Zeta Functions of Representations

by

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Dedicated to Professor Fumihiro Sato on the occasion of his 65th birthday.

Introduction

Special functions have been played an important role in representation theory. Among others, we look at zeta functions attached to representations of groups. The zeta functions for the abelian groups $\mathbb{R}^r$ introduced here are of new kind. Our purpose is to indicate a representation theoretic way to absolute zeta functions extending the previous paper [KO].

We refer to [S] [K] [D] [CC1] [CC2] [CC3] for absolute zeta functions.

1. $\mathbb{R}$

Let $\rho$ be a finite-dimensional representation of $\mathbb{R}$, and $\rho^*$ its contragradient;

$$\begin{align*}
\rho, \rho^* : \mathbb{R} &\rightarrow GL(n), \\
\rho^*(t) = t \rho(t)^{-1}.
\end{align*}$$

We define

$$\begin{align*}
\zeta_{\rho}(s) &= \exp \left( \int_0^\infty \frac{\text{trace}(\rho(t))}{te^{st}} dt \right) \\
&= \exp \left( \frac{1}{\Gamma(w)} \left. \int_0^\infty \frac{\text{trace}(\rho(t))}{te^{st}} t^w dt \right|_{w=0} \right), \\
\zeta_{\rho^*}(s) &= \exp \left( \int_0^\infty \frac{\text{trace}(\rho^*(t))}{te^{st}} dt \right), \\
\varepsilon_R(s) &\overset{\text{def}}{=} \frac{\zeta_{\rho^*}(-s)}{\zeta_\rho(s)}.
\end{align*}$$

THEOREM 1. Let $\rho : \mathbb{R} \rightarrow U(n)$ be a continuous finite-dimensional unitary representation.

(1) \[ \zeta_{\rho}(s) = \det(s - D_\rho)^{-1} \]
with

$$D_\rho = \lim_{t \to 0} \frac{\rho(t) - 1}{t} \in M_n(\mathbb{C}).$$  \tag{2}$$

Note that $D_\rho$ is a skew Hermitian matrix, and can be regarded as an infinitesimal generator of the one-parameter subgroup $\rho$.

$$\zeta^R_\rho(s) = \det(s - D_\rho^*)^{-1} \tag{3}$$

with

$$D_\rho^* = \overline{D_\rho} = -D_\rho. \tag{4}$$

(3) $\zeta^R_\rho(s) = (-1)^n$.

(4) Riemann Hypothesis holds. That is, all the poles of $\zeta_\rho(s)$ are located on the imaginary line $i\mathbb{R}$.

Proof. (1) Since $\rho$ is completely reducible, $\rho$ is a direct sum of (one-dimensional) unitary characters:

$$\rho \cong \chi_1 \oplus \cdots \oplus \chi_n \tag{5}$$

with

$$\chi_k(t) = e^{\lambda_k t}, \quad t \in \mathbb{R} \quad (k = 1, 2, \ldots, n) \tag{6}$$

for some $\lambda_k \in \sqrt{-1}\mathbb{R}$. Then

$$\zeta^R_\rho(s) = \exp \left( \int_0^\infty \frac{e^{\lambda_1 t} + \cdots + e^{\lambda_n t}}{te^{st}} dt \right)$$

$$= \frac{1}{(s - \lambda_1) \cdots (s - \lambda_n)}.$$

Actually,

$$\exp \left( \int_0^\infty \frac{e^{\lambda t}}{te^{st}} dt \right) \overset{def}{=} \exp \left( \frac{\partial}{\partial w} \Gamma(w) \int_0^\infty \frac{e^{\lambda t}}{te^{st}} t^w dt \right) \mid_{w=0}$$

$$= \exp \left( \frac{\partial}{\partial w} (s - \lambda)^{-w} \right) \mid_{w=0}$$

$$= \exp (- \log(s - \lambda))$$

$$= \frac{1}{s - \lambda}.$$

On the other hand

$$D_\rho \cong \lim_{\text{conj } t \to 0} \begin{pmatrix} \frac{\chi_1(t) - 1}{t} & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \frac{\chi_n(t) - 1}{t} \end{pmatrix}.$$
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\[\lim_{t \to 0} \begin{pmatrix} e^{\lambda^t} - 1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda^n t} - 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.\]

Hence

\[\det(s - D_\rho)^{-1} = \frac{1}{(s - \lambda_1) \cdots (s - \lambda_n)}.\] (7)

(2) Similarly,

\[\zeta^R_\rho^*(s) = \frac{1}{(s + \lambda_1) \cdots (s + \lambda_n)}\] (8)

and

\[D_{\rho^*} \cong \begin{pmatrix} -\lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda_n \end{pmatrix}.\] (9)

(3)

\[\varepsilon_\rho^R(s) = \frac{\zeta^R_\rho^*(-s)}{\zeta^R_\rho^*(s)} = \frac{(-s + \lambda_1) \cdots (-s + \lambda_n)}{(s - \lambda_1) \cdots (s - \lambda_n)} = (-1)^n.\]

(4) \(\zeta^R_\rho^*(s) = \infty\) implies \(\text{Re}(s) = 0\). \(\square\)

We note that the unitarity assumption is inevitable in Theorem.

When we put

\[N(u) = \text{trace}(\rho(\log u)),\] (10)

we have a “counting function \(N(u)\)” used in [CC1] [CC2] [KO] to study absolute zeta functions. In general, we admit \(\rho\) to be virtual (not necessarily unitary) representations. For example, let \(\chi\) be the (non-unitary) representation of \(\mathbb{R}\) defined by \(\chi(t) = e^t\), and we define virtual representations of \(\mathbb{R}\) by

\[\rho_{GL(n)} = \chi^{n(n-1)/2}(\chi - 1)(\chi^2 - 1) \cdots (\chi^n - 1),\] (11)

\[\rho_{SL(n)} = \chi^{n(n-1)/2}(\chi^2 - 1) \cdots (\chi^n - 1),\] (12)

then we have

\[\zeta_{GL(n)/F_1}(s) = \zeta^R_{\rho_{GL(n)}}(s),\] (13)

\[\zeta_{SL(n)/F_1}(s) = \zeta^R_{\rho_{SL(n)}}(s).\] (14)
A proof of these formulae is given in [KO].

2. Kurokawa tensor product

For the group $G = \mathbb{R}$, we consider the set of equivalence classes of finite-dimensional unitary (continuous) representations of $G$. Actually, they form a category, where a morphism is a (continuous) $G$-homomorphism. This category has two binary operations, a direct sum and a tensor product. These operations make the category to be a semi-ring, that is, a ring which may not have an additive negative of an object. It satisfies the distribution law, especially. The multiplication is commutative and the 0-dimensional representation is the additive unit, and the 1-dimensional trivial representation is the multiplicative unit, in other words, we have a commutative unital semi-ring.

We also consider another category, whose object is a reciprocal of a monic polynomial in one-variable $s$. The ‘sum’ of two objects in this category is defined to be a product of such rational functions; $f \oplus g = fg$. The ‘product’ of two objects is defined as

$$f \otimes \prod_{j=1}^{n} (s - \mu_j) = \prod_{i=1}^{m} (s - \lambda_i) \prod_{j=1}^{n} (s - \mu_j).$$

(15)

$$f \otimes \prod_{j=1}^{n} (s - \mu_j) = \prod_{i=1}^{m} (s - \lambda_i - \mu_j).$$

(16)

The constant function 1 is considered to be the additive unit, and $1/s$ is considered to be the multiplicative unit. Note that $\oplus$ is defined without using the factorization of polynomials into linear factors, the operation $\otimes$ seems to be not; we only find $f \otimes \prod_{j=1}^{n} (s - \mu_j) = \prod_{j=1}^{n} f(s - \mu_j).$

(17)

These definitions are compatible in the sense that the map $\rho \mapsto \zeta_{\rho}(s)$ gives the semi-ring homomorphism. Note that the contragredient operation $\rho \mapsto \rho^*$ corresponds to the operation

$$f(s) \mapsto (-1)^d f(-s),$$

(18)

where $d$ is the degree of the polynomial $1/f$.

3. $\mathbb{R}^r$

Let $\rho : \mathbb{R}^r \to GL(n, \mathbb{C})$ be a representation of $\mathbb{R}^r$ for $r = 1, 2, 3, \ldots$. In this case we get the zeta function of several variables:

$$\zeta_{\rho} (s_1, \ldots, s_r)$$

$$= \exp \left( \frac{\partial}{\partial w} \frac{1}{\Gamma(w)^r} \int_0^\infty \cdots \int_0^\infty \frac{\text{trace}(\rho(t_1, \ldots, t_r))(t_1 \cdots t_r)^w}{t_1 \cdots t_r e^{s_1 t_1 + \cdots + s_r t_r}} dt_1 \cdots dt_r \bigg|_{w=0} \right).$$

(19)
When we write the character as a sum of irreducible characters as 
\[ \rho = \chi_{\alpha_1}(t_1, \ldots, t_r) + \cdots + \chi_{\alpha_n}(t_1, \ldots, t_r) \]
with \( \chi_{\alpha}(t_1, \ldots, t_r) = e^{\alpha t_1 + \cdots + \alpha t_r} \) for \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r \), we obtain
\[
\frac{1}{\Gamma(w)^r} \int_0^\infty \cdots \int_0^\infty \frac{\text{trace}(\rho(t_1, \ldots, t_r)(t_1 \cdots t_r)^w)}{t_1 \cdots t_r e^{s_1 t_1 + \cdots + s_r t_r}} dt_1 \cdots dt_r
= \sum_{j=1}^n \left( (s_1 - \alpha_1^{(j)}) \cdots (s_r - \alpha_r^{(j)}) \right)^{-w}
\]
and
\[
\zeta_{\rho_s}(s_1, \ldots, s_r) = \prod_{j=1}^n \frac{1}{(s_1 - \alpha_1^{(j)}) \cdots (s_r - \alpha_r^{(j)})}
= \prod_{k \in \{\pm\}} \det(s_k - D_k)^{-1},
\]
with \( D_k = \lim_{t_k \to 0} \frac{\rho(0, \ldots, 0, t_k, 0, \ldots, 0) - 1}{t_k} \) where \( t_k \) is located at the \( k \)-th component of \((0, \ldots, 0, t_k, 0, \ldots, 0)\).

4. Representations of Lie Groups

For a Lie group \( G \) with a (continuous) homomorphism \( \nu : \mathbb{R}^r \to G \) we have the associated zeta function \( \zeta_{\pi_\alpha \circ \nu}(s_1, \ldots, s_r) \) for a representation \( \pi \) of \( G \) under a suitable interpretation of \( \text{trace}(\pi \circ \nu) \). We notice a simple case. Here we use the normalized multiple gamma function and the normalized multiple sine function defined in [KK].

**Theorem 2.** Let \( \pi_\alpha \) be the principal series representation of \( G = SL(2, \mathbb{R}) \) with parameter \( \alpha \in \mathbb{C} \). Let \( \nu : \mathbb{R} \to SL(2, \mathbb{R}) \) be the group homomorphism defined by \( \nu(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \).

\[
\zeta_{\pi_\alpha \circ \nu}(s) = \Gamma_1(s + \alpha) \Gamma_1(s + 1 - \alpha),
\]

\[
\epsilon_{\pi_\alpha \circ \nu}(s) = S_1(s + \alpha) S_1(s + 1 - \alpha).
\]

(2) For \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r \), we consider the tensor product representation \( \pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_r} \) of \( SL(2, \mathbb{R}) \). Then
\[
\zeta_{(\pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_r}) \circ \nu}(s) = \prod_{k \in \{\pm\}^r} \Gamma_r(s + k \cdot \left( \alpha - \frac{1}{2} \right) + \frac{r}{2}),
\]

\[
\epsilon_{(\pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_r}) \circ \nu}(s) = \prod_{k \in \{\pm\}^r} S_r(s + k \cdot \left( \alpha - \frac{1}{2} \right) + \frac{r}{2}).
\]

Here the dot product of vectors is defined \( k \cdot (\alpha - \frac{1}{r}) = \sum_{j=1}^r k_j (\alpha_j - \frac{1}{2}) \) as usual.
Proof. We denote by $\Theta_\alpha$ the (distribution) character of the principal series representation on a split Cartan subgroup. An explicit form of the character formula can be found in the standard textbook, e.g., [Su] [Kn],

$$\Theta_\alpha(u) = \frac{u^{\frac{\alpha}{2}} + u^{-(\alpha - \frac{1}{2})}}{u^{\frac{1}{2}} - u^{-\frac{1}{2}}}$$  \hspace{1cm} (26)

where $u$ in the left-hand side of (26) denote the element $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in SL(2, \mathbb{R})$. The character of the tensor product representation $\pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_r}$ is known to be the product of the characters, and it is written as

$$\Theta_{\alpha_1}(u) \cdots \Theta_{\alpha_r}(u) = u^{-\frac{r}{2}}(1 - u^{-1})^{-r} \sum_{k \in \{\pm\}} u^{-k \cdot (\alpha - \frac{1}{2})}.$$  \hspace{1cm} (27)

Then we compute the integral as in [KO]:

$$Z_{(\pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_r}) \circ \nu}^\mathbb{R}(w, s) = \frac{1}{\Gamma(w)} \int_1^\infty \Theta_{\alpha_1}(u) \cdots \Theta_{\alpha_r}(u) u^{-s} (\log u)^{w-1} \frac{du}{u}$$

$$= \sum_{k \in \{\pm\}} \frac{1}{\Gamma(w)} \int_1^\infty (1 - u^{-1})^{-r} (\log u)^{w-1} u^{-k \cdot (\alpha - \frac{1}{2}) - \frac{r}{2} - s - 1} \frac{du}{u}$$

$$= \sum_{k \in \{\pm\}} \zeta_{\rho}(w; s + k \cdot \left(\alpha - \frac{1}{2}\right) + \frac{r}{2}).$$  \hspace{1cm} (28)

This proves the formulae in the statement (2). \qed

5. $\mathbb{Z}$

This case is a classical one. In fact, it is a Selberg zeta function of a circle. We describe it in comparison with Section 1. Let $\rho$ be a representation of $\mathbb{Z}$, $\rho^*$ the contragredient representation of $\rho$;

$$\rho, \rho^* : \mathbb{Z} \to GL(n), \hspace{1cm} (29)$$

$$\rho^*(m) = \tau \rho(m)^{-1} \text{ for } m \in \mathbb{Z}. \hspace{1cm} (30)$$

We define

$$\zeta_\rho(s) \mathbb{Z} \overset{\text{def}}{=} \exp \left( \sum_{m=1}^{\infty} \frac{\text{trace}(\rho(m))}{me^{sm}} \right), \hspace{1cm} (31)$$

$$\zeta_{\rho^*}(s) \overset{\text{def}}{=} \exp \left( \sum_{m=1}^{\infty} \frac{\text{trace}(\rho^*(m))}{me^{sm}} \right), \hspace{1cm} (32)$$

$$\varepsilon_\rho(s) \overset{\text{def}}{=} \frac{\zeta_{\rho^*}(-s)}{\zeta_\rho(s)}. \hspace{1cm} (33)$$

Theorem 3. Let $\rho : \mathbb{Z} \to U(n)$ be a unitary representation.

(1) $\zeta_\rho^\mathbb{Z}(s) = \text{det}(1 - \rho(1)e^{-s})^{-1}$. 


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(2) $\zeta_{\rho}^Z(s) = \det(1 - \rho(1)^{-1} e^{-s})^{-1}$.

(3) $\varepsilon_{\rho}^Z(s) = (-1)^n \det(\rho(1)) e^{-ns}$.

(4) Riemann Hypothesis holds.

Proof. (1)

\[ \zeta_{\rho}^Z(s) = \exp \left( \text{trace} \sum_{m=1}^{\infty} \frac{\rho(1)^m}{m e^{sm}} \right) \]
\[ = \det \exp \left( - \log(1 - \rho(1)^{-1} e^{-s}) \right) \]
\[ = \det(1 - \rho(1)^{-1} e^{-s})^{-1}. \]

(2)

\[ \zeta_{\rho^*}^Z(s) = \exp \left( \text{trace} \sum_{m=1}^{\infty} \frac{\rho(1)^{-m}}{m e^{sm}} \right) \]
\[ = \det(1 - \rho(1)^{-1} e^{-s})^{-1}. \]

(3)

\[ \varepsilon_{\rho}^Z(s) = \frac{\zeta_{\rho^*}^Z(-s)}{\zeta_{\rho}^Z(s)} \]
\[ = \frac{\det(1 - \rho(1)^{-1} e^{s})^{-1}}{\det(1 - \rho(1)^{-1} e^{-s})^{-1}} \]
\[ = (-1)^n \det(\rho(1)) e^{-ns}. \]

(4) $\zeta_{\rho}^Z(s) = \infty$ implies $\Re(s) = 0$ by (1).

\[ \square \]

References


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