Dimension Formula for the Spaces of Jacobi Forms of Degree Two

by

Ryuji TSUSHIMA

(Received September 28, 2015)
(Revised May 12, 2016)

1. Introduction

Let $S_g = \{ Z \in M_g(C) \mid tZ = Z, \ \text{Im} \ Z > 0 \}$ be the Siegel upper half plane of degree $g$ and let $\Gamma_g = \text{Sp}(g, Z)$. If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, we denote $(AZ + B)(CZ + D)^{-1}$ by $M \langle Z \rangle$. Let $e(z)$ denote $\exp(2\pi iz)$.

DEFINITION 1.1. Let $\mu: GL(g, C) \rightarrow GL(n, C)$ be an irreducible holomorphic representation of $GL(g, C)$ and $m$ a positive integer. A holomorphic mapping $f(Z, W)$ of $S_g \times C^g$ to $C^n$ is called a holomorphic Jacobi form of type $\mu$ and index $m$ with respect to $\Gamma_g$, if it satisfies the following transformation formulas (a), (b) and a regularity condition at infinity (c).

(a) $f(M \langle Z \rangle, t(CZ + D)^{-1}W) = e(mtW(CZ + D)^{-1}CW)\mu(CZ + D)f(Z, W)$, for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$.

(b) $f(Z, W + Z\lambda + \kappa) = e(-m(\vec{t}\lambda Z\lambda + 2\vec{t}\lambda W))f(Z, W)$, for any $\lambda, \kappa \in Z^g$. If $f$ satisfies (a) and (b), $f$ has a Fourier expansion of the form:

$$f(Z, W) = \sum_{N, r} c(N, r) e(\text{Tr}(NZ) + \vec{t}rW),$$

where $N$ runs over the symmetric half integral matrices of degree $g$ and $r$ runs over the integral $g$-vectors. The regularity condition at infinity is:

(c) $c(N, r) = 0$ unless $4mN - r\vec{t}r$ is semi-positive.

DEFINITION 1.2. Let $\mu$ and $m$ be as above. A mapping $f(Z, W)$ of $S_g \times C^g$ to $C^n$ which is real analytic in the real part and the imaginary part of $Z$ and holomorphic in $W$ is called a skew-holomorphic Jacobi form of type $\mu$ and index $m$ with respect to $\Gamma_g$, if it satisfies the above transformation formula (b) and the following transformation formula $(a')$ and has a Fourier-Jacobi expansion $(c')$.

$(a') \ f(M \langle Z \rangle, t(CZ + D)^{-1}W)$
\[ f(Z, W) = e(m'W(CZ + D)^{-1}CW) |\det(CZ + D)| \left( \frac{\mu(CZ + D)\det(CZ + D)}{\det(CZ + D)} \right) f(Z, W), \]

for any \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g. \)

\[ (c') \quad f(Z, W) = \sum_{N, r} c(N, r) e(\text{Tr} \left( N\bar{Z} + \frac{i}{2m} \left( r^t r \right) Y \right) + r^t W), \]

where \( Y = \text{Im } Z \) and \( N \) runs over the symmetric half integral matrices of degree \( g \) and \( r \) runs over the integral vectors such that \( 4mN - r^t r \) is semi-negative.

**Remark 1.3.** If \( g \geq 2 \), then the assumption (c) is superfluous ([Sh2] and [Z]) and similarly the assumption \( (c') \) is superfluous.

**Definition 1.4.** A holomorphic Jacobi form \( f \) is called a **holomorphic Jacobi cusp form** if \( c(N, r) = 0 \) unless \( 4mN - r^t r \) is positive definite in the Fourier-Jacobi expansion \( (c) \). A skew-holomorphic Jacobi form \( f \) is called a **skew-holomorphic Jacobi cusp form** if \( c(N, r) = 0 \) unless \( 4mN - r^t r \) is negative definite in the Fourier-Jacobi expansion \( (c') \).

We denote the complex vector space of holomorphic Jacobi forms of type \( \mu \) and index \( m \) with respect to \( \Gamma_g \) by \( J_{\mu, m}(\Gamma_g) \) and its subspace of holomorphic Jacobi cusp forms by \( J_{\mu, m}^{cusp}(\Gamma_g) \), respectively. We denote the complex vector space of skew-holomorphic Jacobi forms of type \( \mu \) and index \( m \) with respect to \( \Gamma_g \) by \( J_{\mu, m}^{sk}(\Gamma_g) \) and its subspace of skew-holomorphic Jacobi cusp forms by \( J_{\mu, m}^{sk, cusp}(\Gamma_g) \), respectively. For a subgroup \( \Gamma' \) of finite index of \( \Gamma_g \) we define \( J_{\mu, m}(\Gamma') \), \( J_{\mu, m}^{cusp}(\Gamma') \), \( J_{\mu, m}^{sk}(\Gamma') \) and \( J_{\mu, m}^{sk, cusp}(\Gamma') \) similarly. If \( \mu = \text{det}^k \), we denote \( J_{\mu, m}(\Gamma), J_{\mu, m}^{cusp}(\Gamma), J_{\mu, m}^{sk}(\Gamma) \) and \( J_{\mu, m}^{sk, cusp}(\Gamma) \) by \( J_{k, m}(\Gamma), J_{k, m}^{cusp}(\Gamma), J_{k, m}^{sk}(\Gamma) \) and \( J_{k, m}^{sk, cusp}(\Gamma) \), respectively. It is known that \( J_{\mu, m}(\Gamma) \) and \( J_{\mu, m}^{sk}(\Gamma) \) are finite dimensional.

Now we assume that \( g = 2 \). Then \( \mu \) is equivalent to \( \text{det}^k \text{Sym}_j \) \((j, k \geq 0)\), where \( \text{det} \) is the alternating tensor representation of degree two and \( \text{Sym}_j \) is the symmetric tensor representation of degree \( j \) of \( GL(2, \mathbb{C}) \), respectively. In this case we denote \( J_{\mu, m}(\Gamma_2), J_{\mu, m}^{cusp}(\Gamma_2), J_{\mu, m}^{sk}(\Gamma_2) \) and \( J_{\mu, m}^{sk, cusp}(\Gamma_2) \) by \( J_{k, m}(\Gamma_2), J_{k, m}^{cusp}(\Gamma_2), J_{k, m}^{sk}(\Gamma_2) \) and \( J_{k, m}^{sk, cusp}(\Gamma_2) \), respectively.

The purpose of this paper is to compute the dimension of \( J_{k, m}^{cusp}(\Gamma_2) \) and \( J_{k, m}^{sk, cusp}(\Gamma_2) \) by the formula of Riemann-Roch (the holomorphic Lefschetz fixed point formula) and vanishing theorems for higher cohomology groups. We computed the Euler-Poincaré characteristic explicitly for general \( j \) by the formula of Riemann-Roch. We proved the vanishing theorem for the case when \( j = 0 \) (Theorem 2.5 below). Therefore our results give dimension formulas for \( J_{k, m}^{cusp}(\Gamma) \) and \( J_{k, m}^{sk, cusp}(\Gamma) \). But we have not proved the vanishing theorem for the case when \( j > 0 \). Hence our results on the dimension of \( J_{k, m}^{cusp}(\Gamma_2) \) and \( J_{k, m}^{sk, cusp}(\Gamma_2) \)

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\(^1\)The author used the notation \( J_{j, k,m}(\Gamma_2) \) instead of \( J_{k, j,m}(\Gamma_2) \) before. But we use the notation of T. Ibukiyama to avoid confusion.
still remain as conjectures in general. Recently T. Ibukiyama proved the conjecture is true for the case of $J_{k,j,1}(\Gamma_2)$ by a different method ([Ib5]).

2. The Methods and the Results

DEFINITION 2.1. Let $a$ and $b$ be rational $g$-vectors. The theta function $\theta_{a,b}(Z, W)$ with characteristic $(a, b)$ is a holomorphic function on $\mathbb{S}_g \times \mathbb{C}^g$ defined by

$$\sum_{q \in \mathbb{Z}^g} e((1/2)(q + a)Z(q + a) + (q + a)(W + b)).$$

For any integral $g$-vector $r$, we have

$$\theta_{a+r,b}(Z, W) = \theta_{a,b}(Z, W).$$

Hence $\theta_{a,0}(Z, W)$ depends only on $a \mod \mathbb{Z}^g$. So we assume $a$ is an element of $\mathbb{Q}^g / \mathbb{Z}^g$. If $a$ runs over a complete set of representatives of $1/2\mathbb{Z}^g / \mathbb{Z}^g$, then the set $\{\theta_{a,0}(2mZ, 2mW)\}$ form a basis of theta functions of degree $2m$. Therefore if $f$ is a holomorphic Jacobi form of type $\mu$ and index $m$, there exist uniquely determined $n$-dimensional column-vectors $f_r(Z)$

$$r \in \frac{1}{2m} \mathbb{Z}^g / \mathbb{Z}^g$$

on $\mathbb{S}_g$ satisfying

$$f(Z, W) = \sum_r f_r(Z) \theta_{r,0}(2mZ, 2mW).$$

$f_r(Z)$’s are holomorphic with respect to $Z$ ([Sh2]). If $f$ is a skew-holomorphic Jacobi form of type $\mu$ and index $m$, $f$ is also represented by a linear combination of $\theta_{r,0}(2mZ, 2mW)$’s. In this case the coefficients $f_r(Z)$’s are anti-holomorphic ([A1]). Namely, the complex conjugate $\overline{f_r(Z)}$’s are holomorphic.

We define an $n \times (2m)^g$ matrix:

$$F_{\text{mat}}(Z) = (f_r(Z)),$$

and a column-vector of dimension $(2m)^g$:

$$\Theta(Z, W) = (\theta_{r,0}(2mZ, 2mW)).$$

If $f$ is a holomorphic Jacobi form, then by definition we have

\begin{equation}
F_{\text{mat}}(M \langle Z \rangle) \Theta(M \langle Z \rangle, (CZ + D)^{-1}W) = e(m'W(CZ + D)^{-1}CW) \mu(CZ + D) F_{\text{mat}}(Z) \Theta(Z, W).
\end{equation}

If $f$ is a skew-holomorphic Jacobi form, then we have

\begin{equation}
F_{\text{mat}}(M \langle Z \rangle) \Theta(M \langle Z \rangle, (CZ + D)^{-1}W) = e(m'W(CZ + D)^{-1}CW) |\det(CZ + D)| \left( \frac{\mu(CZ + D)}{\det(CZ + D)} \right) F_{\text{mat}}(Z) \Theta(Z, W).
\end{equation}

We need the following transformation formula for the theta functions ([Si]).
THEOREM 2.2. Let \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \). Then for any \( r \in \frac{1}{2m} \mathbb{Z}/\mathbb{Z} \), we have
\[
\theta_{r,0}(2mM(Z), 2m'(CZ + D)^{-1}W) = \det(CZ + D)^{1/2} \Phi(m'W(CZ + D)^{-1}CW) \times \sum_s u_{rs}(M) \theta_{s,0}(2mZ, 2mW),
\]
where \( s \) runs over a complete set of representatives of \( \frac{1}{2m} \mathbb{Z}/\mathbb{Z} \) and \((u_{rs}(M))\) is a unitary matrix of degree \( (2m)^g \) depending only on \( M \) and the choice of \( \det(CZ + D)^{1/2} \).

Let \( u(M) = (u_{rs}(M)) \). Then by the theorem we have
\[
\Theta(M(Z), t(CZ + D)^{-1}W) = \det(CZ + D)^{1/2} \Phi(CZ + D)^{-1}CW)u(M)\Theta(Z, W).
\]

Let \( f \) be a holomorphic Jacobi form. From (1) and (2) we have
\[
F_{\text{mat}}(M(Z)) = \det(CZ + D)^{-1/2} \mu(CZ + D) F_{\text{mat}}(Z)^t u(M).
\]
Namely, \( F_{\text{mat}}(Z) \) is an \( n \times (2m)^g \) matrix valued holomorphic automorphic form and this is naturally identified to an \( n \cdot (2m)^g \)-dimensional vector valued holomorphic automorphic form \( F(Z) \) with respect to the automorphy factor:
\[
\det(CZ + D)^{-1/2} \mu(CZ + D) \otimes u(M).
\]

Let \( f \) be a skew-holomorphic Jacobi form. From (1') and (2) we have
\[
\overline{F_{\text{mat}}(M(Z))} = \det(CZ + D)^{-1/2} \mu(CZ + D) \overline{F_{\text{mat}}(Z)^t u(M)}.
\]
Namely, \( \overline{F_{\text{mat}}(Z)} \) is an \( n \times (2m)^g \) matrix valued holomorphic automorphic form and this is naturally identified to an \( n \cdot (2m)^g \)-dimensional vector valued holomorphic automorphic form \( \overline{F(Z)} \) with respect to the automorphy factor:
\[
\det(CZ + D)^{-1/2} \mu(CZ + D) \otimes u(M).
\]

So we have the following proposition ([Sh2]).

PROPOSITION 2.3. By the mapping: \( f(Z, W) \mapsto F(Z), J_{\mu,m}(\Gamma_g) \) is mapped isomorphically to the space of the vector valued holomorphic automorphic forms with respect to the above automorphy factor (3) and \( \Gamma_g \). \( F(Z) \) is a cusp form if and only if \( f(Z, W) \) is a holomorphic Jacobi cusp form.

By the anti-linear mapping: \( f(Z, W) \mapsto \overline{F(Z)}, J^s_{\mu,m}(\Gamma_g) \) is mapped isomorphically to the space of the vector valued holomorphic automorphic forms with respect to the above automorphy factor (3') and \( \Gamma_g \). \( \overline{F(Z)} \) is a cusp form if and only if \( f(Z, W) \) is a skew-holomorphic Jacobi cusp form.

Let \( \Gamma_g(N) \) be the principal congruence subgroup of level \( N \) of \( \Gamma_g \). Namely,
\[
\Gamma_g(N) = \{ M \in \Gamma_g \mid M \equiv 1_{2g} \pmod{N} \}.
\]
\( \Gamma_g(N) \) is a normal subgroup of \( \Gamma_g \). If \( N \geq 3 \), \( \Gamma_g(N) \) acts on \( \mathcal{S}_g \) without fixed points and the quotient space \( X_g(N) := \Gamma_g(N)\backslash\mathcal{S}_g \) is a (non-compact) manifold. \( X_g(N) \) is an open set of a projective variety \( \overline{X}_g(N) \) which was constructed by I. Satake ([Sa], Satake
compactification). If \( g \geq 2 \), \( \mathcal{X}_g(N) \) has singularities along its cusps: \( \mathcal{X}_g(N) - X_g(N) \). Cusps of \( \mathcal{X}_g(N) \) are (as a set) a disjoint union of copies of \( X_{g^i}(N) \)'s (\( 0 \leq g^i < g \)). A desingularization \( \tilde{\mathcal{X}}_g(N) \) of \( \mathcal{X}_g(N) \) was constructed by J.-I. Igusa and Y. Namikawa \((g = 2, 3, 4)\) ([Ig], [Nm]) and more generally by D. Mumford and others ([AMRT], Toroidal compactification).

Let \( N \geq 3 \) and let \( \mathcal{V}_\mu \) be \( \mathfrak{S}_g \times \mathbb{C}^n \). \( \Gamma_g(N) \) acts on \( \mathcal{V}_\mu \) as follows:

\[
\mathcal{V}_\mu := \Gamma_g(N) / \mathcal{V}_\mu \text{ is non-singular and is a vector bundle over } X_g(N). \quad \mathcal{V}_\mu \text{ is extendable to a vector bundle over } \tilde{\mathcal{X}}_g(N) \text{ ([M], cf. proof of Theorem 2.4 below).}
\]

Let \( \Theta(Z) = \sum_{\eta \in \mathbb{Z}} e^{(i \eta Z \eta)} \) and let

\[
\Gamma_0^g(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_f \mid C \equiv O \pmod{4} \right\}.
\]

If \( M \in \Gamma_0^g(4) \), then \( J(M, Z) := \Theta(M \langle Z \rangle) / \Theta(Z) \) is holomorphic on \( \mathfrak{S}_g \) and satisfies

\[
J(M, Z)^2 = \det(CZ + D) \psi(\det D),
\]

where \( \psi(\det D) = \left( \frac{-4}{\det D} \right) \). If \( M \in \Gamma_g(4) \), we may assume that \( \det(CZ + D)^{1/2} = J(M, Z) \). Then \( u(M) \) becomes a representation of \( \Gamma_g(4) \).

In the following we assume that the level is divisible by 4 and replace \( N \) by \( 4N \). Let \( \mathcal{E}_m \) be \( \mathfrak{S}_g \times \mathbb{C}(2m)^g \). \( \Gamma_g(4N) \) acts on \( \mathcal{E}_m \) as follows:

\[
M(Z, v) = (M \langle Z \rangle, u(M)v).
\]

Let \( \mathcal{H}_g \) be \( \mathfrak{S}_g \times \mathbb{C} \). \( \Gamma_g(4N) \) acts on \( \mathcal{H}_g \) as follows:

\[
M(Z, v) = (M \langle Z \rangle, J(M, Z)v).
\]

\( H_g := \Gamma_g(4N) / \mathcal{H}_g \) is a line bundle over \( X_g(4N) \). \( H_g \) is extendable to a line bundle \( \mathcal{H}_g \) over \( \tilde{\mathcal{X}}_g(4N) \) ([T5] Theorem 1.8). We denote the pullback of \( \mathcal{H}_g \) by the natural morphism of \( \tilde{\mathcal{X}}_g(4N) \) to \( \tilde{\mathcal{X}}_g(4N) \) by \( \mathcal{H}_g \).

Now we have

**Theorem 2.4.** If \( m \mid N \), \( E_m \) is extendable to a vector bundle \( \widetilde{E}_m \) over \( \tilde{\mathcal{X}}_g(4N) \). \( \widetilde{E}_m \) is a flat vector bundle and the Chern class \( c_i(\widetilde{E}_m) \) \((i \geq 1)\) is 0.

**Proof.** In the following we assume that the level is divisible by \( 4m \) and replace \( 4N \) by \( 4mN \). To avoid general terminology we prove the theorem in case \( \tilde{\mathcal{X}}_g(4mN) \) is the Voronoi compactification ([Nm]). Although it is not known whether the Voronoi compactification is smooth or not when \( g \geq 5 \), the essential point of the proof is the same.

Let \( \mathfrak{G}_g \) be the vector space of real quadratic forms of degree \( g \). \( \mathfrak{G}_g^+ \) be the set of positive definite real quadratic forms of degree \( g \) and \( \mathfrak{G}_g^+ \) the “rational” closure of \( \mathfrak{G}_g^+ \) which is, by definition, the convex hull of the set of non-negative integral quadratic forms. Let \( \mathcal{T}_g \) be \( g(g + 1)/2 \)-dimensional algebraic torus and \( \mathcal{T}_g \subset X_g \) the torus embedding associated with the Delone-Voronoi decomposition ([V]) which is a rational partial polyhedral decomposition of \( \mathfrak{G}_g^+ \).
Let $0 \leq g'' < g$, $g' = g - g''$ and let $F$ be a rational boundary component of degree $g''$ of $G$. The "partial compactification in the direction of $F''$" is constructed as follows. Let $\mathfrak{g}_g^*$ be the parabolic compactification of $F$ and $\mathfrak{g}_g^*$ the center of the unipotent radical of $\mathfrak{g}_g^*$. Namely, $\mathfrak{g}_g^*$ is isomorphic to the group which consists of the elements of the form:

$$\begin{pmatrix}
A' & O & B' & * \\
* & U & * & * \\
C' & O & D' & * \\
O & O & O & 1
\end{pmatrix}, \quad \begin{pmatrix}
A' & B' \\
* & * \\
C' & D'
\end{pmatrix} \in Sp(g', \mathbb{R}), \quad U \in GL(g'', \mathbb{R})$$

and $\mathfrak{g}_g^*$ is the subgroup which consists of the elements of the form:

$$\begin{pmatrix}
1_{g'} & O & O & O \\
O & 1_{g''} & O & S \\
O & O & 1_{g'} & O \\
O & O & O & 1_{g''}
\end{pmatrix}, \quad S = ^tI$$

$\mathfrak{g}_g^*$ is a normal subgroup of $\mathfrak{g}_g^*$. Let $\mathfrak{g}_g^*(4mN) = \mathfrak{g}_g^* \cap \Gamma_g(4mN)$ and $\mathfrak{g}_g^*(4mN) = \mathfrak{g}_g^* \cap \Gamma_g(4mN)$.

For a matrix $M = (m_{ij})$ we denote the matrix $(e(m_{ij}))$ by $e(M)$. Let $Z = M g' (\mathbb{C})$ and $T_g, g'' = \mathfrak{g}_g' \times Z \times T_{g''}$. Let $e_g''$ be the map of $\mathfrak{g}_g''$ to $T_{g''}$ which is defined by

$$e_g'' = \left( \begin{array}{cc}
Z_1 & Z_{12} \\
Z_{12} & Z_2
\end{array} \right) \in \mathfrak{g}_g''.$$  

The image of $e_g''$ which we denote by $T_{g, g''}$ is biholomorphic to $\mathfrak{g}_g''(4mN) \setminus \mathfrak{g}_g''$. Let $\mathcal{X}_{g, g''} = \mathfrak{g}_g'' \times Z \times \mathcal{X}_{g''}$. $T_{g, g''}$ is an open subset of $\mathcal{X}_{g, g''}$. Let $\mathcal{X}^v_{g, g''}$ be the interior of the closure of $T_{g, g''}$ in $\mathcal{X}_{g, g''}$. The action of $\mathfrak{g}_g''(4mN)/\mathfrak{g}_g''(4mN)$ on $T_{g, g''}$ is extended onto $\mathcal{X}^v_{g, g''}$. $\mathfrak{g}_g''(4mN)/\mathfrak{g}_g''(4mN)$ is called the partial compactification and the smooth compactification $\overline{\mathcal{X}}_{g}(4mN)$ is constructed by gluing the partial compactifications.

Now let $M \in \mathfrak{g}_g''(4mN)$. Then it is easily verified that

$$\theta_{r, 0}(M(Z), W) = \theta_{r, 0}(Z, W).$$

Namely, the transformation matrix $u(M)$ is the identity matrix. Hence we have

$$\mathfrak{g}_g''(4mN) \times \mathcal{E}_m = \mathfrak{g}_g''(4mN) \setminus (\mathfrak{g}_g'' \times C^{(2m)^9}) = (\mathfrak{g}_g''(4mN) \setminus \mathfrak{g}_g'') \times C^{(2m)^9}.$$  

The product space $T_{g, g''} \times C^{(2m)^9}$ is naturally extended to a product space $\mathcal{X}^v_{g, g''} \times C^{(2m)^9}$. The action of $\mathfrak{g}_g''(4mN)/\mathfrak{g}_g''(4mN)$ on $T_{g, g''} \times C^{(2m)^9}$ is extended onto $\mathcal{X}^v_{g, g''} \times C^{(2m)^9}$. Since $\mathfrak{g}_g''(4mN)/\mathfrak{g}_g''(4mN)$ acts on $\mathcal{X}^v_{g, g''}$ without fixed points, the quotient space $(\mathfrak{g}_g''(4mN)/\mathfrak{g}_g''(4mN)) \setminus (\mathcal{X}^v_{g, g''} \times C^{(2m)^9})$ is a vector bundle on $(\mathfrak{g}_g''(4mN)/\mathfrak{g}_g''(4mN)) \times \mathcal{X}^v_{g, g''}$. The extension $\mathfrak{E}_m$ is constructed by gluing them.

Next we prove that $\mathfrak{E}_m$ is flat. Let $h = h(Z)$ be the matrix-valued function on $\mathfrak{g}_g''$ such that $h(Z)$ is identically the identity matrix of degree $(2m)^9$. Let $(Z, u), (Z, v) \in \mathfrak{E}_m$. We define a metric on $\mathfrak{E}_m$ by $\langle u, v \rangle = \overline{u} h u$. Since $u(M)$ is a unitary matrix, this metric is invariant with respect to $\Gamma_g$. Hence it induces a metric on $T_{g, g''} \times C^{(2m)^9}$ which is also represented by the identity matrix $h$. The constant matrix-valued function $h$ is
extended onto $X^o_{g', g''}$ as a constant function and defines a metric on $X^o_{g', g''} \times \mathbb{C}^{(2m)g}$. This metric is invariant with respect to $\mathfrak{P} g''(4mN)/\mathfrak{B} g''(4mN)$ and induces a metric on $(\mathfrak{P} g''(4mN)/\mathfrak{B} g''(4mN)) \setminus (X^o_{g', g''} \times \mathbb{C}^{(2m)g})$. When we glue the partial compactifications, a metric on $\tilde{E}_m$ is induced from these metrics on the partial compactifications. It is independent of the choice of the partial compactifications. This metric is represented by a constant matrix. Therefore its curvature is zero. □

The space of holomorphic Jacobi forms $J_{\mu, m}(\Gamma_g(4mN))$ is canonically identified with the space

$$\Gamma(\tilde{X}_g(4mN), \mathcal{O}(\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m)),$$

which is the space of the global holomorphic sections of $\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m$. Let $D := \tilde{X}_g(4mN) - X_g(4mN)$ be the divisor at infinity. $D$ is a divisor with simple normal crossings. $J'^{\text{cusp}}_{\mu, m}(\Gamma_g(4mN))$ is canonically identified with the space

$$S_{\mu, m}(\Gamma_g(4mN)) := \Gamma(\tilde{X}_g(4mN), \mathcal{O}(\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m - D)).$$

$\mathcal{O}(\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m - D)$ is the sheaf of germs of holomorphic sections which vanish along $D$ and this is isomorphic to $\mathcal{O}(\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m \otimes [D]^{\otimes(-1)})$, where $[D]$ is the line bundle associated with $D$. Since $K_{\tilde{X}_g(4mN)} \otimes [D]$ is isomorphic to $\tilde{H}^{\otimes(2g+2)}$, the above sheaf is isomorphic to $\mathcal{O}(\tilde{H}^{\otimes(-2g-3)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m \otimes K_{\tilde{X}_g(4mN)})$, where $K_{\tilde{X}_g(4mN)}$ is the canonical line bundle of $\tilde{X}_g(4mN)$. Let $\tilde{E}_m$ be the complex conjugate vector bundle of $E_m$. $\tilde{E}_m$ is also extendable to a flat vector bundle $\tilde{E}_m$ over $\tilde{X}_g(4mN)$ which is the complex conjugate vector bundle of $\tilde{E}_m$. By the anti-linear mapping the space of skew-holomorphic Jacobi cusp forms $J'^{\text{sk, cusp}}_{\mu, m}(\Gamma_g(4mN))$ is identified with the space

$$S'^{\text{sk}}_{\mu, m}(\Gamma_g(4mN)) := \Gamma(\tilde{X}_g(4mN), \mathcal{O}(\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m - D)).$$

The sheaf $\mathcal{O}(\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m - D)$ is isomorphic to $\mathcal{O}(\tilde{H}^{\otimes(-2g-3)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m \otimes K_{\tilde{X}_g(4mN)})$ similarly as above.

In case when $\mu = \det^k$ we can prove

**Theorem 2.5.** If $\mu = \det^k$ and if $k \geq g + 2$ and $p > 0$, then

$$H^P(\tilde{X}_g(4mN), \mathcal{O}(\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m - D))$$

and

$$H^P(\tilde{X}_g(4mN), \mathcal{O}(\tilde{H}^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m - D))$$

are 0-spaces.

**Proof.** If $\mu = \det^k$, we have $V_\mu \simeq H^{\otimes 2k}_g$. Since $\overline{\mathfrak{P}}_g$ is an ample line bundle on $\overline{X}_g(4mN)$ ([B]), we can assume $\overline{X}_g(4mN)$ is a subvariety of $\mathfrak{P}_r(\mathbb{C})$ and $\overline{H}^{\otimes M}_g$ is isomorphic to the restriction of $\mathcal{O}_{\mathfrak{P}_r(\mathbb{C})}(1)$ for large $M$. The Fubini-Study metric on $\mathcal{O}_{\mathfrak{P}_r(\mathbb{C})}(1)$ induces a metric on $\overline{H}^{\otimes M}_g$ and a metric on $\overline{H}_g$. This metric induces a metric on $\tilde{H}_g$ and a metric on $\tilde{H}_g^{\otimes(-2g-3)} \otimes \tilde{V}_\mu \simeq H^{\otimes(2k-2g-3)}_g$ which is positive on the smooth part $X_g(4mN)$
in the sense of Kodaira ([Kd]) if \( k \geq g + 2 \). This metric and the metric on \( \widetilde{E}_m \) which is defined before induce a metric on \( \widetilde{H}^{(2k-2g-3)}_g \otimes \widetilde{E}_m \). This metric is positive on the smooth part \( X_g(4mN) \) in the sense of Nakano ([Nk]) if \( k \geq g + 2 \). This is similarly proved as in [Gr] p.209, although the positivity in [Gr] is different from the positivity in the sense of Nakano. Hence from the vanishing theorem of Nakano ([Nk] and [SS] Corollary 6.24) we have

\[
H^p(\widetilde{X}_g(4mN), \mathcal{O}(\widetilde{H}^{(2k-2g-3)}_g \otimes \widetilde{E}_m \otimes K_{\widetilde{X}_g(4mN)})) \simeq \{0\}
\]

for \( p > 0 \). This and the above isomorphism of the sheaves prove the assertion. Another case is similarly proved.

\[\square\]

Since the Chern characters of \( \widetilde{E}_m \) and \( \widetilde{E}_m \) are \( (2m)^g \), from the above vanishing theorem and the theorem of Riemann-Roch-Hirzebruch we have

**Corollary 2.6.** If \( \mu = \det \) and \( k \geq g + 2 \), then

\[
\dim J^{\text{cusp}}_{\mu,m}(\Gamma_g(4mN)) = \dim S_{\mu,m}(\Gamma_g(4mN)) = (2m)^g \dim S_{k-1/2}(\Gamma_g(4mN)),
\]

and

\[
\dim J^{\text{cusp}}_{k,m}(\Gamma_g(4mN)) = \dim S^{k}_{\mu,m}(\Gamma_g(4mN)) = (2m)^g \dim S_{k-1/2}(\Gamma_g(4mN)),
\]

where \( S_{k-1/2}(\Gamma_g(4mN)) \) is the space of Siegel cusp forms of weight \( k - 1/2 \) (namely, the space of the automorphic forms with respect to the automorphy factor \( J(M, Z)^{2k-1} \)).

We return to the case of general \( \mu \). Let \( (f_1, f_2, \ldots, f_g) \) be the signature of \( \mu \). We present

**Conjecture 2.7.** If \( f_g \geq g + 2 \) and \( p > 0 \), then

\[
H^p(\widetilde{X}_g(4mN), \mathcal{O}(\widetilde{H}^{(-1)}_g \otimes \widetilde{E}_m - D))
\]

and

\[
H^p(\widetilde{X}_g(4mN), \mathcal{O}(\widetilde{H}^{(-1)}_g \otimes \widetilde{E}_m - D))
\]

are 0-spaces.

\( M \in \Gamma_g \) acts on \( S_{\mu,m}(\Gamma_g(4mN)) \) as follows:

\[
MF(M(Z)) = \det(CZ + D)^{-1/2} \mu(CZ + D) \otimes u(M) F(Z).
\]

Since \( \Gamma_g(4mN) \) acts trivially, \( \Gamma_g/\Gamma_g(4mN) \) acts on \( S_{\mu,m}(\Gamma_g(4mN)) \). Hence the dimension of \( J^{\text{cusp}}_{\mu,m}(\Gamma_g) \simeq S_{\mu,m}(\Gamma_g) \) is calculated as an invariant subspace of \( S_{\mu,m}(\Gamma_g(4mN)) \) by using the holomorphic Lefschetz fixed point formula ([AS]) if Conjecture 2.7 is true.

We recall the holomorphic Lefschetz fixed point formula. Let \( X \) be a compact complex manifold and \( V \) a holomorphic vector bundle of rank \( n \) on \( X \), and let \( G \) be a finite group of automorphisms of the pair \( (X, V) \). For \( g \in G \) let \( X^g \) be the set of fixed points of \( g \). \( X^g \) is a disjoint union of submanifolds of \( X \). Let

\[
X^g = \sum_{\alpha} X^g_{\alpha}
\]
be the irreducible decomposition of $X^g$, and let
\[ N^g_a = \sum_\theta N^g_a(\theta) \]
denote the normal bundle of $X^g_a$ decomposed according to the eigenvalues $e^{i\theta}$ of $g$. We put
\[ {\cal U}^\theta(N^g_a(\theta)) = \prod_\beta \left( \frac{1 - e^{-x_\beta - i\theta}}{1 - e^{-i\theta}} \right)^{-1}, \]
where the Chern class of $N^g_a(\theta)$ is
\[ c(N^g_a(\theta)) = \prod_\beta (1 + x_\beta). \]
Let $T(X^g_a)$ be the Todd class of $X^g_a$. Let $V|X^g_a$ be the restriction of $V$ to $X^g_a$ and $ch(V|X^g_a)(g)$ the Chern character of $V|X^g_a$ with $g$-action (see below). Put
\[ \tau(g, X^g_a) = \left\{ \frac{ch(V|X^g_a)(g) \cdot \prod_\beta {\cal U}^\theta(N^g_a(\theta)) \cdot T(X^g_a)}{\det(1 - g(N^g_a))} \right\}[X^g_a] \]
and
\[ \tau(g) = \sum_\alpha \tau(g, X^g_a). \]
We have
\[ \text{THEOREM 2.8 ([AS]).} \]
\[ \sum_{i \geq 0} (-1)^i \text{Tr}(g | H^i(X, {\cal O}(V))) = \tau(g). \]

To use the Lefschetz fixed point formula we have to classify the fixed points (sets). We classify (the irreducible components of) the fixed points of $G$ in the following sense. Let $\Phi_1$ and $\Phi_2$ be the fixed points (sets). $\Phi_1$ and $\Phi_2$ is called equivalent if there is an element of $G$ which maps $\Phi_1$ biholomorphically to $\Phi_2$. Let $\Phi$ be one of fixed points (sets) and let $C(\Phi) = \{ g \in G \mid g(x) = x \text{ for any } x \in \Phi \}$.

Let $g \in C(\Phi)$ and $H = \langle g \rangle$ the subgroup of $C(\Phi)$ which is generated by $g$ and let $\tilde{H}$ be the character group of $H$. Let $\chi \in \tilde{H}$ and let $(V|\Phi)_\chi$ be the subbundle of $V|\Phi$ consisting of the vectors $v \in V|\Phi$ such that $g(v) = \chi(g)v$. $V|\Phi$ is a direct sum of subbundles $(V|\Phi)_\chi$'s. We define the Chern character with $g$-action as follows:
\[ ch(V|\Phi)(g) := \sum_\chi \chi(g) \cdot ch((V|\Phi)_\chi). \]

Now we return to our case where we consider the action of $G = \Gamma_g/\Gamma_g(4mN)$ on $X = \tilde{X}_g(4mN)$ and $V = \tilde{H}_g^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m \otimes [D]^{\otimes(-1)}$ or $V = \tilde{H}_g^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m \otimes [D]^{\otimes(-1)}$. Let $\Phi$ be an irreducible component of fixed points (sets). We can prove
\[ \text{PROPOSITION 2.9.} \] The direct summands $(\tilde{E}_m|\Phi)_\chi$ of $\tilde{E}_m|\Phi$ are also flat vector bundles.
Proof. First we prove the case when $\Phi$ intersects the quotient space $X_g(4mN)$. Let $\Phi^o$ be $\Phi \cap X_g(4mN)$ and $\tilde{\Phi}$ the inverse image of $\Phi^o$ by the natural map of $\mathfrak{S}_g$ to $X_g(4mN)$. Let $\Gamma_g(4mN)$ be the quotient space of $\mathbb{E}_m|\Phi^o$ by $N(\tilde{\Phi})$. If $Z \in \tilde{\Phi}$, we denote the fiber of $\mathbb{E}_m|\tilde{\Phi}$ at $Z$ by $(\mathbb{E}_m|\tilde{\Phi})_Z$. Let $P \in C(\Phi)$ and $H = \langle P \rangle$. $P$ is also regarded as an element of $C(\tilde{\Phi})$. Let $\chi \in H$ and let $(\mathbb{E}_m|\tilde{\Phi})_{Z,\chi}$ be the subspace of $(\mathbb{E}_m|\tilde{\Phi})_Z$ consisting of vectors such that $u(P)v = \chi(P)v$. The union $\bigcup_{Z \in \tilde{\Phi}} (\mathbb{E}_m|\tilde{\Phi})_{Z,\chi}$ constitutes a subbundle of $\mathbb{E}_m|\Phi$, which we denote by $(\mathbb{E}_m|\tilde{\Phi})_{\chi}$. We prove that $(\mathbb{E}_m|\tilde{\Phi})_{\chi}$ is a $N(\tilde{\Phi})$-invariant subbundle of $\mathbb{E}_m|\Phi$. Let $(Z, v) \in (\mathbb{E}_m|\tilde{\Phi})_{Z,\chi}$ and $M \in N(\tilde{\Phi})$. Then $M(Z, v) := (M(Z), u(M)v) \in (\mathbb{E}_m|\tilde{\Phi})_{M(Z)}$ satisfies
\[
P(M(Z), u(M)v) = \frac{P(M(Z), u(P)v)}{u(M)v} = \frac{(M(Z), u(M)v)}{(M(Z), u(M)v)}.
\]
Hence $M(Z, v) \in (\mathbb{E}_m|\tilde{\Phi})_{M(Z),\chi}$. $(\mathbb{E}_m|\Phi^o)_{\chi}$ is defined as a quotient space of $(\mathbb{E}_m|\tilde{\Phi})_{\chi}$ by $N(\tilde{\Phi})$. $(\mathbb{E}_m|\Phi)_{\chi}$ is the closure of $(\mathbb{E}_m|\Phi^o)_{\chi}$ in $\mathbb{E}_m|\Phi$. The restriction of the metric on $\mathbb{E}_m|\Phi$ defines an $N(\tilde{\Phi})$-invariant metric on $(\mathbb{E}_m|\tilde{\Phi})_{\chi}$ and induces a metric on $(\mathbb{E}_m|\Phi^o)_{\chi}$. This metric is extended onto $(\mathbb{E}_m|\tilde{\Phi})_{\chi}$ similarly as before and its curvature is zero.

Next we assume that $\Phi$ is in the boundary of $\hat{X}_g(4mN)$. The image of $\Phi$ by the natural map of $\hat{X}_g(4mN)$ to $\hat{X}_g(4mN)$ is in some copy of $\hat{X}_{g'}(4mN)$ $(g' < g)$. We take the smallest $g'$. Namely, $\Phi$ is over a cusp of degree $g'$. Let $\Phi^o$ be the intersection of $\Phi$ and $(\mathcal{Y}_{g''}(4mN)/2\mathcal{Y}_{g''}(4mN)) \setminus \chi_{g''}$ which is the partial compactification in the direction of $\hat{X}_g(4mN)$. Let $\tilde{\Phi}$ be the inverse image of $\Phi^o$ in $\chi_{g''}$. $\tilde{E}_m$ on $(\mathcal{Y}_{g''}(4mN)/2\mathcal{Y}_{g''}(4mN)) \setminus \chi_{g''}$ is a quotient of the trivial bundle $\chi_{g''} \times \mathbb{C}^{2m}$. Replacing $\mathfrak{S}_g$ with $\chi_{g''}$ we can prove the assertion similarly as before.

Hence for $M \in C(\Phi)$ we have
\[
\text{ch}(\mathcal{E}_m|\Phi)(M) = \sum_{\chi} \chi(M) \text{ch}(\mathcal{E}_m|\Phi)_{\chi} = \sum_{\chi} \chi(M) \text{rank} (\mathcal{E}_m|\Phi)_{\chi} = \text{Tr}u(M).
\]
Let $V = \tilde{H}_g^{\otimes(-1)} \otimes \tilde{V}_{\mu} \otimes [D]^{\otimes(-1)}$. Since the Chern character with $M$-action is also a ring homomorphism of the ring of the holomorphic vector bundles to the cohomology ring as in the case of the usual Chern character, we have

$$ch(V|\Phi)(M) = ch(\tilde{E}_m|\Phi)(M) \cdot ch(\tilde{H}_g^{\otimes(-1)} \otimes \tilde{V}_{\mu} \otimes [D]^{\otimes(-1)}|\Phi)(M)$$

$$= \text{Tr} u(M) \cdot ch(\tilde{H}_g^{\otimes(-1)} \otimes \tilde{V}_{\mu} \otimes [D]^{\otimes(-1)}|\Phi)(M).$$

Let

$$ch_1(V|\Phi)(M) = ch(\tilde{H}_g^{\otimes(-1)} \otimes \tilde{V}_{\mu} \otimes [D]^{\otimes(-1)}|\Phi)(M)$$

and

$$\tau_1(M, \Phi) = \left\{ \frac{ch_1(V|\Phi)(M) \cdot \prod_{\theta} U^\theta(NM(\theta)) \cdot T(\Phi)}{\det(1 - M)(NM)^*} \right\}[\Phi].$$

Then we have

$$ch(V|\Phi)(M) = \text{Tr} u(M) \cdot ch_1(V|\Phi)(M)$$

and

$$\tau(M, \Phi) = \text{Tr} u(M) \cdot \tau_1(M, \Phi).$$

Let $\mu : GL(g, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$ be as above and let $S_\mu(\Gamma_g)$ be the complex vector space of Siegel cusp forms of type $\mu$ with respect to $\Gamma_g$ which is, by definition, the space of the holomorphic $\mathbb{C}^n$-valued function on $\mathfrak{g}$ satisfying the transformation formula

$$f( M \langle Z \rangle ) = \mu(CZ + D) f(Z), \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$$

and the cusp condition. Elements of $S_\mu(\Gamma_g)$ correspond to holomorphic sections of $\tilde{V}_{\mu} \otimes [D]^{\otimes(-1)}$. Therefore we can use the results on the Chern character $ch(\tilde{V}_{\mu} \otimes [D]^{\otimes(-1)})(M)$ when we computed the dimension of $S_\mu(\Gamma_2)$ ([T2], [T3]). Although we have another factor $\tilde{H}_2^{\otimes(-1)}$ now, the results in the case of $S_{\mu}(\Gamma_2)$ are applicable. Note that in the definition of $\tau_1(M, \Phi)$ the terms except $ch_1(V|\Phi)(M)$ depend only on the base space $X$.

Therefore applying the data in [T2] and [T3], we can compute $\tau_1(M, \Phi)$ from $\tau_0(M, \Phi)$ in Theorem 4.1 below. So what we have to do is

(a) to determine $\det(CZ + D)^{1/2} u(M)$ for $M \in C(\Phi)$,
(b) to evaluate the Gaussian sums which appear in $\text{Tr} u(M)$, and
(c) to execute a terrible computation.

Before we state our main theorem, we define a Mathematica code. This code is available from the home page of the author.

**DEFINITION 2.10.** Let $k, j \geq 0, m \geq 1$ and $sk = 0$ or $1$. We define $\text{Jacobi2}[k,j,m,sk]$ as follows:

\begin{verbatim}
Jacobi2[k_,j_,m_,sk_] := Block[{a,lk,lj,lj1,lj2,lj3,x,y,m2,m3,m4,j2,r,p,s,S1e,S1,S2e,S2,S3,SS,SSS},
s = 1 - 2*sk;
mod[x_,y_] := Mod[x,y] + 1;
m2 = Mod[m,2];
\end{verbatim}
\[ m_3 = \text{Mod}[m, 3]; \]
\[ m_4 = \text{Mod}[s \cdot m, 4]; \]
\[ j_2 = \text{Mod}[j, 2]; \]
\[ r = 0; \]
\[ \text{While}[\text{EvenQ}[m/2^r], r++; ]; \]
\[ p = m/2^r; \]
\[ S1e = 4 \cdot \text{Sum}[\text{Mod}[s \cdot x^2, m], \{x, 1, m-1\}]; \]
\[ S1 = S1e + \text{Sum}[\text{Mod}[s \cdot (2 \cdot x - 1)^2, 4 \cdot m], \{x, 1, m\}]; \]
\[ S2e = 16 \cdot \text{Sum}[\text{Mod}[s \cdot x^2, m]^2, \{x, 1, m-1\}]; \]
\[ S2 = S2e + \text{Sum}[\text{Mod}[s \cdot (2 \cdot x - 1)^2, 4 \cdot m]^2, \{x, 1, m\}]; \]
\[ S3 = \text{Sum}[\text{Mod}[s \cdot x^2, 4 \cdot m]^3, \{x, 1, 2 \cdot m-1\}]; \]
\[ SSS = \text{Sum}[\text{Mod}[s \cdot x^2, 4 \cdot m] \cdot \text{Mod}[s \cdot y^2, 4 \cdot m] \cdot \text{Mod}[s \cdot (x-y)^2, 4 \cdot m], \{x, 1, 2 \cdot m-1\}, \{y, 1, 2 \cdot m-1\}]; \]
\[ a = m^2 \cdot (j+1) \cdot ((2 \cdot j+2 \cdot k-3) \cdot (j+2 \cdot k-4) \cdot (2 \cdot k-5)/2^8 \cdot 3^3 \cdot 5 \cdot (j+2 \cdot k-4) / 2^4 / 3 \cdot 2 + 1/2^3 / 3); \]
\[ a = a \cdot (j+1) \cdot ((6 \cdot k+3 \cdot j-40) \cdot S1 / 2^6 / 3^2 + (-2 \cdot k-j+14) \cdot S2 / m / 2^8 / 3 + S3 / m^2 / 2^8 / 3^2); \]
\[ a = a \cdot (j+1) \cdot (S1^2 / m^2 / 2^7 - S1 \cdot S2 / m^3 / 2^8 + SSS / m^3 / 2^8 / 3); \]
\[ (* \text{contribution of } \varphi_1 *) \]
\[ (* \text{contribution of } \varphi_{15}(r) *) \]
\[ (* \text{contribution of } \varphi_{22}(1, r, t) *) \]
\[ (* \text{contribution of } \varphi_{25}(1, r, t) *) \]
\[ l_1 = (1, 1); \]
\[ a = a \cdot l_1[(\text{Mod}[j, 2])] \cdot (j+1) \cdot ((2 \cdot j + 2 \cdot k - 3) \cdot (j+2 \cdot k-4) \cdot (2 \cdot k-5)/2^8 / 3^3 / 5 + (4-2 \cdot k -j) / 2^4 / 3^2 + 1/2^3 / 3); \]
\[ a = a \cdot l_1[(\text{Mod}[j, 2])] \cdot (j+1) \cdot (-m^4 / 3^2 / 5^2 + (2-2 \cdot k-j) \cdot m^4 / 2^8 / 3 + (6 \cdot k+3 \cdot j-40) \cdot m^4 / 2^6 / 3^2); \]
\[ (* \text{contribution of } -\varphi_1 *) \]
\[ (* \text{contribution of } -\varphi_{15}(r) *) \]
\[ (* \text{contribution of } -\varphi_{22}(1, r, t) *) \]
\[ (* \text{contribution of } -\varphi_{25}(1, r, t) *) \]
\[ l_2 = (1, 1); \]
\[ a = a \cdot s \cdot l_1[(\text{Mod}[k, 2])] \cdot m \cdot ((2 \cdot j + 2 \cdot k - 3) \cdot (2 \cdot k-5) / 2^7 / 3^2 - (j+2 \cdot k-4) / 2^4 / 3^1 / 2^3 / 7 / 3); \]
\[ a = a \cdot s \cdot l_1[(\text{Mod}[k, 2])] \cdot (j+1) \cdot (S1 + (j+2 \cdot k-16) / m / 2^7 / 3 + S1 \cdot m^4 / m / 2^7 + m^4 \cdot m \cdot (j+2 \cdot k-1) / 2^7 / 3); \]
\[ (* \text{contribution of } \varphi_2 *) \]
\[ (* \text{contribution of } \pm \varphi_{16}(r) *) \]
\[ (* \text{contribution of } \varphi_{23}(4, r, t) *) \]
\[ , a = a \cdot s \cdot l_1[(\text{Mod}[k, 2])] \cdot m \cdot (2 \cdot j + 2 \cdot k - 3) \cdot (2 \cdot k-5) / 2^7 / 3 + (10-6 \cdot k-3 \cdot j) / 2^5 / 3^7 / 2^4 / 3 + s \cdot l_1[(\text{Mod}[k, 2])] \cdot (4 \cdot S3 + 16-2 \cdot m^4) \cdot m \cdot S1 \cdot 2 \cdot (j+2 \cdot k-14+2 \cdot m^4) \cdot m \cdot S1 \cdot e - 8 \cdot S2 \cdot S2e / m^2 / 2^8; \]
\[ (* \text{contribution of } \varphi_3 *) \]
\[ (* \text{contribution of } \pm \varphi_{17}(r) *) \]
\[ (* \text{contribution of } \varphi_{22}(3, r, t) *) \]
\[ (* \text{contribution of } \varphi_{24}(4, r, t) *) \]
Dimension Formula for the Spaces of Jacobi Forms of Degree Two

\[* contribution of $\pm \varphi_{25}(4,r,s,t) \]*
\[ a = a - s \cdot \text{lk}([\mod[k,2]])(j+1) \cdot S1e/m/2^7 \]

\(* contribution of $\pm \varphi_2(r) \)*
\[ \text{lj} = (1,-1) ; \]
\[ \text{If}[j_2 = 0] \]
\[ a = a + \text{lj}([\mod[j/2,2]]) \cdot (j+2-k-4)/2^5 \cdot 3 \cdot \text{lj}([\mod[j/2,2]]) \]
\[ \cdot (\mod[m,4]/2^5, a = a + 0) \]

\(* contribution of $\varphi_3 \)*
\(* contribution of $\psi_{23}(2,r,t) \)*

\[ \text{lj} = (1,-1) ; \]
\[ \text{If}[j_2 = 0] \]
\[ a = a + \text{lj}([\mod[j/2,2]]) \cdot \text{If}[m_2 = 0, 1, 0] \cdot (j+2-k-4)/2^5 \cdot 3 \cdot \text{lj}([\mod[j/2,2]]) \]
\[ \cdot (\text{If}[m = 2 = 0, 1, -1] \cdot m_4/2^5 - \text{If}[m = 0, 1, 0] \cdot 2^3) \]
\[ a = a + 0 \]

\(* contribution of $\varphi_4 \)*

\(* contribution of $\psi_{24}(2,r,t) \)*

\[ \text{lj} = (1,-1,0) ; \]
\[ a = a + \text{lj}([\mod[j/3,3]]) \cdot \text{If}[m_3 = 0, 3, 1] \cdot (j+2-k-4)/2^2/3^2 \]
\[ a = a - \text{lj}([\mod[j/3,3]]) \cdot (\text{If}[m_3 = 0, 3, 1] \cdot 2^2/3 - \text{If}[m_3 = 0, 1, 0] \cdot \text{Mod}[s_4/m, 3, 3]/3^2) \]

\(* contribution of $\varphi_5 \)*

\(* contribution of $\psi_{25}(2,r,t) \)*

\[ \text{lj} = (1,1,0,1,-1,0) ; \]
\[ a = a + \text{lj}([\mod[j/4,4]]) \cdot (j+2-k-4)/2^2/3^2 \]
\[ a = a - \text{lj}([\mod[j/4,4]]) \cdot 2^2/3^2 \]

\(* contribution of $-\varphi_6 \)*

\(* contribution of $-\psi_{26}(2,r,s,t) \)*

\[ \text{lj} = (-1-j, 4-j-2* k, 1+j, -4+j+2* k), (4-j-2* k, 4-j-2* k, -4+j+2* k, -4+j+2* k), (4-j-2* k, 4-j-2* k, -4+j+2* k, 1+j), (-1-j, -1-j, 1+j, 1+j) ; \]
\[ \text{lj} = ((0, -1, 0, 1), (-1, 1, 0, 1), (1, 1, 0, 0), (0, 0, 0, 0)) ; \]
\[ \text{If}[m_2 = 0] \]
\[ a = a + s_4 \cdot \text{lj}([\mod[j,4], \mod[k,4]]) \cdot m_4/2^5 \cdot 3 \cdot s_4 \cdot \text{lj}([\mod[j,4], \mod[k,4]]) \cdot (S1/m-4*m)/2^5 \]
\[ a = a + 0 \]

\(* contribution of $\varphi_7(1) \)*
\(* contribution of $\varphi_7(2) \)*

\(* contribution of $\varphi_{18}(1,r) \)*
\[ \text{lj} = (-4+j+2* k, -1-j, 4-j-2* k, 1+j), (4-j-2* k, -4+j+2* k, -4+j+2* k, 4-j-2* k), (1+j, 4-j-2* k, -1-j, 4-j+2* k), (-1-j, 1+j, 1+j, 1-j); \]
\[ \text{lj} = ((1, 0, 1, 0), (-1, 1, 1, 1), (0, 1, 0, 1), (0, 0, 0, 0)) ; \]
\[ \text{If}[m_2 = 0] \]
\[ a = a + s_4 \cdot \text{lj}([\mod[j,4], \mod[k,4]]) \cdot 2^5 \cdot 3 \cdot s_4 \cdot \text{lj}([\mod[j,4], \mod[k,4]]) \cdot (m_4-4)/2^5 \]
\[ a = a + 0 \]

\(* contribution of $-\varphi_7(1) \)*
\(* contribution of $-\varphi_{18}(1, r) \)*
\[ \text{lj} = ((0, -1, 0, 1), (-1, -1, 1, 1), (0, 1, 0, 1), (0, 0, 0, 0)) ; \]
\[ a = a + s_4 \cdot \text{lj}([\mod[j,4], \mod[k,4]]) \cdot \text{Sum}[\text{Mod}[s_4, (2+y+m^2)^2, 8*m] - \text{Mod}[s_4, (2+y^2+m^2)^2, 8*m], (y, 0, m-1)]/m/2^6 \]

\(* contribution of $\varphi_{19}(1,r) \)*
\[ \text{lj} = ((1, 0, -1, 0), (-1, 1, 1, 1), (0, 0, 0, 0), (0, 0, 0, 1)); \]
\[ a = a + \text{If}[m_2 = 0, -1, 0] \cdot \text{lj}([\mod[j,4], \mod[k,4]]) \cdot 2^4 \]
\[ a = a + \text{lj}([\mod[j,4], \mod[k+s*m,4]]) \cdot (\text{Mod}[s_4, m] - \text{Mod}[s_4, m+8, 4, 8])/2^6 ; \]
(* contribution of \(\varphi_{19}(1,r)\) and \(\varphi_{19}(2,r)\) *)

\[lj = \{(3-2*j+2*k,-5+2*k,8-2*j+4*k),(8-2*j-4*k,-5+2*k,-3+2*j+2*k),(2+2*j,0,2+2*j)\};\]
\[lj1 = \{(-1,-1,2),(-2,1,1),(0,0,0)\};\]
\[lj2 = \{(-13+2*j+6*k,11-4*j-6*k,2+2*j),(2+2*j,11-4*j-6*k,-13+2*j+6*k),(2+2*j,-4-4*j,2+2*j)\};\]
\[lj3 = \{(1,1,0),(-1,1,0),(0,0,0)\};\]

\[If[m3 != 0,\]
\[a = a + If[EvenQ[r], -1, 1] * lj[[mod[j,3], mod[k,3]]] * m / 2^4 / 3^3 + \]
\[If[EvenQ[r], -1, 1] * lj1[[mod[j,3], mod[k,3]]] * m / 2^4 / 3^2;\]

(* contribution of \(\varphi_{8}(1)\) and \(\varphi_{8}(2)\) *)

(* contribution of \(\varphi_{20}(1,r)\) and \(\varphi_{20}(2,r)\) *)

\[lj = \{(3-2*j+2*k,-5+2*k,8-2*j+4*k),(8-2*j-4*k,-5+2*k,-3+2*j+2*k),(2+2*j,0,2+2*j),\]
\[(-2-2*j,0,2+2*j)\};\]
\[a = a + lj[[mod[j,6], mod[k,3]]] / 2^4 / 3^3;\]
\[lj = \{(1,1,-2),(-2,1,1),(0,0,0),(-1,-1,2),(-2,-1,1),(0,0,0)\};\]
\[a = a + lj[[mod[j,6], mod[k,3]]] * (m4 - 4) / 2^4 / 3^2;\]

(* contribution of \(\varphi_{8}(3)\) and \(\varphi_{8}(4)\) *)

(* contribution of \(\varphi_{20}(3,r)\) and \(\varphi_{20}(4,r)\) *)

\[lj = \{(3-2*j+2*k,-5+2*k,8-2*j+4*k),(8-2*j-4*k,-5+2*k,-3+2*j+2*k),\]
\[(-2-2*j,0,2+2*j)\};\]
\[a = a + s * lj[[mod[j,6], mod[k,6]]] * m / 2^4 / 3^2;\]
\[lj = \{(1,1,0),(-2,1,1,1,1),(-2,0,2,2,0,2),(-1,1,2,1,1,1),\}
\[(0,0,0,0,0,0);\]
\[a = a + s * lj[[mod[j,6], mod[k,6]]] * (m4 - 4) / 2^4 / 3^3;\]

(* contribution of \(\varphi_{8}(3)\) and \(\varphi_{8}(4)\) *)

\[lj = \{(3-2*j+2*k,-5+2*k,8-2*j+4*k),(8-2*j-4*k,-5+2*k,-3+2*j+2*k),\]
\[(-2-2*j,0,2+2*j)\};\]
\[a = a + s * lj[[mod[j,6], mod[k,6]]] * m / 2^4 / 3^2;\]
\[lj = \{(1,1,0),(-2,1,1,1,1),\}
\[(0,0,0,0,0,0);\]
\[a = a + s * lj[[mod[j,6], mod[k,6]]] * (m4 - 4) / 2^4 / 3^3;\]

(* contribution of \(\varphi_{8}(3)\) and \(\varphi_{8}(4)\) *)

(* contribution of \(\varphi_{20}(3,r)\) and \(\varphi_{20}(4,r)\) *)

\[lj = \{(3-2*j+2*k,-5+2*k,8-2*j+4*k),(8-2*j-4*k,-5+2*k,-3+2*j+2*k),\]
\[(-2-2*j,0,2+2*j)\};\]
\[a = a + s * lj[[mod[j,6], mod[k,6]]] * m / 2^4 / 3^2;\]
Dimension Formula for the Spaces of Jacobi Forms of Degree Two

\[ a = s \cdot \text{If}[\text{Mod}[p, 3] == 1, 1, -1] \cdot \text{If}[\text{EvenQ}[r], -1, 1] \cdot \text{lj}[\text{mod}[j, 6], \text{mod}[k, 6]] / 2^4 / 3^2 \]
\[ a = a + \text{lj}[\text{mod}[j, 6], \text{mod}[k, 6]] / 2^4 / 3^2 \cdot \text{mj}[\text{mod}[j, 6], \text{mod}[k, 6]] \cdot (m4 - 4) / 2^4 / 3^2 \]

(* contribution of \(-\varphi_8(3)\) and \(-\varphi_8(4)\) *)

(* contribution of \(-\varphi_{20}(3, r)\) and \(-\varphi_{20}(4, r)\) *)

\[ lj = \{(-1, -1, 2), (-2, 1, 1), \{0, 0, 0\}\}; \]
\[ lj1 = \{(1, -1, 0), (0, -1, 1), \{0, 0, 0\}\}; \]
\[ lj2 = \{(1, -1, 0), (0, -1, 1), \{0, 0, 0\}\}; \]

If \([m3] = 0\)

\[ a = a + s \cdot \text{Sum}[\text{If}[\text{Mod}[y, 3] == 0, 2, -1] \cdot \text{lj}[\text{mod}[j, 3], \text{mod}[k, 3]] \cdot \text{Mod}[s \cdot y^2 + 4 \cdot m, 12 \cdot m] + \text{lj}[\text{mod}[j, 3], \text{mod}[k + 1, 3]] \cdot \text{Mod}[s \cdot y^2 + 8 \cdot m, 12 \cdot m], \{y, 0, 2 \cdot m - 1\}] / m / 2^4 / 3^3 \]
\[ a = a + s \cdot \text{Sum}[\text{If}[\text{Mod}[y, 3] == 0, 0, 1] \cdot \text{lj1}[\text{mod}[j, 3], \text{mod}[k, 3]] \cdot \text{Mod}[s \cdot y^2 + 8 \cdot m, 12 \cdot m], \{y, 0, 2 \cdot m - 1\}] / m / 2^4 / 3^2 \]

(* contribution of \(\varphi_{21}(1, r)\) and \(\varphi_{21}(2, r)\) *)

\[ lj = \{(1, 1, 0, 1), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), \{0, 0, 0\}\}; \]
\[ a = a + \text{lj}[\text{mod}[j, 6], \text{mod}[k, 3]] / 2^3 / 3^2; \]
\[ a = a + \text{lj}[\text{mod}[j, 6], \text{mod}[k + s \cdot m, 3]] \cdot \text{Mod}[s \cdot m, 12] + \text{lj}[\text{mod}[j, 6], \text{mod}[k + s \cdot m + 1, 3]] \cdot \text{Mod}[s \cdot y^2 + 8 \cdot m, 12 \cdot m], \{y, 0, 2 \cdot m - 1\}] / m / 2^3 / 3^2 \]

(* contribution of \(\varphi_{21}(1, r)\) and \(\varphi_{21}(2, r)\) *)

\[ lj = \{(0, 0, 1, 1); \]
\[ a = a + \text{lj}[\text{mod}[j, 4]] \cdot \text{lk}[\text{mod}[k, 2]] \cdot (j + 1) \cdot \text{If}[m2 == 0, 1, 0] / 2^6 \]

(* contribution of \(\pm \varphi(1)\) *)

\[ lk = (0, 0, 1, 1); \]
\[ a = a + \text{lk}[\text{mod}[k, 2]] / 2^3 / 3^2 \]

(* contribution of \(\varphi(2)\) and \(\varphi(3)\) *)

\[ lk = (1, -1, 1); \]
\[ a = a + \text{lk}[\text{mod}[k, 2]] / 2^3 / 3^2 \]

(* contribution of \(\varphi_{10}(1)\) and \(\varphi_{10}(2)\) *)

\[ lj = \{(0, 1, 0, 1), \{0, 0, 0\}\}; \]
\[ a = a + \text{lj}[\text{mod}[j + 2 \cdot j, 3]] \cdot \text{If}[m3 == 0, -3, 1] / 2^3 / 3^3 \]

(* contribution of \(\varphi_{10}(3)\) *)

\[ lj = \{(1, -1, 0, 1), \{0, 0, 0\}\}; \]
\[ a = a + \text{lj}[\text{mod}[j + 2 \cdot j, 3]] \cdot \text{If}[m3 == 0, -3, 1] / 2^3 / 3^3 \]

(* contribution of \(\varphi_{10}(1)\) and \(\varphi_{10}(2)\) *)

\[ lj = \{(1, 1, 0, 1), \{0, 0, 0\}\}; \]
\[ a = a + \text{lj}[\text{mod}[j, 6], \text{mod}[k, 3]] / 2^3 / 3^3 \]

(* contribution of \(\varphi_{10}(1)\) and \(\varphi_{10}(2)\) *)

\[ lj = \{0, 0, 1, 0, 0\}; \]
\[ a = a + \text{lk}[\text{mod}[k, 2]] / 2^3 / 3^2 \]

(* contribution of \(\varphi_{10}(3)\) *)

\[ a = a + \text{lk}[\text{mod}[k, 2]] / 2^3 / 3^2 \]
(* contribution of $\pm \varphi_{10}(3)\) *)
\[lj={{1,-1,-2,-1,1,2},(-2,-1,1,2,1,-1),(1,2,1,-1,-2,-1)}\]
\[lj1={{1,1,0,-1,0,1},(0,-1,-1,0,1,1),(-1,0,1,1,0,-1)}\]
\[a=a+s*If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]*lj[[mod[j,3],mod[k,6]]]/2^2/3^2\]
\[a=a+lj1[[mod[j,3],mod[k,6]]]/2^2/3\]
\[a=a+0;\]

(* contribution of $\varphi_{10}(4)$ and $\varphi_{10}(5)$ *)
\[lj1={{1,-1,-2,-1,1,2},(-2,-1,1,2,1,-1),(1,2,1,-1,-2,-1)}\]
\[lj1={{1,1,0,-1,0,1},(0,-1,-1,0,1,1),(-1,0,1,1,0,-1)}\]
\[a=a+lj1[[mod[j,3],mod[k,6]]]/2^2/3^2\]
\[a=a+0;\]

(* contribution of $\varphi_{10}(6)$ and $\varphi_{10}(7)$ *)
\[lj1={{1,1,0,-1,0,1,0,-1,1,0,1,1},{1,0,1,1,0,1,0,1,1,0,1,1}}\]
\[lj1={{1,1,0,-1,0,1,0,-1,1,0,1,1},{1,0,1,1,0,1,0,1,1,0,1,1}}\]
\[a=a-lj1[[mod[j/2,6],mod[k,3]]]/2^3/3\]
\[a=a+0;\]

(* contribution of $\varphi_{11}(i)$ (i=1,2,3,4) *)
\[lk={1,-1}\]
\[lj={{0,-1,1,-1,0,-1,1,0,1,1,0,1,1},{1,1,0,1,1,0,1,1,0,1,1,0,1,1}}\]
\[a=a+ljk[[mod[j,3],mod[k,12]]]*If[Mod[p,3]==1,1,-1]*If[EvenQ[r],1,1]/2^3/3-s*lj1[[mod[j,4],mod[k,12]]]*If[Mod[p,3]==1,1,-1]*If[EvenQ[r],1,1]/2^3/3\]
\[a=a+0;\]

(* contribution of $\varphi_{11}(i)$ (i=1,2,3,4) *)
\[lk={1,-1}\]
\[lj={{0,-1,1,-1,0,-1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1}}\]
\[a=a+ljk[[mod[j,6],mod[k,12]]]/2^3/3-ljk[[mod[j,4],mod[k,12]]]/2^3/3\]
\[a=a+0;\]

(* contribution of $-\varphi_{11}(i)$ (i=1,2,3,4) *)
\[lk={1,-1}\]
\[lj={1,0,-2,0,1,0};\]
Dimension Formula for the Spaces of Jacobi Forms of Degree Two

\[ \{0, -1, 0, 1, 0, 0\}; \]
\[ \text{If}[m3] = 0, \]
\[ a = a + a[lj[[\text{mod}[j, 6]]]]*lk[[\text{mod}[k, 2]]]*\text{If}[\text{Mod}[p, 3] == 1, 1, -1]*\text{If}[\text{EvenQ}[r], -1, 1]/2^2/3^2; \]
\[ a = a + lj[[\text{mod}[j, 6]]]*lk[[\text{mod}[k, 2]]]/2^2/3; \]
\[ (* \text{contribution of } \pm \phi_{12} \text{ } \ast) \]
\[ lk = \{1, -1\}; \]
\[ lj = \{0, -1, 0, 0, 1, 0, 0\}; \]
\[ a = a + lj[[\text{mod}[j, 8]]]*lk[[\text{mod}[k, 2]]]*\text{If}[m2 == 0, 1, 0]/2^3; \]
\[ (* \text{contribution of } \pm \phi_{13} \text{ } \ast) \]
\[ lj = \{(0, 1, 0, -1, 0), (0, -1, 0, 0, 1, -1), (0, 0, 0, 1, 0, -1), (0, 0, 0, 0, 1, 0)\}; \]
\[ lj1 = \{(2, 1, 0, -1, -2), (-2, 1, -1, 2), (-2, 0, -1, 0, -1, 0)\}; \]
\[ \text{If}[\text{Mod}[m, 5] != 0, \]
\[ a = a + lj1[[\text{mod}[j, 5], \text{mod}[k, 5]]]/2/5; \]
\[ (* \text{contribution of } \phi_{14}(i) \text{ } \ast) \]
\[ lj = \{(0, 1, 0, -1, 0), (0, 1, -1, 0, 0), (0, 0, 0, 1, -1), (0, 0, 1, 0, -1), (0, 0, 0, 0, 0), (0, -1, 0, 0, 0), (0, -1, 1, 0, 0), (0, 0, 0, -1, 0), (0, 0, 0, 1, 0)\}; \]
\[ a = a + lj[[\text{mod}[j, 10], \text{mod}[k, 5]]]/2/5; \]
\[ (* \text{contribution of } -\phi_{14}(i) \text{ } \ast) \]
\[ \text{Return}[a]; \]

The main theorem of this paper is the following

**Theorem 2.11.** The Euler-Poincaré characteristic

\[ \chi(\tilde{X}_g(4mN), O(\tilde{H}_g(1) \otimes \tilde{V}_m - D)) \Gamma_2/\Gamma_2(4mN) \]

is given by \text{Jacobi2}[k, j, m, 0] and the Euler-Poincaré characteristic

\[ \chi(\tilde{X}_g(4mN), O(\tilde{H}_g(1) \otimes \tilde{V}_m - D)) \Gamma_2/\Gamma_2(4mN) \]

is given by \text{Jacobi2}[k, j, m, 1].

From this theorem and the vanishing theorem we have

**Corollary 2.12.** If \( k \geq 4 \), the dimension of \( J^{\text{cusp}}_{k,m}(\Gamma_2) \) is given by \text{Jacobi2}[k, 0, m, 0] and the dimension of \( J^{\text{sk,cusp}}_{k,m}(\Gamma_2) \) is given by \text{Jacobi2}[k, 0, m, 1].

### 3. The Transformation Formula of Theta Functions

Let \( \Phi \) be an irreducible component of fixed points (sets) of \( \Gamma_2/\Gamma_2(4mN) \) acting on \( \tilde{X}_2(4mN) \). In this section we determine \( \det(CZ + D)^{1/2} \) and the unitary matrices \( u(M) \) for \( M \) which fixes points on \( \Phi \). First we classify the fixed points (sets). The fixed points (sets) in the quotient space \( X_2(4mN) \) were classified in [Go] and the fixed points (sets) in the divisor at infinity were classified in [T1]. In total there are 25 kinds of fixed points (sets). We use the same notations \( \Phi_1, \Phi_2, \ldots, \Phi_{25} \) as in [T1] for the representatives of the
fixed points (sets). They are given by the followings. We denote $e(1/4)$, $e(1/3)$, $e(1/5)$, $(1 + 2\sqrt{-3})/3$ by $i$, $\rho$, $\omega$, $\eta$, respectively. $Z$, $Z_1$, $Z_2$ move over $\mathcal{G}_1$, $W$ moves over $C$ and
\[
\begin{pmatrix} Z_1 \\ Z_2 \\
 Z_{12} \
 \end{pmatrix}
\]
moves over $\mathcal{G}_2$.

$\Phi_1 : \left\{ \begin{pmatrix} Z_1 \\ Z_12 \\
 Z_{12} \
 \end{pmatrix} \right\}$, $\Phi_2 : \left\{ \begin{pmatrix} Z_1 \\ 0 \\
 0 \
 \end{pmatrix} \right\}$, $\Phi_3 : \left\{ \begin{pmatrix} Z_1 \\ 1/2 \\
 1/2 \
 \end{pmatrix} \right\}$,
$\Phi_4 : \left\{ \begin{pmatrix} Z \\ 0 \\
 0 \
 \end{pmatrix} \right\}$, $\Phi_5 : \left\{ \begin{pmatrix} Z \\ 1/2 \\
 1/2 \
 \end{pmatrix} \right\}$, $\Phi_6 : \left\{ \begin{pmatrix} Z \\ Z/2 \\
 Z \
 \end{pmatrix} \right\}$,
$\Phi_7 : \left\{ \begin{pmatrix} i \\ 0 \\
 0 \
 \end{pmatrix} \right\}$, $\Phi_8 : \left\{ \begin{pmatrix} \rho \\ 0 \\
 0 \
 \end{pmatrix} \right\}$, $\Phi_9 : \left\{ \begin{pmatrix} i \\ 0 \\
 0 \
 \end{pmatrix} \right\}$,
$\Phi_{10} : \left\{ \begin{pmatrix} \rho \\ 0 \\
 0 \
 \end{pmatrix} \right\}$, $\Phi_{11} : \left\{ \begin{pmatrix} \rho \\ 0 \\
 0 \
 \end{pmatrix} \right\}$, $\Phi_{12} : \left\{ \begin{pmatrix} \sqrt{3}/2 \\ 1 \\
 1/2 \
 \end{pmatrix} \right\}$,
$\Phi_{13} : \left\{ \begin{pmatrix} \eta \\ (\eta - 1)/2 \\
 \eta \
 \end{pmatrix} \right\}$, $\Phi_{14} : \left\{ \begin{pmatrix} \omega \\ \omega + \omega^3 \\
 \omega + \omega^3 - \omega^4 \
 \end{pmatrix} \right\}$, $\Phi_{15} : \left\{ \begin{pmatrix} Z \\ W \\
 W \
 \end{pmatrix} \right\}$,
$\Phi_{16} : \left\{ \begin{pmatrix} Z \\ 0 \\
 0 \
 \end{pmatrix} \right\}$, $\Phi_{17} : \left\{ \begin{pmatrix} Z \\ 1/2 \\
 1/2 \
 \end{pmatrix} \right\}$, $\Phi_{18} : \left\{ \begin{pmatrix} i \\ 0 \\
 0 \
 \end{pmatrix} \right\}$,
$\Phi_{19} : \left\{ \begin{pmatrix} i \\ (i + 1)/2 \\
 i 
 \end{pmatrix} \right\}$, $\Phi_{20} : \left\{ \begin{pmatrix} \rho \\ 0 \\
 0 \
 \end{pmatrix} \right\}$, $\Phi_{21} : \left\{ \begin{pmatrix} \rho/2 \\ (\rho + 2)/3 \\
 \infty \
 \end{pmatrix} \right\}$, $\Phi_{22} : \left\{ \begin{pmatrix} \infty \\ W \\
 W \
 \end{pmatrix} \right\}$, $\Phi_{23} : \left\{ \begin{pmatrix} \infty \\ 0 \\
 0 \
 \end{pmatrix} \right\}$,
$\Phi_{24} : \left\{ \begin{pmatrix} \infty \\ 1/2 \\
 1/2 
 \end{pmatrix} \right\}$, $\Phi_{25} : \left\{ \begin{pmatrix} \infty \\ \infty \\
 \infty 
 \end{pmatrix} \right\}$.

$\Phi_1$, $\ldots$, $\Phi_{14}$ are in the quotient space $X_2(4mN)$, $\Phi_{15}$, $\ldots$, $\Phi_{21}$ are over a one-dimensional cusp and $\Phi_{22}$, $\ldots$, $\Phi_{25}$ are over a zero-dimensional cusp.

**Remark 3.1.** Although we represented $\Phi_7$ by $\left\{ \begin{pmatrix} i \\ 0 \\
 Z 
 \end{pmatrix} \right\} \subset \mathcal{G}_2$ symbolically, $\Phi_7$ means the closure of the image of $\left\{ \begin{pmatrix} i \\ 0 \\
 Z 
 \end{pmatrix} \right\}$ to $X_2(4mN)$. The same applies to the other cases. Let $N_1$ be $4mN$. Strictly speaking $\Phi_{17}$ should be represented as
\[
\left\{ \begin{pmatrix} Z \\ 1/2 \\
 \infty 
 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} Z \\ (N_1(Z + 1)/2 + 1)/2 \\
 \infty 
 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} Z \\ (N_1Z + 1)/2 \\
 \infty 
 \end{pmatrix} \right\}.
\]
This appears as a boundary of $\Phi_3$ and is a four fold cover of a one-dimensional cusp.

**Definition 3.2.** Let $\Phi$ be one of the 25 kinds of fixed points (sets) and let
\[
C(\Phi, \Gamma_2/\Gamma_2(4mN)) = \{ M \in \Gamma_2/\Gamma_2(4mN) \mid M(Z) = Z \text{ for any } Z \in \Phi \},
\]
\[
N(\Phi, \Gamma_2/\Gamma_2(4mN)) = \{ M \in \Gamma_2/\Gamma_2(4mN) \mid M \text{ maps } \Phi \text{ into } \Phi \}.
\]
C(Φ, Γ₂/Γ₂(4mN)) and N(Φ, Γ₂/Γ₂(4mN)) are the isotropy group and the stabilizer group of Φ, respectively. ϕ ∈ C(Φ, Γ₂/Γ₂(4mN)) is called a proper element of C(Φ, Γ₂/Γ₂(4mN)), if Φ is an irreducible component of the fixed points set of ϕ.

To obtain the dimension formula we have to know the order of N(Φ, Γ₂/Γ₂(4mN)). We list Φ, the order of C(Φ, Γ₂/Γ₂(4mN)) and the order of N(Φ, Γ₂/Γ₂(4mN)) in this order in the following theorem.

**Theorem 3.3.** The orders of the isotropy groups and the stabilizer groups of Φ₁, Φ₂, ..., Φ₂₅ are as follows. For the sake of simplicity we denote 4mN by N₁. ∏ means the product when p runs over the prime divisors of 4mN.

(1) Φ₁ 2 \(N₁^{10} \prod (1 - p^{-2})(1 - p^{-4})\)
(2) Φ₂ 4 \(2N₁^{6} \prod (1 - p^{-2})^2\)
(3) Φ₃ 4 \((8/3)N₁^{6} \prod (1 - p^{-2})^2\)
(4) Φ₄ 8 \(4N₁^{3} \prod (1 - p^{-2})\)
(5) Φ₅ 8 \((16/3)N₁^{3} \prod (1 - p^{-2})\)
(6) Φ₆ 12 \(\begin{cases} 12N₁^{3} \prod (1 - p^{-2}), & \text{if } 3 \nmid 4mN \\ 9N₁^{3} \prod (1 - p^{-2}), & \text{if } 3 \mid 4mN \end{cases}\)
(7) Φ₇ 8 \(4N₁^{3} \prod (1 - p^{-2})\)
(8) Φ₈ 12 \(6N₁^{3} \prod (1 - p^{-2})\)
(9) Φ₉ 32 32
(10) Φ₁₀ 72 72
(11) Φ₁₁ 24 24
(12) Φ₁₂ 24 24
(13) Φ₁₃ 48 48
(14) Φ₁₄ 10 10
(15) Φ₁₅ 2N₁ 2N₁^{6} \prod (1 - p^{-2})
(16) Φ₁₆ 4N₁ 2N₁^{3} \prod (1 - p^{-2})
(17) Φ₁₇ 4N₁ \((8/3)N₁^{3} \prod (1 - p^{-2})\)
(18) Φ₁₈ 8N₁ 8N₁
(19) Φ₁₉ 8N₁ 8N₁
Proof. Due to [T1] Theorem 2.2, Theorem 3.1 and Theorem 4.1.

Next we recall the transformation formula of theta functions and apply this formula to the proper elements of \( C(\Phi, \Gamma_2/\Gamma_2(4mN)) \). The transformation formula was proved in more general form ([Si]). Let \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \). If \( C = O \), we can compute the transformation formula directly. So we assume that \( C \neq O \). First we assume that \( C \) is regular. If \( P \) is a square matrix, we denote the vector which is composed of the diagonal elements of \( P \) by \( P_0 \). We denote \( Sx \) by \( S \). 

THEOREM 3.4 ([Si] Satz 6). Let \( r = \frac{a}{2m} \) and \( s = \frac{b}{2m} \). Assume that \( C \) is regular.

Then

\[
\theta_{r,0}(2mM(Z), 2m^i(CZ + D)^{-1}W) = \left| i^{-1}(Z + C^{-1}D)/2m \right|^{1/2} e(m^iW(CZ + D)^{-1}CW) \\times \sum_{s \in (1/2m)Z^g/Z^g} \rho_s e((-A^iB[a] + 2^iAAb)/4m) \theta_{s,0}(2mZ, 2mW),
\]

where

\[
\rho_s = \sum_{h \in Z^g/CZ^g} e(mAC^{-1}[h] + i(b - iAa)C^{-1}h + C^{-1}D(b - iAa)/4m)
\]

and we choose the branch of \( |i^{-1}(Z + C^{-1}D)/2m|^{1/2} \) so that this is in the right plane when \( Y = \text{Re} i^{-1}Z \) tends to the infinity.

REMARK 3.5. In the following cases, \( C \) is unimodular if \( C \) is regular. Hence we have \( \rho_s = e(C^{-1}D[b - iAa]/4m) \).

Let \( G_0 \) be a \( 2g \times g \) matrix with integral coefficients. Let \( V \) be a regular matrix of degree \( g \) with integral coefficients such that \( G_0V^{-1}k \) is a vector with integral coefficients if and only if \( k \) is so. \( V \) is determined up to a multiplication by unimodular matrix from the left. Put

\[
G_0 = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}
\]

where \( G_1 \) and \( G_2 \) are square matrices of degree \( g \). Then we denote \( V \) by \( (G_1, G_2) \). We have \( (G_1V^{-1}, G_2V^{-1}) = 1_g \).
Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2$ and let $V = (2mA, C)$. Then we have $(2m)1_g, C = V$ ([Si] Hilfssatz 3) and $2mV^{-1}$ has integral coefficients. Let $F = 2mA V^{-1}$ and $G = CV^{-1}$. There exist square matrices of degree $g$ with integral coefficients $H$ and $G$ such that

\[ iHF + iKG = 1_g. \]

Assume that $C$ is regular and let

\[ R = 2mA C^{-1}, \quad n = (tFG)0, \quad \rho = \sum_{h \in \mathbb{Z}^g/\mathbb{C} \mathbb{Z}^g} e((R[h + Hn/2] + i(h + Hn/2)Kn)/2), \]

\[ N = 2m^l H A'i BH + iKC'BH + iHB'C'K + iKD'C'K/2m. \]

Then we have

Theorem 3.6 ([Si] Satz 7). The Gaussian sum

\[ \rho_s = \sum_{h \in \mathbb{Z}^g/\mathbb{C} \mathbb{Z}^g} e(mAC^{-1}[h] + i(b - 1Aa)C^{-1}h + C^{-1}D[b - 1Aa]/4m) \]

is nonzero if and only if the vector $k$ which satisfies

\[ b - 1Aa = iV(k + n/2) \]

has integral coefficients and in this case it holds that

\[ \rho_s = \rho e((N[k + n/2] + i16/256 KKK)/2) \]

Therefore the transformation formula which was a sum with respect to $s$ is reduced to a sum with respect to $k$ which runs over $\mathbb{Z}^g/2m^l V^{-1} \mathbb{Z}^g$.

Let $x$ be an integer and let $M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} 1_g & (16mx)1_g \\ (16m)1_g & (1 + 256m^2x)1_g \end{pmatrix}$. Then we have $V_2 = (2mA_2, C_2) = (2m)1_g$, $F_2 = 2mA_2 V_2^{-1} = 1_g$ and $G_2 = C_2 V_2^{-1} = (8)1_g$. Hence $H_2 = 1_g$ and $K_2 = O$ satisfy

\[ iH_2F_2 + iK_2G_2 = 1_g. \]

We define $N_2$ from $M_2$, $H_2$ and $K_2$ similarly as above. Then we have $N_2 = (32m^2x)1_g$ and $n_2 = (iF_2G_2)0 = (8, 8, \ldots, 8)$. The vector $k$ which satisfies

\[ b - 1A_2a = iV_2(k + n_2/2) = 2m(k + n_2/2) \]

has integral coefficients if and only if $b \equiv 1A_2a \equiv a \pmod{2m}$ and in this case we may assume that $k + n_2/2 = 0$. Since $R_2 = 2mA_2 C_2^{-1} = (1/8)1_g$, we have

\[ \rho_2 = \sum_{h \in \mathbb{Z}^g/\mathbb{C} \mathbb{Z}^g} e(1_g[h + n_2/2]/16) = (1 + i)^g(4m)^g. \]

Hence $\rho_r = \rho_2$ and $\rho_s = 0$, if $s \neq r \pmod{2m}$. Let $W' = i(C_2Z + D_2)^{-1}W$. Since $B_2 \equiv O \pmod{16m}$, we have the transformation formula for $M_2$:

\[ \theta_{r,0}(2mM_2(Z), 2mW') = |i^{-1}(Z + C_2^{-1}D_2)/2m|^{1/2} e(mW'C_2W)\rho_2 \theta_{r,0}(2mZ, 2mW). \]
Next we consider a general transformation formula for \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \) where \( C \) satisfies \( 0 < \text{rank } C < g \). Let \( M_2 \) be as above and let \( M_1 = M M_2^{-1} \). Let \( C_1 \) be the lower-left \( g \times g \) submatrix of \( M_1 \). Then \( C_1 = (1 + 256m^2x)C - 16mD \). We can choose an integer \( x, 0 \leq x \leq g \) so that \( C_1 \) is regular. Then we apply the transformation formula for \( M_1 \) to \( \theta_r, 0 (2mM_2 (Z), 2mW') \) and finally we obtain the transformation formula for \( M = M_1 M_2 \).

Let \( g = 2 \) and let

\[
M = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

This is indexed as \( \varphi_7(1) \) in the following theorem and fixes the points

\[
\begin{pmatrix} i \\ 0 \\ z \end{pmatrix}, \quad z \in S_1.
\]

As an example we show the calculation of the transformation formula for \( M \). \( x = 1 \) satisfies our condition and \( M_2 = \begin{pmatrix} 1/2 & (16m)_{12} \\ (16m)_{12} & (1 + 256m^2)_{12} \end{pmatrix} \). Then

\[
M_1 = MM_2^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} 16m & 0 & -1 & 0 \\ 0 & 1 + 256m^2 & 0 & -16m \\ 1 + 256m^2 & 0 & -16m & 0 \\ 0 & -16m & 0 & 1 \end{pmatrix}.
\]

We have

\[
V_1 = ((2m)_{12}, C_1) = \begin{pmatrix} 1 & 0 \\ 0 & 2m \end{pmatrix},
\]

\[
F_1 = 2mA_1 V_1^{-1} = \begin{pmatrix} 32m & 0 \\ 0 & 1 + 256m^2 \end{pmatrix},
\]

\[
G_1 = C_1 V_1^{-1} = \begin{pmatrix} 1 + 256m^2 & 0 \\ 0 & -8 \end{pmatrix}.
\]

Hence

\[
H_1 = \begin{pmatrix} -8 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 32m^2 \end{pmatrix}
\]

satisfy

\[
^tH_1 F_1 + ^tK_1 G_1 = 1_g.
\]

Moreover we have

\[
n_1 = (^tF_1 G_1)_0 = (1 + 256m^2)^t(32m^2, -8) \equiv ^t(0, 0) \pmod{8},
\]

\[
R_1 = \begin{pmatrix} 32m^2 & 0 \\ 1 + 256m^2 & -1 + 256m^2 \\ 0 & -8 \end{pmatrix},
\]

\[
\rho_1 = \sum_{h \in \mathbb{Z} / C_1 \mathbb{Z}^2} e((R_1[h + H_1 n_1/2] + ^t(h + H_1 n_1/2) K_1 n_1))/2)
\]
We define $N_1$ from $M_1$, $H_1$ and $K_1$ similarly as before. Then $N_1$ has even integral coefficients. The vector $k$ which satisfies $b - A_1 a = V_1 (k + n_1 / 2)$ has integral coefficients if and only if $a_2 \equiv b_2$ (mod 2m). Hence we have $\rho_3 = \rho_1 e((N_1 [k + n_1 / 2] + n_1^t H_1 K_1 k) / 2) = \rho_1$, if $a_2 \equiv b_2$ (mod 2m) and $\rho_3 = 0$, otherwise. Let $W'' = (C_1 M_2 (Z) + D_1)^{-1} W' = \rho (C Z + D)^{-1} W$. We have the transformation formula for $M_1$:

$$\theta_{r,0}(2m M (Z), 2m W'') = i^{-1} (M_2 (Z) + C_1^{-1} D_1) / 2m^{1/2} e(m' W'' C_1 W') \times \sum_{b_1 \in \mathbb{Z}/2m \mathbb{Z}, \ b_2 \equiv a_2 \ (\text{mod} \ 2m)} \rho_1 e((-A_1^t B_1 [a] + 2^t a B_1 b) / 4m) \theta_{s,0}(2m M_2 (Z), 2m W') .$$

We have $A_1 \equiv A$, $B_1 \equiv B$ (mod 16m) and

$$i' W'' C_1 W' + i' W' C_2 W = i' W'' C W$$

and from the transformation formulas for $M_1$ and $M_2$ we have

$$\theta_{r,0}(2m M (Z), 2m W'') = i^{-1} (M_2 (Z) + C_1^{-1} D_1) / 2m^{1/2} i^{-1} (Z + C_2^{-1} D_2) / 2m^{1/2} e(m' W'' C W) \times \sum_{b_1 \in \mathbb{Z}/2m \mathbb{Z}, \ b_2 \equiv a_2 \ (\text{mod} \ 2m)} \rho_1 \rho_2 e((-A^t B [a] + 2^t a B b) / 4m) \theta_{s,0}(2m Z, 2m W) .$$

Now we have

$$i^{-1} (M_2 (Z) + C_1^{-1} D_1) / 2m^{1/2} i^{-1} (Z + C_2^{-1} D_2) / 2m^{1/2} \rho_1 \rho_2 = i^{-1} C_1^{-1} / 2m^{1/2} i^{-1} C_2^{-1} / 2m^{1/2} |C_1 M_2 (Z) + D_1|^{1/2} |C_2 Z + D_2|^{1/2} \rho_1 \rho_2$$

and

$$(C_1 M_2 (Z) + D_1)(C_2 Z + D_2) = C Z + D = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} .$$

Hence choosing the branches of the square roots properly, the above value is turned to

$$\frac{1 + i}{\sqrt{2}} \frac{1 - i}{\sqrt{2}} .$$

We choose the sign of $\det(C Z + D)^{1/2}$ so that $\det(C Z + D)^{1/2} = \frac{1 + i}{\sqrt{2}}$. Then the unitary matrix $u(\varphi(1)) = (u_{r,s}(\varphi(1)))$ in the transformation formula is given by

$$u_{r,s}(\varphi(1)) = \begin{cases} \frac{1 - i}{\sqrt{2}} e(-a_1 b_1 / 2m), & \text{if } a_2 \equiv b_2 \ (\text{mod} \ 2m), \\ 0, & \text{otherwise.} \end{cases}$$

Other cases are similarly calculated.

We indexed the proper elements of $C(\Phi_1, \Gamma_2/\Gamma_2(4m N))$, $C(\Phi_2, \Gamma_2/\Gamma_2(4m N))$, $C(\Phi_25, \Gamma_2/\Gamma_2(4m N))$ as $\varphi_1, \varphi_2, \ldots, \varphi_{25}(1, r, s, t), \ldots, \varphi_{25}(6, r, s, t)$ in [T1]. We use the
same notations. In [T1] we considered the action of \( \Gamma_2(1)/\pm \Gamma_2(N) \) on \( S_k(\Gamma_2(N)) \) because the action of \( \varphi \in \Gamma_2 \) is equal to that of \(-\varphi\). But the actions of \( \varphi \) on \( J_{p,m}^{\text{cusp}}(\Gamma_2(4mN)) \) and on \( J_{p,m}^{s,cusp}(\Gamma_2(4mN)) \) are different from the actions of \(-\varphi\). Hence we have to consider the actions of \( \Gamma_2(1)/\Gamma_2(4mN) \) and in this paper \( \varphi \)'s are elements of \( \Gamma_2(1)/\Gamma_2(4mN) \).

We list \( \varphi \)'s explicitly in the following theorem to make their sign clear. Let \( \varphi_1, \varphi_2 \in C(\Phi, \Gamma_2(4mN)) \). If there exists an element \( P \) of \( N(\Phi, \Gamma_2(4mN)) \) such that \( \varphi_2 = P^{-1}\varphi_1 P \), then we have \( \tau(\varphi_1, \Phi) = \tau(\varphi_2, \Phi) \). Hence their contributions to the dimension formula are same to each other. We classify the elements of \( C(\Phi, \Gamma_2(4mN)) \) by the conjugacy in the above sense. We denote by \( e(\varphi) \) the number of the elements of \( C(\Phi, \Gamma_2(4mN)) \) which are conjugate to \( \varphi \). If \( e(\varphi) > 1 \), we show it under \( \varphi \) in the following theorem. Let \( a \) be an integer. We put \( \delta(a) = 1, \) if \( a \equiv 0 \text{ (mod } 2m) \) and \( \delta(a) = 0, \) otherwise.

**Theorem 3.7.** The transformation formulas for the proper elements of \( C(\Phi_i, \Gamma_2(4mN)) \) \((1 \leq i \leq 25)\) are as follows. We list the notation of \( \varphi, \) itself, \( \det(CZ + D)^{1/2} \) and \( u_{rs}(\varphi) \) in this order. We put \( r = \frac{a}{2m}, \) \( s = \frac{b}{2m} \) where \( a = (a_1, a_2), \) \( b = (b_1, b_2) \in Z^2/2mZ^2. \) We mark \( \varphi \)'s such that \( \varphi \) is conjugate to \(-\varphi\) by *.  

\[
\begin{align*}
(1) & \quad \varphi_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 1 \quad \delta(a_1 - b_1)\delta(a_2 - b_2) \\
& \quad -\varphi_1 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad 1 \quad \delta(a_1 + b_1)\delta(a_2 + b_2) \\
(2) & \quad \varphi_2^* \quad (e = 2) &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad i \quad -i\delta(a_1 + b_1)\delta(a_2 - b_2) \\
& \quad \varphi_3^* \quad (e = 2) &= \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad i \quad -i\delta((a_1a_2)/2m) \times \delta(a_1 + b_1)\delta(a_2 - b_2) \\
(4) & \quad \varphi_4^* \quad (e = 2) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 1 \quad \delta(a_1 - b_2)\delta(a_2 + b_1) \\
& \quad \varphi_5^* \quad (e = 2) &= \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad 1 \quad \delta((a_1a_2)/2m) \times \delta(a_1 - b_2)\delta(a_2 + b_1) \\
\end{align*}
\]
(6) \[ \varphi_6 \]

\[
\begin{pmatrix}
-1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]

\[ \rho \]

\[ \delta(a_1 + b_2)\delta(a_1 - a_2 + b_1) \]

(7) \[ \varphi_7 \]

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[ \frac{1 + i}{\sqrt{2}} \]

\[ \frac{1 + i}{2\sqrt{m}} e(-a_1b_1/2m) \]

\[ \times \delta(a_2 - b_2) \]

(8) \[ \varphi_8 \]

\[
\begin{pmatrix}
-1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[ \rho \]

\[ \frac{1}{2\sqrt{m}} e(-a_1b_1/2m) \]

\[ \times \delta(a_2 - b_2) \]
\[-\varphi_8(1) \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rho \begin{cases} \frac{-i-1}{2\sqrt{m}}e(b_1(b_1 - 2a_1)/4m) \\ \times \delta(a_2 + b_2) \end{cases}\]

\[-\varphi_8(2) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rho^2 \begin{cases} \frac{-i-1}{2\sqrt{m}}e(a_1(a_1 - 2b_1)/4m) \\ \times \delta(a_2 + b_2) \end{cases}\]

\[-\varphi_8(3) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rho \begin{cases} \frac{i}{2\sqrt{m}}e(b_1(b_1 + 2a_1)/4m) \\ \times \delta(a_2 + b_2) \end{cases}\]

\[-\varphi_8(4) \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} -i\rho^2 \begin{cases} \frac{-i-1}{2\sqrt{m}}e(a_1(a_1 + 2b_1)/4m) \\ \times \delta(a_2 + b_2) \end{cases}\]

\[\varphi_9(1) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} i \begin{cases} \frac{i}{2m}e((-a_1b_1 - a_2b_2)/2m) \end{cases}\]

\[-\varphi_9(1) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} -i \begin{cases} \frac{i}{2m}e((a_1b_1 + a_2b_2)/2m) \end{cases}\]

\[\varphi_9(2)^* \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{cases} \frac{1-i}{\sqrt{2}} \end{cases} \frac{1}{2\sqrt{m}}e(-a_2b_2/2m) \times \delta(b_1 - a_2)\]

\[\varphi_9(3)^* \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{cases} \frac{1+i}{\sqrt{2}} \end{cases} \frac{1}{2\sqrt{m}}e(a_2b_1/2m) \times \delta(b_2 - a_1)\]

\[\varphi_{10}(1) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \rho^2 \begin{cases} \frac{i}{2m}e((b_1(b_1 + 2a_1) + b_2(b_2 + 2a_2))/4m) \end{cases}\]

\[\varphi_{10}(2) \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rho \begin{cases} \frac{i}{2m}e((-a_1(a_1 + 2b_1) - a_2(a_2 + 2b_2))/4m) \end{cases}\]
### Dimension Formula for the Spaces of Jacobi Forms of Degree Two

<table>
<thead>
<tr>
<th>Case</th>
<th>Matrix</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\varphi_{10}(1)$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -1 \ 1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$-\rho^2 \frac{i}{2m} \left(e((b_1(b_1-2a_1) + b_2(b_2-2a_2))/4m\right)$</td>
</tr>
<tr>
<td>$-\varphi_{10}(2)$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 1 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; -1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$-\rho \frac{i}{2m} \left(e((-a_1(a_1-2b_1) - a_2(a_2-2b_2))/4m\right)$</td>
</tr>
<tr>
<td>$\varphi_{10}(3)$ ($e = 2$)</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; -1 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$i \frac{i}{2m} \left(e((b_1(b_1-2a_1) - a_2(a_2+2b_2))/4m\right)$</td>
</tr>
<tr>
<td>$-\varphi_{10}(3)$ ($e = 2$)</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ -1 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; -1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$-i \frac{i}{2m} \left(e((b_1(b_1+2a_1) - a_2(a_2-2b_2))/4m\right)$</td>
</tr>
<tr>
<td>$\varphi_{10}(4)^* ($ ($e = 6$)</td>
<td>$\begin{pmatrix} 0 &amp; -1 &amp; 0 &amp; -1 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$i\rho^2 \frac{1+i}{2\sqrt{m}} \left((b_1-a_2) \times e((-a_1^2 - 2a_1b_2)/4m\right)$</td>
</tr>
<tr>
<td>$\varphi_{10}(5)^* ($ ($e = 6$)</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \ -1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$-i\rho \frac{1-i}{2\sqrt{m}} \left((b_2-a_1) \times e((b_1^2 + 2a_2b_1)/4m\right)$</td>
</tr>
<tr>
<td>$\varphi_{10}(6)^* ($ ($e = 2$)</td>
<td>$\begin{pmatrix} -1 &amp; 0 &amp; -1 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$i\rho \frac{1}{2m} \left(e((-a_1(a_1+2b_1) - a_2(a_2-2b_2))/4m\right)$</td>
</tr>
<tr>
<td>$\varphi_{10}(7)^* ($ ($e = 2$)</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; -1 \ 0 &amp; 0 &amp; 0 &amp; -1 \ -1 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$-i\rho^2 \frac{1}{2m} \left(e((b_1(b_1+2a_1) + b_2(b_2-2a_2))/4m\right)$</td>
</tr>
<tr>
<td>$\varphi_{10}(8)^* ($ ($e = 6$)</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; -1 \ 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$-\rho \frac{1+i}{2\sqrt{m}} \left((b_1-a_2) \times e((b_1^2 - 2a_1b_2)/4m\right)$</td>
</tr>
<tr>
<td>$\varphi_{10}(9)^* ($ ($e = 6$)</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \ -1 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$-\rho^2 \frac{1-i}{2\sqrt{m}} \left((b_2-a_1) \times e((-a_2^2 - 2a_2b_1)/4m\right)$</td>
</tr>
</tbody>
</table>
(11) \( \varphi_{11}(1) \left( \begin{array}{cccc}
-1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \right) \) \( \kappa^7 \frac{i}{2m} e \left( \frac{-a_1(a_1 + 2b_1) - 2a_2b_2}{4m} \right) \)

\( \varphi_{11}(2) \left( \begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{array} \right) \) \( \kappa^{17} \frac{i}{2m} e \left( \frac{b_1(b_1 + 2a_1) + 2a_2b_2}{4m} \right) \)

\( \varphi_{11}(3) \left( \begin{array}{cccc}
-1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{array} \right) \) \( \kappa \frac{1}{2m} e \left( \frac{-a_1(a_1 + 2b_1) + 2a_2b_2}{4m} \right) \)

\( \varphi_{11}(4) \left( \begin{array}{cccc}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
-1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \right) \) \( \kappa^{23} \frac{1}{2m} e \left( \frac{b_1 + 2a_1b_1 - 2a_2b_2}{4m} \right) \)

\( \varphi_{11}(1) \left( \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \right) \) \( \kappa^7 \frac{i}{2m} e \left( \frac{a_1(a_1 - 2b_1) + 2a_2b_2}{4m} \right) \)

\( \varphi_{11}(2) \left( \begin{array}{cccc}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \right) \) \( \kappa^{17} \frac{i}{2m} e \left( \frac{b_1(b_1 - 2a_1) - 2a_2b_2}{4m} \right) \)

\( \varphi_{11}(3) \left( \begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \right) \) \( \kappa \frac{1}{2m} e \left( \frac{-a_1(a_1 - 2b_1) - 2a_2b_2}{4m} \right) \)

\( \varphi_{11}(4) \left( \begin{array}{cccc}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
\end{array} \right) \) \( \kappa^{23} \frac{1}{2m} e \left( \frac{b_1(b_1 - 2a_1) + 2a_2b_2}{4m} \right) \)

(12) \( \varphi_{12} \left( \begin{array}{cccc}
0 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 \\
1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} \right) \) \( i \frac{i}{2m} e \left( \frac{-a_1b_1 - a_2b_1 - a_2b_2}{2m} \right) \)

\( \varphi_{12} \left( \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{array} \right) \) \( -i \frac{i}{2m} e \left( \frac{a_1b_1 + a_2b_1 + a_2b_2}{2m} \right) \)
Dimension Formula for the Spaces of Jacobi Forms of Degree Two

(13) \[ \begin{align*}
\varphi_{13} (e = 6) & \begin{pmatrix} 0 & -1 & -1 & 0 \\
-1 & 1 & 0 & -1 \\
1 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix} i \frac{e^{i}}{2m} e((−2a_2(a_1 + b_2) − b_1(b_1 + 2a_1 + 2a_2))/4m) \\
-\varphi_{13} (e = 6) & \begin{pmatrix} 0 & 1 & 1 & 0 \\
1 & -1 & 0 & 1 \\
-1 & 1 & 1 & 0 \\
0 & -1 & 0 & 0 \\
\end{pmatrix} -i \frac{e^{i}}{2m} e((−2a_2(a_1 - b_2) − b_1(b_1 - 2a_1 - 2a_2))/4m)
\end{align*} \]

(14) \[ \begin{align*}
\varphi_{14}(1) & \begin{pmatrix} 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 \\
\end{pmatrix} \omega^3 \frac{e^{i}}{2\sqrt{m}} \delta(b_2 - a_1) e((−a_1^2 + 2(a_1b_1 + a_1b_2 + a_2b_1))/4m) \\
\varphi_{14}(2) & \begin{pmatrix} -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 \\
1 & -1 & -1 & 0 \\
\end{pmatrix} \omega \frac{e^{i}}{2m} e((a_1(a_1 + 2b_1 + 2b_2) + b_2(b_2 + 2a_2 + 2b_1))/4m) \\
\varphi_{14}(3) & \begin{pmatrix} 0 & -1 & -1 & 0 \\
-1 & 0 & 0 & -1 \\
1 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix} \omega^4 \frac{e^{i}}{2m} e((−2a_2(2a_1 + a_2 + 2b_2) − b_1(2a_1 + 2a_2 + b_1))/4m) \\
\varphi_{14}(4) & \begin{pmatrix} 0 & 0 & -1 & -1 \\
1 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} \omega^2 \frac{e^{i}}{2\sqrt{m}} \delta(b_1 - a_2) e((a_2^2 − 2(a_1b_1 + a_1b_2 + a_2b_1))/4m) \\
-\varphi_{14}(1) & \begin{pmatrix} 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 \\
\end{pmatrix} \omega^3 -\frac{e^{i}}{2\sqrt{m}} \delta(b_2 + a_1) e((−a_1^2 + 2(a_1b_1 + a_1b_2 + a_2b_1))/4m) \\
-\varphi_{14}(2) & \begin{pmatrix} 1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 \\
-1 & 1 & 1 & 0 \\
\end{pmatrix} \omega \frac{e^{i}}{2m} e((a_1(a_1 − 2b_1 − 2b_2) + b_2(b_2 − 2a_2 + 2b_1))/4m) \\
-\varphi_{14}(3) & \begin{pmatrix} 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
-1 & 1 & 1 & 0 \\
0 & -1 & 0 & 0 \\
\end{pmatrix} \omega^4 \frac{e^{i}}{2m} e((−2a_2(a_1 + a_2 − 2b_2) + b_1(2a_1 + 2a_2 − b_1))/4m) \\
-\varphi_{14}(4) & \begin{pmatrix} 0 & 0 & 1 & 1 \\
-1 & 1 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} \omega^2 -\frac{e^{i}}{2\sqrt{m}} \delta(b_1 + a_2) e((a_2^2 + 2(a_1b_1 + a_1b_2 + a_2b_1))/4m) \end{align*} \]
\[
\begin{align*}
\varphi_{15}(r) & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 1 \ e(a^2_r/4m) \delta(a_1 - b_1) \delta(a_2 - b_2) \\
-\varphi_{15}(r) & = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -r \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + 1 \ e(a^2_r/4m) \delta(a_1 + b_1) \delta(a_2 + b_2) \\
\varphi_{16}(r) & = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + i \ -i e(a^2_r/4m) \delta(a_1 + b_1) \delta(a_2 - b_2) \\
-\varphi_{16}(r) & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + i \ -i e(a^2_r/4m) \delta(a_1 - b_1) \delta(a_2 + b_2) \\
\varphi_{17}(r) & = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & r \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + i \ -i e((2a_1a_2 + a^2_2r)/4m) \times \delta(a_1 + b_1) \delta(a_2 - b_2) \\
-\varphi_{17}(r) & = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & -r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + i \ -i e((2a_1a_2 + a^2_2r)/4m) \times \delta(a_1 - b_1) \delta(a_2 + b_2) \\
\varphi_{18}(1, r) & = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & r \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ 1 + \frac{i}{\sqrt{2}} \ e((2a_1b_1 + a^2_2r)/4m) \times \delta(a_2 - b_2) \\
\varphi_{18}(2, r) & = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & r \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ 1 + \frac{i}{\sqrt{2}} \ e((-2a_1b_1 + a^2_2r)/4m) \times \delta(a_2 - b_2) \\
-\varphi_{18}(1, r) & = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -r \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ 1 + \frac{i}{\sqrt{2}} \ e((-2a_1b_1 + a^2_2r)/4m) \times \delta(a_2 + b_2) \\
-\varphi_{18}(2, r) & = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -r \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ 1 + \frac{i}{\sqrt{2}} \ e((2a_1b_1 + a^2_2r)/4m) \times \delta(a_2 + b_2) \\
\end{align*}
\]
<table>
<thead>
<tr>
<th>Dimension Formula for the Spaces of Jacobi Forms of Degree Two</th>
</tr>
</thead>
</table>
| \( \varphi_{19}(1, r) \) | \[
\begin{pmatrix}
0 & 0 & 1 & 1 \\
-1 & 1 & 0 & r \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
| \( \frac{1+i}{\sqrt{2}} \frac{1+i}{2\sqrt{m}} e((2a_1b_1 + 2a_1b_2 + a_2^2 r)/4m) \times \delta(a_2 - b_2) \) |
| \( \varphi_{19}(2, r) \) | \[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & -1 & r \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
| \( \frac{1+i}{\sqrt{2}} \frac{1+i}{2\sqrt{m}} e((-2a_1b_1 - 2a_2b_1 + a_2^2 r)/4m) \times \delta(a_2 - b_2) \) |
| \( -\varphi_{19}(1, r) \) | \[
\begin{pmatrix}
0 & 0 & -1 & -1 \\
1 & -1 & 0 & -r \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
| \( \frac{1+i}{\sqrt{2}} \frac{1+i}{2\sqrt{m}} e((-2a_1b_1 - 2a_2b_1 + a_2^2 r)/4m) \times \delta(a_2 + b_2) \) |
| \( -\varphi_{19}(2, r) \) | \[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & -1 & 1 & -r \\
-1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
| \( \frac{1+i}{\sqrt{2}} \frac{1+i}{2\sqrt{m}} e((2a_1b_1 + 2a_2b_1 + a_2^2 r)/4m) \times \delta(a_2 + b_2) \) |
| \( \varphi_{20}(1, r) \) | \[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & r \\
-1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
| \( \rho \frac{1+i}{\sqrt{2}} \frac{1+i}{2\sqrt{m}} e((b_1(1 + 2a_1) + a_2^2 r)/4m) \times \delta(a_2 - b_2) \) |
| \( \varphi_{20}(2, r) \) | \[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & r \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
| \( \rho^2 \frac{i+1}{2\sqrt{m}} e((-a_1(1 + 2b_1) + a_2^2 r)/4m) \times \delta(a_2 - b_2) \) |
| \( \varphi_{20}(3, r) \) | \[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & r \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
| \( i\rho \frac{i+1}{2\sqrt{m}} e((b_1(1 - 2a_1) + a_2^2 r)/4m) \times \delta(a_2 - b_2) \) |
| \( \varphi_{20}(4, r) \) | \[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & r \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
| \( -i\rho^2 \frac{1+i}{\sqrt{2}} \frac{1+i}{2\sqrt{m}} e((-a_1(a_1 - 2b_1) + a_2^2 r)/4m) \times \delta(a_2 - b_2) \) |
| \( -\varphi_{20}(1, r) \) | \[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -1 & 0 & -r \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
| \( \rho \frac{1+i}{\sqrt{2}} \frac{1+i}{2\sqrt{m}} e((b_1(1 - 2a_1) + a_2^2 r)/4m) \times \delta(a_2 + b_2) \) |
| \( -\varphi_{20}(2, r) \) | \[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & -1 & 0 & -r \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
| \( \rho^2 \frac{i+1}{2\sqrt{m}} e((-a_1(a_1 - 2b_1) + a_2^2 r)/4m) \times \delta(a_2 + b_2) \) |
\begin{align*}
-\varphi_{20}(3, r) & \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & -1 & 0 & -r \\
-1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \
& \quad i \rho \frac{i - 1}{2\sqrt{m}} e((b_1(b_1 + 2a_1) + a_2^2 r)/4m) \times \delta(a_2 + b_2) \\
-\varphi_{20}(4, r) & \begin{pmatrix}
-1 & 0 & -1 & 0 \\
0 & -1 & 0 & -r \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \
& \quad -i \rho^2 \frac{-1 + i}{2\sqrt{m}} e((-a_1(a_1 + 2b_1) + a_2^2 r)/4m) \times \delta(a_2 + b_2) \\
\varphi_{21}(1, r) & \begin{pmatrix}
-1 & 0 & -1 & 1 \\
0 & 1 & -1 & r \\
1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix} \
& \quad \rho \frac{i - 1}{2\sqrt{m}} e((-a_1^2 - 2(a_1 + a_2)b_1 + a_2^2 r)/4m) \times \delta(a_2 - b_2) \\
\varphi_{21}(2, r) & \begin{pmatrix}
0 & 0 & -1 & 1 \\
1 & -1 & 1 & r \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix} \
& \quad \rho^2 \frac{i - 1}{2\sqrt{m}} e((-a_1^2 + 2(a_1 + a_2)b_1 + a_2^2 r)/4m) \times \delta(a_2 - b_2) \\
-\varphi_{21}(1, r) & \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & -1
\end{pmatrix} \
& \quad \rho \frac{i + 1}{2\sqrt{m}} e((b_1^2 - 2(b_1 + b_2)a_1 + a_2^2 r)/4m) \times \delta(a_2 + b_2) \\
-\varphi_{21}(2, r) & \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix} \
& \quad \rho^2 \frac{i + 1}{2\sqrt{m}} e((-a_1^2 + 2(a_1 + a_2)b_1 + a_2^2 r)/4m) \times \delta(a_2 + b_2) \\
\varphi_{22}(1, r, t) & \begin{pmatrix}
1 & 0 & r & 0 \\
0 & 1 & 0 & t \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \
& \quad 1 e((a_1^2 r + a_2^2 t)/4m) \times \delta(a_1 - b_1) \delta(a_2 - b_2) \\
-\varphi_{22}(1, r, t) & \begin{pmatrix}
-1 & 0 & -r & 0 \\
0 & -1 & 0 & -t \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \
& \quad 1 e((a_1^2 r + a_2^2 t)/4m) \times \delta(a_1 + b_1) \delta(a_2 + b_2) \\
\varphi_{22}(3, r, t)^* & \begin{pmatrix}
0 & 1 & 0 & r \\
1 & 0 & t & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \
& \quad i e((a_1^2 r + a_2^2 t)/4m) \times \delta(a_1 - b_2) \delta(a_2 - b_1) \\
\varphi_{23}(2, r, t)^* & \begin{pmatrix}
0 & 1 & 0 & r \\
0 & 1 & 0 & -t \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix} \
& \quad 1 e((a_1^2 r + a_2^2 t)/4m) \times \delta(a_1 - b_2) \delta(a_2 + b_1)
\end{align*}
We assume $r \not\equiv 0 \pmod{4mN}$ for $\varphi_{15}(r), \varphi_{16}(r), \varphi_{17}(r), \pm\varphi_{18}(i, r)$ ($i = 1, 2$) and $\pm\varphi_{20}(i, r)$ ($i = 1, 2, 3, 4$).

We assume $r \not\equiv 0 \pmod{4mN}$ and $t \not\equiv 0 \pmod{4mN}$ for $\pm\varphi_{21}(1, r, t)$, $\varphi_{23}(4, r, t)$ and $\varphi_{24}(4, r, t)$.

We assume $r + t \not\equiv 0 \pmod{4mN}$ for $\varphi_{22}(3, r, t)$, $\varphi_{23}(2, r, t)$ and $\varphi_{24}(2, r, t)$.

We assume $r + s \not\equiv 0 \pmod{4mN}$, $t + s \not\equiv 0 \pmod{4mN}$ and $s \not\equiv 0 \pmod{4mN}$ for $\pm\varphi_{25}(1, r, s, t)$.

<table>
<thead>
<tr>
<th>$\varphi_{23}(4, r, t)^*1$</th>
<th>$\begin{pmatrix} -1 &amp; 0 &amp; -r &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; t \ 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</th>
<th>$-i\mathbf{e}(a_1^2r + a_2^2t)/4m$ $\times\delta(a_1 + b_1)\delta(a_2 - b_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_{24}(2, r, t)^*$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; -1 &amp; r \ -1 &amp; 0 &amp; -t &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; -1 &amp; 0 \end{pmatrix}$</td>
<td>$\mathbf{e}(2a_1a_2 + a_1^2r + a_2^2t)/4m$ $\times\delta(a_1 - b_2)\delta(a_2 + b_1)$</td>
</tr>
<tr>
<td>$\varphi_{24}(4, r, t)^*1$</td>
<td>$\begin{pmatrix} -1 &amp; 0 &amp; -r &amp; 1 \ 0 &amp; 1 &amp; -1 &amp; t \ 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$-i\mathbf{e}(2a_1a_2 + a_1^2r + a_2^2t)/4m$ $\times\delta(a_1 + b_1)\delta(a_2 - b_2)$</td>
</tr>
<tr>
<td>$\varphi_{25}(1, r, s, t)$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; r &amp; s \ 0 &amp; 1 &amp; s &amp; t \ 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\mathbf{e}(a_1^2r + 2a_1a_2s + a_2^2t)/4m$ $\times\delta(a_1 - b_1)\delta(a_2 - b_2)$</td>
</tr>
<tr>
<td>$\varphi_{25}(4, r, s, t)^*2$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; s &amp; r \ 1 &amp; 0 &amp; t &amp; s \ 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$-i\mathbf{e}(a_1^2r + 2a_1a_2s + a_2^2t)/4m$ $\times\delta(a_1 - b_1)\delta(a_2 - b_1)$</td>
</tr>
</tbody>
</table>

*1 $\varphi_{23}(4, r, t)$ is conjugate to $\varphi_{23}(4, t, r)$. $\varphi_{24}(4, r, t)$ is conjugate to $\varphi_{24}(4, t, r)$.

*2 $\varphi_{25}(4, r, s, t)$ is conjugate to $\varphi_{25}(4, r, -s, t)$. 

\( \varphi_{25}(2, r, s, t) \) 
\( (e = 2) \) 
\( \begin{pmatrix} -1 & 1 & s & -r & -s \\ 1 & 0 & t & -s & -t \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \) 
\( \mathbf{e}(a_1^2r + 2a_1a_2s + a_2^2t)/4m \times\delta(a_1 + b_2)\delta(a_1 - a_2 + b_1) \)

\( \varphi_{25}(2, r, s, t) \) 
\( (e = 2) \) 
\( \begin{pmatrix} 1 & 1 & -s & r & s \\ -1 & 0 & -t & s & t \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \) 
\( \mathbf{e}(a_1^2r + 2a_1a_2s + a_2^2t)/4m \times\delta(a_1 - b_2)\delta(a_1 - a_2 - b_1) \)

\( \varphi_{25}(4, r, s, t) \) 
\( (e = 3) \) 
\( \begin{pmatrix} 0 & 1 & s & r \\ 1 & 0 & t & s \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \) 
\( -i\mathbf{e}(a_1^2r + 2a_1a_2s + a_2^2t)/4m \times\delta(a_1 - b_2)\delta(a_2 - b_1) \)
We assume $r + s + t \not\equiv 0 \pmod{4mN}$ for $\pm \varphi_{25}(2, r, s, t)$.
We assume $r + 2s + t \not\equiv 0 \pmod{4mN}$ and $s \not\equiv 0 \pmod{4mN}$ for $\varphi_{25}(4, r, s, t)$.

The Gaussian sums which appear as the traces of the unitary matrices in the cases (1), (2), . . . , (14) are calculated as follows.

**COROLLARY 3.8.** Let $m = 2^r p$, where $p$ is an odd integer. Then the traces of the unitary matrices are as follows. $(\frac{p}{3})$ and $(\frac{p}{5})$ mean the Legendre symbol. “≡ 0” means “≡ 0 (mod 2m)”.

(1)

$$\text{Tr}_u(\varphi_1) = \sum_{a \in \mathbb{Z}/2m\mathbb{Z}} 1 \equiv 4m^2$$

$$\text{Tr}_u(-\varphi_1) = \sum_{2a_1 \equiv 2a_2 \equiv 0} 1 \equiv 4$$

(2)

$$\text{Tr}_u(\varphi_2) = \sum_{a_2} \sum_{2a_1 \equiv 0} i = 4mi$$

(3)

$$\text{Tr}_u(\varphi_3) = \sum_{a_2} \sum_{2a_1 \equiv 0} i e((-a_1a_2)/2m) = \sum_{a_2=0}^{2m-1} i \left(1 + (-1)^{a_2}\right) = 2mi$$

(4)

$$\text{Tr}_u(\varphi_4) = \sum_{2a_1 \equiv a_1-a_2 \equiv 0} 1 \equiv 2$$

(5)

$$\text{Tr}_u(\varphi_5) = \sum_{2a_1 \equiv a_1-a_2 \equiv 0} e((-a_1a_2)/2m) = 1 + (-1)^m = \begin{cases} 2, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases}$$

(6)

$$\text{Tr}_u(\varphi_6) = \sum_{3a_1 \equiv a_1+a_2 \equiv 0} 1 = \begin{cases} 1, & \text{if } 3 \nmid m \\ 3, & \text{if } 3 \mid m \end{cases}$$

$$\text{Tr}_u(-\varphi_6) = \sum_{a_1 \equiv a_2 \equiv 0} 1 \equiv 1$$

(7)

$$\text{Tr}_u(\varphi_7(1)) = \frac{1 + i}{2\sqrt{m}} \sum_{a_2} \sum_{a_1=0}^{2m-1} e\left(\frac{a_1^2}{2m}\right) = \begin{cases} 2\sqrt{2mi}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases}$$

$$\text{Tr}_u(\varphi_7(2)) = \frac{1 - i}{2\sqrt{m}} \sum_{a_2} \sum_{a_1=0}^{2m-1} e\left(\frac{-a_1^2}{2m}\right) = \begin{cases} -2\sqrt{2mi}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases}$$
\[ \text{Tr}_u(-\varphi_7(1)) = \frac{1+i}{2\sqrt{m}} \sum_{2a_2=0}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( -\frac{a_1^2}{2m} \right) = \begin{cases} 2\sqrt{2}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases} \]

\[ \text{Tr}_u(-\varphi_7(2)) = \frac{1-i}{2\sqrt{m}} \sum_{2a_2=0}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( \frac{a_1^2}{2m} \right) = \begin{cases} 2\sqrt{2}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases} \]

(8)

\[ \text{Tr}_u(\varphi_8(1)) = \frac{i-1}{2\sqrt{m}} \sum_{a_2}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( -\frac{3a_1^2}{4m} \right) = \begin{cases} 2\sqrt{3}m, & \text{if } 3 \mid m \\ 2\left( \frac{p}{q} \right)(-1)^{r+1}m, & \text{if } 3 \nmid m \end{cases} \]

\[ \text{Tr}_u(\varphi_8(2)) = \frac{i+1}{2\sqrt{m}} \sum_{a_2}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( \frac{3a_1^2}{4m} \right) = \begin{cases} -2\sqrt{3}m, & \text{if } 3 \mid m \\ 2\left( \frac{p}{q} \right)(-1)^{r+1}m, & \text{if } 3 \nmid m \end{cases} \]

\[ \text{Tr}_u(\varphi_8(3)) = \frac{i+1}{2\sqrt{m}} \sum_{a_2}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( \frac{a_1^2}{4m} \right) = -2mi \]

\[ \text{Tr}_u(\varphi_8(4)) = \frac{i-1}{2\sqrt{m}} \sum_{a_2}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( \frac{-a_1^2}{4m} \right) = 2mi \]

\[ \text{Tr}_u(-\varphi_8(1)) = \frac{i-1}{2\sqrt{m}} \sum_{a_2}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( \frac{a_1^2}{4m} \right) = -2 \]

\[ \text{Tr}_u(-\varphi_8(2)) = \frac{i+1}{2\sqrt{m}} \sum_{a_2}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( -\frac{a_1^2}{4m} \right) = -2 \]

\[ \text{Tr}_u(-\varphi_8(3)) = \frac{i+1}{2\sqrt{m}} \sum_{a_2}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( -\frac{3a_1^2}{4m} \right) = \begin{cases} -2\sqrt{3}, & \text{if } 3 \mid m \\ 2\left( \frac{p}{q} \right)(-1)^{r+1}i, & \text{if } 3 \nmid m \end{cases} \]

\[ \text{Tr}_u(-\varphi_8(4)) = \frac{i-1}{2\sqrt{m}} \sum_{a_2}^{2m-1} \sum_{a_1=0}^{2m-1} e\left( \frac{3a_1^2}{4m} \right) = \begin{cases} -2\sqrt{3}, & \text{if } 3 \mid m \\ -2\left( \frac{p}{q} \right)(-1)^{r+1}i, & \text{if } 3 \nmid m \end{cases} \]

(9)

\[ \text{Tr}_u(\varphi_9(1)) = \frac{i}{2m} \sum_{a_1=a_2=0}^{2m-1} e\left( \frac{a_1^2 + a_2^2}{2m} \right) = \begin{cases} -2, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases} \]

\[ \text{Tr}_u(-\varphi_9(1)) = \frac{-i}{2m} \sum_{a_1=a_2=0}^{2m-1} e\left( \frac{-a_1^2 - a_2^2}{2m} \right) = \begin{cases} -2, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases} \]

\[ \text{Tr}_u(\varphi_9(2)) = \frac{1-i}{2\sqrt{m}} \sum_{a_1=0}^{2m-1} e\left( \frac{a_1^2}{2m} \right) = \begin{cases} \sqrt{2}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases} \]
\( \text{Tr}(\varphi_9(3)) = \frac{1 + i}{2\sqrt{m}} \sum_{a_1=0}^{2m-1} e \left( \frac{-a_1^2}{2m} \right) = \begin{cases} \sqrt{2}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases} \)

(10)

\( \text{Tr}(\varphi_{10}(1)) = -i \frac{2m}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{-3a_1^2 - 3a_2^2}{4m} \right) = \begin{cases} -3, & \text{if } 3 \mid m \\ 1, & \text{if } 3 \nmid m \end{cases} \)

\( \text{Tr}(\varphi_{10}(2)) = i \frac{2m}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{3a_1^2 + 3a_2^2}{4m} \right) = \begin{cases} -3, & \text{if } 3 \mid m \\ 1, & \text{if } 3 \nmid m \end{cases} \)

\( \text{Tr}(\varphi_{10}(-1)) = i \frac{2m}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{a_1^2 + a_2^2}{4m} \right) = -1 \)

\( \text{Tr}(\varphi_{10}(-2)) = -i \frac{2m}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{-a_1^2 - a_2^2}{4m} \right) = -1 \)

\( \text{Tr}(\varphi_{10}(3)) = i \frac{2m}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{a_1^2 + 3a_2^2}{4m} \right) = \begin{cases} -\sqrt{3}, & \text{if } 3 \mid m \\ i \left( \frac{\phi}{2} \right)^{-1r}, & \text{if } 3 \nmid m \end{cases} \)

\( \text{Tr}(\varphi_{10}(-3)) = -i \frac{2m}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{-3a_1^2 - a_2^2}{4m} \right) = \begin{cases} -\sqrt{3}, & \text{if } 3 \mid m \\ -i \left( \frac{\phi}{2} \right)^{-1r}, & \text{if } 3 \nmid m \end{cases} \)

\( \text{Tr}(\varphi_{10}(4)) = 1 - i \frac{2m}{2m} \sum_{a_1=0}^{2m-1} e \left( \frac{3a_1^2}{4m} \right) = \begin{cases} \sqrt{3}, & \text{if } 3 \mid m \\ -i \left( \frac{\phi}{2} \right)^{-1r}, & \text{if } 3 \nmid m \end{cases} \)

\( \text{Tr}(\varphi_{10}(5)) = 1 + i \frac{2m}{2m} \sum_{a_1=0}^{2m-1} e \left( \frac{-3a_1^2}{4m} \right) = \begin{cases} \sqrt{3}, & \text{if } 3 \mid m \\ i \left( \frac{\phi}{2} \right)^{-1r}, & \text{if } 3 \nmid m \end{cases} \)

\( \text{Tr}(\varphi_{10}(6)) = -1 \frac{2m}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{3a_1^2 - a_2^2}{4m} \right) = \begin{cases} -\sqrt{3}, & \text{if } 3 \mid m \\ -i \left( \frac{\phi}{2} \right)^{-1r}, & \text{if } 3 \nmid m \end{cases} \)

\( \text{Tr}(\varphi_{10}(7)) = -1 \frac{2m}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{-3a_1^2 + a_2^2}{4m} \right) = \begin{cases} -\sqrt{3}, & \text{if } 3 \mid m \\ -i \left( \frac{\phi}{2} \right)^{-1r}, & \text{if } 3 \nmid m \end{cases} \)

\( \text{Tr}(\varphi_{10}(8)) = 1 - i \frac{2m}{2m} \sum_{a_1=0}^{2m-1} e \left( \frac{a_1^2}{4m} \right) = 1 \)

\( \text{Tr}(\varphi_{10}(9)) = 1 + i \frac{2m}{2m} \sum_{a_1=0}^{2m-1} e \left( \frac{-a_1^2}{4m} \right) = 1 \)
Dimension Formula for the Spaces of Jacobi Forms of Degree Two

(11)

\[ \text{Tr} \, \varphi_{11}(1) = \frac{i}{2m} \sum_{a_1 = a_2 = 0}^{2m - 1} e \left( \frac{3a_1^2 + 2a_2^2}{4m} \right) = \begin{cases} -\sqrt{6}, & \text{if } 6 \mid m \\ i \left( \frac{q}{m} \right) (-1)^r \sqrt{\gamma}, & \text{if } 6 \mid m^2 - 4 \\ 0, & \text{if } m \text{ is odd} \end{cases} \]

\[ \text{Tr} \, \varphi_{11}(2) = \frac{-i}{2m} \sum_{a_1 = a_2 = 0}^{2m - 1} e \left( \frac{-3a_1^2 - 2a_2^2}{4m} \right) = \begin{cases} -\sqrt{6}, & \text{if } 6 \mid m \\ -i \left( \frac{q}{m} \right) (-1)^r \sqrt{\gamma}, & \text{if } 6 \mid m^2 - 4 \\ 0, & \text{if } m \text{ is odd} \end{cases} \]

\[ \text{Tr} \, \varphi_{11}(3) = \frac{1}{2m} \sum_{a_1 = a_2 = 0}^{2m - 1} e \left( \frac{3a_1^2 - 2a_2^2}{4m} \right) = \begin{cases} \sqrt{6}, & \text{if } 6 \mid m \\ -i \left( \frac{q}{m} \right) (-1)^r \sqrt{\gamma}, & \text{if } 6 \mid m^2 - 4 \\ 0, & \text{if } m \text{ is odd} \end{cases} \]

\[ \text{Tr} \, \varphi_{11}(4) = \frac{1}{2m} \sum_{a_1 = a_2 = 0}^{2m - 1} e \left( \frac{-3a_1^2 + 2a_2^2}{4m} \right) = \begin{cases} \sqrt{6}, & \text{if } 6 \mid m \\ i \left( \frac{q}{m} \right) (-1)^r \sqrt{\gamma}, & \text{if } 6 \mid m^2 - 4 \\ 0, & \text{if } m \text{ is odd} \end{cases} \]

(12)

\[ \text{Tr} \, \varphi_{12} = \frac{i}{2m} \sum_{a_1 = a_2 = 0}^{2m - 1} e \left( \frac{a_1^2 + a_1 a_2 + a_2^2}{2m} \right) = \begin{cases} -\sqrt{3}, & \text{if } 3 \mid m \\ -i \left( \frac{q}{m} \right) (-1)^r, & \text{if } 3 \nmid m \end{cases} \]

\[ \text{Tr} \, \varphi_{12} = \frac{-i}{2m} \sum_{a_1 = a_2 = 0}^{2m - 1} e \left( \frac{-a_1^2 - a_1 a_2 - a_2^2}{2m} \right) = \begin{cases} -\sqrt{3}, & \text{if } 3 \mid m \\ i \left( \frac{q}{m} \right) (-1)^r, & \text{if } 3 \nmid m \end{cases} \]

(13)

\[ \text{Tr} \, \varphi_{13} = \frac{i}{2m} \sum_{a_1 = a_2 = 0}^{2m - 1} e \left( \frac{3a_1^2 + 4a_1 a_2 + 2a_2^2}{4m} \right) = \begin{cases} -\sqrt{2}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd} \end{cases} \]
\[
\text{Tr} u(-\varphi_{13}) = -\frac{i}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( -\frac{a_1^2 - 2a_2^2}{4m} \right) = \begin{cases} 
-\sqrt{2}, & \text{if } m \text{ is even} \\
0, & \text{if } m \text{ is odd} 
\end{cases} 
\]

(14)

\[
\text{Tr} u(-\varphi_{14}(1)) = -\frac{1+i}{2\sqrt{m}} \sum_{a_1=a_2=0}^{2m-1} e \left( -\frac{5a_1^2}{4m} \right) = \begin{cases} 
-\sqrt{5}, & \text{if } 5 \mid m \\
- \left( \frac{\sqrt{5}}{5} \right) (-1)^r, & \text{if } 5 \nmid m 
\end{cases} 
\]

\[
\text{Tr} u(-\varphi_{14}(2)) = \frac{i}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( -\frac{3a_1^2 - 4a_1a_2 - 3a_2^2}{4m} \right) = \begin{cases} 
\sqrt{5}, & \text{if } 5 \mid m \\
- \left( \frac{\sqrt{5}}{5} \right) (-1)^r, & \text{if } 5 \nmid m 
\end{cases} 
\]

\[
\text{Tr} u(-\varphi_{14}(3)) = -\frac{i}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( -\frac{3a_1^2 + 4a_1a_2 + 3a_2^2}{4m} \right) = \begin{cases} 
\sqrt{5}, & \text{if } 5 \mid m \\
- \left( \frac{\sqrt{5}}{5} \right) (-1)^r, & \text{if } 5 \nmid m 
\end{cases} 
\]

\[
\text{Tr} u(-\varphi_{14}(4)) = -\frac{1-i}{2\sqrt{m}} \sum_{a_1=a_2=0}^{2m-1} e \left( -\frac{5a_1^2}{4m} \right) = \begin{cases} 
-\sqrt{5}, & \text{if } 5 \mid m \\
- \left( \frac{\sqrt{5}}{5} \right) (-1)^r, & \text{if } 5 \nmid m 
\end{cases} 
\]

\[
\text{Tr} u(-\varphi_{14}(1)) = -\frac{1+i}{2\sqrt{m}} \sum_{a_1=0}^{2m-1} e \left( -\frac{a_1^2}{4m} \right) = -1 
\]

\[
\text{Tr} u(-\varphi_{14}(2)) = \frac{i}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( \frac{a_1^2 + a_2^2}{4m} \right) = -1 
\]

\[
\text{Tr} u(-\varphi_{14}(3)) = -\frac{i}{2m} \sum_{a_1=a_2=0}^{2m-1} e \left( -\frac{a_1^2 - a_2^2}{4m} \right) = -1 
\]

\[
\text{Tr} u(-\varphi_{14}(4)) = -\frac{1-i}{2\sqrt{m}} \sum_{a_1=0}^{2m-1} e \left( \frac{a_1^2}{4m} \right) = -1 
\]

4. The Computation of the Dimension Formula

In this section we assume that \( g = 2 \). In the case when \( \mu = \text{Sym}_j \) which is the \( j \)-th symmetric tensor representation of \( GL(2, \mathbb{C}) \) we denote \( \tilde{V}_\mu \) by \( \tilde{V}_j \). Let \( \mu \) be \( \det^k \text{Sym}_j \). Then \( \tilde{V}_\mu \) is isomorphic to \( \tilde{H}_2^{\otimes 2k} \otimes \tilde{V}_j \). Let \( \Phi \) be an irreducible component of fixed points (sets) of \( \Gamma_2/\Gamma_2(4mN) \) acting on \( \tilde{X}_2(4mN) \) and \( M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \) a proper element of the isotropy group \( C(\Phi, \Gamma_2/\Gamma_2(4mN)) \). Let \( V = \tilde{H}_2^{\otimes (-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m \otimes [D]^{\otimes (-1)} \).

From the result in §2 we have

\[
ch(V|\Phi)(M) = \text{Tr} u(M) ch(\tilde{H}_2^{\otimes (-1)} \otimes \tilde{V}_\mu \otimes [D]^{\otimes (-1)}|\Phi)(M) 
= \text{Tr} u(M) ch(\tilde{H}_2^{\otimes (2k-1)} \otimes \tilde{V}_j \otimes [D]^{\otimes (-1)}|\Phi)(M) 
= \text{Tr} u(M) ch(\tilde{H}_2^{\otimes (2k-1)}|\Phi)(M) \cdot ch(\tilde{V}_j \otimes [D]^{\otimes (-1)}|\Phi)(M) 
\]
Then we have

\[ \chi(V|\Phi)(M) = \chi(\tau|\Phi)(M)(CZ + D)^{k-1/2} \]

and

\[ \chi_0(V|\Phi)(M) = \chi(\tau_0|\Phi)(M)(CZ + D)^{k-1/2} \]

By \( mN \), we have

\[ \dim \{ \chi(V|\Phi)(M) \} = \dim \{ \chi(\tau_0|\Phi)(M) \} \]

In §3 we determined \( u(M) \) and \( |CZ + D|^{1/2} \). Hence it suffices to determine \( \tau_0(M, \Phi) \).

This is easily determined from the results in the case of Siegel modular forms. It suffices to replace \( N \) by \( 4mN \), delete \( \det(CZ + D)^k \) and replace \( k \) by \( k - 1/2 \) in [T3] Theorem 3.2. Here is a point of difference. In the case of Siegel modular forms \( j \) was even. But we have to consider the case when \( j \) is odd in the case of Jacobi forms.

**Theorem 4.1.** Let \( \rho = \epsilon(1/3), \omega = \epsilon(1/5), \sigma = \epsilon(1/12) \) and \( \zeta = \epsilon(1/4mN) \).

For the sake of simplicity we denote \( 4mN \) by \( N_1 \). \( p \) runs over the primes of \( 4mN \) in the product \( \prod \). \( \text{Tr}_p \) and \( \text{Tr}_i \) means the trace map of \( \mathbb{Q}(\rho) \) and \( \mathbb{Q}(i) \) to \( \mathbb{Q} \), respectively.

\( \tau_0(\varphi_1, \Phi_1), \tau_0(\varphi_2, \Phi_2), \ldots, \tau_0(\varphi_{25}(4, r, s, t), \Phi_{25}) \) are as follows. Since \( \tau_0(-\varphi, \Phi) = (-1)^j \tau_0(\varphi, \Phi) \), we list \( \tau_0(\varphi, \Phi) \) only.

1. \[ \tau_0(\varphi_1, \Phi_1) = 2^{-10}3^{-3}5^{-1}(j + 1)(2k - 5)(2j + 2k - 3)(j + 2k - 4)N_1^{10} \]
   \[ -240(j + 1)(j + 2k - 4)N_1^8 + 1440(j + 1)N_1^7 \prod(1 - p^{-2}) \]
2. \[ \tau_0(\varphi_2, \Phi_2) = 2^{-10}3^{-2}(1 + (1/j)(2k - 5)(2j + 2k - 3)N_1^5 \]
   \[ -24(j + 2k - 4)N_1^4 + 144N_1^4 \prod(1 - p^{-2}) \]
3. \[ \tau_0(\varphi_3, \Phi_3) = 2^{-7}3^{-2}(1 + (1/j)(2k - 5)(2j + 2k - 3)N_1^5 \]
   \[ -12(j + 2k - 4)N_1^4 + 48N_1^5 \prod(1 - p^{-2}) \]
4. \[ \tau_0(\varphi_4, \Phi_4) = 2^{-6}3^{-1}(j + (1/j)(2k - 4)N_1^3 - 12N_1^2) \prod(1 - p^{-2}) \]
5. \[ \tau_0(\varphi_5, \Phi_5) = 2^{-5}3^{-1}(j + (1/j)(2k - 4)N_1^3 - 8N_1^2) \prod(1 - p^{-2}) \]
6. \[ \tau_0(\varphi_6, \Phi_6) = \text{Tr}_p(\rho^j(1 - r^2))(j + 2k - 4)N_1^3 - 9N_1^2 \]
   \[ \times \begin{cases} 2^{-1}3^{-3} \prod(1 - p^{-2}), & \text{if } 3 \nmid N_1 \\ 2^{-3}3^{-2} \prod(1 - p^{-2}), & \text{if } 3 \mid N_1 \end{cases} \]
7. \[ \tau_0(\varphi_7(1, \Phi_7) = 2^{-4}3^{-1}(j)(2k - 5)N_1^2 - 12N_1^2) \prod(1 - p^{-2}) \]
   \[ + 2^{-6}3^{-1}(j)(2j + 2k - 3)N_1^3 - 12N_1^2) \prod(1 - p^{-2}) \]
\[ \tau_0(\varphi_7(2), \Phi_7) = \text{the conjugate of } \tau_0(\varphi_7(1), \Phi_7) \text{ over } \mathbb{Q} \]

(8) \[ \tau_0(\varphi_8(1), \Phi_8) = 2^{-4}3^{-3}(\rho^2)^j(1 - \rho^2)((2k - 5)N_1^3 - 12N_1^2) \prod(1 - p^{-2}) \\
+ 2^{-4}3^{-3}(1 - \rho)((2j + 2k - 3)N_1^3 - 12N_1^2) \prod(1 - p^{-2}) \]

\[ \tau_0(\varphi_8(2), \Phi_8) = \text{the conjugate of } \tau_0(\varphi_8(1), \Phi_8) \text{ over } \mathbb{Q} \]

(9) \[ \tau_0(\varphi_8(3), \Phi_8) = 2^{-4}3^{-2}(1 - \rho^2)^j(1 - \rho^2)((2k - 5)N_1^3 - 12N_1^2) \prod(1 - p^{-2}) \\
+ 2^{-4}3^{-2}(1 - \rho)((2j + 2k - 3)N_1^3 - 12N_1^2) \prod(1 - p^{-2}) \]

\[ \tau_0(\varphi_8(4), \Phi_8) = \text{the conjugate of } \tau_0(\varphi_8(3), \Phi_8) \text{ over } \mathbb{Q} \]

(10) \[ \tau_0(\varphi_9(1), \Phi_9) = -2^{-3}(i)^j(j + 1) \]

\[ \tau_0(\varphi_9(2), \Phi_9) = -2^{-3}(1 + (-1)^j) \left( \frac{1 + i}{\sqrt{2}} \right)^j (1 + i) \]

(11) \[ \tau_0(\varphi_9(3), \Phi_9) = -2^{-3}(1 + (-1)^j) \left( \frac{1 - i}{\sqrt{2}} \right)^j (1 - i) \]

(12) \[ \tau_0(\varphi_{10}(1), \Phi_{10}) = 3^{-2}(\rho^2)^j(2\rho + 1)(j + 1) \]

\[ \tau_0(\varphi_{10}(2), \Phi_{10}) = 3^{-2}(\rho)^j(2\rho^2 + 1)(j + 1) \]

\[ \tau_0(\varphi_{10}(3), \Phi_{10}) = 2^{-1}3^{-1}((-\rho^2)^j + 1) \]

\[ \tau_0(\varphi_{10}(4), \Phi_{10}) = 2^{-1}3^{-1}(\rho^2)^j + 1) \]

\[ \tau_0(\varphi_{10}(5), \Phi_{10}) = 2^{-1}3^{-1}(\rho)^j + 1) \]

\[ \tau_0(\varphi_{10}(6), \Phi_{10}) = 2^{-1}3^{-1}(\rho)^j + 1) \]

\[ \tau_0(\varphi_{10}(7), \Phi_{10}) = 2^{-1}3^{-1}(\rho)^j + 1) \]

\[ \tau_0(\varphi_{10}(8), \Phi_{10}) = 2^{-1}3^{-1}(i\rho)^j + 1) \]

\[ \tau_0(\varphi_{10}(9), \Phi_{10}) = 2^{-1}3^{-1}(i\rho^2)^j + 1) \]

(13) \[ \tau_0(\varphi_{11}(1), \Phi_{11}) = 2^{-1}3^{-1}((\sigma^4)^j + 1) - (\sigma^3)^j + 1) \]

\[ \tau_0(\varphi_{11}(2), \Phi_{11}) = 2^{-1}3^{-1}((\sigma^8)^j + 1) - (\sigma^9)^j + 1) \]

\[ \tau_0(\varphi_{11}(3), \Phi_{11}) = 2^{-1}3^{-1}((\sigma^4)^j + 1) - (\sigma^9)^j + 1) \]

\[ \tau_0(\varphi_{11}(4), \Phi_{11}) = 2^{-1}3^{-1}((\sigma^8)^j + 1) - (\sigma^3)^j + 1) \]

(14) \[ \tau_0(\varphi_{12}(1), \Phi_{12}) = 2^{-1}3^{-1}(\rho)^j + 1) \]

\[ \tau_0(\varphi_{12}(2), \Phi_{12}) = 2^{-1}3^{-1}(\rho^2)^j + 1) \]

\[ \tau_0(\varphi_{12}(3), \Phi_{12}) = 2^{-1}3^{-1}(\rho)^j + 1) \]

\[ \tau_0(\varphi_{12}(4), \Phi_{12}) = 2^{-1}3^{-1}(\rho^2)^j + 1) \]

\[ \tau_0(\varphi_{13}(1), \Phi_{13}) = 2^{-3}(\omega)^j - (\omega^3)^j \]

\[ \tau_0(\varphi_{13}(2), \Phi_{13}) = 2^{-3}(\omega^4)^j - (\omega^3)^j \]

\[ \tau_0(\varphi_{13}(3), \Phi_{13}) = 2^{-3}(\omega)^j - (\omega^3)^j \]

\[ \tau_0(\varphi_{13}(4), \Phi_{13}) = 2^{-3}(\omega^4)^j - (\omega^3)^j \]

(15) \[ \tau_0(\varphi_{14}(1), \Phi_{14}) = 2^{-3}(j + 1) \]

\[ \tau_0(\varphi_{14}(2), \Phi_{14}) = 2^{-3}(j + 1)^2 \]

\[ \tau_0(\varphi_{14}(3), \Phi_{14}) = 2^{-3}(j + 1)^3 \]

\[ \tau_0(\varphi_{14}(4), \Phi_{14}) = 2^{-3}(j + 1)^4 \]
The Euler-Poincaré characteristic $\chi(\tilde{X}_g(4mN), O(\tilde{H}_g^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{E}_m - D)) / \Gamma_2 / \Gamma_2(4mN)$ is calculated as

$$\sum_{\Phi} \sum_\varphi \frac{\tau(\varphi, \Phi)}{|N(\Phi, \Gamma_2(1)/\Gamma_2(4mN))|} = \sum_{\Phi} \sum_\varphi \frac{\text{Tr} u(\varphi) |CZ + D|^{k-1/2} \tau_0(\varphi, \Phi)}{|N(\Phi, \Gamma_2(1)/\Gamma_2(4mN))|},$$

where $\tau_0(\varphi, \Phi)$ is calculated as

$$\tau_0(\varphi_{16}(r), \Phi_{16}) = 2^{-6}3^{-1} \left( \frac{(1+(-1)^j)(2(2k-4)N_1)}{(1-\zeta^r)} \right) \left( \frac{8(2k-4)N_1}{(1-\zeta^r)} + \frac{4}{(1-\zeta^r)^2} \right) \times N_1^2 \prod(1 - p^{-2}),$$

$$\tau_0(\varphi_{17}(r), \Phi_{17}) = 2^{-4}3^{-1} \left( \frac{(1+(-1)^j)(2(2k-4)N_1)}{(1-\zeta^r)} \right) \left( \frac{8(2k-4)N_1}{(1-\zeta^r)} + \frac{4}{(1-\zeta^r)^2} \right) \times N_1^2 \prod(1 - p^{-2}),$$

$$\tau_0(\varphi_{18}(1, r), \Phi_{18}) = 2^{-2}((i)^j - i)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{19}(1, r), \Phi_{19}) = 2^{-2}((i)^j - i)(\exp(\pi i (2r - 1)/N_1) - 1)^{-1}$$

and

$$\tau_0(\varphi_{19}(2, r), \Phi_{19}) = 2^{-2}((i)^j + i)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{20}(1, r), \Phi_{20}) = 2^{-2}(\rho^2 - 1)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{20}(2, r), \Phi_{20}) = 2^{-2}(\rho^2 - 1)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{20}(3, r), \Phi_{20}) = 3^{-1}(1 + \rho^2)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{20}(4, r), \Phi_{20}) = 3^{-1}(1 + \rho^2)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{21}(1, r), \Phi_{21}) = 3^{-2}(\rho^2 - 1)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{21}(2, r), \Phi_{21}) = 3^{-2}(\rho^2 - 1)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{22}(1, r, t), \Phi_{22}) = \frac{(j+1)}{(\zeta^r-1)(\zeta^{rt}-1)} \left( \frac{2}{(\zeta^r-1)} + \frac{2}{(\zeta^{rt}-1)} + 3 \right)$$

and

$$\tau_0(\varphi_{23}(2, r, t), \Phi_{23}) = 2^{-2}((i)^j + (-i)^j)(\zeta^{rt} - 1)^{-1}$$

and

$$\tau_0(\varphi_{23}(4, r, t), \Phi_{23}) = 2^{-2}(1 + (-1)^j)(\zeta^r - 1)^{-1}$$

and

$$\tau_0(\varphi_{24}(2, r, t), \Phi_{24}) = 2^{-2}((i)^j + (-i)^j)(\zeta^{rt} - 1)^{-1}$$

and

$$\tau_0(\varphi_{24}(4, r, t), \Phi_{24}) = 2^{-2}(1 + (-1)^j)(\zeta^{rt} - 1)^{-1}$$

and

$$\tau_0(\varphi_{25}(1, r, s, t), \Phi_{25}) = (j + 1)(\zeta^{r+s} - 1)^{-1}(\zeta^{r+s} - 1)^{-1}(\zeta^{s} - 1)^{-1}$$

and

$$\tau_0(\varphi_{25}(2, r, s, t), \Phi_{25}) = 3^{-1}\text{Tr}_\rho((\rho^2)^j(1 - \rho))((\zeta^{r+s} - 1)^{-1}$$

and

$$\tau_0(\varphi_{25}(4, r, s, t), \Phi_{25}) = 2^{-1}(1 + (-1)^j)(\zeta^{r+s} - 1)^{-1}(\zeta^{s} - 1)^{-1}$$
where \( \Phi \) runs over the representatives of the fixed points (sets) and \( \varphi \) runs over the proper elements in \( C(\Phi, \Gamma_2(1)/\Gamma_2(4mN)) \) ([T1] Theorem 5.2). This is also represented as

\[
\sum_{\varphi} \langle \tau \rangle_{\Phi} C(\Phi, \Gamma_2(1)/\Gamma_2(4mN))^k = \frac{\text{Tr} u(\varphi) |CZ + D|^{k-1/2} \tau_0(\varphi, \Phi)}{|N(\Phi, \Gamma_2(1)/\Gamma_2(4mN))|},
\]

where \( \varphi \) runs over the representatives of the proper elements in \( C(\Phi, \Gamma_2(1)/\Gamma_2(4mN)) \) with respect to the conjugacy described before Theorem 3.7. When we calculate \( \chi(\tilde{X}_g(4mN), \mathcal{O}(\tilde{H}_g^{\otimes(-1)} \otimes \tilde{V}_\mu \otimes \tilde{\tau}_m - D))_{\Gamma_2/\Gamma_2(4mN)} \), we replace \( \text{Tr} u(\varphi) \) by \( \text{Tr} u(\varphi) \). \(|CZ + D|, \tau_0(\varphi, \Phi) \) and \(|N(\Phi, \Gamma_2(1)/\Gamma_2(4mN))| \) are given in Theorem 3.7, Theorem 4.1 and Theorem 3.3, respectively. \( \text{Tr} u(\varphi) \) are given in Corollary 3.8 for \( \Phi_1, \Phi_2, \ldots, \Phi_{14} \). Therefore the contributions of \( \Phi_1, \Phi_2, \ldots, \Phi_{14} \) are calculated by using these data. The proper elements of \( \varphi \in C(\Phi, \Gamma_2(1)/\Gamma_2(4mN)) \) for \( \Phi_{15}, \Phi_{16}, \ldots, \Phi_{25} \) has some index (for example \( \tau \) in \( \varphi_{15}(r) \)). In these cases we do not evaluate the trace of \( u(\varphi) \) but evaluate the summation of \( u(\varphi) \tau_0(\varphi, \Phi) \) with respect to \( r \).

Let \( w = 1 \) in the case of holomorphic Jacobi forms and \( w = -1 \) in the case of skew-holomorphic Jacobi forms. For an integer \( a \) we denote by \( \langle a \rangle \) the integer which satisfies \( \langle a \rangle \equiv a \) (mod 4m) and \( 0 \leq \langle a \rangle \leq 4m - 1 \). Since strange sums such as

\[
\sum_{x=1}^{2m-1} \sum_{y=1}^{2m-1} \left( wx^2 \right) \left( wy^2 \right) \left( w(x - y)^2 \right)
\]

which we denoted by SSS appear in the Mathematica code, we show the details of the evaluations with respect \( r \) in the following. We show only for the cases of unipotent elements, namely for the cases of \( \varphi_1, \varphi_{15}(r), \varphi_{22}(1, r, t) \) and \( \varphi_{25}(1, r, s, t) \). We omit the evaluations for the other cases, since they are essentially routine works.

To evaluate the sum of this type we use the following

**Lemma 4.2.** Let \( N_1 = 4mN \) and let \( \xi \) be \( e(1/N_1) \).

(1) \[
\sum_{r=1}^{N_1-1} \frac{\xi^{-arN}}{(1 - \xi^r)^2} = \frac{N_1 - 1}{2} - \frac{N_1 \langle a \rangle}{4m}.
\]

(2) \[
\sum_{r=1}^{N_1-1} \frac{\xi^{-arN}}{(1 - \xi^r)^3} = -\frac{(N_1 - 1)(N_1 - 5)}{12} - \frac{1}{2} \left( \frac{\langle a \rangle^2 N_1^2}{m^2} - \frac{(N_1 - 2)N_1 \langle a \rangle}{4m} \right).
\]

(3) \[
\sum_{r=1}^{N_1-1} \frac{\xi^{-arN}}{(1 - \xi^r)^3} = -\frac{(N_1 - 1)(N_1 - 3)}{8}
\]

\[
-\frac{1}{12} \left( \frac{2N_1^3 \langle a \rangle^3}{4m^3} - \frac{(3N_1 - 9)N_1^2 \langle a \rangle^2}{4m^2} + \frac{(N_1^2 - 9N_1 + 12)N_1 \langle a \rangle}{4m} \right).
\]

By Theorem 3.3 (1), Corollary 3.8 (1) and Theorem 4.1 (1) the contributions of \( \varphi_1 \) to the case of holomorphic Jacobi forms and to the case of skew-holomorphic Jacobi forms
are equal to
\[
\frac{\text{Tr} u(\varphi_1) | CZ + D^{k-1/2} \tau_0(\varphi_1, \Phi_1)}{\left| N(\Phi_1, T_2(1)/T_2(N_1)) \right|} = \frac{\text{Tr} u(\varphi_1) | CZ + D^{k-1/2} \tau_0(\varphi_1, \Phi_1)}{\left| N(\Phi_1, T_2(1)/T_2(N_1)) \right|}
\]
\[
= 2^{-8}3^{-3}5^{-1}m^2 \left\{ (j + 1)(2k - 5)(2j + 2k - 3)(j + 2k - 4) - \frac{240(j + 1)(j + 2k - 4)}{N_1^2} + \frac{1440(j + 1)}{N_1^3} \right\} .
\]

Let \( w \) and \( \zeta \) be as above. We present the case of holomorphic Jacobi forms and the case of skew-holomorphic Jacobi forms in one expression. We put
\[
\text{Tr} u(\varphi)^w = \begin{cases} 
\text{Tr} u(\varphi), & \text{if } w = 1, \\
\text{Tr} u(\varphi), & \text{if } w = -1.
\end{cases}
\]

By Theorem 3.7 (15) we have
\[
\text{Tr} u(\varphi_{15}(r))^w = \sum_{a_1, a_2 = 0}^{2m - 1} \text{e} \left( -wa_2^2r/4m \right) = 2m \sum_{a_2 = 0}^{2m - 1} \zeta^{-wa_2^2rN} .
\]

Hence from Theorem 3.3 (15) and Theorem 4.1 (15) the summation of the contributions of \( \varphi_{15}(r) \) \( (1 \leq r \leq N_1 - 1) \) is equal to
\[
\sum_{r=1}^{N_1-1} \frac{\text{Tr} u(\varphi_{15}(r))^w | CZ + D^{k-1/2} \tau_0(\varphi_{15}(r), \Phi_{15})}{\left| N(\Phi_{15}, T_2(1)/T_2(N_1)) \right|}
\]
\[
= 2^{-3}3^{-1}m(j + 1)N_1^{-6} \sum_{a_2 = 0}^{2m - 1} \sum_{r=1}^{N_1-1} \zeta^{-wa_2^2rN} \left\{ \frac{(9 - (j + 2k - 4)N_1)}{(1 - \zeta^r)} + \frac{(j + 2k - 4)N_1 - 6}{(1 - \zeta^r)^2} - \frac{4}{(1 - \zeta^r)^3} \right\}
\]
\[
= -2^{-4}3^{-2}m^2(j + 1)(j + 2k - 4) + 2^{-6}3^{-2} \sum_{a_2} \left( wa_2^2 \right) (j + 1)(3j + 6k - 10)
\]
\[
- 2^{-8}3^{-1}m^{-1} \sum_{a_2} \left( wa_2^2 \right)^2 (j + 1)(j + 2k - 2) + 2^{-8}3^{-2}m^{-2} \sum_{a_2} \left( wa_2^2 \right)^3 (j + 1)
\]
\[
+ 2^{-2}3^{-1}N_1^{-1}m^2(j + 1) - 2^{-4}N_1^{-1} \sum_{a_2} \left( wa_2^2 \right) (j + 1)
\]
\[
+ 2^{-6}N_1^{-1}m^{-1} \sum_{a_2} \left( wa_2^2 \right)^2 (j + 1)
\]
\[
+ 2^{-4}3^{-2}N_1^{-2}m^2(j + 1)(j + 2k - 10) + 2^{-5}3^{-1}N_1^{-2} \sum_{a_2} \left( wa_2^2 \right) (j + 1)
\]
\[
- 2^{-3}3^{-1}N_1^{-3}m^2(j + 1) .
\]
Similarly, by Theorem 3.7 (22) we have
\[
\text{Tr} u(\varphi_{22}(1, r, t)) = \sum_{a_1, a_2=0}^{2m-1} \epsilon(-w(a_1^2 r + a_2^2 t)/4m) = \sum_{a_1, a_2=0}^{2m-1} \zeta^{-wa_1^2 r} \zeta^{-wa_2^2 t}.
\]
Hence from Theorem 3.3 (22) and Theorem 4.1 (22) the summation of the contributions of \( \varphi_{22}(1, r, t) \) (1 \( \leq r, t \leq N_1 - 1 \)) is equal to
\[
\sum_{r, t=1}^{N_1-1} \frac{\text{Tr} u(\varphi_{22}(1, r, t)) |CZ + D^{k-1/2} \tau_0(\varphi_{22}(1, r, t), \Phi_{22})|}{|N(\Phi_{22}, \Gamma_2(1)/\Gamma_2(N_1))|} = 2^{-3} N_1^{-3} \sum_{a_1, a_2=0}^{2m-1} \sum_{r, t=1}^{N_1-1} \zeta^{-wa_1^2 r} \zeta^{-wa_2^2 t} (j + 1)(\zeta^{-1} + 1)^2 \left( \frac{2}{(\xi' - 1)} + \frac{2}{(\xi' - 1)} + 3 \right)
\]
\[
= 2^{-2} N_1^{-1} m^2 (j + 1) - 2^{-2} 3^{-1} \sum_a \left( wa^2 \right)^2 (j + 1) + 2^{-6} m^{-2} (j + 1) \left( \sum_a \left( wa^2 \right)^2 \right)^2
\]
\[
+ 2^{-6} m^{-1} \sum_a \left( wa^2 \right)^2 (j + 1) - 2^{-8} m^{-3} (j + 1) \sum_a \left( wa^2 \right)^2 \sum_a \left( wa^2 \right)^2
\]
\[
- 2^{-3} 3^{-1} 5 N_1^{-1} m^2 (j + 1) + 2^{-3} N_1^{-1} \sum_a \left( wa^2 \right)^2 (j + 1)
\]
\[
- 2^{-7} N_1^{-1} m^{-2} (j + 1) \left( \sum_a \left( wa^2 \right)^2 \right)^2
\]
\[
- 2^{-6} N_1^{-1} m^{-1} \sum_a \left( wa^2 \right)^2 (j + 1) + 2^{-1} 3^{-1} N_1^{-2} m^2 (j + 1)
\]
\[
- 2^{-3} 3^{-1} N_1^{-2} \sum_a \left( wa^2 \right)^2 (j + 1) - 2^{-3} 3^{-1} N_1^{-3} m^2 (j + 1).
\]

Similarly, by Theorem 3.7 (25) we have
\[
\text{Tr} u(\varphi_{25}(1, r, s, t)) = \sum_{a_1, a_2=0}^{2m-1} \epsilon(-w(a_1^2 r + 2a_1 a_2 s + a_2^2 t)/4m)
\]
\[
= \sum_{a_1, a_2=0}^{2m-1} \zeta^{-wa_1^2 r} \zeta^{-wa_2^2 (s-t)} \zeta^{-wa_1^2 (r+s)} \zeta^{-wa_2^2 (s+t)} \zeta^{-w(a_1-a_2)^2 (-s-t)}.
\]
Hence from Theorem 3.3 (25) and Theorem 4.1 (25) the summation of the contributions of \( \varphi_{25}(1, r, s, t) \) (\( r + s \neq 0 \) (mod \( N_1 \)), \( -s \neq 0 \) (mod \( N_1 \)), \( s + t \neq 0 \) (mod \( N_1 \))) is equal to
\[
\sum_{r, s, t} \frac{\text{Tr} u(\varphi_{25}(1, r, s, t)) |CZ + D^{k-1/2} \tau_0(\varphi_{25}(1, r, s, t), \Phi_{25})|}{|N(\Phi_{25}, \Gamma_2(1)/\Gamma_2(N_1))|} = 2^{-2} 3^{-1} N_1^{-3} \sum_{a_1, a_2=0}^{2m-1} \sum_{r, s, t} \left( j + 1 \right) \frac{\zeta^{-wa_1^2 r}}{(\zeta^{r-s} - 1)} \cdot \frac{\zeta^{-wa_2^2 (s-t)}}{(\zeta^{s+t} - 1)} \cdot \frac{\zeta^{-w(a_1-a_2)^2 (-s-t)}}{(\zeta^{s-t} - 1)}
\]
\[= -2^{−3}3^{−1}m^2(j + 1) + 2^{−5} \sum_a \left(wa^2\right)(j + 1) - 2^{−7}m^{−2}(j + 1) \left(\sum_a \left(wa^2\right)\right)^2 + 2^{−8}3^{−1}m^{−3}(j + 1) \sum_{a_1, a_2} \left(wa_1^2\right)\left(wa_2^2\right)\left(w(a_1 - a_2)^2\right) + 2^{−3}N_1^{−1}m^2(j + 1) - 2^{−4}N_1^{−1} \sum_a \left(wa^2\right)(j + 1) + 2^{−7}N_1^{−1}m^{−2}(j + 1) \left(\sum_a \left(wa^2\right)\right)^2 - 2^{−3}N_1^{−2}m^2(j + 1) + 2^{−5}N_1^{−2} \sum_a \left(wa^2\right)(j + 1) + 2^{−3}3^{−1}N_1^{−3}m^2(j + 1).\]

From the computations above the total of the contributions of the unipotent elements is equal to
\[2^{−8}3^{−3}5^{−1}m^2(j + 1)(2k - 5)(2j + 2k - 3)(j + 2k - 4) - 2^{−4}3^{−2}m^2(j + 1)(j + 2k - 4) + 2^{−3}3^{−1}m^2(j + 1) + 2^{−3}m^2(j + 1)(3j + 6k - 40) \sum_a \left(wa^2\right) - 2^{−8}3^{−1}m^{−1}(j + 2k - 14) \sum_a \left(wa^2\right)^2 + 2^{−8}3^{−2}m^{−2}(j + 1) \sum_a \left(wa^2\right)^3 + 2^{−7}m^{−2}(j + 1) \left(\sum_a \left(wa^2\right)\right)^2 - 2^{−8}m^{−3}(j + 1) \sum_a \left(wa^2\right) \sum_a \left(wa^2\right)^2 + 2^{−8}3^{−1}m^{−3}(j + 1) \sum_{a_1, a_2} \left(wa_1^2\right)\left(wa_2^2\right)\left(w(a_1 - a_2)^2\right)\].

**Remark 4.3.** In the contributions of \(\varphi_1, \varphi_{15}(r), \varphi_{22}(1, r, t)\) and \(\varphi_{25}(1, r, s, t)\) the terms which include \(N_1^{−1}, N_1^{−2}\) and \(N_1^{−3}\) appear. When we take the total they cancel out as easily verified. Since \(\dim J_{k,j,m}^{\text{cusp}}(\Gamma_2)\) and \(\dim J_{k,j,m}^{\text{sk,cusp}}(\Gamma_2)\) are independent on \(N\), we may choose \(N = 1\) and \(N = 4m\). But we use general \(N\) because we can check mistakes of the computation by this cancellation.

**Theorem 5.1.**
\[\sum_{k=0}^{\infty} \dim J_{k,1}^{\text{cusp}}(\Gamma_2) t^k = \sum_{k=0}^{\infty} \text{Jacobi2}[k, 0, 1, 0] t^k + t^3 = \frac{t^{10} + t^{12} + t^{14} + 2t^{16} + t^{18} + t^{21} - t^{26} + t^{27} - t^{28} + t^{29} + t^{35}}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})},\]

**5. Applications**
\[ \sum_{k=0}^{\infty} \dim J_{k,1}^{J,k,cusp}(\Gamma_2) t^k = \sum_{k=0}^{\infty} \text{Jacobi2}[k,0,1,1] t^k + 1 + t^3 = \frac{h(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}, \]

where \( h(t) \) is
\[ t^{11} + 2t^{13} + t^{15} + 2t^{19} + t^{21} - 2t^{23} + t^{24} - t^{25} + t^{26} - t^{29} + t^{30} - t^{31} + t^{32} + t^{35}. \]

**Proof.** Let \( \varphi_4 \) be the Eisenstein series of degree 2 and weight 4. If \( f \in J_{k,1}^{cusp}(\Gamma_2) \), then \( \varphi_4 f \in J_{k,1}^{cusp}(\Gamma_2) \). Since \( \dim J_{k,1}^{cusp}(\Gamma_2) = \text{Jacobi2}[k,0,1,0] = 0 \) for \( k = 4, 5, 6, 7 \), we have \( \dim J_{k,1}^{cusp}(\Gamma_2) = 0 \) for \( k = 0, 1, 2, 3 \). On the other hand we have \( \text{Jacobi2}[3,0,1,0] = -1 \). Hence the equality of the first line and the second line in (1) holds. We can prove the equality of the first line and the second line in (2) similarly. \( \square \)

We use the following theorem concerning the surjectivity of \( \Phi \)-operator ([A2], [D] and [H2]).

**THEOREM 5.2.** We define \( J_{k,m}(\Gamma_0) = J_{k,m}^{J,k,cusp}(\Gamma_0) = C \) and assume \( m \) is square free.

1. If \( k \) is even and \( k \geq 6 \), then
   \( \dim J_{k,m}(\Gamma_2) = \dim J_{k,m}^{cusp}(\Gamma_2) + \dim J_{k,m}^{cusp}(\Gamma_1) + \dim J_{k,m}^{cusp}(\Gamma_0) \).

2. If \( k \) is odd and \( k \geq 7 \), then
   \( \dim J_{k,m}^{J,k}(\Gamma_2) = \dim J_{k,m}^{J,k,cusp}(\Gamma_2) + \dim J_{k,m}^{J,k,cusp}(\Gamma_1) + \dim J_{k,m}^{J,k,cusp}(\Gamma_0) \).

Let \( \Gamma_0^{2}(4) \), \( J(M,Z) \) and \( \psi \) be as in §2. We denote \( J(M,Z) \) by \( \det^{1/2} \). A vector valued Siegel modular form \( F \) of weight \( \det^{k-1/2} \text{Sym}_j \) with character \( \psi^l \) (\( l = 0 \) or 1) is defined to be a \( C^{j+1} \)-valued holomorphic function \( F \) on \( \mathfrak{S}_2 \) which satisfies
\[ F(M(Z)) = \psi(\det(D)) J(M,Z) \text{Sym}_j(CZ + D) F(Z) \]
for any \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{2}(4) \). Let \( M_{k-1/2,j}(\Gamma_0^{2}(4), \psi^l) \) denote the space of these functions. “Plus spaces” \( M_{k-1/2,j}^{+}(\Gamma_0^{2}(4), \psi^l) \) was defined in [Kh], [EZ], [Ib1], [H1b], [H1] and [Ki] which is a subspace of \( M_{k-1/2,j}(\Gamma_0^{2}(4), \psi^l) \). And the following isomorphisms were proved.

**THEOREM 5.3.** There exist the following isomorphisms as Hecke modules.
\[ J_{k,j,1}(\Gamma_g) \simeq M_{k-1/2,j}^{+}(\Gamma_0^{2}(4), \psi^k), \]
\[ J_{k,j,1}^{J,k}(\Gamma_g) \simeq M_{k-1/2,j}^{+}(\Gamma_0^{2}(4), \psi^{k+1}). \]
COROLLARY 5.4.

\[ \sum_{k=0}^{\infty} \dim J_{k,1}(\Gamma_2) t^k = \frac{t^4 + t^6}{(1-t^4)(1-t^6)}, \]

\[ \sum_{k=0}^{\infty} \dim J_{k,1}^{\pm}(\Gamma_2) t^k = \frac{t + t^3}{(1-t^4)(1-t^6)}. \]

Proof. The first assertion for \( k \geq 6 \) and the second assertion for \( k \geq 7 \) are proved by Theorem 5.1 and Theorem 5.2. The author computed the dimensions of \( M_{k-1/2,j}(\Gamma_0^2(4)) \) and \( M_{k-1/2,j}(\Gamma_0^2(4), \psi) \) in [T5]. If \( j = 0 \), we denote \( M_{k-1/2,j}(\Gamma_0^2(4), \psi^j) \) and \( M_{k-1/2,j}(\Gamma_0^2(4), \psi^j) \) by \( M_{k-1/2,j}(\Gamma_0^2(4), \psi^j) \) and \( M_{k-1/2,j}(\Gamma_0^2(4), \psi^j) \), respectively. The structures of the modules \( \bigoplus_{k=0}^{\infty} M_{k-1/2,j}(\Gamma_0^2(4)) \) and \( \bigoplus_{k=0}^{\infty} M_{k-1/2,j}(\Gamma_0^2(4), \psi) \) were determined explicitly in [Hib] Theorem 1.3 and Theorem 1.5 and also the structures of their submodules \( \bigoplus_{k=0}^{\infty} M_{k-1/2,j}(\Gamma_0^2(4)) \) and \( \bigoplus_{k=0}^{\infty} M_{k-1/2,j}(\Gamma_0^2(4), \psi) \) were determined explicitly in [Hib] Theorem 1.8 and Theorem 1.10. They determined the structures of the modules \( \bigoplus_{k=0}^{\infty} M_{k-1/2}(\Gamma_0^2(4)) \) and \( \bigoplus_{k=0}^{\infty} M_{k-1/2}(\Gamma_0^2(4), \psi) \) by using this corollary and Theorem 5.3. But strictly speaking this corollary was a conjecture in case when Theorem 5.2 is not applicable. The cases of low weight in the following were verified by [Hib] and Theorem 5.3. We have

\[ \sum_{k=0}^{\infty} \dim J_{k,1}(\Gamma_1) t^k = \frac{t^4 + t^6}{(1-t^4)(1-t^6)}, \]

\[ \sum_{k=0}^{\infty} \dim J_{k,1}^{\pm}(\Gamma_1) t^k = \frac{t + t^3}{(1-t^4)(1-t^6)}. \]

Since \( \dim J_{4,1}(\Gamma_2) = \dim M_{7/2}(\Gamma_0^2(4)) = 1 \) by [Hib], \( \Phi \)-operator is surjective in this case and we have

\[ \sum_{k=0}^{\infty} \dim J_{k,1}(\Gamma_2) t^k = \sum_{k=0}^{\infty} \dim J_{k,1}^{cusp}(\Gamma_2) t^k + \frac{t^4 + t^6}{(1-t^4)(1-t^6)}. \]

Since \( \dim J_{1,1}^k(\Gamma_2) = \dim M_{1/2}(\Gamma_0^2(4), \psi) = 1 \), \( \dim J_{3,1}^k(\Gamma_2) = \dim M_{3/2}(\Gamma_0^2(4), \psi) = 0 \) and \( \dim J_{5,1}^k(\Gamma_2) = \dim M_{5/2}(\Gamma_0^2(4), \psi) = 1 \) by [Hib], \( \Phi \)-operators are surjective when \( k = 1 \) and \( k = 5 \) but not surjective when \( k = 3 \) and we have

\[ \sum_{k=0}^{\infty} \dim J_{k,1}^{cusp}(\Gamma_2) t^k = \sum_{k=0}^{\infty} \dim J_{k,1}^{cusp}(\Gamma_2) t^k + \frac{t + t^3}{(1-t^4)(1-t^6)} - 3. \]

\[ \square \]

T. Ibukiyama determined the structures of \( \bigoplus_{k=0}^{\infty} J_{k,1}(\Gamma_2) \) and \( \bigoplus_{k=0}^{\infty} J_{2k,2}(\Gamma_2) \) explicitly in [Ib3] Theorem 5.1 and Theorem 6.1, respectively.
We denote the subspace of $M_{k-1/2,j}^+(I_0^2(4), \psi^k)$ and $M_{k,j}(I_2)$ which consist of cusp forms by $S_{k-1/2,j}^+(I_0^2(4), \psi^k)$ and $S_{k,j}(I_2)$, respectively. The dimension of $S_{k,j}(I_2)$ was computed in [T2]. T. Ibukiyama also compared the dimension of $\dim S_{k-1/2,j}^+(I_0^2(4), \psi)$ and the dimension of $S_{j+3,2k-6}(I_2)$ by using Theorem 5.3 and assuming the vanishing theorems (Conjecture 2.7). From numerical computation for many explicitly constructed examples he presented the following conjecture which is an analogy of Shimura correspondence ([Sh1]).

CONJECTURE 5.5 ([Ib2], [Ib4]). There exists an isomorphism

$$S_{k-1/2,j}^+(I_0^2(4), \psi) \simeq S_{j+3,2k-6}(I_2)$$

which preserves L-functions.

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Ryuji TSUSHIMA
Department of Mathematics
School of Science and Technology, Meiji University
1–1–1 Higashimita, Tama-ku, Kawasaki-shi,
Kanagawa 214–8571, Japan
E-mail: tsushima@isc.meiji.ac.jp
home page: http://www.isc.meiji.ac.jp/~tsushima/english.html