Existence of a Regular Unimodular Triangulation of the Edge Polytopes of Finite Graphs

by

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Abstract. In this paper, we give several criteria for the edge polytope of a graph to possess a regular unimodular triangulation in terms of some simple data of the graph. We further apply our criteria to several examples of graphs and show that their edge polytopes possess a regular unimodular triangulation.

Introduction

Let $G$ be a finite connected simple graph and $P_G$ be the edge polytope of $G$. The combinatorial structure of $P_G$, especially which type of triangulations $P_G$ admits, is an interesting problem, and many research studies have been done on this topic (see [5, Chapter 5] and references therein). In [7], Ohsugi obtained a necessary and sufficient condition for $P_G$ to possess a regular unimodular triangulation (there exists a monomial order such that the initial ideal of the toric ideal of the graph $G$ is generated by squarefree monomials). However, this condition is not so easy to apply to a given graph just by looking at the graph.

In this paper, for a graph $G$, we will give several criteria for the existence of a regular unimodular triangulation of $P_G$ in terms of some simple data of the graph. We also apply our criteria to some examples and show that their edge polytopes possess a regular unimodular triangulation.

The contents of this paper are as follows. In Section 1, we review the definitions of and some basic results on the graphs in [7]. In Section 2, we give several slightly different criteria for $P_G$ to possess a regular unimodular triangulation. In Section 3, we show some examples to which our criteria are applicable.

1. Preliminaries

A matrix $A = (a_{ij})_{1 \leq i \leq d, 1 \leq j \leq n} \in \mathbb{Z}^{d \times n}$ is called a configuration matrix if there exists $c \in \mathbb{R}^d$ such that $a_j \cdot c = 1, \ 1 \leq j \leq n$ where $a_j$ is a column vector of $A$.

Let $\mathcal{A} = [a_1, \ldots, a_n] \in \mathbb{Z}^{d \times n}$ be a configuration matrix. Let $\Delta$ be a collection of simplices whose vertices belong to a configuration matrix $\mathcal{A}$. Then, $\Delta$ called a covering of
A if
\[
\text{CONV}(\mathcal{A}) = \bigcup_{F \in \Delta} F
\]
holds. In addition, if a covering \(\Delta\) of a configuration matrix \(\mathcal{A}\) is a simplicial complex, then it is called a \textit{triangulation} of \(\mathcal{A}\). For a configuration matrix \(\mathcal{A} = [a_1, \ldots, a_n] \in \mathbb{Z}^{d \times n}\), let
\[
\mathbb{Z}\mathcal{A} = \left\{ \sum_{i=1}^{n} z_i a_i : z_i \in \mathbb{Z} \right\} \subset \mathbb{Z}^d.
\]
Let \(B \subset \{a_1, \ldots, a_n\}\) be the vertex set of a maximal simplex \(\sigma \in \Delta\) in a covering (triangulation) \(\Delta\) of \(\mathcal{A}\). Suppose that the rank of a configuration matrix \(\mathcal{A} \in \mathbb{Z}^{d \times n}\) is equal to \(d\). Let \(\delta\) be the greatest common divisor of all \(d \times d\) minors of \(\mathcal{A}\). Then, the \textit{normalized volume} of \(\sigma\) is defined by
\[
\text{VOL}(\sigma) = \frac{|\det(B)|}{\delta}.
\]
A covering (triangulation) \(\Delta\) of \(\mathcal{A}\) is said to be \textit{unimodular} if the normalized volume of any maximal simplex in \(\Delta\) is equal to 1. For a configuration matrix \(\mathcal{A} = [a_1, \ldots, a_n] \in \mathbb{Z}^{d \times n}\) and a vector \(w = [w_1, \ldots, w_n] \in \mathbb{Q}^n\), let \(\Delta_w\) be the set of all convex polytopes \(\text{CONV}([a_{i_1}, \ldots, a_{i_r}])\) satisfying the following condition:
\[
\text{There exists } c \in \mathbb{Q}^d \text{ such that }\begin{align*}
a_j \cdot c &= w_j, \quad j \in \{i_1, \ldots, i_r\}, \\
a_j \cdot c &< w_j, \quad j \notin \{i_1, \ldots, i_r\}.
\end{align*}
\]
A triangulation \(\Delta\) of a configuration matrix \(\mathcal{A}\) is said to be \textit{regular} if there exists \(w \in \mathbb{Q}^d\) such that \(\Delta = \Delta_w\).

Let \(t_1, t_2, \ldots, t_d\) be variables. Let \(\mathcal{A} = (a_{ij})_{1 \leq i \leq d, 1 \leq j \leq n} \in \mathbb{Z}^{d \times n}\) be a configuration matrix. To each column vector
\[
a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{dj} \end{pmatrix},
\]
we associate the monomial
\[
t^{a_j} = t_1^{a_{1j}} t_2^{a_{2j}} \cdots t_d^{a_{dj}}
\]
with allowing negative powers. Let \(K\) be a field and let \(K[x] = K[x_1, x_2, \ldots, x_n]\) be a polynomial ring in \(n\) variables over \(K\). If \(f = f(x_1, x_2, \ldots, x_n) \in K[x]\), then we define \(\pi(f)\) by setting
\[
\pi(f) = f(t^{a_1}, t^{a_2}, \ldots, t^{a_n}).
\]
Let
\[
K[\mathcal{A}] = \{\pi(f) : f \in K[x]\}.
\]
We say that \(K[\mathcal{A}]\) is the \textit{toric ring} of \(\mathcal{A}\). In general, a configuration matrix \(\mathcal{A}\) satisfies \(\mathbb{Z}_{\geq 0}\mathcal{A} \subset \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_{\geq 0}\mathcal{A}\). The toric ring \(K[\mathcal{A}]\) is said to be \textit{normal} if it satisfies \(\mathbb{Z}_{\geq 0}\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_{\geq 0}\mathcal{A}\).

With respect to the normality of the toric ring \(K[\mathcal{A}]\), the existence of unimodular triangulations and unimodular coverings of \(\mathcal{A}\) plays an important role.
The even closed walk of the chord two cycles, where one is an odd cycle and the other is an even cycle. We call the even cycle require that

\[ \text{(4)} \]

is an FHM graph that has at least one pair of disjoint odd cycles. A graph such that any pair of disjoint odd cycles has a bridge. A

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\[ \text{ρ}(\text{e}) \]

is an odd cycle contained in

\[ \text{G} \]

. Then, \( \text{c} \) divides \( \text{C} \) into two cycles, where one is an odd cycle and the other is an even cycle. We call the even cycle the even closed walk of the chord \( \text{c} \) in \( \text{C} \). In the even closed walk \( \text{Γ} \) of the chord \( \text{c} \) in \( \text{C} \), we require that \( \text{c} \) be an even-numbered edge of \( \text{Γ} \).

Let \( \text{G} \) be a finite connected simple graph on the set of vertex \{1, \ldots, \text{d}\}. Let \( \text{K}[\text{t}_1, \ldots, \text{t}_\text{d}] \) denote the polynomial ring in \( \text{d} \) indeterminates over a field \( \text{K} \) and let \( \text{K}[\text{G}] \) be the subalgebra of \( \text{K}[\text{t}_1, \ldots, \text{t}_\text{d}] \) generated by all quadratic monomials \( \text{t}_i \text{t}_j \) such that \{i, j\} is an edge of \( \text{G} \). The affine semigroup ring \( \text{K}[\text{G}] \) is called the edge ring of \( \text{G} \).

Let \( \text{C}_1, \text{C}_2 \) be a pair of disjoint odd cycles in \( \text{G} \) (namely, the odd cycles \( \text{C}_1 \) and \( \text{C}_2 \) have no common vertex) and \( \text{b} \) be a bridge of this pair. Here, a bridge \( \text{b} \) of the pair \( \text{C}_1, \text{C}_2 \) is an edge \( \text{b} = \{i, j\} \), where \( i \) is a vertex of \( \text{C}_1 \) and \( j \) is a vertex of \( \text{C}_2 \) or vice versa. Then, the even closed walk of \( \text{b} \) in \( \text{C}_1, \text{C}_2 \) is the closed walk \( \text{C}_1, \text{b}, \text{C}_2, -\text{b} \). In this notation, \( -\text{b} \) means the oppositely directed edge of \( \text{b} \) and the cycle \( \text{C}_1 \) starts from the vertex \( \text{C}_1 \cap \text{b} \) and ends at the same vertex. The same holds for \( \text{C}_2 \). We note that, in the even closed walk \( \text{Γ} \) of the bridge \( \text{b} \) in \( \text{C}_1, \text{C}_2 \), \( \text{b} \) appears twice as an even-numbered edge of \( \text{Γ} \).

A Fulkerson–Hoffman–McAndrew (FHM) graph ([4]) is a finite connected simple graph such that any pair of disjoint odd cycles has a bridge. A fundamental FHM graph ([4]) is an FHM graph that has at least one pair of disjoint odd cycles.

It is also known that the normality of edge polytopes is characterized by the following condition.

**Proposition 1.2** ([10, Corollary 2.3]). Let \( \text{G} \) be a finite connected simple graph. Then the following conditions are equivalent:

(i) the edge ring \( \text{K}[\text{G}] \) is normal;

(ii) the edge polytope \( \text{PG} \) possesses a unimodular covering;

(iii) the graph \( \text{G} \) is an FHM graph.

The following is a basic fact about the fundamental FHM graph ([10, Corollary 2.3], [7, Proposition 3.4], [8], and [11]).
Let $G$ be a finite connected simple graph.

(i) If the edge polytope $P_G$ possesses a regular unimodular triangulation, then $G$ is an FHM graph.

(ii) If $G$ possesses no pair of disjoint odd cycles, then $P_G$ possesses a regular unimodular triangulation.

(iii) There exists an example of an edge polytope $P_G$ of a fundamental FHM graph $G$ that possesses no regular unimodular triangulation.

Thus, we focus on the fundamental FHM graph hereafter. We will review the necessary and sufficient condition for $P_G$ to have a regular unimodular triangulation.

Let $G$ be a fundamental FHM graph. Suppose $G$ possesses $p$ pairs of disjoint odd cycles $\Pi_1 = (C_1, C'_1), \ldots, \Pi_p = (C_p, C'_p)$. For each $i$ ($1 \leq i \leq p$), let $\{b_j^i \mid 1 \leq j \leq q_i\}$ be the set of bridges of $\Pi_i$ and the chords of $C_i$ or $C'_i$. Let $\Gamma_j^i = (e_{i_1}^i e_{i_2} \ldots e_{i_{2s+1}})$ be the even closed walk of $b_j^i$, where the bridge or chord is even numbered.

Now, we define the open half-space $H_{b_j^i}$ by

$$H_{b_j^i} := \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^{r} x_{2k-1} > \sum_{k=1}^{r} x_{2k}\right\}.$$  \hfill (1)

Furthermore, we set $W := \bigcap_{i=1}^{p} \left( \bigcup_{j=1}^{q_i} H_{b_j^i} \right)$. The following result is our starting point.

Proposition 1.4 (cf. [7, Theorem 3.5]). The edge polytope $P_G$ possesses a regular unimodular triangulation if and only if $W \neq \emptyset$.

## 2. Criteria for the existence of a regular unimodular triangulation

Let $G$ be a fundamental FHM graph. In this section, we will give four criteria for the edge polytope $P_G$ to possess a regular unimodular triangulation in terms of the simple data of the graph $G$. Our criteria are based on the existence of special bridges in each pair of disjoint odd cycles. Let $\Pi_1, \ldots, \Pi_p$ be all the pairs of disjoint odd cycles in $G$ as before and $\{b_1^1, \ldots, b_p^p\}$ be the set of bridges, where $b^i$ is the bridge of $\Pi_i$. Let $\Gamma_j^i := (e_{i_1}^i e_{i_2} \ldots e_{i_{2s+1}} b^i, e_{j_1}, \ldots, e_{j_{2t+1}}, -b^i)$.

Now, we define

$$\alpha_i := |\{b_1^1, \ldots, b_p^p\} \cap \{e_{i_2}, e_{i_4}, \ldots, e_{i_{2s}}, e_{j_2}, e_{j_4}, \ldots, e_{j_{2t}}\}|,$$

$$\beta_i := |\{b_1^1, \ldots, b_p^p\} \cap \{e_{i_1}, e_{i_3}, \ldots, e_{i_{2s+1}}, e_{j_1}, e_{j_3}, \ldots, e_{j_{2t+1}}\}|.$$

Furthermore, we set $a_i := 2 + \alpha_i - \beta_i$.

Theorem 2.1. Work with the same notation as above. The edge polytope of a fundamental FHM graph $G$ possesses a regular unimodular triangulation if it has a set of bridges $\{b_1^1, \ldots, b_p^p\}$ ($b^i$ is the bridge of $\Pi_i$) that satisfies the following condition: For each $i$, $a_i \geq 0$ holds, and furthermore, the number of $\Gamma_j^i$'s such that $a_i = 0$ is at most 2.
Proof. We first rewrite $W$ in Proposition 1.4 by the distributive law as follows:

$$W = \bigcap_{i=1}^{p} \left( \bigcup_{j=1}^{q_i} H_{b_{ij}} \right) = \bigcup_{j_1, \ldots, j_p} (H_{b_{i1}} \cap \cdots \cap H_{b_{ip}}),$$

where $j_k$ satisfies $1 \leq j_k \leq q_k$. We set

$$C_b = C_{b_1^1, \ldots, b_p^p} := H_{b_{i1}} \cap \cdots \cap H_{b_{ip}}$$

and call $C_b$ as the open cone of $b = \{b_1^1, \ldots, b_p^p\}$. Thus, $W \neq \phi$ is equivalent to that there is a set of bridges $b = \{b_1^1, \ldots, b_p^p\} (b^i$ is a bridge of $\Pi_i)$ such that $C_b$ is non-empty.

For each $i$, let $I_i$ be the even closed walk of $b^i$ and $f_i > 0$ be the inequality (1) defined by $b^i$. We denote by the same $f_i$ an $n$-dimension vector that consists of the coefficients of the left-hand side (LHS) of the inequality $f_i > 0$. We note if the bridge $b^i$ is equal to an edge $e_j$, if the $j$th component $f_i[j]$ of the vector $f_i$ is $-2$, and if the other edge $e_k$ is contained in $\Gamma_i$, $f_i[k] = +1$ (respectively $-1$) if $e_k$ is an odd (respectively even)-numbered edge of $\Gamma_i$. The other components of $f_i$ are 0.

We define the standard weight vector $w \in \mathbb{R}^n$ of $C_b$ as follows. If there exists $i$ such that $f_i[k] = -2$, then we set $w[k] := -1$. The other components of $w$ are 0. We note if $a_i$ is equal to $(f_i, w)$ (inner product) for each $i$.

(i) Suppose $a_i > 0$ for any $i$. Since $(f_i, w) > 0$ for any $i$, then $w \in W$
(ii) Suppose $a_j = 0$ and $a_i > 0$ ($i \neq j$). Let $b^j$ be a bridge of $\Gamma_j$ and $b^i = e_i$. Let $w' := w + (-1/10 \cdot e_i)$, where $e_i$ is a unit vector. Now, we consider $(f_j, w') = (f_j, w) + (f_j, -1/10 \cdot e_i)$. By the assumption, $(f_j, w) = a_j = 0$. Moreover, we obtain $(f_j, -1/10 \cdot e_i) = 1/5$. Therefore, $(f_j, w') = (f_j, w) + (f_j, -1/10 \cdot e_i) = 1/5 > 0$. On the other hand, let $b^k$ be a bridge of $\Gamma_i$ and $b^k = e_m$. Let $w' := w + (-1/10 \cdot e_m)$. Next, we consider $(f_i, w') = (f_i, w) + (f_i, -1/10 \cdot e_m)$. By the assumption, $(f_i, w) = a_i > 0$. Moreover, we obtain $(f_i, -1/10 \cdot e_m) = 1/5$. Therefore, $(f_i, w') = (f_i, w) + (f_i, -1/10 \cdot e_m) > 0$.

(iii) Suppose $a_j = a_k = 0$ and $a_i > 0$ ($i \neq j, i \neq k$). There exists at least an edge $e_l$ in $\Gamma_j$ that is not contained in $\Gamma_k$. On the other hand, there exists at least an edge $e_m$ in $\Gamma_k$ that is not contained in $\Gamma_j$. Let $v$ be a vector that satisfies the following condition: $v[l] = 1/10$ (respectively $-1/10$) if $e_l$ is odd numbered (respectively even numbered) in $\Gamma_j$. $v[m] = 1/10$ (respectively $-1/10$) if $e_m$ is odd numbered (respectively even numbered) in $\Gamma_k$. The other components of $v$ are 0. Let $w' := w + v$. Now, we consider $(f_i, w') = (f_i, w) + (f_i, v)$. By the assumption, $(f_i, w) = a_i > 0$. On the other hand, we obtain $(f_i, v) \geq 3/10$. Here, since $a_i \in \mathbb{Z}_{>0}$, then $(f_i, w) = a_i \geq 1$. Therefore, $(f_i, w') = (f_i, w) + (f_i, v) \geq 7/10 > 0$. Next, we consider $(f_j, w') = (f_j, w) + (f_j, v)$. By the assumption, $(f_j, w) = a_j = 0$. Moreover, we obtain $(f_j, v) = 1/10$ or $1/5$. Therefore, $(f_j, w') = (f_j, w) + (f_j, v) > 0$. Work with the same discussion as above. We obtain $(f_k, w') = (f_k, w) + (f_k, v) > 0$.

We have the following corollaries.
COROLLARY 2.2. Work with the same notation as above. The edge polytope of a fundamental FHM graph $G$ possesses a regular unimodular triangulation if it has a set of bridges $\{b^1, \ldots, b^p\}$ ($b^i$ is the bridge of $\Pi_i$) that satisfies the following condition: For each $i$, $a_i > 0$ holds.

COROLLARY 2.3. The edge polytope of a fundamental FHM graph $G$ possesses a regular unimodular triangulation if it has a set of bridges $\{b^1, \ldots, b^p\}$ ($b^i$ is the bridge of $\Pi_i$) that satisfies the following condition: for each even closed walk $\Gamma_i$ of $b^i$, the number of the other bridges $b^j$ contained in $\Gamma_i$ is at most 2, and, furthermore, the number of $\Gamma_i$'s that contain exactly two other bridges is at most 2.

COROLLARY 2.4. The edge polytope of a fundamental FHM graph $G$ possesses a regular unimodular triangulation if it has a set of bridges $\{b^1, \ldots, b^p\}$ ($b^i$ is the bridge of $\Pi_i$) that satisfies the following condition: each even closed walk of the bridge $b^i$ contains at most one other bridge $b^j$.

We note that the narrowest condition is Corollary 2.4, whereas the broadest is Theorem 2.1. However, Corollary 2.4 is the easiest to check graphically.

REMARK 2.5. (i) In Theorem 2.1, if there exist more than two $i$'s such that $a_i = 0$, the following result holds. Suppose $a_i = 0$ for $i = i_1, \ldots, i_r$ ($r \geq 3$) and $a_i > 0$ for the other $i$'s. Let $H \subset \mathbb{R}^n$ be the hyperplane defined by $\sum_{j=1}^{n} w[j] x_j = 0$. If the convex cone generated by $f_{i_1}, \ldots, f_{i_r}$ in $H$ is strongly convex i.e., for a convex cone $P$, $P \cap -P = \{0\}$, $W$ is not empty. The proof is the same as that of Theorem 2.1. Namely, thanks to this condition, we can vary $w$ slightly to get a new weight $w'$ such that $(f_i, w') > 0$ for any $i$. However, this condition is not clear at all just by looking at the graph.

(ii) More generally, let $C(f_1, \ldots, f_p)$ be an open cone in $\mathbb{R}^n$ defined by $p$ linear homogeneous inequalities $f_i > 0$ ($1 \leq i \leq p$). Then, $C(f_1, \ldots, f_p) \neq \phi$ holds if and only if the dual cone $C(f_1, \ldots, f_p)^\vee = \mathbb{R}_{\geq 0} f_1 + \cdots + \mathbb{R}_{\geq 0} f_p$ of $C(f_1, \ldots, f_p)$ is strongly convex ($f_i$ is the coefficient vector of the LHS of the inequality). It is difficult to determine whether $C(f_1, \ldots, f_p)^\vee$ is strongly convex or not just by looking at the graph.

(iii) The edge polytope of the following graph does not possess the regular unimodular triangulations (Example 3.2 in [6]). Moreover, there exist three $i$'s such that $a_i = 0$. Therefore, we cannot improve the condition of Theorem 2.1 such that “the number of $\Gamma_i$’s such that $a_i = 0$ is at most 3”

![Graph Image](image-url)
3. Applications

We first apply our criteria to the complete graph $G = K_6$ with six vertices. It is known that $P_{K_d}$ possesses a regular unimodular triangulation for any $d$ (see [12]). Moreover, it is known that an edge polytope of a gap-free graph or a complete multipartite graph possesses a regular unimodular triangulation too (see [2] and [9]).

REMARK 3.1. The complete graph $K_6$ satisfies the condition of Corollary 2.2, but does not satisfy the condition of Corollary 2.3.

We finally show several other examples that satisfy our criteria.

EXAMPLE 3.2. The following five types of graphs satisfy the condition of Corollary 2.4. More precisely, in the graphs $A_{m,n}$, $B_{m,n}$, and $C_{m_1,m_2,n_1,n_2}$, all the pairs of disjoint odd cycles (triangles) have a bridge $b$ in common, and, thus, there are no other bridges contained in the even closed walk of $b$.

$D_{m_1,m_2,m_3,m_4}$ has a set of bridges $\{b_1, b_2\}$ where any disjoint pair has a bridge in this set, and the even closed walk of $b_i$ ($i = 1, 2$) contains (exactly) one other bridge. $E_{m_1,m_2,m_3}$ has a set of three bridges $\{b_1, b_2, b_3\}$ where any disjoint pair has a bridge in this set, and there are no other bridges contained in the even closed walk of $b_i$ ($i = 1, 2, 3$).

EXAMPLE 3.3. The following two types of graphs satisfy the condition of Corollary 2.2, but not that of Corollary 2.3. $F_{m_1,m_2,m_3,m_4}$ has a minimal set of six bridges $\{b_i \mid 1 \leq i \leq 6\}$.
\( i \leq 6 \) where any disjoint pair has a bridge in this set, and \( G_{m_1, m_2, m_3, m_4, m_5} \) has a minimal set of ten bridges \( \{ b_i \mid 1 \leq i \leq 10 \} \).

**Example 3.4.** The following graph satisfies the condition of Theorem 2.1. Moreover, there exist just two \( i \)'s such that \( a_i = 0 \). The following graph has a minimal set of seven bridges \( \{ b_i \mid 1 \leq i \leq 7 \} \) where any disjoint pair has a bridge in this set. When \( \Gamma_1 = (e_{24}, e_{16}, e_{15}, b_7, b_1, b_5, b_6, b_2, -b_7) \), then \( \alpha_1 = 1 \) and \( \beta_1 = 3 \). Therefore, \( a_1 = 2 + \alpha_1 - \beta_1 = 2 + 1 - 3 = 0 \). On the other hand, by symmetry, when \( \Gamma_2 = (e_{22}, e_{11}, e_{12}, b_6, b_5, b_1, b_7, e_{24}, b_4, -b_6) \), then \( a_2 = 0 \).
4. The algorithm and program

We have implemented a program for the computer algebra system Magma [1] that determines whether a given fundamental FHM graph satisfies our criteria. By using the program “cycle 12.c” (see “http://sloppyjoe9.wixsite.com/mysite/program”), we tested 10 fundamental FHM graphs in appendix A.

<table>
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<tr>
<th>Graph number</th>
<th>Theorem 2.1</th>
<th>Corollary 2.2</th>
<th>Corollary 2.3</th>
<th>Corollary 2.4</th>
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A. Example of 10 fundamental FHM graphs

Graph 1

Graph 2

Graph 3

Graph 4

Graph 5

Graph 6
Existence of a Regular Unimodular Triangulation of the Edge Polytopes of Finite Graphs

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