2k$ for $\Gamma^{\tau}$, where $\Gamma^{\tau}$ is the group generated by the third powers of all elements in $\Gamma(1)$. The commutator subgroup $\Gamma(1)^{\tau}$ is the intersection of $\Gamma^2$ and $\Gamma^3$, where $\Gamma^2$ is the group generated by the second powers of all elements in $\Gamma(1)$ (see [4]).

3. For the dimension of the linear space of all modular forms of dimension $-2k$ with respect to a Fuchsian group, refer to Gunning ([1], p. 25).
\[ F_{6n+4}^{(2)}(\tau) = \sum_{m=0}^{[n/3]} a^{(2)}(m) \Delta^{3m+1} (\tau) G_{6(n-2m)}(\tau), \]

where \(a^{(2)}(0) = 1.\) ([n/2] denotes the greatest integer less than or equal to \(n/2.\))

Next we shall determine the linear coefficients in each case. But we only consider the case (iii), as the others may be obtained in a similar way.

Put

\[ \Delta^{3m+1} (\tau) = q^{3m+1} \cdot \prod_{i=1}^{\infty} (1-q^i)^{4(3m+1)} = q^{3m+1} \sum_{r=0}^{\infty} \alpha_m^{(2)}(r) q^r, \]

and

\[ G_{6(n-2m)}(\tau) = 1 - \frac{12(n-2m)}{B_{6(n-2m)}} \sum_{r=1}^{\infty} \sigma_{6(n-2m)-1}(r) q^r, \]

where \(B_{6(n-2m)}\) is the \(6(n-2m)\)-th Bernoulli number, \(\sigma_{6(n-2m)-1}(r) = \sum_{d|r} d^{6(n-2m)-1},\) and \(q = e^{2\pi i \tau}.\) Then

\[ \Delta^{3m+1} (\tau) G_{6(n-2m)}(\tau) \]

\[ = q^{3m+1} \sum_{r=0}^{\infty} \alpha_m^{(2)}(r) q^r \left(1 - \frac{12(n-2m)}{B_{6(n-2m)}} \sum_{r=1}^{\infty} \sigma_{6(n-2m)-1}(r) q^r\right) \]

\[ = q^{3m+1} \left(\sum_{r=0}^{\infty} \alpha_m^{(2)}(r) q^r - \frac{12(n-2m)}{B_{6(n-2m)}} \sum_{r=0}^{\infty} \alpha_m^{(2)}(r) q^r \sum_{r=1}^{\infty} \sigma_{6(n-2m)-1}(r) q^r\right) \]

\[ = q^{3m+1} \left\{1 + \sum_{r=1}^{\infty} \left(\alpha_m^{(2)}(r) - \frac{12(n-2m)}{B_{6(n-2m)}} \sum_{r_1 \geq 0, r_2 > 0} \alpha_m^{(2)}(r_1) \sigma_{6(n-2m)-1}(r_2)\right) q^r\right\}. \]

Hence, it follows that

\[ F_{6n+4}^{(2)}(\tau) = \sum_{m=0}^{[n/3]} a^{(2)}(m) \sum_{r=0}^{\infty} \gamma_m^{(2)}(r) q^{3r+3m+1}, \]

where \(q = e^{2\pi i \tau},\) \(\gamma_m^{(2)}(0) = 1,\) and for \(r \geq 1\)

\[ \gamma_m^{(2)}(r) = \alpha_m^{(2)}(r) - \frac{12(n-2m)}{B_{6(n-2m)}} \sum_{r_1 \geq 0, r_2 > 0} \alpha_m^{(2)}(r_1) \sigma_{6(n-2m)-1}(r_2) \]

On the other hand, since \(F_{6n+4}^{(2)}(\tau)\) is the function corresponding to the Dirichlet series \(L_{6n+4}(s)\) by Mellin’s transformation,

\[ F_{6n+4}^{(2)}(\tau) = \sum_{\mu = 1 \pmod{\sqrt{-3}}} \mu^{6n+4} e^{2\pi i \mu \tau/3}. \]

Therefore

\[ \sum_{m=0}^{[n/3]} a^{(2)}(m) \sum_{r=0}^{\infty} \gamma_m^{(2)}(r) q^{3r+3m+1} = \sum_{\mu = 1 \pmod{\sqrt{-3}}} \mu^{6n+4} q^{N(\mu)} \quad (q = e^{2\pi i \tau}), \]
and hence we have
\[ \sum_{m=0}^{t} a^{(5)}(m) \gamma^{(5)}(t-m) = \sum_{N \equiv 1 \pmod{3}} \mu_{1}^{N+4}, \]
where \( t = 1, 2, 3, \ldots \).

Then we assert

**Theorem 2.** We have the following relations:

(i) \[ \sum_{m=0}^{t} a^{(1)}(m) \gamma^{(1)}(t-m) = \sum_{N \equiv 1 \pmod{3}} \mu_{1}^{N+4}, \]

(ii) \[ \sum_{m=0}^{t} b^{(1)}(m) \beta^{(1)}(t-m) = \sum_{N \equiv 1 \pmod{3}} \mu_{1}^{N+1}, \]

(iii) \[ \sum_{m=0}^{t} a^{(2)}(m) \gamma^{(2)}(t-m) = \sum_{N \equiv 1 \pmod{3}} \mu_{1}^{N+3}, \]

where

\[ \gamma^{(1)}_{m}(r) = a_{m}^{(1)}(r) - \frac{4(3n-6m+2)}{B_{1}(3n-6m+2)} \sum_{r \geq 0, r_{1} \geq 0, r_{2} \geq 0} \alpha_{m}^{(1)}(r_{1}) \sigma_{2(3n-6m+2)-1}(r_{2}), \quad \gamma^{(1)}_{m}(0) = 1, \]

\[ \beta^{(1)}_{m}(r) = a_{m}^{(1)}(r) - \frac{12(n-2m)}{B_{1}(n-2m)} \sum_{r \geq 0, r_{1} \geq 0, r_{2} \geq 0} \alpha_{m}^{(1)}(r_{1}) \sigma_{2(n-2m)-1}(r_{2}), \quad \beta^{(1)}_{m}(0) = 1, \]

\[ \gamma^{(2)}_{m}(r) = a_{m}^{(2)}(r) - \frac{12(n-2m)}{B_{1}(n-2m)} \sum_{r \geq 0, r_{1} \geq 0, r_{2} \geq 0} \alpha_{m}^{(2)}(r_{1}) \sigma_{2(n-2m)-1}(r_{2}), \quad \gamma^{(2)}_{m}(0) = 1, \]

and \( \alpha_{m}^{(1)}(r) \) and \( \alpha_{m}^{(2)}(r) \) denote the coefficients of the \( q \)-expansion \( q = e^{2\pi i r} \) of \( \Delta^{(2m+1)\frac{a}{b}}(\tau) \) and \( \Delta^{(2m+1)\frac{b}{a}}(\tau) \) respectively.

**Examples:**

<table>
<thead>
<tr>
<th>( k )</th>
<th>Linear relation</th>
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<tbody>
<tr>
<td>1</td>
<td>( F^{(3)}_{2}(\tau) = \Delta^{3}(\tau) )</td>
</tr>
<tr>
<td>2</td>
<td>( F^{(3)}_{4}(\tau) = \Delta^{3}(\tau) )</td>
</tr>
<tr>
<td>3</td>
<td>( F^{(5)}<em>{6}(\tau) = \Delta^{5}(\tau)G</em>{6}(\tau) )</td>
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<tr>
<td>4</td>
<td>( F^{(1)}<em>{8}(\tau) = \Delta^{8}(\tau)G</em>{6}(\tau) )</td>
</tr>
<tr>
<td>5</td>
<td>( F^{(3)}<em>{10}(\tau) = \Delta^{3}(\tau)G</em>{10}(\tau) )</td>
</tr>
<tr>
<td>6</td>
<td>( F^{(5)}<em>{12}(\tau) = \Delta^{5}(\tau)G</em>{10}(\tau) )</td>
</tr>
<tr>
<td>7</td>
<td>( F^{(1)}<em>{14}(\tau) = \Delta^{1}(\tau)G</em>{10}(\tau) + \alpha\Delta^{7}(\tau) )</td>
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In the table, \( \alpha = \left\{ \frac{(2-\sqrt{-3})^{3} + (2+\sqrt{-3})^{3} + 62756}{691} \right\} \).

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References


