Complex Pisot Numeration Systems

by

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Abstract. Using a real number \( \beta > 1 \), a positive real number \( x \) can be represented by 
\[ x = \sum_{n=k_0}^{\infty} b_n \beta^{-n}, \quad b_n \in \{0, 1, 2, \ldots, \lfloor \beta \rfloor \}, \]
which is so-called the \( \beta \)-expansion. This primitive numerical representation of the real number \( x \) proposes various topics in the field of number theory, ergodic theory, dynamical system theory, and tiling theory, etc. In particular, for an algebraic integer \( \beta (> 1) \), many properties of the \( \beta \)-transformation are studied in [A], [Bl], [Re], [P], [IT], [Sol], [Sch], [T], etc. However, it seems to be unclear whether for a complex number \( z \) there exists the algorithm which induces the complex \( \lambda \)-expansion 
\[ z = \sum_{n=k_0}^{\infty} a_n \lambda^{-n}, \quad a_n \in \Gamma \]
where \( \Gamma \) is the finite digit set of \( \mathbb{Z}[\lambda] \) by using a complex algebraic integer \( \lambda \). In this paper, by using a complex Pisot number \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), \( |\lambda| > 1 \), we give the algorithm which induces the complex Pisot \( \lambda \)-expansion.

0. Introduction

The purpose of this paper is to show the algorithm to produce the complex \( \lambda \)-expansion of 
\[ z \in \mathbb{C}, \quad z = \sum_{n=k_0}^{\infty} a_n \lambda^{-n}, \quad a_n \in \Gamma \]
where \( \Gamma \) is the finite digit set of \( \mathbb{Z}[\lambda] \). For this purpose, we introduce a complex Pisot number \( \lambda \).

**Definition 0.1.** A complex number \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) is a complex Pisot number if \( \lambda \) is the algebraic integer of the minimal polynomial 
\[ p(x) = x^d - k_1 x^{d-1} - \cdots - k_{d-1} x - k_d, \]
where \( k_i \in \mathbb{Z} \), \( 1 \leq i \leq d \), whose roots \( \lambda, \lambda_1, \lambda_2, \ldots, \lambda_d \) satisfy
\[ |\lambda| > 1 > |\lambda_i| \quad (3 \leq i \leq d). \quad (0.1) \]

If \( k_d = \pm 1 \), \( \lambda \) is said to be unimodular. In this paper, we assume that \( \lambda \) is an unimodular complex Pisot number. Let \( A \) be the \( d \times d \) integer matrix whose characteristic polynomial coincides with \( p(x) \) and \( \lambda \) is a complex Pisot number of \( p(x) \). We call \( A \) the complex Pisot matrix of \( \lambda \).

We consider that a complex Pisot matrix \( A \) of \( \lambda \) is the linear transformation on the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \), therefore \( A \) has the 2-dimensional \( A \)-invariant expanding plane \( P_e \) and the \((d - 2)\)-dimensional \( A \)-invariant contracting plane \( P_c \). Using \( P_e \) and \( P_c \), \( \mathbb{R}^d \) is decomposed into \( P_e \) and \( P_c \), i.e., \( \mathbb{R}^d = P_e \oplus P_c \). Then, let us define the projection \( \pi_e : \mathbb{R}^d \to P_e \) (resp. \( \pi_c : \mathbb{R}^d \to P_c \)) along \( P_c \) (resp. \( P_e \)) by \( \pi_e x = x_1 \) (resp. \( \pi_c x = x_2 \)) for \( x = x_1 + x_2 \in \mathbb{R}^d \), where \( x_1 \in P_e \) and \( x_2 \in P_c \).
DEFINITION 0.2. For a complex Pisot number \( \lambda \), we assume that we can find the finite family of compact sets \( P = \{ \gamma_j \} \) of \( P_e \) with the finite integer vector sequence \( f_k^{(j)} \) and the finite index sequence \( V_k^{(j)} \) where \( I \) is an index set, satisfying

\[
\begin{align*}
\text{(N1)} & \quad \mu_e (\gamma_j) > 0, \text{ cl}(\text{int}(\gamma_j)) = \gamma_j, \text{ and } \mu_e (\partial \gamma_j) = 0, \quad \mu_e \text{ is the Lebesgue measure on } P_e, \text{ int}(Y) \text{ and cl}(Y) \text{ are the interior and the closure of a set } Y \text{ respectively, and } \partial Y := Y \setminus \text{int}(Y); \\
\text{(N2)} & \quad \text{for each } j \in I, \text{ the following set equation holds: } \\
\quad A\gamma_j & = \bigcup_{k=1}^{l_j} \left( \gamma_{V(j)}^{(k)} + \pi_e f_k^{(j)} \right) \quad \text{(disjoint)} \\
\text{(N3)} & \quad \gamma := \bigcup_{j \in I} \gamma_j \quad \text{(disjoint)}.
\end{align*}
\]

Then, we say that the pair \( (A, P) \) is the complex Pisot numeration system of \( \lambda \).

Note. In this paper, the index set \( I \) is chosen as \( I = \{ 1, 2, 3 \}, \{ 1 \land 2, 1 \land 3, 2 \land 3 \}, \) etc.

From the complex Pisot numeration system \( (A, P) \) of \( \lambda \), we obtain the numerical expression of \( x \in \gamma \) by

\[
x = \sum_{n=1}^{\infty} A^{-n} \left( \pi_e f_{k_{n-1}}^{(j_{n-1})} \right)
\]

where the double positive integer sequence \( \left( (n_k^0), (n_k^1), \cdots \right) \) is given by the following process: for \( x_0 = x \in \gamma_{j_0} \subset \gamma \), there exists \( (n_k^0) \) such that \( Ax_0 \in \gamma_{V(j_0)}^{(k_0)} + \pi_e f_{k_0}^{(j_0)} \) by (0.2) and then, put \( x_1 := Ax_0 - \pi_e f_{k_0}^{(j_0)} \in \gamma_{V(j_0)}^{(k_0)} \) and \( j_1 := V_{k_0}^{(j_0)} \). Using \( x_1 \in \gamma_{j_1} \) and the existence \( (n_k^1) \) such that \( Ax_1 \in \gamma_{V(j_1)}^{(k_1)} + \pi_e f_{k_1}^{(j_1)} \), we obtain \( x_2 := Ax_1 - \pi_e f_{k_1}^{(j_1)} \in \gamma_{j_2} \) and \( j_2 := V_{k_1}^{(j_1)} \), and so on.

Moreover, there exists the linear map \( \phi_e : P_e \rightarrow \mathbb{C} \) satisfying

\[
\phi_e (Ax) = 2 \phi_e (x) \quad \text{and } \phi_e (\pi_e e_1) = 1,
\]

so the numerical expression (0.3) can be represented as the complex Pisot \( \lambda \)-expansion by

\[
z = \phi_e (x) = \sum_{n=1}^{\infty} a_{(j_{n-1})^-} \lambda^{-n} \in \phi_e (\gamma) \subset \mathbb{C},
\]

where \( a_{(j_{n-1})^-} = \phi_e \left( \pi_e f_{k_{n-1}}^{(j_{n-1})} \right) \). The precise definitions of these expressions (0.3), (0.4), and \( \phi_e \) are found in the section 1.
In this paper, we discuss when we can find the complex Pisot numeration system 
\((A, \mathcal{P})\) of \(\lambda\), in other words, for the complex Pisot matrix \(A\), we give the way how to find the finite family of compact sets \(\mathcal{P} = \{\gamma_j\}_{j \in I}\) satisfying (N1), (N2), (N3) in Definition 0.2.

We introduce three classes of the complex Pisot number which has the complex Pisot numeration system. The first class introduced in the section 2 is that the inverse of the complex Pisot matrix \(A\) of \(\lambda\) is the non-negative \(3 \times 3\) integer matrix. The second class introduced in the section 3 is that the complex Pisot matrix of \(\lambda\) is the \(3 \times 3\) companion matrix whose characteristic polynomial is \(p(x) = x^3 - ax^2 - bx \pm 1\). In this class, we give the sufficient condition of \(a, b \in \mathbb{Z}\) for existence of the complex Pisot numeration system of \(\lambda\). And the third class introduced in the section 4 is that the complex Pisot matrix of \(\lambda\) is the \(4 \times 4\) companion matrix whose characteristic polynomial is \(p(x) = x^4 - ax^3 - bx^2 - cx \pm 1\). In this class, we give the sufficient condition of \(a, b, c \in \mathbb{Z}\) for existence of the complex Pisot numeration system of \(\lambda\).

1. Complex Pisot expansions

1.1. Expanding transformations

Let us start to give the precise definition of the \(\lambda\)-expansion in this section again. For this purpose, let us start to give the following definition.

**DEFINITION 1.1.** Let \((A, \mathcal{P})\) be an complex Pisot numeration system of \(\lambda\) and let \(B \subset P_e\) be the union of the boundary set of each compact set \(\gamma_j\), i.e., \(B := \bigcup_{j \in I} \partial \gamma_j\). We define the expanding transformation \(T_A : \gamma \setminus B \rightarrow \gamma \setminus B\) by

\[
T_A(x) := Ax - \pi e f^{(j_0)}_{k_0} \quad \text{if} \quad x \in \operatorname{int}(\gamma_{j_0}) \quad \text{and} \quad Ax \in \operatorname{int}(\gamma_{j_0}^{(k_0)}) + \pi e f^{(j_0)}_{k_0},
\]

and for \(T_A(x) \in \operatorname{int}(\gamma_{j_0}^{(k_0)})\), the iteration of \(T_A\) is defined by

\[
T_A^n(x) := AT_A^{n-1}(x) - \pi e f^{(j_{n-1})}_{k_{n-1}} \quad \text{if} \quad \begin{cases} 
T_A^{n-1}(x) \in \operatorname{int}(\gamma_{j_{n-1}}) \\
\text{and} \\
AT_A^{n-1}(x) \in \operatorname{int}(\gamma_{j_{n-1}}^{(k_{n-1})}) + \pi e f^{(j_{n-1})}_{k_{n-1}} 
\end{cases}.
\]

If \(AT_A^{n-1}(x) - \pi e f^{(j_{n-1})}_{k_{n-1}} \in B\), then the iteration will be stopped. By the definition of the null set \(Nu := \bigl\{x \in \gamma \mid \exists n : T_A^{n-1}(x) \in B\bigr\}\), the iteration \(T_A^n\) is well-defined for all \(n\) for \(\mu_e\)-almost all \(x \in \gamma\).

From Definition 1.1, for \(\mu_e\)-almost all \(x \in \gamma\), there uniquely exists the sequence \(w(x) := \left(\left\lceil \frac{h_0}{4}\right\rceil, \left\lceil \frac{h_1}{4}\right\rceil, \ldots, \left\lceil \frac{h_n}{4}\right\rceil, \ldots\right)\) satisfying \(T_A^n(x) \in \operatorname{int}(\gamma_{j_n})\) and \(AT_A^n(x) \in \operatorname{int}(\gamma_{j_{n+1}}) + \pi e f^{(j_{n+1})}_{k_{n+1}}\). \(x\) can be represented by

\[
x = \sum_{n=1}^{\infty} A^{-n} \left(\pi e f^{(j_{n-1})}_{k_{n-1}}\right).
\]
We call (1.5) the numerical representation of \( x \).

1.2. \( \lambda \)-expansion

**Lemma 1.2.** Let \( \lambda \) be a unimodular complex Pisot number, let \( A \) be a complex Pisot matrix of \( \lambda \), and let \( u_1, u_2 \) be the eigenvectors corresponding to \( \lambda, \bar{\lambda} \) respectively. Put \( v_1 := \frac{u_2 + u_1}{\sqrt{2}}, v_2 := \frac{u_2 - u_1}{\sqrt{2}} \), then \( \{ v_1, v_2 \} \) is a base of \( P_e \) satisfying

\[
A v_1 = c v_1 + d v_2, \quad A v_2 = -d v_1 + c v_2
\]

where \( \lambda = c + di \). Moreover, there exists a linear map \( \phi_e : L(v_1, v_2) (= P_e) \to \mathbb{C} \) satisfying the following properties:

1. \( \phi_e (Ax) = \lambda \phi_e (x) \) for \( x \in L(v_1, v_2) \);
2. \( \phi_e (\pi_e e_1) = 1 \).

**Proof.** It is easy to see that for \( L(v_1, v_2) \ni y = y_1 v_1 + y_2 v_2, y_1, y_2 \in \mathbb{R} \), we have

\[
A [v_1 \ v_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [v_1 \ v_2] \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.
\]

Let us define

\[
\phi_e : L(v_1, v_2) \to \mathbb{C}, \quad y = y_1 v_1 + y_2 v_2 \mapsto \alpha_e (y_1 + y_2 i),
\]

where \( \alpha_e \in \mathbb{C} \) is a constant. Then, we see that \( \phi_e \) is the linear map and that \( \phi_e \) satisfies \( \phi_e (Ay) = \lambda \phi_e (y) \). Moreover, if we choose the constant \( \alpha_e = \frac{1}{x_1^{(1)} v_1 + x_2^{(1)} v_2} \), then \( \phi_e \) satisfies the properties (1) and (2). \( \square \)

Hence, we obtain the following representation: for \( \phi \circ \mu_e \)-almost all \( z \in \phi_e (\gamma) \subset \mathbb{C} \), \( z \) can be represented by

\[
z = \sum_{n=1}^{\infty} a_{(j_n k_n)} \lambda^{-n}
\]

where \( a_{(j_n k_n)} = \phi_e \left( \pi_e f_{(j_n k_n)} \right) \). It is the \( \lambda \)-expansion with the finite digits

\[
\left\{ \phi_e \left( \pi_e f_{(j k)} \right) \mid j \in I, 1 \leq k \leq l_j \right\}.
\]

By the way, if \( A \) is given by

\[
A = \begin{bmatrix}
0 & \cdots & 0 & O \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & k_d \\
O & 0 & \cdots & k_2 \\
& & & 0 & k_1
\end{bmatrix},
\]

(1.7)

which is called a companion matrix of $p(x)$, then we see that $a_{\langle \lambda \rangle} \in \mathbb{Z}[\lambda]$. In fact, from

$$
\phi_e(\pi_e e_k) = \phi_e(\pi_e A^{k-1} e_1)
$$

by Lemma 1.2 (1) $\lambda^{k-1} \phi_e(\pi_e e_1) \Rightarrow \lambda^{k-1}$,

we know that for $[m_1 m_2 \cdots m_d] \in \mathbb{Z}^d$,

$$
\phi_e(\pi_e [m_1 m_2 \cdots m_d]) = m_1 + m_2 \lambda + \ldots + m_d \lambda^{d-1} \in \mathbb{Z}[\lambda].
$$

Therefore, if the complex Pisot matrix $A$ is isomorphic to the companion matrix $C$ of $\lambda$, i.e.,

$$
\exists B \in GL(d, \mathbb{Z}) : B^{-1} AB = C,
$$

then we can find $\phi_e$ such that $\phi_e(\pi_e u) \in \mathbb{Z}[\lambda]$ for $u \in \mathbb{Z}^d$, that is, we obtain the representation (1.6) in the sense of $a_{\langle \lambda \rangle} \in \mathbb{Z}[\lambda]$.

1.3. The graph of the admissible edge sequence

Let us define the directed multigraph $G_{\lambda} = (V, E, i, t)$ consisting of a finite set $V$ of vertices, a countable set of directed edges $E$ and two functions $i, t : E \to V$. For each edge $e \in E$, $i(e)$ is the initial vertex of $e$ and $t(e)$ is the terminal vertex of $e$. From the finite sequence $\{V_k^{(j)}\}_{1 \leq k \leq j}$, $V_k^{(j)} \in I$ in Definition 0.2 (N2), we define $V, E, i, t$ as follows:

$$
V := I, \ E := \left\{(j_k) \big| j \in V, 1 \leq k \leq l_j \right\},
$$

$$
i : E \to V, \ t : E \to V, \ \psi : (j_k) \mapsto j, \ \psi : (j_k) \mapsto V_k^{(j)}.
$$

From the directed multigraph $G_{\lambda}$, we obtain the one-sided edge-admissible symbolic space $\Omega_\lambda^{(j)} (1 \leq j \leq N)$:

$$
\Omega_\lambda^{(j)} = \left\{(j_0 k_0 j_1 k_1) \cdots \big| j_0 = j \in V, \ t(j_p k_p) = i(j_{p+1} k_{p+1}) \right\}
$$

$$
= \left\{(j_0 k_0 j_1 k_1) \cdots \big| j_0 = j \in V, \ t(j_p k_p) = j_{p+1} \right\}.
$$

Moreover, we know that for $\mu_e$-almost all $x \in \gamma$, the sequence $w(x) = (j_0 k_0) (j_1 k_1) \cdots$ given by (1.5) is the admissible sequence of $G_{\lambda}$.

Let us define the labeling $L : E \to \pi_e \mathbb{Z}^d$ and the map $\varphi : \Omega_\lambda^{(j)} \to P_e$ by

$$
L \left(\left(\frac{j}{k}\right)\right) := \pi_e j_k^{(j)}, \quad \varphi \left(\left(\frac{j_0}{k_0} \frac{j_1}{k_1} \cdots\right)\right) := \sum_{n=1}^{\infty} A^{-n} \left(\pi_e j_{k_{n-1}}^{(j_{n-1})}\right),
$$

then we have the following proposition.

**Proposition 1.3.** If $G_{\lambda}$ is irreducible, then $\varphi \left(\Omega_\lambda^{(j)}\right) = \gamma_j$ for $j \in I$. 
Theorem 2.1 ([AI], [IR], [FFIW]). Let \( \sigma \) be an unimodular Pisot substitution of three letters and \( M_\sigma \) the incidence matrix of \( \sigma \). Then atomic surfaces \( \{ \delta_i \}_{i=1,2,3} \) satisfy the following properties:

1. \( \mu_\varepsilon (\delta_i) > 0 \), \( \delta_i = \text{cl} (\text{int} (\delta_i)) \), and \( \mu_\varepsilon (\partial \delta_i) = 0 \);
2. \( M_\sigma^{-1} \delta_i = \bigcup_{j=1}^{\infty} \bigcup_{k:W_k^{(i)} = i} \left( \delta_j + M_\sigma^{-1}(\varepsilon f (P_k^{(i)})) \right) \) (disjoint)
where \( \sigma(j) = W_1^{(j)}W_2^{(j)}\cdots W_l^{(j)} \) and \( P_k^{(j)} \) is the prefix of \( W_k^{(j)} \), i.e.,
\[
P_k^{(j)} = W_1^{(j)}W_2^{(j)}\cdots W_{k-1}^{(j)};
\]

(3) If \( \sigma \) satisfies the strong coincidence condition, i.e., there exist \( n \) and \( k \) such that \( \sigma^n(i), i = 1, 2, 3 \) have the same \( k \)-th letter and their prefixes of the length \( k-1 \) of \( \sigma^n(i) \) have the same image under the abelianization map \( f \), then \( \delta = \bigcup_{j=1}^3 \delta_j \) is disjoint.

The formula (2) in Theorem 2.1 says that the set \( A\delta_i \) is generated by the union of the set \( (\delta_j + \text{translation}) \). Therefore, we can rewrite the formula (2) as (2') as follows: there exists the finite integer vector sequence \( \{f(i)h\}_{1 \leq h \leq l_i} \) such that
\[
\{f_1^{(i)}, \ldots, f_l_i^{(i)}\} = \bigcup_{j=1}^3 \bigcup_{k: W_k^{(j)} = i} \left( \delta_j + \pi eA \left( f \left( P_k^{(j)} \right) \right) \right).
\]

Hence, by this rewriting, we see that the pair \((A, P)\), which is constructed by the matrix \( A = M_{\sigma}^{-1} \) and the family of compact sets \( P = \{\delta_i\}_{1 \leq i \leq 3} \), is the complex Pisot numeration system.

**Remark 2.2.** In the next section, by using \( E_2(\theta) \), the compact set \( \gamma_i \) will be introduced by
\[
\gamma_i := \lim_{n \to \infty} M_\sigma^{-n} \pi eE_2(\theta)^n \left( e_i \wedge \sigma^{-1} \right)
\]
where \( \theta \) is the mirror image of the inverse of \( \sigma \), i.e., \( \theta := (\sigma^{-1})^\leftarrow \) and \( \sigma \) is a substitution (see [AI], [SAI], [E]). We see that \( \gamma_1 = -\delta_1 \) holds.

**Example 2.3** (Rauzy substitution: [Ra], [AI], [IK]). Let \( \sigma \) be \( 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1 \) and the incidence matrix of \( \sigma \) \( M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \). Then, \( A = M_{\sigma}^{-1} \) satisfies the complex Pisot condition, i.e., \( \lambda_1 = -0.771845 + 1.11514i, \lambda_2 = -0.771845 - 1.11514i, \lambda_3 = 0.543689 \). Moreover, the family of compact sets \( P = \{\delta_1, \delta_2, \delta_3\} \), which is given by the projection method, i.e.,
\[
\delta_i = \text{cl} \left( \{\pi e f(s_1s_2\cdots s_{k-1}) : \exists k \in \mathbb{N}, s_k = i \} \right),
\]
satisfies not only the following set equations:
\[
A\delta_1 = \delta_1 \cup \delta_2 \cup \delta_3, \quad A\delta_2 = \delta_1 + \pi e \delta_3, \quad A\delta_3 = \delta_2 + \pi e \delta_3
\]
(see Figure 1), i.e., the property (N2), but also the properties (N1) and (N3) of Definition 0.2 where \( w = s_1 s_2 \cdots = \lim_{n \to \infty} \sigma^n (1) \) is the fixed point of \( \sigma \). Therefore, we see that \( (A, P) \) is the complex Pisot numeration system of \( \lambda \).

On this example, the directed multigraph \( G_\lambda \) and the labeled graph \( G'_\lambda \) are given by Figure 2.

Therefore, \( \phi \circ \mu \)-almost all \( z \in \phi_e (\gamma) \) can be represented by (1.6): \( z = \sum_{n=1}^{\infty} a(j_{n-1}) \lambda^{-n} \) where \( a(j_{n-1}) \) given by the property \( \phi_e (Ax) = \lambda \phi_e (x) \) and \( \phi_e (\pi_e e_3) = \phi_e (\pi_e A e_1) = \lambda \phi_e (\pi_e e_1) = \lambda \) as follows:

\[
a(j_{n-1}) = \left\{ \begin{array}{ll} 0 & \text{if } (j_{n-1}) = (1) \\ \lambda & \text{if } (j_{n-1}) = (2) \text{ or } (3) \end{array} \right.
\]

3. Complex Pisot numeration systems from \( 3 \times 3 \) unimodular complex Pisot companion matrices

3.1. Classifying of \( 3 \times 3 \) unimodular complex Pisot companion matrices

In this section, we give the complex Pisot numeration system generated by a \( 3 \times 3 \) unimodular complex Pisot companion matrix.

Let \( A \) be the \( 3 \times 3 \) companion matrix whose characteristic polynomial is \( p_\pi (x) = x^3 - ax^2 - bx \pm 1, a, b \in \mathbb{Z} \). Let us consider two types of matrices, called (type -1) and

\[
\Phi = \{ \delta_i \}_{i=1,2,3} \text{ and } \mathcal{A}.\]
(type $+1$) respectively, as follows:

$$A_+ = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix} : \text{(type } -1\text{)}, \quad A_- = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix} : \text{(type } +1\text{)}$$

$$p_-(x) = x^3 - ax^2 - bx + 1, \quad p_+(x) = x^3 - ax^2 - bx - 1.$$  

For each matrix, we will examine the property of algebraic integers $\lambda_1, \lambda_2, \lambda_3$ of $p_\mp(x)$.

**Proposition 3.1 (for type $-1$).** The roots $\lambda_1, \lambda_2, \lambda_3$ of $p_-(x) = x^3 - ax^2 - bx + 1$ satisfy the conditions:

$$-1 < \lambda_3 < 0, \quad \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}, \quad |\lambda_1| = |\lambda_2| > 1 > |\lambda_3| \tag{3.9}$$

if and only if the coordinate of $a$ and $b$ satisfies the following:

1. $-a + b < 0$;
2. (i) $a^2 + 3b \leq 0$ or (ii) if $a^2 + 3b > 0$ then $27 - 4a^3 - 18ab - a^2b^2 - 4b^3 > 0$

(see Figure 3). Moreover, $Re(\lambda_1) > 0$ (resp. $Re(\lambda_1) < 0$) if and only if

3. $a \geq 0$ (resp. $a < 0$),

where “$Re(z)$” means the real part of $z \in \mathbb{C}$.

**Proposition 3.2 (for type $+1$).** The roots $\lambda_1, \lambda_2, \lambda_3$ of $p_+(x) = x^3 - ax^2 - bx - 1$ satisfy the conditions:

$$0 < \lambda_3 < 1, \quad \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}, \quad |\lambda_1| = |\lambda_2| > 1 > |\lambda_3| \tag{3.10}$$

if and only if the coordinate of $a$ and $b$ satisfies the following:

1. $a + b > 0$;
2. (i) $a^2 + 3b \leq 0$ or (ii) if $a^2 + 3b > 0$, then $27 + 4a^3 + 18ab - a^2b^2 - 4b^3 > 0$

(see Figure 3). Moreover, $Re(\lambda_1) < 0$ (resp. $Re(\lambda_1) > 0$) if and only if

3. $a \leq 0$ (resp. $a > 0$).

Before we prove Propositions 3.1 and 3.2, we prepare the following lemma.

![Figure 3. The condition of $(a, b)$ satisfying (3.9) and (3.10).](image-url)
Therefore we know that \( \lambda_1 + \lambda_2 \neq 0 \), i.e., \( \lambda_1 \) and \( \lambda_2 \) are not purely imaginary numbers.

**Proof.** From the relation between \( \lambda_1 \) and \( \lambda_2 \), the real parts of \( \lambda_1 \) and \( \lambda_2 \) is \( \frac{\lambda_1 + \lambda_2}{2} \). So, \( \lambda_1 \) and \( \lambda_2 \) are purely imaginary numbers if and only if \( \lambda_1 + \lambda_2 = 0 \). From the relation between \( p_\mp (x) \) and roots, we know that \( \lambda_1 + \lambda_2 + \lambda_3 = a \). Suppose that \( \lambda_1, \lambda_2 \) are purely imaginary numbers, then, \( a = \lambda_3 \). However, from the assumption \(-1 < \lambda_3 < 0\) of (3.9) or \( 0 < \lambda_3 < 1 \) of (3.10), it contradicts the fact that \( a \) is an integer. Therefore \( \lambda_1 + \lambda_2 \neq 0 \), i.e., \( \lambda_1 \) and \( \lambda_2 \) are not purely imaginary numbers.

**Proof of Proposition 3.1.** About (1), (2): It is easy to see that the roots of \( p_−(x) \) satisfy the condition (3.9) if and only if

(i) \( p_−(−1) < 0 \);

(ii) \( (i-1) D \leq 0 \) or \( (ii-2) \) if \( D > 0 \) then \( p_−(x) p_−(t) > 0 \),

where \( D \) is the discriminant of \( p'_−(x) = 3x^2 - 2ax - b \) and \( s, t \) are the roots of \( p'_−(x) \).

The conditions (i) and (ii) are explicitly given by (I) and (II) respectively:

(I) \( -a + b < 0 \);

(II) \( (II-1) D = a^2 + 3b \leq 0 \)

or \( (II-2) \) if \( D > 0 \) then \( p_−(s) p_−(t) = \frac{1}{27} (27 - 4a^3 - 18ab - a^2b^2 - 4b^3) > 0 \)

(see Figure 3).

About (3): Put \( x^3 - ax^2 - bx + 1 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = 0 \). Then, we know that \( \lambda_1 + \lambda_2 = a - \lambda_3 \). From Lemma 3.3, we see that \( a - \lambda_3 > 0 \) implies \( a > 0 \). Conversely, we know that \( a \geq 0 \) implies \( a - \lambda_3 > 0 \).

We get the proof of Proposition 3.2 analogously.

COROLLARY 3.4 (for type \(-1\)). For the roots \( \lambda_1, \lambda_2, \lambda_3 \) of \( p_−(x) = x^3 - ax^2 - bx + 1 \),

(1) the condition \( p_−(1) = 2 - a - b > 0 \) is the necessary condition of (3.9);

(2) the condition \( p_−(−1) = −a + b < 0 \) is the necessary condition of (3.9).

Therefore, from \( a, b \in \mathbb{Z} \), we see that \( b \leq 0 \) is the necessary condition of (1), (2).

COROLLARY 3.5 (for type \(+1\)). For the roots \( \lambda_1, \lambda_2, \lambda_3 \) of \( p_+(x) = x^3 - ax^2 - bx - 1 \),

(1) the condition \( p_+(1) = a - b > 0 \) is the necessary condition of (3.10);

(2) the condition \( p_+(−1) = −2 - a + b < 0 \) is the necessary condition of (3.10).

Therefore, from \( a, b \in \mathbb{Z} \), we see that \( b \leq 0 \) is the necessary condition of (1), (2).

We call \( A_\mp \) the unimodular complex Pisot companion matrices if the characteristic polynomial of \( A_\mp \) coincides with \( p_\mp (x) \) and the roots \( \lambda_1, \lambda_2, \lambda_3 \) of \( p(x) \) satisfying the condition (3.9) in Proposition 3.1 or (3.10) in Proposition 3.2, i.e.,

(type \(-1\)) : \( -1 < \lambda_3 < 0 \), \( \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R} \), \( |\lambda_1| = |\lambda_2| > 1 \),

(type \(+1\)) : \( 1 > \lambda_3 > 0 \), \( \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R} \), \( |\lambda_1| = |\lambda_2| > 1 \).
Let $u_i$, $(1 \leq i \leq 3)$ be the eigenvectors of $\lambda_i$ respectively. Put $v_1 := \frac{u_3 - u_1}{\lambda_3 - \lambda_1}$, $v_2 := \frac{u_3 - u_2}{\lambda_3 - \lambda_2}$, and $v_3 := u_3$, then, by Lemma 1.2, we obtain the following properties:

$$A[v_1 \ v_2] = [v_1 \ v_2] \begin{bmatrix} c & -d \\ d & c \end{bmatrix},$$

where $\lambda_1 = c + di$, $\pi_e = \mathcal{L}(v_1, v_2)$, and $\pi_c = \mathcal{L}(v_3)$. Let $\mathcal{P}_e : \mathbb{R}^3 \to \pi_e$ be the projection along $\pi_e$ and let us denote the counter clockwise angle between $\pi_e e_i$ and $\pi_e e_j$ by $\arg(i \wedge j)$. Then we have the following lemmas.

**Lemma 3.6 (for type $-1$).** If $a \geq 0$, then $0 < \arg(1 \wedge 2), \arg(2 \wedge 3) < \frac{\pi}{2}$, and if $a < 0$, then $\frac{\pi}{2} < \arg(1 \wedge 2), \arg(2 \wedge 3) < \pi$ (see Figure 4).

**Proof.** Assume that $a \geq 0$, then we see that $\text{Re}(\lambda(\pi_e e_1)) > 0$ by Proposition 3.1 (3). On the other hand, we know that $\phi_e(\pi_e e_2) = \phi_e(\lambda e_1) = \lambda$, $\phi_e(\pi_e e_3) = \lambda_2^2$. Therefore, it is clear that $0 < \arg(\lambda) < \frac{\pi}{2}$ and $0 < \arg(\lambda_2^2) < \pi$. Moreover, we also know that $\phi_e : \mathcal{P}_e \to \mathbb{C}$ is linear and bijective. Therefore, we see that $0 < \arg(1 \wedge 2), \arg(2 \wedge 3) < \frac{\pi}{2}$. The case of $a < 0$ is proved analogously. \hfill \Box

**Lemma 3.7 (for type $+1$).** If $a \leq 0$, then $\frac{\pi}{2} < \arg(1 \wedge 2), \arg(2 \wedge 3) < \pi$ and if $a > 0$, then $0 < \arg(1 \wedge 2), \arg(2 \wedge 3) < \frac{\pi}{2}$ (see Figure 4).

We get the proof by the analogous discussion of Lemma 3.6.

From Lemmas 3.6 and 3.7, we classify the characteristic polynomial $p_\pi(x)$ into four classes such that

<table>
<thead>
<tr>
<th></th>
<th>(type $-1$)</th>
<th>(type $+1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; -1 \ 1 &amp; 0 &amp; b \ 0 &amp; 1 &amp; a \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; b \ 0 &amp; 1 &amp; a \end{bmatrix}$</td>
</tr>
<tr>
<td>$p(x)$</td>
<td>$x^3 - ax^2 - bx + 1$</td>
<td>$x^3 - ax^2 - bx - 1$</td>
</tr>
<tr>
<td>Signature of $\lambda_3$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>The distribution of eigenvalues of $A$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>Signature of $\text{Re}(\lambda_3)$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a \geq 0$</td>
<td>$a &lt; 0$</td>
</tr>
<tr>
<td>$\pi_e e_i$</td>
<td>$V_0$</td>
<td>$V_1$</td>
</tr>
<tr>
<td>Name</td>
<td>$(\pm 1, 0)$</td>
<td>$(\pm 1, 1)$</td>
</tr>
</tbody>
</table>

Let $A_\pi$ be the $3 \times 3$ unimodular complex Pisot companion matrix, let $\lambda_3$ be the real eigenvalue of $A_\pi$, and let $v^* = [v^*_1 \ v^*_2 \ v^*_3]$ and $v = [v_1 \ v_2 \ v_3]$ be the row and column eigenvectors of $\lambda_3$, i.e.,

$$v^* A = \lambda_3 v^*, \quad A_\pi v = \lambda_3 v.$$

Then, $v^*$ and $v$ are explicitly given by

$$v^* = \begin{bmatrix} 1 & \lambda_3 & \frac{1}{\lambda_3} \end{bmatrix}, \quad v = \begin{bmatrix} 1 & \lambda_3 - a & 1 \end{bmatrix}.$$

(3.11)
Therefore, by using (3.11), we have the following lemma.

**Lemma 3.8.**

where \( \text{sgn} (v) = + \) if \( v > 0 \) and \( \text{sgn} (v) = - \) if \( v < 0 \).

And \( P_e \) is characterized by \( \psi^* \) as follows.

**Lemma 3.9.** \( P_e = \{ x \in \mathbb{R}^3 \mid \langle x, \psi^* \rangle = 0 \} \).

Then, we have the following table:

<table>
<thead>
<tr>
<th>Name</th>
<th>type (-1,0)</th>
<th>type (-1,1)</th>
<th>type (+1,0)</th>
<th>type (+1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; -1 \ 1 &amp; 0 &amp; b \ 0 &amp; 1 &amp; a \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; b \ 0 &amp; 1 &amp; a \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; b \ 0 &amp; 1 &amp; a \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; b \ 0 &amp; 1 &amp; a \end{bmatrix} )</td>
</tr>
<tr>
<td>( p(x) )</td>
<td>( x^3 - ax^2 - bx + 1 )</td>
<td>( x^3 - ax^2 - bx - 1 )</td>
<td>( x^3 - ax^2 - bx - 1 )</td>
<td></td>
</tr>
<tr>
<td>Signature of ( \lambda_3 )</td>
<td>-</td>
<td>+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The distribution of eigenvalues of ( A )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Signature of ( \text{Re} (\lambda_1) )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( a )</td>
<td>( a \geq 0 )</td>
<td>( a &lt; 0 )</td>
<td>( a &gt; 0 )</td>
<td>( a \leq 0 )</td>
</tr>
<tr>
<td>Distributions of ( { \pi x e_i }_{i=1,2,3} )</td>
<td>( V_0 )</td>
<td>( V_1 )</td>
<td>( V_0 )</td>
<td>( V_1 )</td>
</tr>
<tr>
<td>Signature of ( \psi^* )</td>
<td>( (+, -, +) )</td>
<td>( (+, +, +) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Image of ( { e_i }_{i=1,2,3} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Signature of ( \psi )</td>
<td>( (+, -, +) )</td>
<td>( (+, +, +) )</td>
<td>( (+, -, +) )</td>
<td>( (+, +, +) )</td>
</tr>
</tbody>
</table>

### 3.2. Stepped planes and quasi-periodic tilings of \( P_e \)

For the 2-dimensional \( A \)-invariant expanding plane \( P_e \), we introduce the stepped plane in this section.
For \( x \in \mathbb{R}^3 \) and \( i, j \in \{ \pm 1, \pm 2, \pm 3 \} \), let us define the 2-dimentional unit face \((x, i \wedge j)\) given by
\[
(x, i \wedge j) := \left\{ x + \lambda (\text{sgn} (i)) e_{|i|} + \mu (\text{sgn} (j)) e_{|j|} \mid 0 \leq \lambda, \mu \leq 1 \right\}
\]
(see Figure 5).

Using Lemma 3.8 and Lemma 3.9, let us define the stepped plane of \( P_\varepsilon \) as follows.

**Definition 3.10** (for type \((-1, 0)\) and type \((-1, 1)\), i.e., in the case of \( v^+ = (+, -, +) \)). Let us define the sets of unit faces \( S^-_e, S^-_\varepsilon \) of \( P_\varepsilon \) as follows:
\[
S^-_e := \left\{ (x, (-2) \wedge 1) \mid x \in \mathbb{Z}^3, \langle x, v^+ \rangle \geq 0, \langle x - e_3, v^+ \rangle < 0 \right\}
\]
\[
\cup \left\{ (x, 1 \wedge 3) \mid x \in \mathbb{Z}^3, \langle x, v^+ \rangle \geq 0, \langle x + e_2, v^+ \rangle < 0 \right\}
\]
\[
\cup \left\{ (x, 3 \wedge (-2)) \mid x \in \mathbb{Z}^3, \langle x, v^+ \rangle \geq 0, \langle x - e_1, v^+ \rangle < 0 \right\},
\]
\[
S^-_\varepsilon := \left\{ (x, (-2) \wedge 1) \mid x \in \mathbb{Z}^3, \langle x, v^+ \rangle > 0, \langle x - e_3, v^+ \rangle \leq 0 \right\}
\]
\[
\cup \left\{ (x, 1 \wedge 3) \mid x \in \mathbb{Z}^3, \langle x, v^+ \rangle > 0, \langle x + e_2, v^+ \rangle \leq 0 \right\}
\]
\[
\cup \left\{ (x, 3 \wedge (-2)) \mid x \in \mathbb{Z}^3, \langle x, v^+ \rangle > 0, \langle x - e_1, v^+ \rangle \leq 0 \right\}.
\]

**Definition 3.11** (for type \((+1, 0)\) and type \((+1, 1)\), i.e., in the case of \( v^+ = (+, +, +) \)). Let us define the sets of unit faces \( S^+_e, S^+_\varepsilon \) of \( P_\varepsilon \) as follows:
\[
S^+_e := \left\{ (x, i \wedge j) \mid x \in \mathbb{Z}^3, \{i, j, k\} = \{1, 2, 3\}, i \wedge j \in \{1 \land 2, 3 \land 1, 2 \land 3\}, \right\}
\]
\[
\langle x, v^+ \rangle \geq 0, \langle x - e_k, v^+ \rangle < 0 \right\},
\]
\[
S^+_\varepsilon := \left\{ (x, i \wedge j) \mid x \in \mathbb{Z}^3, \{i, j, k\} = \{1, 2, 3\}, i \wedge j \in \{1 \land 2, 3 \land 1, 2 \land 3\}, \right\}
\]
\[
\langle x, v^+ \rangle > 0, \langle x - e_k, v^+ \rangle \leq 0 \right\}.
\]

**Definition 3.12.** We define the family of finite sets of unit faces \( G^\varepsilon_\lambda \) of \( S^\varepsilon_\lambda \), called the patch, which is generated as a finite formal sum of unit faces as follows:
\[
G^\varepsilon_\lambda := \sum_{\lambda \in \Lambda} \left\{ (x, i \wedge j)_\lambda \mid \#_{\lambda} < +\infty, (x, i \wedge j)_\lambda \in S^\varepsilon_\lambda, (x_\lambda, i \wedge j)_\lambda \neq (x_\lambda, i \wedge j)_{\lambda'} \text{ if } \lambda \neq \lambda' \right\}
\]
(see Figure 6). The other cases \( G_\geq, G_\leq, G_< \) are defined analogously.

**Figure 6.** Examples of patches.

**Definition 3.13.** Using \( S_\geq \), we define the surfaces \( S_\geq \) of \( S_\geq \) called the stepped plane of \( P_e \) as follows:

\[
S_\geq := \bigcup_{(x, i \land j) \in S_\geq} (x, i \land j)
\]

(see Figure 7). \((x, i \land j) \in S_\geq\) is called the unit face of the stepped plane located at \( x \). The other cases \( S_\leq, S_+, S_- \) are defined analogously.

**Figure 7.** Stepped planes \( S_\geq \) and \( S_\leq \).

**Remark 3.14.** (1) \( \pi_e S_\geq, \pi_e S_\leq, \pi_e S_+, \pi_e S_- = P_e \).

(2) The fact that \( x \in P_e \) and \( x \in \mathbb{Z}^3 \) implies \( x = 0 \). Since \( v^* \equiv [1 \lambda \lambda^2] \) is rationally independent, i.e., if \( l + m\lambda + n\lambda^2 = 0 \) for some \( m, n \), then \((l, m, n) = (0, 0, 0)\).

(3) We see that \( S_\leq \supset \{(0, (-2) \land 1), (0, 1 \land 3), (0, 3 \land (-2))\} \) and \( S_\leq \supset \{(e_3, (-2) \land 1), (-e_2, 1 \land 3), (e_1, 3 \land (-2))\} \). Moreover, we have \( S_\leq \setminus S_\geq = \{(0, (-2) \land 1), (0, 1 \land 3), (0, 3 \land (-2)), (e_3, (-2) \land 1), (-e_2, 1 \land 3), (e_1, 3 \land (-2))\} \) (see Figure 8).
(4) We see that \(S^+_3 \supset \{0, 1 \wedge 2\}, \{0, 3 \wedge 1\}, \{0, 2 \wedge 3\}\) and \\
\(S^+_5 \supset \{(e_3, 1 \wedge 2), (e_2, 3 \wedge 1), (e_1, 2 \wedge 3)\}\). Moreover, we have \\
\(S^+_3 \setminus S^+_5 = \{(0, 1 \wedge 2), (0, 3 \wedge 1), (0, 2 \wedge 3), (e_3, 1 \wedge 2), (e_2, 3 \wedge 1), (e_1, 2 \wedge 3)\}\). \\
(see Figure 8).

(5) For unit faces of \(S_3^+\) and \(S_5^+\), we consider the rearrangement such as \\
\((x, (-2) \wedge 1) = (x - e_2, 1 \wedge 2), (x, 3 \wedge (-2)) = (x - e_2, 2 \wedge 3)\) . (3.12) \\
Then, using the rearrangement (3.12), \(S^+_3\) and \(S^+_5\) are rewritten by \\
\(S^+_3 := \{(z, 1 \wedge 2) \mid [-e_2, v^*] \leq [z, v^*] < [-e_2 + e_3, v^*]\}\) \\
\(\cup \{z, 1 \wedge 3) \mid 0 \leq [z, v^*] < [-e_2, v^*]\}\) \\
\(\cup \{(z, 2 \wedge 3) \mid [-e_2, v^*] \leq [z, v^*] < [-e_2 + e_1, v^*]\}\) , \\
\(S^+_5 := \{(z, 1 \wedge 2) \mid [-e_2, v^*] < [z, v^*] \leq [-e_2 + e_3, v^*]\}\) \\
\(\cup \{z, 1 \wedge 3) \mid 0 < [z, v^*] \leq [-e_2, v^*]\}\) \\
\(\cup \{(z, 2 \wedge 3) \mid [-e_2, v^*] < [z, v^*] \leq [-e_2 + e_1, v^*]\}\) .

For the characterization of the faces which generate the stepped plane, we prepare the following notations.

**NOTATION 1** (for type \((-1, 0)\), type \((-1, 1)\)). For the set of unit faces of \(S_3^+\), \(S_5^+\) \\
which generate the stepped plane \(\mathcal{P}_3^+, \mathcal{P}_5^+\) respectively, let us denote the segments \(I_{i \wedge j}^+\) \\
\((i \wedge j)\) of \(L(v) = P_c, i \wedge j \in V_0\) as follows:

\(I_{i \wedge 2}^+ := [\pi_c(-e_2), \pi_c(-e_2 + e_3)]_c\) \\
\(I_{i \wedge 3}^+ := [\pi_c(0), \pi_c(-e_2)]_c\) \\
\(I_{2 \wedge 3}^+ := [\pi_c(-e_2), \pi_c(-e_2 + e_1)]_c\) \\
\(I_{1 \wedge 2}^+ := [\pi_c(-e_2), \pi_c(-e_2 + e_3)]_c\) \\
\(I_{1 \wedge 3}^+ := [\pi_c(-e_2), \pi_c(-e_2 + e_1)]_c\) \\
\(I_{1 \wedge 2}^+ := [\pi_c(-e_2), \pi_c(-e_2 + e_3)]_c\) \\
\(I_{1 \wedge 3}^+ := [\pi_c(-e_2), \pi_c(-e_2 + e_1)]_c\).
FIGURE 9. $I_\vartriangleleft (i \wedge j), i \wedge j \in V_0$.

FIGURE 10.

\[ := \{ \alpha \pi_c (-e_2) | 0 < \alpha \leq 1 \}, \]
\[ I_\uparrow (2 \wedge 3) := (\pi_c (-e_2), \pi_c (-e_2 + e_1))], \]
\[ := \{ \alpha \pi_c (-e_2 + e_1) + (1 - \alpha) \pi_c (-e_2) | 0 < \alpha \leq 1 \} \]

(see Figure 9).

NOTATION 2 (for type $(+1, 0)$, type $(+1, 1)$). For the set of unit faces $S^-_\vartriangleleft$, $S^-_\uparrow$, which generate the stepped plane $S^-_{\vartriangleleft}$, $S^-_{\uparrow}$ respectively, let us denote the segments of $I^-_\vartriangleleft (i \wedge j), I^-_\uparrow (i \wedge j)$ of $L (v) = P_e, i \wedge j \in V_1$ as follows:

\[ I^-_\vartriangleleft (i \wedge j) := [\pi_c z, \pi_c x), \]
\[ I^-_\uparrow (i \wedge j) := (\pi_c z, \pi_c x), \]
\[ I^\| (i \wedge j) := (\pi_c z, \pi_c x), \]

where $\{i, j, k\} = \{1, 2, 3\}$.

Using $I^-_\vartriangleleft (i \wedge j), I^-_\uparrow (i \wedge j), i \wedge j \in V_0$ and $I^\| (i \wedge j), i \wedge j \in V_1$, we can characterize the faces of stepped plane.

LEMMA 3.15. Under the assumption $\text{sgn} (v) = \text{sgn} (v^*)$, i.e., under type $(-1, 0)$ or type $(+1, 1)$, $(z, i \wedge j) \in S^-_\vartriangleleft$, $(x, i \wedge j) \in S^-_\uparrow$, $(x, i \wedge j) \in S^\|$, if and only if $\pi_c z \in I^-_\vartriangleleft (i \wedge j)$ (resp. $\pi_c x \in I^-_\uparrow (i \wedge j)$, $\pi_c x \in I^\| (i \wedge j)$) (see Figure 10).

Proof. From the definition of $S^-_\vartriangleleft$, it is clear that $S^-_\vartriangleleft \ni (z, 1 \wedge 2)$ if and only if $(-e_2, v^*) \leq (z, v^*) < (-e_2 + e_3, v^*)$. Let $\pi_c : \mathbb{R}^3 \rightarrow L (v^*)$ be the projection along $P_e$. Then, from the fact that $\pi_c (z) = (z, v^*) v^*$, we can write that $(z, v^*) \in [(e_2, v^*), (-e_2 + e_3, v^*)]$. 

\[ \pi_c (z) = (z, v^*) v^* \in [(e_2, v^*), (-e_2 + e_3, v^*)] \]
characterized by $sgn$ are the quasi-periodic tilings of $e^1, v^1$.) on $\mathcal{L}(v^1)$ where $|v^1| = 1$. From $z = \pi_e z + \pi_v z$, $(\pi_e z, v^1) \in [(\pi_e (-e_2), v^1), (\pi_e (-e_2 + e_1), v^1))$ on $\mathcal{L}(v^1)$. Moreover, $sgn(v) = sgn(v^1)$, we see that $\pi_e z \in [\pi_e (-e_2), \pi_e (-e_2 + e_1))$. We can prove the other cases analogously.

Hereafter, let us discuss only type $(-1, 0)$ or type $(+1, 1)$ cases whose classes are characterized by $sgn(v) = sgn(v^1)$.

**Definition 3.16.** Let us define the set of projected unit faces of $S_e^2$ as follows:

$$T^2_e := \{ \pi_e (z, i \wedge j) : (z, i \wedge j) \in S_e^2 \}.$$  

The other cases of $T^2_e$, $T^2_{e+}$, $T^2_{e-}$ are defined analogously.

**Definition 3.17.** Using $T^2_e$, the tiling of $P_e$ is defined by

$$\mathcal{T}^2_e := \bigcup_{\pi_e(z, i \wedge j) \in T^2_e} \pi_e (z, i \wedge j).$$

The other cases $\mathcal{T}^2_{e-}$, $\mathcal{T}^2_{e+}$, $\mathcal{T}^2_{e-}$ are defined analogously.

Then, by the property $sgn(v) = sgn(v^1)$, we have the following proposition.

**Proposition 3.18.** If $sgn(v) = sgn(v^1)$, then $T^2_e$, $T^2_v$ are the quasi-periodic tilings of $P_e$ by proto-tiles $\{\pi_e (0, i \wedge j) : i \wedge j \in \{1 \wedge 2, 1 \wedge 3, 2 \wedge 3\} = V_0\}$ and $T^2_v$, $T^2_v$ are the quasi-periodic tilings of $P_v$ by proto-tiles $\{\pi_e (0, i \wedge j) : i \wedge j \in \{1 \wedge 2, 3 \wedge 1, 2 \wedge 3\} = V_1\}$.

The proof is obtained by Theorem 3.8 in [IO2] analogously.

**Remark 3.19.** We are interested in the projection $\pi_e S_e^2$ (resp. $\pi_e S_e^2$, $\pi_e S_e^2$, $\pi_e S_e^2$). But in the cases of type $(-1, 1)$ and type $(+1, 0)$, $\pi_e S_e^2$ (resp. $\pi_e S_e^2$, $\pi_e S_e^2$, $\pi_e S_e^2$) are not tilings but coverings because of the property $sgn(v^1) \neq sgn(v)$. Therefore, we will discuss only the case of $sgn(v^1) = sgn(v)$, i.e., type $(-1, 0)$ and type $(+1, 1)$. It is unclear about the case of $sgn(v^1) \neq sgn(v)$ now. We will try to introduce the existence of the numeration system and the tiling property in the different paper.

Finally, we give the definition of the positive oriented face as follows.

**Definition 3.20.** The unit face $(x, i \wedge j)$ located at $x$ is positive oriented if $i \wedge j \in V_0$ for type $(-1, 0)$ and if $i \wedge j \in V_1$ for type $(+1, 1)$.

**Remark 3.21.** In the case of type $(-1, 0)$, let us assume that $i \wedge j = (-2) \wedge 1$ for the unit face $(x, i \wedge j)$. Then, we can rearrange it as $(x - e_2, 1 \wedge 2)$, so it is the positive oriented by Definition 3.20. Thus if the rearrangement face is positive oriented, we also say that the non-arrangement face is positive oriented.

### 3.3. 2-dimensional extension $E_2(\sigma)$

Let us introduce the automorphisms $\sigma_-$ and $\sigma_+$ on the free group $F(1, 2, 3)$ whose incidence matrices are $A_-$ and $A_+$ respectively as follows:

$\begin{align*}
\sigma_- : & \quad 1 \rightarrow 2, \quad 2 \rightarrow 3, \quad 3 \rightarrow 3^{e_1 -1} 2^b, \\
\sigma_+ : & \quad 1 \rightarrow 2, \quad 2 \rightarrow 3, \quad 3 \rightarrow 12^b 3^a.
\end{align*}$
Using the automorphism \( \sigma (= \sigma_- \text{ or } \sigma_+) \), let us introduce the 2-dimensional extension \( E_2(\sigma) \) of \( \sigma \) on the family of patches generated by the symbolic faces of the set \( V_\epsilon (= V_0 \text{ or } V_1) \): for each \( i \wedge j \in V_\epsilon \),

\[
E_2(\sigma)(0, i \wedge j) := (0, \sigma(i) \wedge \sigma(j))
\]

\[
:= \sum_{1 \leq k \leq l_i} \left( f \left( P_k^{(i)} \right) + f \left( P_k^{(j)} \right) \right) \cdot W(i)^{k} \wedge W(j)^{k} \quad \text{(3.13)}
\]

\[
E_2(\sigma)(x, i \wedge j) := \lambda x + E_2(\sigma)(0, i \wedge j)
\]

\[
E_2(\sigma) \left( \sum_{k} (x, i \wedge j)_{\lambda} \right) := \sum_{k} E_2(\sigma)(x, i \wedge j)_{\lambda}
\]

where \( f : F(1, 2, 3) \rightarrow \mathbb{Z}^3 \) is the homomorphism satisfying \( f(\epsilon) = 0, f(i) = e_1, \sigma(i) = W_1^{(i)} W_2^{(i)} \cdots W_{l_i}^{(i)} \). \( W_k^{(i)} \) is the prefix of \( W_k^{(i)} \), i.e., \( P_k^{(i)} = W_1^{(i)} \cdots W_{k-1}^{(i)} \) and \( y + (0, i \wedge j) = (y, i \wedge j) \) (see [AFHI]).

If 2-dimensional extension \( E_2(\sigma) \) given by (3.13) satisfies the property that all faces of \( E_2(\sigma)(0, i \wedge j) \) are positive orientated, we say that \( E_2(\sigma) \) has the positive orientation property (the POP-property for simplicity).

Then, we have the following proposition.

**Proposition 3.22.** \( E_2(\sigma_-) \) and \( E_2(\sigma_+) \) have the POP-property.

**Proof.** \( E_2(\sigma_-)(0, i \wedge j), i \wedge j \in V_0 \) can be explicitly given by

\[
E_2(\sigma_-)(0, 1 \wedge 2) = (0, 2 \wedge 3)
\]

\[
E_2(\sigma_-)(0, 1 \wedge 3) = (0, 2 \wedge 3^0) + \left( f \left( 3^0 \right), 2 \wedge 1^{-1} \right)
\]

\[
\overset{(a)}{=} \left( \sum_{k=1}^{a} ((k-1) e_3, 2 \wedge 3) \right) + (ae_3 - e_1, 1 \wedge 2)
\]

\[
E_2(\sigma_-)(0, 2 \wedge 3) = \left( f \left( 3^0 \right), 3 \wedge 1^{-1} \right) + \left( f \left( 3^0 \right) + f \left( 1^{-1} \right), 3 \wedge 2^b \right)
\]

\[
\overset{(a)}{=} (ae_3 - e_1, 1 \wedge 3) + \sum_{k=1}^{b} (ae_3 - e_1 - ke_2, 2 \wedge 3).
\]

Here, the technical manner \((a)\) means that the “rearrangement” is used. It is clear that all faces of \( E_2(\sigma_-)(0, i \wedge j), i \wedge j \in V_0 \) are positive oriented, i.e., \( E_2(\sigma_-) \) has the POP-property. We get the proof of \( E_2(\sigma_+) \) analogously. \( \square \)

### 3.4. Invariant stepped plane generated by \( E_2(\sigma_\pm) \)

For \( t \in \mathbb{R}^3 \), let us consider the plane \( P_e(0) = P_e + t \), the stepped plane \( \mathcal{S}^{\pm}_e(0) = \mathcal{S}^{\pm}_e + t \) of \( P_e(t) \). Moreover, the element \((z, i \wedge j) \in S^{\pm}_e(t) = S^{\pm}_e + t\) can be characterised by \( \pi(z, i \wedge j) \in I^{\pm}_e(i \wedge j) \) from Lemma 3.15 where \( I^{\pm}_e(i \wedge j) := I^{\pm}_e(i \wedge j) + \pi_e t \). The cases of \( \mathcal{S}^{\pm}_e(t), S^{\pm}_e(t), \) and \( I^{\pm}_e(i \wedge j)(t) \) is defined analogously.

Now, let us consider the existence problem of the \( E_2(\sigma) \)-invariant stepped plane.
DEFINITION 3.23 (for type $(-1,0)$). Let $s \in \mathbb{R}^3$ be the solution satisfying
\[ s + e_3 - e_2 = As + a e_3 - e_1, \] (3.14)
which is given by $s = \left\lceil \frac{b - 1}{2-a-b}, \frac{(a-1)(b-1)}{2-a-b}, \frac{a-1}{2-a-b} \right\rceil$. Using $s$, let us consider the plane $P_\epsilon(s)$ and $S^\epsilon_\gamma(s)$ of $P_\epsilon(s)$, moreover let us define the $\mathcal{U}_\epsilon^x(s)$ and $\mathcal{U}_\gamma^x(s)$, which are called the seed, as follows:
\[
\mathcal{U}_\epsilon^x(s) := (s - e_2, 1 \wedge 2) + (s, 1 \wedge 3) + (s - e_2, 2 \wedge 3), \\
\mathcal{U}_\gamma^x(s) := (s - e_2 + e_3, 1 \wedge 2) + (s - e_2, 1 \wedge 3) + (s - e_2 + e_1, 2 \wedge 3).
\]
Then, from the definition of $I^x(i \wedge j)$, $i \wedge j \in V_0$, $\pi_\epsilon(s - e_2) \in I^x(1 \wedge 2)(s)$, $\pi_\epsilon(s - e_2) \in I^x(2 \wedge 3)(s)$. Therefore, we see that $\mathcal{U}_\epsilon^x(s) \in G^\epsilon_\gamma(s)$ where $G^\epsilon_\gamma(s) = G^\epsilon_\gamma + s$. By the analogous discussion, we get $\mathcal{U}_\gamma^x(s) \in G^\gamma_\gamma(s)$, where $G^\gamma_\gamma(s) = G^\gamma_\gamma + s$.

DEFINITION 3.24 (for type $(+1,1)$). Let us define the $\mathcal{U}_\epsilon^x$ and $\mathcal{U}_\gamma^x$, which are called the seed, as follows:
\[
\mathcal{U}_\epsilon^x := (0, 1 \wedge 2) + (0, 1 \wedge 3) + (0, 2 \wedge 3), \\
\mathcal{U}_\gamma^x := (e_3, 1 \wedge 2) + (e_2, 3 \wedge 1) + (e_1, 2 \wedge 3).
\]
Then, it is easy to see that
\[
\mathcal{U}_\epsilon^x \in G^\epsilon_\gamma, \\
\mathcal{U}_\gamma^x \in G^\gamma_\gamma
\]
by the analogous discussion above.

REMARK 3.25. We usually treat $\mathcal{U}_\epsilon^x(s), \mathcal{U}_\epsilon^x(s), \mathcal{U}_\gamma^x, \mathcal{U}_\gamma^x$ as patches, i.e., $\mathcal{U}_\epsilon^x(s) \in G^\epsilon_\gamma(s)$, but we sometimes treat them three distinct unit faces, i.e., $\mathcal{U}_\epsilon^x(s) \subset S_\gamma^\epsilon(s)$.

LEMMA 3.26 (for type $(-1,0)$). Using $s$ satisfying (3.14), we get the following relations:
\[
E_2(\sigma_-) \mathcal{U}_\epsilon^x(s) > \mathcal{U}_\gamma^x(s), \\
E_2(\sigma_-) \mathcal{U}_\gamma^x(s) > \mathcal{U}_\epsilon^x(s)
\]
where $\delta > \gamma$ means that the patch $\gamma$ is the subpatch of the patch $\delta$. In other words, $\delta > \gamma$ means that if $(z, i \wedge j) \in \gamma$, then $(z, i \wedge j) \in \delta$.

LEMMA 3.27 (for type $(+1,1)$). The following relations hold:
\[
E_2(\sigma_+) \mathcal{U}_\epsilon^x > \mathcal{U}_\gamma^x, \\
E_2(\sigma_+) \mathcal{U}_\gamma^x > \mathcal{U}_\epsilon^x.
\]
The proofs of Lemmas 3.26 and 3.27 are given by checking of $E_2(\sigma_-) \mathcal{U}_\epsilon^x(s), E_2(\sigma_-) \mathcal{U}_\gamma^x(s), E_2(\sigma_+) \mathcal{U}_\gamma^x(s)$ explicitly.

PROPOSITION 3.28 (for type $(-1,0)$). Using $s$ satisfying (3.14), let us consider two seeds $\mathcal{U}_\epsilon^x(s)$ and $\mathcal{U}_\gamma^x(s)$. Then, the following properties hold:
\begin{enumerate}
\item $\mathcal{U}_\epsilon^x(s) \in G^\epsilon_\gamma(s), \mathcal{U}_\gamma^x(s) \in G^\gamma_\gamma(s)$;
\item $E_2(\sigma_-) \mathcal{U}_\epsilon^x(s) > \mathcal{U}_\gamma^x(s), E_2(\sigma_-) \mathcal{U}_\gamma^x(s) > \mathcal{U}_\epsilon^x(s)$;
\item $E_2(\sigma_-) \mathcal{U}_\epsilon^x(x) - \mathcal{U}_\gamma^x(s) = E_2(\sigma_-) \mathcal{U}_\gamma^x(s) - \mathcal{U}_\epsilon^x(s)$;
\item $S^\epsilon_\gamma(s) \setminus \mathcal{U}_\gamma^x(s) \ni (z, i \wedge j)$ implies $E_2(\sigma_-)(z, i \wedge j) \in G^\gamma_\gamma(s)$;
\item $S^\epsilon_\gamma(s) \setminus \mathcal{U}_\epsilon^x(s) \ni (z, i \wedge j)$ implies $E_2(\sigma_-)(z, i \wedge j) \in G^\epsilon_\gamma(s)$;
\end{enumerate}
Lemma 2.3 in [IO2].

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(5) \((z, i \land j), (z', (i \land j))' \in S^2_\preceq (s) \text{ or } S^2_\succeq (s), (z, i \land j) \neq (z', (i \land j))'\) imply \(\exists \left(\begin{array}{l}
(w, k \land l) : (w, k \land l) \in E_2(\sigma_-)(z, i \land j) \text{ and } (w, k \land l) \in E_2(\sigma_-)
\end{array}\right)
\)\((z', (i \land j))'.\)

(6) For any \((z, i \land j) \in S^2_\preceq \setminus U^2_\preceq (s),\) there exists \((y, k \land l) \in S^2_\preceq (s) \setminus U^2_\preceq (s)\) such that \(E_2(\sigma_-)(y, k \land l) \ni (z, i \land j).\)

Proof. (1) is clear from the definition of \(U^2_\preceq (s)\) and \(U^2_\succeq (s).\) (2) is proved in Lemma 3.26. (3) is clear from the definition of \(E_2(\sigma_-).\) For (4), let us assume that \(S^2_\preceq (s) \setminus U^2_\preceq (s) \ni (z, 1 \land 2),\) i.e., by Lemma 3.15, let us assume that \(\pi_e z \in (\pi_e (s - e_2), \pi_e (s - e_2 + e_1)) = \text{int} \left(\begin{array}{l}
(I^2_\preceq (1 \land 2) (s))\left(\begin{array}{l}
\end{array}\right)\end{array}\right).\) From \(E_2(\sigma_-)(z, 1 \land 2) = (A z, 2 \land 3),\) we want to show that \(\pi_e (A z) \in (\pi_e (s - e_2), \pi_e (s - e_2 + e_1)) = \text{int} \left(\begin{array}{l}
(I^2_\preceq (2 \land 3) (s))\left(\begin{array}{l}
\end{array}\right)\end{array}\right).\) From the assumption \(\pi_e z \in (\pi_e (s - e_2), \pi_e (s - e_2 + e_1))\) and from the properties that \(A\) is the linear contracting map on \(P_e\) and \(\det A = -1,\) we have

\[
\pi_e (A z) \in (\pi_e (s - e_2 + e_3), \pi_e (s - e_2)) \quad \text{c} \subset (\pi_e (0, \pi_e e_1) \quad \text{c}.
\]

Therefore, we get \(\pi_e A z \in \text{int} \left(\begin{array}{l}
(I^2_\preceq (2 \land 3) (s))\left(\begin{array}{l}
\end{array}\right)\end{array}\right).\) We prove the other cases, i.e., \((z, 1 \land 3), (z, 2 \land 3) \in S^2_\preceq (s) \setminus U^2_\preceq (s),\) analogously. In particular, we use the properties such that \(\exists (w, k \land l) \in S^2_\preceq (s) : (w, k \land l) \in E_2(\sigma_-)(z, i \land j) \cap E_2(\sigma_-)(z', (i \land j)').\) It is enough to consider \(k \land l = 2 \land 3.\) Let consider the case that \(i \land j = 1 \land 2\) and \(i \land j = 1 \land 3.\) From the definition of \(E_2(\sigma_-),\) we assume that there exists \(j (1 \leq j \leq a) : A z = A z' + (j - 1) e_3.\) By the way, \((z, 1 \land 2) \in S^2_\preceq (s) \left(z', 1 \land 3\right) \in S^2_\preceq (s)\) so \(\pi_e z \in I^2_\preceq (1 \land 2) (s)\) and \(\pi_e z \in I^2_\preceq (1 \land 3) (s) \text{ from } 1 \leq j \leq a, a \geq 0, \text{ we know that } 0 \leq j - 1 \leq a - 1.\) This fact contradicts the assumption that there exists \(j (1 \leq j \leq a) : A z = A z' + (j - 1) e_3.\) We prove other cases, i.e., \(\left\{z, i \land j, (z', (i \land j)')\right\} \in \left\{\left\{(z, 1 \land 2), (z', 1 \land 2)\right\}, \left\{(z, 1 \land 2), (z', 2 \land 3)\right\}, \left\{(z, 1 \land 3), (z', 1 \land 3)\right\}, \left\{(z, 2 \land 3), (z', 2 \land 3)\right\}\right\}, \text{ analogously. For (6) the proof is obtained by the analogous discussion with Lemma 2.3 in [IO2].}\)

On type \((+1, 1),\) we obtain analogous result.

**PROPOSITION 3.29 (for type \((+1, 1)).** Let us consider two seeds \(U^2_\preceq \) and \(U^2_\succeq \).

Then, the following properties hold:

1. \(U^2_\preceq \in G^2_\preceq, U^2_\succeq \in G^2_\succeq;\)
2. \(E_2(\sigma_+) U^2_\preceq \subseteq U^2_\succeq, E_2(\sigma_+) U^2_\succeq \supseteq U^2_\preceq;\)
3. \(E_2(\sigma_+) U^2_\preceq - U^2_\preceq = E_2(\sigma_+) U^2_\succeq - U^2_\succeq;\)
4. \(S^2_\preceq \setminus U^2_\preceq \ni (x, i \land j) \text{ implies } E_2(\sigma_+) (x, i \land j) \in G^2_\preceq;\)
\(S^2_\preceq \setminus U^2_\succeq \ni (x, i \land j) \text{ implies } E_2(\sigma_+) (x, i \land j) \in G^2_\succeq;\)
(5) \( (x, i \wedge j), (x', (i \wedge j)) \in S_{2}^{\pm} \) (or \( S_{j}^{\pm} \)), \( (x, i \wedge j) \neq (x', (i \wedge j)) \) imply \( 3(x, k \wedge l) : (w, k \wedge l) \in E_{2}(\sigma_{+}) (x, i \wedge j) \) and \( (w, k \wedge l) \in E_{2}(\sigma_{+}) (x') (i \wedge j) \).

(6) For any \( (x, i \wedge j) \in S_{2}^{\pm} \), there exists \( (y, k \wedge l) \in S_{2}^{\pm} \) such that \( E_{2}(\sigma_{+}) (y, k \wedge l) \neq (x, i \wedge j) \).

\[ \text{COROLLARY 3.30.} \quad E_{2}(\sigma_{\pm}) / U_{\pm} (s) > U_{\pm} (s) \quad \text{and} \quad E_{2}(\sigma_{\pm}) / U_{\pm} (s) > U_{\pm} (s) \quad \text{where} \ s \ \text{is satisfying (3.14)}. \]

Hence, we see that there exist the invariant quasi-periodic tilings of \( P_{c} \) by \( E_{2}(\sigma_{\pm}) \) (see [AI], [FIR]).

3.5. Complex Pisot Numeration Systems from \( 3 \times 3 \) Unimodular Complex Pisot Companion Matrices

Let us discuss only the existence of complex Pisot number system on type \((-1,0)\) and type \((+1,1)\).

Let us define \( T_{\pm}^{\pm}(s), \mathcal{F}_{\pm}^{\pm}(s) \) as

\[ T_{\pm}^{\pm}(s) := T_{\pm}^{\pm} + \pi_{e} s, \quad \mathcal{F}_{\pm}^{\pm}(s) := \mathcal{F}_{\pm}^{\pm} + \pi_{e} s \]

where \( s \) is the solution of (3.14). Then from the definition of \( S_{2}^{\pm}(s) \) and the property of \( \sgn(x) = \sgn(x^{*}) \), we see that \( \mathcal{F}_{\pm}^{\pm}(s) \) is a quasi-periodic tiling of \( P_{c} \) with prototiles \( (\pi_{e} (0, i \wedge j) | i \wedge j \in V_{0}) \). Since the non-periodicity of the tiling comes from the irreducibility of \( p(x) \) and the quasi-periodicity of the tiling comes from the fact that the tiling is constructed by the projection \( \pi_{e} \) of the \( s \)-translated stepped plane \( \mathcal{F}_{\pm}^{\pm}(s) \). The other cases \( T_{\pm}^{\pm}(s), \mathcal{F}_{\pm}^{\pm}(s), T_{\pm}^{+}, T_{\pm}^{-}, \) and \( \mathcal{F}_{\pm}^{+}, \mathcal{F}_{\pm}^{-} \) are discussed analogously.

Let \( \pi_{e} G_{\pm}^{\pm}(s) \) (resp. \( \pi_{e} G_{\pm}^{\pm}(s) \)) be the family of patches of \( T_{\pm}^{\pm}(s) = T_{\pm}^{\pm} + \pi_{e} s \) (resp. \( T_{\pm}^{\pm}(s) \)), we proved that the operator \( \pi_{e} E_{2}(\sigma_{\pm}) \) is the tiling substitution from \( \pi_{e} G_{\pm}^{\pm}(s) \) (resp. \( \pi_{e} G_{\pm}^{\pm}(s) \)) to \( \pi_{e} G_{\pm}^{\pm}(s) \) (resp. \( \pi_{e} G_{\pm}^{\pm}(s) \)) and that the operator \( \pi_{e} E_{2}(\sigma_{\pm}) \) is the tiling substitution from \( \pi_{e} G_{\pm}^{\pm}(s) \) (resp. \( \pi_{e} G_{\pm}^{\pm}(s) \)) to \( \pi_{e} G_{\pm}^{\pm}(s) \) (resp. \( G_{\pm}^{\pm} \) ) in the subsection 3.4.

Finally, we arrive at the following theorem.

\[ \text{THEOREM 3.31.} \quad (1) \quad \text{Let us define} \quad \gamma_{i \wedge j, -} \quad \text{by} \]

\[ \gamma_{i \wedge j, -} := \lim_{n \to \infty} A^{-n} \pi_{e} E_{2}(\sigma_{\pm})^{n} (s_{i \wedge j, i \wedge j}) \]

where \( s_{i \wedge j, i \wedge j} \in U_{\pm} (s) \) or \( U_{\pm} (s) \). Moreover, we assume that there exist the new seed \( U_{\pm}^{\pm}(s') \) such that

(i) \( U_{\pm}^{\pm}(s') = U_{\pm}^{\pm}(s) + u \) for some \( u \in \mathbb{Z}^{\pm} \);

(ii) \( E_{2}(\sigma_{\pm})^{2} U_{\pm}^{\pm}(s') > U_{\pm}^{\pm}(s') \);

(iii) \( \bigcup_{n=1}^{\infty} E_{2}(\sigma_{\pm})^{2n} U_{\pm}^{\pm}(s') = P_{c} \).

Then \( (A_{-}, \mathcal{P}_{-}), \mathcal{P}_{-} = \{ \gamma_{i \wedge j, -} | i \wedge j \in V_{0} \} \) satisfies the properties for the complex Pisot Numeration System, i.e., (N1), (N2), (N3) in Definition 0.2 hold.

(2) Let us define \( \gamma_{i \wedge j, +} \) by

\[ \gamma_{i \wedge j, +} := \lim_{n \to \infty} A^{-n} \pi_{e} E_{2}(\sigma_{+})^{n} (x_{i \wedge j, i \wedge j}) \]
where \((x_{i \land j}, i \land j) \in \mathcal{U}_s^x\) or \(\mathcal{U}_s^z\). Moreover, we assume that there exist the new seed \(\hat{\mathcal{U}}_i^x\) such that

(i) \(\hat{\mathcal{U}}_i^x(x') = \mathcal{U}_i^x + x'\) for some \(x' \in \mathbb{Z}^3\);

(ii) \(E_2(\sigma_+)\hat{\mathcal{U}}_i^x(x') = \hat{\mathcal{U}}_i^x(x')\);

(iii) \(\bigcup_{n=1}^{\infty} E_2(\sigma_+)^n \hat{\mathcal{U}}_i^x(x') = P_\infty\).

Then \((A_+, \mathcal{P}_+), \mathcal{P}_+ = \{\gamma_{i \land j} \mid i \land j \in V_1\}\) satisfies the properties for the complex Pisot numeration system, i.e., \((N1), (N2), (N3)\) in Definition 0.2 hold.

**Proof.** For (1). The property \((N1)\) for \(\gamma_{i \land j}' = \lim_{n \to \infty} A^{-n} E_2(\sigma_-)^n (x_{i \land j}, i \land j)\), \((s_{i \land j}, i \land j) \in \mathcal{U}_i^x\) holds from the assumption (ii) and (iii). Therefore, we see that the property \((N1)\) holds for \(\gamma_{i \land j}\) (c.f. Theorem 1.5 in [EIR] using Theorem 5.3 in [LW]). We prove the property \((N2)\) analogously with the proof of Corollary 2 in [AI]. From the relation \(E_2(\sigma_-)(0, i \land j)\) given by the proof of Proposition 3.22, we obtain the set equations of \(\{\gamma_{i \land j} \mid \mathcal{P}_+ = \{\gamma_{i \land j} \mid i \land j \in V_0\}\}\) by

\[ A_{\gamma_{i \land j}} = \bigcup_{1 \leq k \leq k_i} \left( \gamma_{i \land j} w_k^j + f(P_k^i) + f(P_k^j) \right) \]

On the other hand, \(\gamma_{i \land j}\) is written by the translation of \(\hat{\mathcal{U}}_i^x\). Therefore, we obtain that \(A_{\gamma_{i \land j}}\) is written by the sum of the translation of the elements by \(\{\gamma_{i \land j} \mid i \land j \in V_0\}\). To prove the property \((N3)\), we must show that \(E_2(\sigma_+)\) satisfies the strongly coincidence condition (see [AI]). By the way, the strongly coincidence condition is geometrically given by the following: there exist \(n, i \land j, y, \) and \(t \in \mathbb{R}^3\) (resp. \(n', (i \land j)', y', \) and \(t'\)) such that

\[ E_2(\sigma_+)^n(y, i \land j) > t + \mathcal{U}_i^x(s) \] (resp. \(E_2(\sigma_+)^n (y', (i \land j)') > t' + \mathcal{U}_i^z\)).

This conditions holds in the case of \(n = 2, i \land j = 2 \land 3, y = s - e_2, \) and \(t = 0\) in Lemma 3.26. (2) is proved analogously and see Remark 3.33. □

**Corollary 3.32.** Let \(\mathcal{T}_i^x(s) := \{\gamma_{i \land j} + \pi \mathbf{e} \mid i \land j \in V_0, (\mathbf{e}, i \land j) \in \mathcal{T}_i^x(s)\}\). Then, \(\mathcal{T}_i^x(s)\) is a quasi-periodic self-similar tiling of \(P_\infty\) by the linear transformation \(A\). The other cases \(\mathcal{T}_i^z(s)\), \(\mathcal{T}_i^x(s)\), \(\mathcal{T}_i^z(s)\) are discussed analogously.

**Remark 3.33.** For type \((+1, 1)\) in Theorem 3.31, we have obtained the numeration system \((A_+, \mathcal{P} = \{\gamma_{i \land j} \mid i \land j \in V_1\}\) where

\[ \gamma_{i \land j} = \lim_{n \to \infty} A_{\sigma_+}^{-n} \pi \mathbf{e} E_2(\sigma_+)^n (x_{i \land j}, i \land j) \]

for \((x_{i \land j}, i \land j) \in \mathcal{U}_i^x\) or \(\mathcal{U}_i^z\) where \(\sigma_+\) is given by \(\sigma_+ : 2 \to 3\) and \(A_{\sigma_+} =

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & b \\
0 & 1 & a
\end{bmatrix}
\]

Let us consider the automorphism \(\tau = \left(\sigma_+\right)^{-1}\), which is the inverse
of the mirror image of $\sigma_+$, then,

$$1 \mapsto 1^{-b} 2^{-a} 3$$

$$\tau : 2 \mapsto 1$$

$$3 \mapsto 2.$$  

From the condition of $(a, b)$ satisfying the complex Pisot condition (3.10), i.e., (1), (2), (3) in Proposition 3.2, it is clear that the automorphism $\tau$ is a substitution. Therefore we know that the limit set $\gamma_{\land, j,+}$ is also obtained as $\gamma_{\land, j,+} = -\delta_k$, $\{1, 2, 3\} = \{i, j, k\}$ by Theorem 2.1, in other words, the numeration system produced from the class of type $(+1, 1)$ is the numeration system produced from the unimodular Pisot substitution (c.f. Remark 2.2 discussed in the section 2).

**EXAMPLE 3.34.** Let us consider the case $(a, b) = (1, 0)$ of type $(-1, 0)$, i.e., $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and its characteristic polynomial is $x^3 - x^2 + 1$. The eigenvalues of $A$ are

$$\lambda_1 = 0.877439 + 0.744862i,$$  

$$\lambda_2 = 0.877439 - 0.744862i,$$  

$$\lambda_3 = -0.754878 \quad (\text{see Figure 11})$$

and $V_0$ is given by $V_0 = \{1 \land 2, 1 \land 3, 2 \land 3\}$.

![Figure 11](image)

Let the automorphism $\sigma : \begin{bmatrix} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 31^{-1} \end{bmatrix}$, then $E_2(\sigma)$ is given by

$$E_2(\sigma)(0, 1 \land 2) = (0, (1 \land \sigma)(2)) = (0, 2 \land 3)$$

$$E_2(\sigma)(0, 1 \land 3) = (0, 2 \land 31^{-1}) \equiv (0, 2 \land 3) + (e_3, 2 \land 1^{-1})$$

$$E_2(\sigma)(0, 2 \land 3) = (0, 3 \land 31^{-1}) \equiv (0, 3 \land 3) + ((e_3 - e_1), 1 \land 3)$$

where $(\ast)$ means the rearrangement and it satisfies the POP-property (see Figure 12). Let $U^-(-e_1)$ be

$$U^-(-e_1) := ((-e_1 - e_2 + e_3), 1 \land 2) + ((-e_1 - e_2), 1 \land 3) + (-e_2, 2 \land 3),$$

then $U^-(-e_1)$ satisfies

$$E_2(\sigma)^2 U^-(-e_1) \succ U^-(-e_1) \quad (\text{see Figure 13}).$$
Let us define $\mathcal{T}^\sigma (-e_1)$ as follows: the tiling $\mathcal{T}^\sigma (-e_1)$ generated by the projection $\pi_e$ of the stepped plane $\mathcal{S}^\sigma (-e_1)$ satisfies the following properties:

\[
T^\sigma (-e_1) := \{ \pi_e (z, i \wedge j) \mid (z, i \wedge j) \in E_2(\sigma)^n U^\sigma (-e_1) \text{ for some } n \},
\]

\[
\mathcal{T}^\sigma (-e_1) := \bigcup_{\pi_e (z, i \wedge j) \in T^\sigma (-e_1)} \pi_e (z, i \wedge j) (= P_e)
\]
on this example (see Figure 14). The analogous proof can be obtained by the method of ∗-connected in [AFHI], or C-covered propery in [IO1].
Let us the limit set $\gamma_{i\wedge j} = \lim_{n \to \infty} A^{-n} \pi_\alpha E_2 (\sigma)^n (s_{i\wedge j}, i \wedge j)$ for $(s_{i\wedge j}, i \wedge j) \in \mathcal{U}_- (e_1)$. Then, $\mathcal{P} = \{\gamma_{i\wedge j}\}_{i\wedge j \in \mathcal{V}_0}$ satisfies not only the following set equations

\[
\begin{align*}
A\gamma_{1\wedge 2} &= \gamma_{2\wedge 3} + \pi_\alpha e_1 \\
A\gamma_{1\wedge 3} &= (\gamma_{1\wedge 2} - \pi_\alpha e_3) \cup (\gamma_{2\wedge 3} - \pi_\alpha e_3) \\
A\gamma_{2\wedge 3} &= \gamma_{1\wedge 3} + \pi_\alpha e_2
\end{align*}
\]

(see Figure 15), i.e., the property (N2), but also the properties (N1) and (N3) of Definition 0.2. Therefore, we see that $(A, \mathcal{P})$ is the the complex Pisot numeration system of $\lambda$.

Then, the labeled graph $(V, E, i, t, \mathcal{L})$ on the example is given by Figure 16. From the fact that

\[
\phi (\pi_\alpha e_1) = 1, \quad \phi (\pi_\alpha e_2) = \lambda, \quad \phi (\pi_\alpha e_3) = \lambda^2,
\]

we see that the labeled graph $(V, E, i, t, (\phi \mathcal{L}))$ is given by Figure 16.
Figure 16. The graph of $(a, b) = (1, 0)$ for type $(-1, 0)$.

Therefore, let $\Omega_{i \land j}$ be the label-admissible symbolic space which is starting from the vertex $i \land j$ by the labeled graph $(V, E, i, t, (\phi \mathcal{L}))$ and let its element be $(a_0, a_1, a_2, \ldots)$, $a_i \in \{-1, \lambda, \lambda^2, -\lambda^2\}$, then $z \in \phi_e \left( \bigcup_{i \land j \in V_e \gamma_i \land j} \right)$ is represented by $z = \sum_{n=1}^{\infty} a_n - \lambda^{-n}$ where $a_n = \phi_e \left( \pi_e f_{kn-1} \right)$.

4. Complex Pisot numeration systems from $4 \times 4$ unimodular complex Pisot companion matrices

4.1. Setting

In this section, we discuss how we obtain the complex Pisot numeration system from a $4 \times 4$ unimodular complex Pisot companion matrix (see [AFHI], [FIR], [F]).

Let $A_{\pm}$ be the $4 \times 4$ companion matrix whose characteristic polynomial is $p_{\pm}(x) = x^4 - ax^3 - bx^2 - cx \pm 1$, $a, b, c \in \mathbb{Z}$, i.e.,

$$A_{\pm} = \begin{bmatrix}
0 & 0 & 0 & \pm 1 \\
1 & 0 & 0 & c \\
0 & 1 & 0 & b \\
0 & 0 & 1 & a
\end{bmatrix}.$$  

We assume that the algebraic integers $\lambda_i$ ($1 \leq i \leq 4$) of $p_{\pm}(x)$ satisfy the non-Pisot hyperbolic condition, i.e.,

$$|\lambda_1| \geq |\lambda_2| > 1 > |\lambda_3| \geq |\lambda_4|.$$  

Under the non-Pisot hyperbolic condition, let $u_i (1 \leq i \leq 4)$ be the eigenvectors of $\lambda_i$ respectively and put the corresponding vectors $v_i$ of eigenvectors

$$\begin{cases}
v_1 = \frac{u_2 + u_1}{2}, & v_2 = \frac{u_2 - u_1}{2l} \\
v_1 = u_1, & v_2 = u_2 \\
v_3 = \frac{u_4 + u_3}{2}, & v_4 = \frac{u_4 - u_3}{2l} \\
v_3 = u_3, & v_4 = u_4
\end{cases}$$

if $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_3, \lambda_4 \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_3, \lambda_4 \in \mathbb{R}$.

Then, the linear transformation $A$ has the 2-dimesional $A$-invariant expanding plane $P_e$ spanned by $\{v_1, v_2\}$ and the 2-dimensional $A$-invariant contracting plane $P_c$ spanned by $\{v_3, v_4\}$. Using $P_e$ and $P_c$, $\mathbb{R}^d$ is decomposed by $P_e$ and $P_c$, i.e., $\mathbb{R}^d = P_e \oplus P_c$. Then, let us define the projection $\pi_e : \mathbb{R}^4 \rightarrow P_e$, $\pi_e (xv_1 + yv_2 + zv_3 + wv_4) := xv_1 + yv_2$.
Using the automorphism \( \pi_c : \mathbb{R}^4 \to P_c, \pi_c(xv_1 + yv_2 + zv_3 + wv_4) := zv_3 + wv_4 \) be the projection to \( P_c \). Moreover, \( P_c \circ A = A \circ P_c \) (resp. \( P_c \circ A = A \circ P_c \)) holds.

Using the representation by

\[
\pi_c \in P_c \ (\text{resp.} \ \pi_e \in P_e) \quad \text{of the projected canonical basis } \{e_i\}_{1 \leq i \leq 4} \quad \text{are given by}
\]

\[
\left\{ \begin{array}{l}
\pi_c = x_{1i}v_1 + x_{2i}v_2 \simeq [x_{1i}, x_{2i}] \\
\pi_e = x_{3i}v_3 + x_{4i}v_4 \simeq [x_{3i}, x_{4i}] \\
\end{array} \right.
\]

**DEFINITION 4.1.** The set of the projected canonical basis \( \{\pi_c e_i\}_{1 \leq i \leq 4} \) has the good star property if \( \pi_c e_i = w\pi_c e_j \) for some real number \( w \neq 0 \) implies \( i = j \). We define the good star property for the set \( \{\pi_e e_i\}_{1 \leq i \leq 4} \) analogously.

Using \( \{\pi_c e_i\}_{1 \leq i \leq 4} \) (resp. \( \{\pi_e e_i\}_{1 \leq i \leq 4} \)) with the good star property, we uniquely obtain the proto-tiles set \( V_c \) (resp. \( V_e \)) consisting of six symbolic faces whose orientation are positive denoted by

\[
V_c := \left\{ i \neq j, i, j \in \{1, \ldots, 4\}, \ i \wedge j \text{ is the positive oriented parallelogram} \right\}
\]

\[
\left\{ \begin{array}{l}
\text{generated by } \pi_c e_i \text{ and } \pi_c e_j \text{ where } i \wedge j \text{ is chosen if the counterclockwise angle } \alpha \text{ between } \pi_c e_i \text{ and } \pi_c e_j \text{ satisfies } 0 < \alpha < \pi \\
\end{array} \right.
\]

The case of \( V_c \) is defined by \( \{\pi_e e_i\}_{1 \leq i \leq 4} \) analogously.

Let a pair \((x, i \wedge j) \in \mathbb{Z}^4 \times V_c \) (resp. \( V_c \)) be the positive oriented parallelogram \( i \wedge j \) located at \( x \), i.e.,

\[
(x, i \wedge j) := \{x + \mu e_i + \nu e_j \mid 0 \leq \mu, \nu \leq 1\}
\]

(see Figure 17).

**Figure 17.** \((x, i \wedge j)\).

Let \( \sigma \) (resp. \( \theta \)) be the automorphism on the free group \( F(1, 2, 3, 4) \) given by

\[
\sigma : \left\{ \begin{array}{l}
1 \rightarrow 2 \\
2 \rightarrow 3 \\
3 \rightarrow 4 \\
4 \rightarrow 2^c3^b4^a1^{+1}
\end{array} \right. \quad \text{and} \quad \sigma^{-1} : \left\{ \begin{array}{l}
1 \rightarrow 4^{-1}1^c2^b3^a \\
2 \rightarrow 1 \\
3 \rightarrow 2 \\
4 \rightarrow 3
\end{array} \right.
\]

\[
, \theta := \sigma^{-1} : \left\{ \begin{array}{l}
1 \rightarrow 3^a2^b1^c4^{-1} \\
2 \rightarrow 1 \\
3 \rightarrow 2 \\
4 \rightarrow 3
\end{array} \right.
\]

Using the automorphism \( \sigma \) (resp. \( \theta \)), the 2-dimensional extension \( E_2(\sigma) \) (resp. \( E_2(\theta) \)) on the patches of the symbolic faces of \( V_c \) on \( P_c \) (resp. \( V_e \) on \( P_e \)) is defined analogously with (3.13) in the section 3.
From now on, we try to get the sufficient conditions of \(a, b, c \in \mathbb{Z}\) satisfying the following properties:

(S1) The eigenvalues \(\lambda_i\) of \(A\) satisfy the hyperbolic non-Pisot condition:

\[
|\lambda_1 (= \lambda)| \geq |\lambda_2| > 1 > |\lambda_3| \geq |\lambda_4| ;
\]

(S2) \(\{\pi e_i\}_{1 \leq i \leq 4}\) (resp. \(\{\pi e_i\}_{1 \leq i \leq 4}\)) satisfy the good star property (described later);

(S3) The 2-dimensional extension \(E_2 (\sigma)\) (resp. \(E_2 (\theta)\)) has the POP-property.

4.2. Computer experiments

For the companion matrix \(A_−\):

\[
A_− = \begin{bmatrix}
0 & 0 & 0 & −1 \\
1 & 0 & 0 & c \\
0 & 1 & 0 & b \\
0 & 0 & 1 & a
\end{bmatrix}, \quad −15 \leq a, b, c \leq 15 .
\]

by the computer experiments, we observe the following facts:

(1) For \(−15 \leq a, b, c \leq 15\), the automorphism \(\sigma\) (resp. \(\theta\)) satisfies all of the properties (S1), (S2), (S3) if and only if \(b = 0\), \(c = −a − 1\), \(−a\), \(−a + 1\).

More precisely,

(2) See the table in the section 4.5;

(3) There is no automorphism \(\sigma\) (resp. \(\theta\)) satisfying all of the properties (S1), (S2), (S3) associated with the companion matrix \(A_+\)

\[
A_+ = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & c \\
0 & 1 & 0 & b \\
0 & 0 & 1 & a
\end{bmatrix}, \quad −7 \leq a, b, c \leq 7 .
\]

The condition that \(b = 0\) and \(c \in \{−a − 1, −a, −a + 1\}\) for the companion matrix \(A_−\) of \(p (x) = x^4 − ax^3 − bx^2 − cx + 1\) seems to be the necessary and sufficient condition satisfying that \(E_2 (\sigma)\) has the POP-property.

4.3. Theorem

We will prove the following theorem in this section.

**THEOREM 4.2.** Let \(A\) be the companion matrix of \(p (x) = x^4 − ax^3 − cx + 1\) by

\[
A_− = \begin{bmatrix}
0 & 0 & 0 & −1 \\
1 & 0 & 0 & c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & a
\end{bmatrix}
\]
satisfying \(a, c \in \mathbb{Z}\) and \(c \in \{−a − 1, −a, −a + 1\}\). Then

(1) the automorphism \(\sigma\) (resp. \(\theta\)) associated with \(A\) (resp. \(A^{−1}\)) satisfies all of the properties (S1), (S2), (S3). In particular, \((a, c)\) satisfies

\[
(a, c) \in \begin{cases}
(−3, 4), (−2, 3), (−2, 2), (−2, 1), (−1, 2), (−1, 1), (−1, 0), (0, 1), \\
(0, −1), (1, 0), (1, −1), (1, −2), (2, −1), (2, −2), (2, −3), (3, −4)
\end{cases}
\]

(4.16)
then, the eigenvalues $\lambda_i$ (1 $\leq$ i $\leq$ 4) of A satisfy the complex Pisot condition:

$|\lambda_1| = |\lambda_2| > 1 > |\lambda_3|, |\lambda_4|$, and $\lambda_2 = \overline{\lambda_1}$.

(see the table in the section 4.5);

(2) Under the assumption that there exists $U_e$ (resp. $U_c$) which is the family of $4C_2 = 6$ pieces symbolic distinct faces satisfying

(i) there exists k (resp. $k'$) such that

$E_2 (\sigma)^k U_e \supset U_e$, $U_e = \bigcup_{i,j \in V_e} (\pi_e x_{i,j}, i \wedge j)$

(resp. $E_2 (\theta)^k U_c \supset U_c$, $U_c = \bigcup_{i,j \in V_c} (\pi_c x_{i,j}, i \wedge j)$)

(ii) $\bigcup_{n=1}^{\infty} E_2 (\sigma)^{nk} U_e = P_e$ (resp. $\bigcup_{n=1}^{\infty} E_2 (\theta)^{nk'} U_c = P_c$),

then, the compact set

$\gamma_{i,j,e} := \lim_{n \to \infty} A^{-n} \pi_e E_2 (\sigma)^n (x_{i,j}, i \wedge j)$

(resp. $\gamma_{i,j,e} := \lim_{n \to \infty} A^{-n} \pi_c E_2 (\theta)^n (x_{i,j}, i \wedge j)$)

for each $(x_{i,j}, i \wedge j) \in U_e$ (resp. $U_c$).

Then $P = \{ \gamma_{i,j,e} \}_{i,j \in V_e}$ (resp. $\{ \gamma_{i,j,e} \}_{i,j \in V_c}$) satisfies a set equation and the properties (N1), (N2), (N3), then (A, $\mathcal{P}$) has the complex Pisot numeration system.

To prove Theorem 4.2, we prepare a few lemmas.

**Lemma 4.3.** For $p (x) = x^4 - ax^3 - cx + 1$, $a, c \in \mathbb{Z}$, $c \in \{-a - 1, -a, -a + 1\}$,

(1) if $a \geq 5$, then $p (x)$ has two real roots in the interval $(-1, 0)$;

(2) if $a \leq -5$, then $p (x)$ has two real roots in the interval $(0, 1)$.

**Proof.** For (1). It is easy to see that $p (0) = 1$, $p (-1) = 2 + a + c \geq 1$, $p \left( \frac{1}{2} \right) = \frac{1}{16} (17 + 2a + 8c) < 0$. Therefore, the statement holds. (2) can be obtained analogously by the observation of $p (0)$, $p \left( \frac{1}{2} \right) < 0$, and $p (1) \geq 1$. \(\square\)

**Lemma 4.4.** For $p (x) = x^4 - ax^3 - cx + 1$, $a, c \in \mathbb{Z}$, $c \in \{-a - 1, -a, -a + 1\}$, the roots $\lambda_i$ (1 $\leq$ i $\leq$ 4) of $p (x)$ are distributed as follows:

(1) if $a \geq 5$, then

$-1 < \lambda_3 < \lambda_4 < 0 < 1 < \lambda_2 < \lambda_1$;

(2) if $a \leq -5$, then

$\lambda_1 < \lambda_2 < -1 < 0 < \lambda_4 < \lambda_3 < 1$.

**Proof.** For (1). By Lemma 4.3, we know that $p (x)$ has two real roots in the interval $(-1, 0)$. Let $q (x) = x^4 - cx^3 - ax + 1$, then $q (x)$ satisfies the condition (2) in Lemma 4.3. Therefore there exist two roots $\mu_1$ and $\mu_2$ of $q (x)$ in $(0, 1)$. From the fact that $\lambda_1 := \frac{1}{\mu_1}$, $\lambda_2 := \frac{1}{\mu_2}$ are the roots of $p (x)$, we see that $\lambda_1, \lambda_2$ satisfy the relation $1 < \lambda_2 < \lambda_1$. (2) is obtained by the analogous discussion. \(\square\)
**Lemma 4.5.** Let \( A \) be the companion matrix whose characteristic polynomial 
\[ p(x) = x^4 - ax^3 - cx + 1, \]
\( a, c \in \mathbb{Z} \), and \( v_j \) \( (1 \leq j \leq 4) \) be the corresponding vectors of the eigenvectors discussed in (4.15). Then the signature of \( x_j, (1 \leq j \leq 4) \) in (4.15) are given by

1. if \( a \geq 5 \), then
\[ (\text{sgn} (x_{11}), \text{sgn} (x_{21}), \text{sgn} (x_{31}), \text{sgn} (x_{41})) = (+, -, -, +); \]
2. if \( a \leq -5 \), then
\[ (\text{sgn} (x_{11}), \text{sgn} (x_{21}), \text{sgn} (x_{31}), \text{sgn} (x_{41})) = (-, +, +, -). \]

**Proof.** From Lemma 4.4, \( v_i \) \( (1 \leq i \leq 4) \) of (4.15) are the eigenvectors themselves. The eigenvector \( v_i \) \( (1 \leq i \leq 4) \) is given by
\[ v_i = \left[ \begin{array}{c} -1/\lambda_i \vspace{2pt} \\ -1/\lambda_i^2 \vspace{2pt} \end{array} \right]. \]

Therefore we have
\[
\begin{align*}
x_{11} &= -\frac{\lambda_1^3 (\lambda_2 \lambda_3 \lambda_4 \lambda^2 - (\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 \lambda^2) \lambda_2 + \lambda_3 + \lambda_4)}{(-\lambda_1 + \lambda_2) (-\lambda_1 + \lambda_3) (-\lambda_1 + \lambda_4)} \\
x_{21} &= -\frac{\lambda_1^3 (\lambda_1 \lambda_4 \lambda^2 - (\lambda_1 \lambda_4 + \lambda_2 \lambda_4 + \lambda_4 \lambda^2) \lambda_1 + \lambda_2 + \lambda_4)}{(-\lambda_2 + \lambda_1) (-\lambda_2 + \lambda_3) (-\lambda_2 + \lambda_4)} \\
x_{31} &= -\frac{\lambda_1^3 (\lambda_3 \lambda_2 \lambda^2 - (\lambda_3 \lambda_2 + \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda^2) \lambda_3 + \lambda_4 + \lambda_3)}{(-\lambda_3 + \lambda_1) (-\lambda_3 + \lambda_2) (-\lambda_3 + \lambda_4)} \\
x_{41} &= -\frac{\lambda_1^3 (\lambda_4 \lambda_2 \lambda^2 - (\lambda_4 \lambda_2 + \lambda_4 \lambda_4 + \lambda_2 \lambda_4 \lambda^2) \lambda_4 + \lambda_2 + \lambda_3)}{(-\lambda_4 + \lambda_1) (-\lambda_4 + \lambda_2) (-\lambda_4 + \lambda_3)}. 
\end{align*}
\]

On the other hand, we know the relations between roots and coefficients for \( p(x) \) such that
\[
\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= a \quad (4.17) \\
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 &= 0 \quad (4.18) \\
\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 &= c \quad (4.19) \\
\lambda_1 \lambda_2 \lambda_3 \lambda_4 &= 1. \quad (4.20)
\end{align*}
\]

By the way,
\[
\begin{align*}
\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 &= \lambda_1 (\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4) \frac{1}{\lambda_1} \\
&\overset{(4.19)}{=} (c - \lambda_2 \lambda_3 \lambda_4) \frac{1}{\lambda_1} \\
&\overset{(4.20)}{=} \left(c - \frac{1}{\lambda_1}\right) \frac{1}{\lambda_1}. 
\end{align*}
\]

Thus we have
\[
\lambda_1^3 \left(\frac{\lambda_2 \lambda_3 \lambda_4 \lambda^2 - (\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 \lambda^2) \lambda_2 + \lambda_3 + \lambda_4}{(-\lambda_1 + \lambda_2) (-\lambda_1 + \lambda_3) (-\lambda_1 + \lambda_4)}\right).
\]
Therefore, $x_{11} = -\frac{1}{\lambda_1}$. By Lemma 4.4, if $a \geq 5$, it is clear that $x_{11} > 0$. For $x_j$, $j = 2, 3, 4$, we can discuss analogously. We get the proof for (2) analogously.

**Lemma 4.6.** For $p(x) = x^4 - ax^3 - cx^2 + 1$, $a, c \in \mathbb{Z}$, $c \in \{-a - 1, -a, -a + 1\}$, the following properties hold:

1. If $a \geq 5$, then $[\pi e_i]_{1 \leq i \leq 4}$ satisfies the good star property and the proto-tiles set $V_c$ is given by

   \[
   V_c = \{1 \wedge 2, 1 \wedge 3, 1 \wedge 4, 2 \wedge 3, 2 \wedge 4, 3 \wedge 4\}
   \]

   called $V(0)$ in the table in the section 4.5. Moreover, $E_2(\sigma)$ has the POP property. On the other hand, $[\pi e_i]_{1 \leq i \leq 4}$ satisfies the good star property and the proto-tiles set $V_c$ is given by

   \[
   V_c = \{2 \wedge 1, 1 \wedge 3, 2 \wedge 1, 2 \wedge 4, 4 \wedge 3\}
   \]

   called $V(2)$ in the table in the section 4.5. Moreover, $E_2(\theta)$ has the POP property.

2. If $a \leq -5$, then $[\pi e_i]_{1 \leq i \leq 4}$ satisfies the good star property and the proto-tiles set $V_c$ is given by

   \[
   V_c = \{2 \wedge 1, 1 \wedge 3, 2 \wedge 1, 2 \wedge 4, 4 \wedge 3\}
   \]

   called $V(2)$ in the table in the section 4.5. Moreover, $E_2(\sigma)$ has the POP property. On the other hand, we see that $[\pi e_i]_{1 \leq i \leq 4}$ satisfies the good star property and the proto-tiles set $V_c$ is given by

   \[
   V_c = \{1 \wedge 2, 1 \wedge 3, 1 \wedge 4, 2 \wedge 3, 2 \wedge 4, 3 \wedge 4\}
   \]

   called $V(0)$ in the section 4.5. Moreover, $E_2(\theta)$ has the POP property.

**Proof.** For (1). From Lemma 4.4, $v_i$ ($1 \leq i \leq 4$) of (4.15) are the eigenvectors themselves and we know that $P_e = L(v_1, v_2)$ and $P_e = L(v_3, v_4)$. From the notation of the inverse matrix of $[v_1, v_2, v_3, v_4]$, we have $\pi e_1 = x_{11} v_1 + x_{21} v_2$ and we know that

\[
\begin{align*}
\pi e_2 &= \pi e A e_1 = \lambda_1 x_{11} v_1 + \lambda_2 x_{21} v_2 \\
\pi e_3 &= \pi e A e_2 = \lambda_2^2 x_{11} v_1 + \lambda_2 x_{21} v_2 \\
\pi e_4 &= \pi e A e_3 = \lambda_2^3 x_{11} v_1 + \lambda_2 x_{21} v_2 .
\end{align*}
\]

Therefore, $[\pi e_i]_{1 \leq i \leq 4}$ satisfies the good star property since $1 < \lambda_2 < \lambda_1$ and $x_{11} \geq 0$, $x_{21} \leq 0$ by Lemma 4.5. So, we obtain Figure 18 and we see that the proto-tiles set is given by

\[
V_c = \{1 \wedge 2, 1 \wedge 3, 1 \wedge 4, 2 \wedge 3, 2 \wedge 4, 3 \wedge 4\} = V(0) .
\]
Now let us operate the 2-dimensional extension $E_2(\sigma)$ to the proto-tiles of $V_e$, then we obtain the following:

$$E_2(\sigma)(0, 1 \wedge 2) = (0, \sigma(1) \wedge \sigma(2)) = (0, 2 \wedge 3)$$
$$E_2(\sigma)(0, 1 \wedge 3) = (0, \sigma(1) \wedge \sigma(3)) = (0, 2 \wedge 4)$$
$$E_2(\sigma)(0, 1 \wedge 4) = \left(\sum_{k=1}^{a} (ce_2 + (k - 1)e_4, 2 \wedge 4)\right) + (ce_2 + ae_4 - e_1, 1 \wedge 2)$$
$$E_2(\sigma)(0, 2 \wedge 3) = (0, \sigma(2) \wedge \sigma(3)) = (0, 3 \wedge 4)$$
$$E_2(\sigma)(0, 2 \wedge 4) = \left(\sum_{k=1}^{c} (-ke_2, 2 \wedge 3)\right) + \left(\sum_{k=1}^{a} (ce_2 + (k - 1)e_4, 3 \wedge 4)\right) + (ce_2 + ae_4 - e_1, 1 \wedge 3)$$
$$E_2(\sigma)(0, 3 \wedge 4) = \left(\sum_{k=1}^{c} (-ke_2, 2 \wedge 4)\right) + (ce_2 + ae_4 - e_1, 1 \wedge 4) .$$

Therefore, we see that $E_2(\sigma)$ has the POP property.

Now let us consider on $P_e$, that is, we have $\pi_e e_1 = x_{31}v_3 + x_{41}v_4$ and we know that

$$\pi_e e_2 = \pi_e A e_1 = \lambda_3 x_{31}v_3 + \lambda_4 x_{41}v_4$$
$$\pi_e e_3 = \pi_e A e_2 = \lambda_3^2 x_{31}v_3 + \lambda_4^2 x_{41}v_4$$
$$\pi_e e_4 = \pi_e A e_3 = \lambda_3^3 x_{31}v_3 + \lambda_4^3 x_{41}v_4 .$$

Therefore $\{\pi_e e_i\}_{1 \leq i \leq 4}$ also satisfies the good star property since $-1 < \lambda_3 < \lambda_4 < 0$ and $x_{31} \leq 0, x_{41} \geq 0$ by Lemma 4.5. So, we obtain Figure 19 and the proto-tiles set is given by

$$V_e = \{2 \wedge 1, 1 \wedge 3, 4 \wedge 1, 3 \wedge 2, 2 \wedge 4, 4 \wedge 3\} (= V(2)) .$$

Operating the 2-dimensional extension $E_2(\theta)$ to the proto-tiles of $V_e$, then we obtain the following:

$$E_2(\theta)(0, 2 \wedge 1) = (0, \theta(2) \wedge \theta(1)) = \left(\sum_{k=1}^{a} ((k - 1)e_3, 1 \wedge 3)\right) + (ae_3 + ce_1 - e_4, 4 \wedge 1) .$$
\[
E_2(\theta)(\mathbf{0}, 1\wedge 3) = \left( \sum_{k=1}^{a-1} ((k-1)e_3, 3 \wedge 2) \right) + \left( \sum_{k=1}^{-c} (ae_3 - ke_1, 2 \wedge 1) \right) + (ae_3 + ce_1 - e_4, 2 \wedge 4)
\]

\[
E_2(\theta)(\mathbf{0}, 4\wedge 1) = \left( \sum_{k=1}^{-c} (ae_3 - ke_1, 1 \wedge 3) \right) + (ae_3 + ce_1 - e_4, 4 \wedge 3)
\]

\[
E_2(\theta)(\mathbf{0}, 3\wedge 2) = (\mathbf{0}, \theta (3) \wedge \theta (2)) = (\mathbf{0}, 2 \wedge 1)
\]

\[
E_2(\theta)(\mathbf{0}, 2\wedge 4) = (\mathbf{0}, \theta (2) \wedge \theta (4)) = (\mathbf{0}, 1 \wedge 3)
\]

\[
E_2(\theta)(\mathbf{0}, 4\wedge 3) = (\mathbf{0}, \theta (4) \wedge \theta (3)) = (\mathbf{0}, 3 \wedge 2)
\]

Therefore, we see that \(E_2(\theta)\) has the POP property.

For the case of (2) \(a \leq -5\), we get the conclusion analogously.

\[\square\]

**Proof of Theorem 4.2.** The first part (1) is obtained by Lemma 4.6 in the case \(a \leq -5\), \(a \geq 5\), and for each \(a (-5 \leq a \leq 5)\), we can check that the proto-tiles set \(V_e, V_c\) and the fact that \(E_2(\sigma)\) has the POP property explicitly (see the table in the section 4.5). The second part (2), mentioned that the family of compact sets \(\{ \gamma_{i \wedge j, e} \}, i \wedge j \in V_e\) (resp. \(\{ \gamma_{i \wedge j, c} \}, i \wedge j \in V_c\)) satisfies (N1), (N2), (N3) of the complex Pisot numeration system property, can be obtained by the analogous proof of Theorem 3.31.

\[\square\]

**4.4. Example of the complex Pisot numeration system**

**Example 4.7.** Let us consider the minimal polynomial

\[p(x) = x^4 - x^3 + 1\]

then its companion matrix \(A\) is given by

\[
A = \begin{bmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]
FIGURE 20.

which is corresponding to \((a, c) = (1, 0)\) in Theorem 4.2.

The eigenvalues of \(A\) satisfy

\[
|\lambda_1| = |\lambda_2| > 1 > |\lambda_3| = |\lambda_4|
\]

and the distributions \([\pi_e e_i]_{1 \leq i \leq 4}\) is as Figure 20. The proto-tiles set \(V_e\) is chosen as

\[
V_e = \{1 \land 2, 1 \land 3, 1 \land 4, 2 \land 3, 2 \land 4, 3 \land 4\}
\]

Let us define the automorphism \(\sigma\) of \(F\) by

\[
\sigma: \begin{cases}
1 \rightarrow 2 \\
2 \rightarrow 3 \\
3 \rightarrow 4 \\
4 \rightarrow 41^{-1}
\end{cases}
\]

and using the automorphism \(\sigma\), we see that the 2-dimensional extension \(E_2(\sigma)\) from the positive orientated face to the “patch” of faces is given by

\[
E_2(\sigma)(0, 1 \land 2) = (0, \sigma(1 \land \sigma(2))) = (0, 2 \land 3)
\]

\[
E_2(\sigma)(0, 1 \land 3) = (0, 2 \land 4)
\]

\[
E_2(\sigma)(0, 1 \land 4) = (0, 2 \land 41^{-1}) = (0, 2 \land 4 + (f(4), 2 \land 1^{-1}))
\]

\[
E_2(\sigma)(0, 2 \land 3) = (0, 3 \land 4)
\]

\[
E_2(\sigma)(0, 2 \land 4) = (0, 3 \land 41^{-1}) = (0, 3 \land 4 + ((e_4 - e_1), 1 \land 3))
\]

\[
E_2(\sigma)(0, 3 \land 4) = (0, 4 \land 41^{-1}) = (0, 4 \land 4 + (f(4), 4 \land 1^{-1})) = ((e_4 - e_1), 1 \land 4)
\]

where \((*)\) is the rearrangement which is introduced in the section 3. Then, we see that \(E_2(\sigma)\) has the POP-property (see Figure 21).

Starting \((-e_3, 3 \land 4)\), we see that

\[
E_2(\sigma)^3(-e_3, 3 \land 4) \succ \pi_e(-e_3, 3 \land 4)
\]

moreover, let

\[
T_e = \{\pi_e(x, i \land j) \mid \pi_e(x, i \land j) \in \pi_e E_2(\sigma)^{3n}(-e_3, 3 \land 4) \text{ for some } n \in \mathbb{N}\},
\]
then \( T_e = \bigcup_{\pi_e(z, i \wedge j) \in T_e} \pi_e(z, i \wedge j) \) is the quasi-periodic tiling of \( P_e \) (see Figure 22).

Now we can find the octagonal patch \( U_e \) satisfying \( E_2(\sigma)U_e \supset U_e \) (see Figure 23):

\[
U_e = \left(\begin{array}{c}
-e_3 - e_1 - e_2, 1 \wedge 2 \\
(e_4 - e_1 - e_3), 1 \wedge 3 \\
(e_4 - e_1 - e_2 - e_3), 2 \wedge 3 \\
(-e_3 - e_1 - e_2), 2 \wedge 4 \\
(-e_3, 3 \wedge 4)
\end{array}\right).
\]

Let us define

\[
\gamma_{i \wedge j} := \lim_{n \to \infty} A^{-n} \pi_e E_2(\sigma)^n (x_{i \wedge j}, i \wedge j) \quad \text{for} \quad (x_{i \wedge j}, i \wedge j) \in U_e.
\]

Then, the family of the compact sets \( \{\operatorname{cl}(\operatorname{int}(\gamma_{i \wedge j}))\}_{i \wedge j \in V_e} \) has the following set equations:

\[
A\gamma_{1 \wedge 2} = \gamma_{2 \wedge 3} + \pi_e(-2e_4 + e_1)
\]
\[
A\gamma_{1 \wedge 3} = \gamma_{2 \wedge 4} + \pi_e e_3
\]
\[
A\gamma_{1 \wedge 4} = (\gamma_{2 \wedge 4} + \pi_e(e_3 + e_1 - e_4)) \cup (\gamma_{1 \wedge 2} + \pi_e e_3)
\]
\[
A\gamma_{2 \wedge 3} = \gamma_{3 \wedge 4} + \pi_e(e_4 + e_2)
\]
\[
A\gamma_{2 \wedge 4} = (\gamma_{3 \wedge 4} + \pi_e(-e_2 - e_4)) \cup (\gamma_{1 \wedge 3} + \pi_e(-e_2 - e_4))
\]
\[
A\gamma_{3 \wedge 4} = \gamma_{1 \wedge 4} + \pi_e e_3
\]

(see Figure 23).

Moreover, we can see that

\[
\operatorname{cl}(\operatorname{int}(\gamma_{i \wedge j})) = \gamma_{i \wedge j}, \quad \mu_e(\partial\gamma_{i \wedge j}) = 0, \quad \text{and} \quad \gamma = \bigcup_{i \wedge j \in V_e} \gamma_{i \wedge j} \text{ is disjoint},
\]

therefore, we see that \((A, \mathcal{P}), \mathcal{P} = \{\gamma_{i \wedge j}\}_{i \wedge j \in V_e}\) is the complex Pisot numeration system.

The labeled graph \((V, E, i, t, \mathcal{L})\) is given by Figure 24.
Figure 22. The quasi-periodic tiling $\mathcal{T}_{\nu}$. 
From the fact that
\[ \phi(\pi e_1) = 1, \quad \phi(\pi e_2) = \lambda, \quad \phi(\pi e_3) = \lambda^2, \quad \phi(\pi e_4) = \lambda^3, \]
the labeled graph \((V, E, i, t, (\phi L))\) is given by Figure 25.

Therefore, let \(\Omega_{i,j}\) be the labeled admissible sequence space which is starting from the vertex \(i \wedge j\) by the labeled graph \((V, E, i, t, (\phi L))\) and its element be \((a_1, a_2, \ldots)\), \(a_i \in \{ -2\lambda^3 + 1, \lambda^2, -\lambda^3 + \lambda^2 + 1, \lambda + 1, -\lambda^3 - \lambda \} \), then \(z \in \phi\left(\bigcup_{i,j \in V} \gamma_{i,j}\right)\) is represented by \(z = \sum_{n=1}^{\infty} a_{n-1} \lambda^{-n}\) where \(a_n = \phi_e\left(\pi e f_{k_{n-1}}^{(j_{n-1})}\right)\).

### 4.5. Appendix: The table

Finally, we will show the table how the eigenvalues \(\lambda_i\) are distributed depending on \(a, c \in \mathbb{Z}\). Notation on the table is as follows:

1. “Comp” (“resp. Real”) means the complex (resp. real) number respectively.
2. \(V (i), i = 0, 1, 2\) are the set of the proto-tiles such that
   - \(V (0) = \{1 \wedge 2, 1 \wedge 3, 1 \wedge 4, 2 \wedge 3, 2 \wedge 4, 3 \wedge 4\}\)
   - \(V (1) = \{1 \wedge 2, 3 \wedge 1, 1 \wedge 4, 2 \wedge 3, 4 \wedge 2, 3 \wedge 4\}\)
   - \(V (2) = \{2 \wedge 1, 1 \wedge 3, 4 \wedge 1, 3 \wedge 2, 2 \wedge 4, 4 \wedge 3\}\).
<table>
<thead>
<tr>
<th>$a$</th>
<th>$c$</th>
<th>$x^4 + ax^3 + cx + 1$</th>
<th>$\lambda_1, \lambda_2$</th>
<th>$\lambda_3, \lambda_4$</th>
<th>Distribution of $\lambda_i$</th>
<th>$V_2$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4$</td>
<td>5</td>
<td>$x^4 + 4x^3 - 5x + 1$</td>
<td>Real</td>
<td>Real</td>
<td>$V(2)$</td>
<td>$V(0)$</td>
<td>$V(2)$ $V(0)$</td>
</tr>
<tr>
<td>$-4$</td>
<td>4</td>
<td>$x^4 + 4x^3 - 4x + 1$</td>
<td>Real</td>
<td>Real</td>
<td>$V(2)$</td>
<td>$V(0)$</td>
<td>$V(2)$ $V(0)$</td>
</tr>
<tr>
<td>$-4$</td>
<td>3</td>
<td>$x^4 + 4x^3 - 3x + 1$</td>
<td>Real</td>
<td>Comp</td>
<td>$V(2)$</td>
<td>$V(0)$</td>
<td>$V(2)$ $V(0)$</td>
</tr>
<tr>
<td>$-3$</td>
<td>4</td>
<td>$x^4 + 3x^3 - 4x + 1$</td>
<td>Comp</td>
<td>Real</td>
<td>$V(1)$</td>
<td>$V(0)$</td>
<td>$V(1)$ $V(0)$</td>
</tr>
<tr>
<td>$-3$</td>
<td>3</td>
<td>$(x^2 + 2x - 1)$</td>
<td>Real</td>
<td>Real</td>
<td>$V(2)$</td>
<td>$V(0)$</td>
<td>$V(2)$ $V(0)$</td>
</tr>
<tr>
<td>$-3$</td>
<td>2</td>
<td>$x^4 + 3x^3 - 2x + 1$</td>
<td>Real</td>
<td>Comp</td>
<td>$V(2)$</td>
<td>$V(0)$</td>
<td>$V(2)$ $V(0)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>3</td>
<td>$x^4 + 2x^3 - 3x + 1$</td>
<td>Comp</td>
<td>Real</td>
<td>$V(1)$</td>
<td>$V(0)$</td>
<td>$V(1)$ $V(0)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>2</td>
<td>$x^4 + 2x^3 - 2x + 1$</td>
<td>Comp</td>
<td>Comp</td>
<td>$V(1)$</td>
<td>$V(0)$</td>
<td>$V(1)$ $V(0)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>2</td>
<td>$x^4 + x^3 - 2x + 1$</td>
<td>Comp</td>
<td>Comp</td>
<td>$V(1)$</td>
<td>$V(0)$</td>
<td>$V(1)$ $V(0)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>1</td>
<td>$x^4 + x^3 - x + 1$</td>
<td>Comp</td>
<td>Comp</td>
<td>$V(1)$</td>
<td>$V(0)$</td>
<td>$V(1)$ $V(0)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>0</td>
<td>$x^4 + x^3 + 1$</td>
<td>Comp</td>
<td>Comp</td>
<td>$V(1)$</td>
<td>$V(0)$</td>
<td>$V(1)$ $V(0)$</td>
</tr>
</tbody>
</table>
On other pairs \((a, c)\), \(c = -a - 1, -a, -a - 1, \{\lambda_i\}_{1 \leq i \leq 4}\) are totally real.
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