Zeta Functions of \((\text{SL}_2 \times \text{SL}_2 \times \text{GL}_2, M_2 \oplus M_2)\) Associated with a Pair of Maass Cusp Forms

by

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§0. Introduction

In [S3], we generalized the theory of zeta functions of prehomogeneous vector spaces ([SS], [S1]) to zeta functions whose coefficients involve periods of automorphic forms under the additional assumption that a prehomogeneous vector space has a symmetric structure. The purpose of the present paper is to make a detailed study of such zeta functions for the prehomogeneous vector space \((G, \rho, V) = (\text{SL}_2 \times \text{SL}_2 \times \text{GL}_2, \rho, M_2 \oplus M_2)\) and a pair of Maass cusp forms \((\Phi_1, \Phi_2)\), an automorphic form on \(\text{SL}_2 \times \text{SL}_2\). Here the representation \(\rho : G \rightarrow \text{GL}(V)\) is defined by

\[
\rho(g_1, g_2, h)(x_1, x_2) = (g_1 x_1 g_2^{-1}, g_1 x_2 g_2^{-1})^t h.
\]

Our main result is that the zeta functions attached to a pair of Maass cusp forms are identified with the convolution of two Dirichlet series obtained as the Mellin transforms of the theta liftings of \(\Phi_1, \Phi_2\).

It is known that \((G, \rho, V)\) is a regular prehomogeneous vector space (of type (15) in the list of [SK, §7]) with only one irreducible relative invariant

\[
P(x) = (\text{tr}(x_1 J x_2 J^{-1}))^2 - 4 \det x_1 \det x_2, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The singular set \(S\) is given by \(S = \{ x \in V | P(x) = 0 \}\) and \(V \setminus S\) is a Zariski-open \(\rho(G)\)-orbit.

Put \(\Gamma_0 = \text{SL}_2(\mathbb{Z}), \Gamma = \Gamma_0 \times \Gamma_0 \times \Gamma_0\) and \(\mathcal{L} = M_2(\mathbb{Z}) \oplus M_2(\mathbb{Z})\). Let \(\Phi_1, \Phi_2 : \Gamma_0 \setminus \mathcal{L} \rightarrow \mathbb{C}\) be even Maass cusp forms, which are eigenfunctions of the non-Euclidean Laplacian on the upper half-plane \(\mathcal{H}\):

\[
y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi_i(z) + \lambda_i (1 - \lambda_i) \Phi_i(z) = 0 \quad (i = 1, 2).
\]

Then the zeta functions we consider in this paper are of the form

\[
\zeta_{\pm}(\Phi_1, \Phi_2; s) = \sum_{x \in \rho(\Gamma) \setminus \mathcal{L}, \pm P(x) > 0} \frac{M_{\Phi_1, \Phi_2}(x)}{|P(x)|^s},
\]

where \(M_{\Phi_1, \Phi_2}(x)\) is a certain period of \(\Phi_1 \times \Phi_2\) over the isotropy subgroup of \(G\) at \(x\).
For \( i = 1, 2 \), let

\[
\Theta(\Phi_i)(z) = \sum_{n \neq 0} \rho_i(n)W_{\frac{1}{2}\text{sgn}(n), \frac{2i-1}{4}}(4\pi |n|y)\text{e}[nx]
\]

be the Fourier expansion of the theta lift \( \Theta(\Phi_i) \) of \( \Phi_i \) (see [KS]). Then our main result is the following:

**THEOREM A.** The zeta functions \( \zeta_{\pm}(\Phi_1, \Phi_2; s) \) coincide with

\[
\xi(2s-1) \sum_{n=1}^{\infty} \frac{\rho_1(\pm n)\rho_2(\pm n)}{n^{s-3/2}}
\]

up to constant factors depending on \( \lambda_1, \lambda_2 \).

The general theory in [S3] provides us a recipe for proving a functional equation satisfied by the zeta functions.

**THEOREM B.** The zeta functions \( \zeta_{\pm}(\Phi_1, \Phi_2; s) \) have analytic continuations to meromorphic functions of \( s \). Moreover, if we put

\[
\xi_{\pm}(\Phi_1, \Phi_2; s) = \pi^{-2s} \Gamma(s + \frac{\lambda_1-\lambda_2-1}{2}) \Gamma(s - \frac{\lambda_1+\lambda_2}{2}) \zeta_{\pm}(\Phi_1, \Phi_2; s),
\]

then they satisfy the following functional equation:

\[
\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} (\Phi_1, \Phi_2; 2-s) = C(\lambda_1, \lambda_2; s) \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} (\Phi_1, \Phi_2; s),
\]

where

\[
C(\lambda_1, \lambda_2; s) = \frac{1}{\sin \pi s \cos \pi \lambda_2} \cdot \frac{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)}{\Gamma(1-\lambda_1-\lambda_2)} \cdot \frac{\sin \pi s \cos \frac{\pi \lambda_1}{2} \cos \frac{\pi \lambda_2}{2}}{\sin \pi s \sin \frac{\pi \lambda_1}{2} \sin \frac{\pi \lambda_2}{2}}.
\]

Note that \( SL_2 \times SL_2 \cong Spin_4 \) and our prehomogeneous vector space is a special case of \( (Spin_m \times GL_m, M_{m,n}) \). The zeta functions of this general prehomogeneous vector space attached to automorphic forms on \( Spin_m \) are expected to be the Koecher-Maass series of the theta liftings of the automorphic forms under consideration. In [BS], Böcherer and Schulze-Pillot considered the Koecher-Maass series of the Yoshida lifting and proved that the series is connected with the convolution of Dirichlet series of two modular forms. Theorem A can be viewed as a Maass form version of their result.

We can apply the method used in this paper also to a pair of holomorphic cusp forms for \( SL_2(\mathbb{Z}) \) on the basis of Shintani’s work [Shn]. If we take the real analytic Eisenstein series \( E(z, \lambda) \) for \( \Phi_1 \) and/or \( \Phi_2 \), then we can obtain a similar result by considering the prehomogeneous vector spaces \( (B_2 \times SL_2 \times GL_2, V) \) or \( (B_2 \times B_2 \times GL_2, V) \), \( B_2 \) being the Borel subgroup of \( SL_2 \). However our method cannot apply to the case of three modular forms; this is a limitation of the theory in [S3].

The key to the proof of Theorem A is a study of the \( G \)-equivariant quotient map

\[
\psi : V \setminus S \longrightarrow X := (V \setminus S)/SL_2.
\]
The space $X$ is given explicitly by

$$X = \left\{ (Y_1, Y_2) \in \text{Sym}_n \times \text{Sym}_n \mid \det Y_1 = \det Y_2 \neq 0 \right\}.$$ 

The quotient map $\psi$ behaves quite well on the set of integral points in $V \setminus S$ and the structure of $X$ is responsible for the convolution structure in the expression of the zeta functions in Theorem A.

In §1, we examine the structure of the prehomogeneous vector space $(G, \rho, V)$ through the quotient mapping $\psi$. In §2, after recalling some necessary results on the theta lifting of Maass forms from [KS], we prove Theorems A and B.

**Notation.**

Let $\text{Sym}_n$ (resp. $\text{M}_{m,n}$) be the set of symmetric (resp. square, rectangular) matrices of size $n$ (resp. $m$, $m$ by $n$), which we consider as the affine space of dimension $n(n + 1)/2$ (resp. $m^2$, $mn$). For a commutative ring $R$, $\text{Sym}_n(R)$ (resp. $\text{M}_{m,n}(R)$) denotes the set of symmetric (resp. square, rectangular) matrices of size $n$ (resp. $m^2$, $mn$) with entries in $R$. We denote by $\text{Disc}(Y)$ the discriminant of $Y \in \text{Sym}_n(R)$, which is defined to be $(-1)^n(n-1)/2 \det Y$. For a real symmetric matrix $Y \in \text{Sym}_n(\mathbb{R})$ we write $\text{sgn}(Y) = (i, j)$ if $Y$ has $i$ positive and $j$ negative eigenvalues. For matrices $A$ and $B$, we put $A[B] = ^tBA B$ if the product is defined. For a real vector space $V$, $S(V)$ is the space of rapidly decreasing functions on $V$.

### §1. Prehomogeneous vector space $(\text{SL}_2 \times \text{SL}_2 \times \text{GL}_2, \text{M}_2 \oplus \text{M}_2)$

Put $G = \text{SL}_2 \times \text{SL}_2 \times \text{GL}_2$ and $V = \text{M}_2 \oplus \text{M}_2$. We define a rational representation $\rho$ of $G$ on $V$ by

$$\rho(g_1, g_2, 1)(x_1, x_2) = (g_1 x_1 g_2^{-1}, g_1 x_2 g_2^{-1}), \quad g_1, g_2 \in \text{SL}_2,$$

$$\rho(1, 1, h)(x_1, x_2) = (x_1, x_2)^h = (ax_1 + bx_2, cx_1 + dx_2), \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2.$$

Then the triple $(G, \rho, V)$ is a regular prehomogeneous vector space equivalent to the prehomogeneous vector space of type (15) with $n = 4$, $m = 2$ in the Sato-Kimura classification ([SK, §7]). For $x = \begin{pmatrix} x \cr d \end{pmatrix} \in \text{M}_2$, we put

$$x^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \det x \cdot x^{-1}$$

and define a symmetric bilinear form $(\cdot, \cdot) : \text{M}_2 \times \text{M}_2 \to \mathbb{C}$ by

$$(x_1, x_2) = \text{tr} x_1 x_2^*.$$ 

Note that $\langle x, x \rangle = 2 \det x$. We consider the mapping $\pi : V \to \text{Sym}_2$ defined by

$$\pi(x) = \pi(x_1, x_2) = \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle \end{pmatrix}.$$

Then we have

$$\pi(\rho(g_1, g_2, h)x) = h \pi(x)^h \quad (1.1)$$
and the discriminant of $\pi(x)$

$$P(x) := \text{Disc} \pi(x) = (x_1, x_2)^2 - (x_1, x_1) (x_2, x_2)$$

is the fundamental relative invariant of $(G, \rho, V)$, which corresponds to the character $\chi(g_1, g_2, h) = (\det h)^2$. The singular set $S$ of $(G, \rho, V)$ is given by $S = \{x \in V | P(x) = 0\}$. In the following we consider $(G, \rho, V)$ as a prehomogeneous vector space defined over $\mathbb{Q}$. The $\mathbb{Q}$-structure is the standard one.

Let $\sigma : M \to \text{Sym}_2$ be the mapping defined by

$$\sigma(A) = JA + \tau AJ^{-1}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

Then, by a straightforward calculation, we have

$$\sigma(I_2) = 0, \quad \sigma(x_1 x_2^*) = -\sigma(x_2 x_1^*) \quad (x_1, x_2 \in M_2), \quad (1.2)$$

$$\text{Disc} \sigma(A) = (tr A)^2 - 4 \det A, \quad (1.3)$$

$$\sigma(g_1 x_1 g_1^{-1}) = \sigma(x_1) [g_1^{-1}] \quad (g_1 \in SL_2). \quad (1.4)$$

We define a mapping $\psi : V = M_2 \oplus M_2 \to \text{Sym}_2 \oplus \text{Sym}_2$ by setting

$$\psi(x) = \psi(x_1, x_2) = (\sigma(x_1 x_2^*), -\sigma(x_1^* x_2)).$$

**Lemma 1.** (1) Put $\psi_1(x) = \sigma(x_1 x_2^*)$, $\psi_2(x) = -\sigma(x_1^* x_2)$. Then we have

$$\text{Disc} \psi_1(x) = \text{Disc} \psi_2(x) = \text{Disc} \pi(x).$$

(2) The identity

$$\psi(\rho(g_1, g_2, h)x) = (\det h) \cdot (\psi_1(x) [g_1^{-1}], \psi_2(x) [g_2^{-1}])$$

holds for any $(g_1, g_2, h) \in G$.

**Proof.** (1) By (1.3) we have

$$\text{Disc} \psi_1(x) = (tr x_1 x_2^*)^2 - 4 \det(x_1 x_2^*)$$

$$= (x_1, x_2)^2 - 2(\det x_1)(2 \det x_2)$$

$$= \text{Disc} \pi(x).$$

Similarly one can prove the identity $\text{Disc} \psi_2(x) = \text{Disc} \pi(x)$.

(2) By (1.4) we have

$$\psi(\rho(g_1, g_2, 1) \cdot (x_1, x_2)) = (\psi_1(x) [g_1^{-1}], \psi_2(x) [g_2^{-1}]).$$

To see the equivariance property for $h \in GL_2$, we note that, for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\psi((x_1, x_2)^t h) = \psi((ax_1 + bx_2, cx_1 + dx_2)) = (\sigma(X_1), -\sigma(X_2)),$$

$$X_1 = (ac \det x_1 + bd \det x_2) I_2 + ad x_1 x_2^* + bc x_2 x_1^*,$$

$$X_2 = (ac \det x_1 + bd \det x_2) I_2 + ad x_1^* x_2 + bc x_2^* x_1.$$

Hence, by (1.2), we obtain $\psi(x^t h) = (\det h) \cdot \psi(x)$. 

$\square$
Let $X = \{(Y_1, Y_2) \in \text{Sym}_2 \oplus \text{Sym}_2 \mid \text{Disc } Y_1 = \text{Disc } Y_2 \neq 0\}$.

Then, by Lemma 1 (1), the image $\psi(V \setminus S)$ is contained in $X$. The $G$-equivariant mapping $\psi : V \setminus S \rightarrow X$ plays a crucial role in the analysis of the structure of our prehomogeneous vector space $(G, \rho, V)$.

**Lemma 2.** For $x \in V \setminus S$, then the projection $p_1 : G \rightarrow \text{SL}_2 \times \text{SL}_2$ induces an isomorphism of the isotropy subgroup $G_x$ at $x$ onto the group

$\{(g_1, g_2) \in \text{SL}_2 \times \text{SL}_2 \mid \psi_1(x)[g_1] = \epsilon \psi_1(x), \ \psi_2(x)[g_2] = \epsilon \psi_2(x), \ \epsilon = \pm 1\}$.

In particular, the identity component $G^o_x$ of $G_x$ is a subgroup of index 2 and is isomorphic to $\text{SO}(\psi_1(x)) \times \text{SO}(\psi_2(x))$.

**Proof.** If $(g_1, g_2, h) \in G$ is in $G_x$, then by Lemma 1 (2) and (1.1) we have $(\psi_1(x), \psi_2(x)) = (\det h \cdot (\psi_1(x)[g_1^{-1}], \psi_2(x)[g_2^{-1}]), \pi(x) = \pi(x)\theta h)$. Hence $\det h = \pm 1$ and $(g_1, g_2)$ is in the group given in the lemma. If $(g_1, g_2, h)$ is in $G^o_x$, then det $h = 1$ and $(g_1, g_2)$ is in $\text{SO}(\psi_1(x)) \times \text{SO}(\psi_2(x))$. It is easy to see that $G_x \cap \ker(p_1)$ is trivial and the projection $p_1$ is injective when restricted to $G_x$. Since $\dim G_x = \dim G - \dim V = 10 - 8 = 2 = \dim(\text{SO}(\psi_1(x)) \times \text{SO}(\psi_2(x)))$, $G^o_x$ is isomorphic to $\text{SO}(\psi_1(x)) \times \text{SO}(\psi_2(x))$. It is easy to see that the index of $G^o_x$ in $G_x$ is equal to 2. Hence $G_x$ is isomorphic to the group given in the lemma.

**Lemma 3.** (1) The space $(X, \psi)$ is a geometric quotient of $V \setminus S$ for the action of $\rho(1, 1) \times \text{SL}_2$.

(2) Let $k/\mathbb{Q}$ be an arbitrary field extension. For $x, x' \in V(k) \setminus S$, $\psi(x) = \psi(x')$ if and only if $x' = x'h$ for some $h \in \text{SL}_2(k)$.

**Proof.** (1) It is obvious that $V \setminus S$ is irreducible and $\psi$ is surjective. Since $X$ is a homogeneous space of $\text{GL}_1 \times \text{SL}_2 \times \text{SL}_2$, $X$ is normal. Hence, by [PV, Theorem 4.2], the first assertion follows immediately from the second.

(2) The ‘if’-part of the assertion is obvious in view of Lemma 1 (2). Let us prove the ‘only if’-part. If $h \in \text{SL}_2$ satisfies $x' = x'h$ for $x, x' \in V(k) \setminus S$, then $h$ is necessarily in $\text{SL}_2(k)$. Hence we need not worry about the $k$-rationality of $h$. Since $V \setminus S$ is an open $G$-orbit, one finds a $(g_1, g_2, h_0) \in G$ such that $x = \rho(g_1, g_2, h)x'$. From the assumption $\psi(x) = \psi(x')$ it follows that $\det h = \pm 1$ and $\psi(x)[g_1] = (\det h \cdot \psi(x))[g_1] = (\det h \cdot \psi(x))[g_1]$. Hence, by Lemma 2, there exists an $h' \in \text{GL}_2$ such that $\det h' = \det h$ and $(g_1, g_2, h')$ is in $G_x$. Thus we have $x = \rho(1, 1, h^{-1}h)x'$ and $\det(h'^{-1}h) = 1$.

For $i, j = 0, 1, 2$, we put

$$V_{ij} = \{x \in V(\mathbb{R}) \setminus S \mid \text{sgn } (\psi_1(x)) = (i, 2 - i), \ \text{sgn } (\psi_2(x)) = (j, 2 - j)\}.$$  \hspace{1cm} (1.5)

Since $\text{Disc } \psi_1(x) = \text{Disc } \psi_2(x)$, the set $V_{ij}$ is empty unless $i \equiv j \pmod{2}$. Let $G^+$ be the identity component of the real Lie group $G(\mathbb{R})$. The group $G^+$ is given by

$$G^+ = \{(g_1, g_2, g_0) \mid g_1, g_2 \in \text{SL}_2(\mathbb{R}), \ g_0 \in \text{GL}_2(\mathbb{R}), \ \det g_0 > 0\}.$$
Lemma 4. The $G^+$-orbit decomposition of $V(\mathbb{R}) \setminus S = \{ x \in V(\mathbb{R}) | P(x) \neq 0 \}$ is given by

$$V(\mathbb{R}) \setminus S = V_{0,0} \cup V_{0,2} \cup V_{1,1} \cup V_{2,0} \cup V_{2,2}.$$ 

Proof. The lemma is an immediate consequence of Lemma 1 (2) and Lemma 3. □

Now we study the integral structure of $G = \psi \circ \rho (\mathbb{C}) \in V(\mathbb{Q})$. Let $\text{Sym}_2^\mathbb{Z}(\mathbb{Z})$ be the set of even integral $2 \times 2$ symmetric matrices and put

$$X_\mathbb{Z} = X \cap (\text{Sym}_2^\mathbb{Z}(\mathbb{Z}) \times \text{Sym}_2^\mathbb{Z}(\mathbb{Z})).$$

Lemma 5. The mapping $\psi : \mathcal{L} \setminus S \to X_\mathbb{Z}$ is surjective.

Proof. Let $(Y_1, Y_2)$ be an arbitrary element in $X_\mathbb{Z}$. Put $d = \text{Disc} Y_1 = \text{Disc} Y_2$. Take $g_i \in GL_2(\mathbb{Q}) (i = 1, 2)$ which diagonalizes $Y_i$. Then $Y_1[g_1] = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix}$ for some $a_1 \in \mathbb{Q}^\times$. By an elementary matrix calculation, we have

$$\frac{a_1}{a_2} Y_2 = Y_1[g], \quad g = g_1 \begin{pmatrix} 1 & 0 \\ 0 & a_1/a_2 \end{pmatrix} g_2^{-1}.$$ 

By multiplying the both sides of the identity by a suitable integer, we see that there exists an $x_1 \in M_2(\mathbb{Z})$ with non-zero determinant satisfying $Y_1[x_1] = \text{det} x_1 \cdot Y_2$. We may assume that the greatest common divisors of all the entries of $x_1$ is 1. Then $x_1$ is written as $x_1 = y_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_2$ for some $y_1, y_2 \in SL_2(\mathbb{Z})$ and some non-zero integer $m$. Put $Y'_i = Y_i[y_1] (i = 1, 2)$. Then we have $Y'_1[\begin{pmatrix} 0 & m \\ 1 & 0 \end{pmatrix}] = m Y'_2$. Hence there exist integers $y_1, y_2, y_3$ satisfying $Y'_1 = \begin{pmatrix} 2 y_1 & y_2 \\ y_2 & 2 y_3 \end{pmatrix}$ and $Y'_2 = \begin{pmatrix} 2 m y_1 & y_2 \\ y_2 & 2 y_3 \end{pmatrix}$. It is easy to check that $(Y'_1, Y'_2) = \psi \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} y_2 & -y_1 \\ y_1 & 0 \end{pmatrix} \end{pmatrix}$. This implies that $x = \rho(y_1, y_2, 1) \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} y_2 & -y_1 \\ y_1 & 0 \end{pmatrix}$ satisfies

$$\psi(x) = (Y_1, Y_2).$$ □

For simplicity we put $G_0 = SL_2(\mathbb{Z})$ and $G = G_0 \times G_0 \times G_0 (\subset G(\mathbb{Q}))$.

Lemma 6. The mapping $\psi : \rho(G) \setminus (\mathcal{L} \setminus S) \to \Gamma_0^2 \setminus X_\mathbb{Z}$ induced by $\psi : \mathcal{L} \setminus S \to X_\mathbb{Z}$ is factorized as

$$\psi : \rho(G) \setminus (\mathcal{L} \setminus S) \xrightarrow{\psi(1)} \rho(\Gamma_0) \setminus (\mathcal{L} \setminus S) \setminus SL_2(\mathbb{Q}) \xrightarrow{\psi(2)} \Gamma_0^2 \setminus X_\mathbb{Z}.$$ 

Here the mapping $\psi(2)$ is a bijection.

Proof. The factorization $\psi = \psi(2) \circ \psi(1)$ and the injectivity of $\psi(2)$ are consequences of Lemma 3. The surjectivity of $\psi(2)$ is a consequence of Lemma 5. □

By Lemma 2, we have a natural isomorphism $G_2^\mathbb{Z} \xrightarrow{\zeta_{\mathbb{Z}}} SO(\psi_1(x)) \times SO(\psi_2(x))$ for any $x \in V(\mathbb{Q}) \setminus S$. Hence we may consider $G_2^\mathbb{Z}$ as a subgroup of $SO(\psi_1(x))(\mathbb{Z}) \times SO(\psi_2(x))(\mathbb{Z})$ of finite index.

For any $\rho(G)$-orbit $\{x\} \in \mathcal{L} \setminus S$, we put

$$\tau(\{x\}) = \sum_{\{v\} \in \psi(1)^{-1}(\psi(1)(\{x\}))} [SO(\psi_1(v))(\mathbb{Z}) \times SO(\psi_2(v))(\mathbb{Z}) : G_2^\mathbb{Z}(\mathbb{Z})]. \quad (1.6)$$
LEMMA 7. For \( x \in \mathcal{L} \setminus S \), put
\[
\tilde{x} = \begin{pmatrix}
x_{11}^1 & x_{12}^1 & x_{11}^2 & x_{12}^2 \\
x_{21}^1 & x_{22}^1 & x_{21}^2 & x_{22}^2 \\
\end{pmatrix} \in M_{4,2}(\mathbb{Z}),
\]
where \( x_{ij}^k \) is the \((i, j)\)-entry of \( x_k \). Let \( a_1, a_2 \) be the elementary divisors of \( \tilde{x} \). Then we have
\[
\tau(\{x\}) = \sharp(\Gamma_0 \setminus T(a_1 a_2)),
\]
where \( T(a_1 a_2) \) is the set of all matrices in \( M_{2}(\mathbb{Z}) \) with determinant \( a_1 a_2 \).

Proof. We put \( \mathcal{L}(x) = \{ h \in SL_2(\mathbb{Q}) | x^h \in \mathcal{L} \} \). Let \( p_1 : G \to SL_2 \times SL_2 \) and \( p_2 : G \to GL_2 \) be the projections defined by \( p_1(g_1, g_2, h) = (g_1, g_2) \) and \( p_2(g_1, g_2, h) = h \).

We define a subgroup \( H_{x, r} \) of \( SO(x, r) )((\mathbb{Q}) \) by
\[
H_{x, r} = p_2(p_1^{-1}(SO(\psi_1(x)) (\mathbb{Z}) \times SO(\psi_2(x))(\mathbb{Z})) \cap G_x^0).
\]

Then we obtain the following one to one correspondence:
\[
\Gamma_0 \setminus \mathcal{L}(x)/H_{x, r} \leftrightarrow \psi(1)^{-1}(\psi(1)((x))).
\]

Since \( p_1 \) induces an isomorphism of \( G_x^0 \) onto \( SO(\psi_1(x)) \times SO(\psi_2(x)) \) and
\[
G_x^0(\mathbb{Z}) = G_x^0 \cap p_1^{-1}(SO(\psi_1(x))(\mathbb{Z}) \times SO(\psi_2(x))(\mathbb{Z})) \cap p_2^{-1}(\Gamma_0),
\]
we obtain
\[
[SO(\psi_1(x))(\mathbb{Z}) \times SO(\psi_2(x))(\mathbb{Z})] : G_x^0(\mathbb{Z}) = [H_{x, r} : H_{x, r} \cap \Gamma_0].
\]

For a \( v = x^h (h \in \mathcal{L}(x)) \), we have
\[
H_{v, r} = hH_{x, r}h^{-1} \quad \text{and} \quad SO(\pi(v)) = hSO(\pi(x))h^{-1}.
\]

Hence
\[
[SO(\psi(v))(\mathbb{Z}) \times SO(\psi_2(v))(\mathbb{Z})] : G_x^0(\mathbb{Z}) = [H_{x, r} : H_{x, r} \cap h^{-1}\Gamma_0h].
\]

Thus we obtain
\[
\tau(\{x\}) = \sum_{h \in \Gamma_0 \setminus \mathcal{L}(x)/H_{x, r}} [H_{x, r} : H_{x, r} \cap h^{-1}\Gamma_0h].
\]

Since there exists a one to one correspondence
\[
H_{x, r} \cap h^{-1}\Gamma_0h \setminus H_{x, r} \leftrightarrow \Gamma_0 \setminus \Gamma_0hH_{x, r},
\]
we have
\[
\tau(\{x\}) = \sharp(\Gamma_0 \setminus \mathcal{L}(x)).
\]

Let \( a_1, a_2 \) be the elementary divisors of \( \tilde{x} \). Then \( \tilde{x} \) can be written as
\[
\tilde{x} = u \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \gamma \quad (u \in GL_4(\mathbb{Z}), \gamma \in \Gamma_0).
For an $h \in \text{SL}_2(\mathbb{Q})$, $h$ belongs to $\mathcal{L}(x)$ if and only if \(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \gamma \gamma^{-1} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \). Hence

\[ \mathcal{L}(x) = T(a_1a_2)\gamma^{-1} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^{-1}. \]

This implies that $\sharp(\Gamma_0 \setminus \mathcal{L}(x)) = \sharp(\Gamma_0 \setminus T(a_1a_2))$. This proves the lemma.

We call a $(Y_1, Y_2) \in X_\mathbb{Z}$ a primitive pair if $\frac{1}{2}(Y_1, Y_2) \not\in X_\mathbb{Z}$ for any $n \geq 2$. We denote by $X^{pr}$ the set of all primitive pairs in $X_\mathbb{Z}$.

We call an $x = (x_1, x_2) \in \mathcal{L}$ primitive if the elementary divisors of $\tilde{x} \in M_{4,2}(\mathbb{Z})$ are $\{1, 1\}$. We denote by $L^{pr}$ the set of all primitive elements of $\mathcal{L}$.

**Lemma 8.** For $x \in \mathcal{L} \setminus S$, $\psi(x)$ is a primitive pair if and only if $x$ is primitive. Hence we have a surjection $\psi : L^{pr} \to X^{pr}$.

**Proof.** Any one of the conditions $\psi(x) \in X^{pr}$ and $x \in L^{pr}$ implies that the greatest common divisor of all the entries of $x_i$ is equal to 1 for every $i = 1, 2$. Hence, if we replace $x$ by $\rho(\gamma_0, \gamma_1, \gamma_2) x$ for appropriate $\gamma_0, \gamma_1, \gamma_2 \in \Gamma_0$, we may assume that $x$ is of the form \(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \\ y_3 & 0 \end{pmatrix}\). Then we have

\[ \psi(x) = \begin{pmatrix} 2y_3 & -y_1 \\ -y_1 & -2ay_2 \end{pmatrix}, \begin{pmatrix} 2ay_3 & -y_1 \\ -y_1 & -2y_2 \end{pmatrix}. \]

Hence $\psi(x) \in X^{pr}$ if and only if

\[ \gcd(y_1, y_2, ay_2, y_3, ay_3) = \gcd(y_1, y_2, y_3) = 1. \]

This is equivalent to the condition that $x \in L^{pr}$.

**Lemma 9.** If $x \in L^{pr} \setminus S$, then we have an isomorphism \(G^*_2(\mathbb{Z}) \cong \text{SO}(\psi_1(x))(\mathbb{Z}) \times \text{SO}(\psi_2(x))(\mathbb{Z})\) and the mapping

\[ \rho(\Gamma) \setminus (L^{pr} \setminus S) \to \Gamma_0^2 \setminus X^{pr} \]

is a bijection.

**Proof.** To see this, it is sufficient to prove that $\tau(\{x\}) = 1$ for $x \in L^{pr} \setminus S$. This is obvious from Lemma 7, since $x$ is primitive.

**Remark.** The integral structure of $\mathcal{L} \setminus S$ was studied deeply by Bhargava [B]. The construction of $\psi$ is closely related to his result. In fact, our $\frac{1}{2}\pi(x), \frac{1}{2}\psi_1(x), \frac{1}{2}\psi_2(x)$ coincide essentially with Bhargava’s $Q_1, Q_2, Q_3$. 


2. Zeta functions attached to a pair of Maass cusp forms

2.1. Maass forms of weight 0 and 1/2

First we recall some basic facts on Maass wave forms of weight 0 and 1/2 following [KS].

Let \( \mathfrak{H} = \{ z \in \mathbb{C} | \text{Im} \, z > 0 \} \), the upper half-plane. Then the group \( \text{SL}_2(\mathbb{R}) \) acts on \( \mathfrak{H} \) by linear fractional transformation. We put \( \Gamma_0 = \text{SL}_2(\mathbb{Z}) \) as in §1. For \( k = 0 \) or \( 1/2 \), put

\[
\Delta_k = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - kiy \frac{\partial}{\partial x}.
\]

Let \( L^2(\Gamma_0 \backslash \mathfrak{H}) \) be the space of measurable functions on \( \Gamma_0 \backslash \mathfrak{H} \) square integrable with respect to the invariant measure \( \frac{dx \, dy}{y^2} \). Put

\[
\mathfrak{S}_0^+ (\Gamma_0 \backslash \mathfrak{H}, \lambda) = \left\{ \phi \in L^2(\Gamma_0 \backslash \mathfrak{H}) \left| \Delta_0 \phi + \lambda(1 - \lambda) \phi = 0, \phi(z) = \phi(-\overline{z}) \right. \right. \left. \int_0^1 \phi(x + iy) \, dx = 0 \right\}.
\]

A function in \( \mathfrak{S}_0^+ (\Gamma_0 \backslash \mathfrak{H}, \lambda) \) is called an even Maass cusp form (of weight 0).

Put \( \Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \left| \begin{array} {c} c \equiv 0 \pmod{4} \end{array} \right. \right\} \). For \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \), we define the automorphy factor \( J(\gamma, z) \) by

\[
J(\gamma, z) = \epsilon_d^{-1} \left( \frac{cz + d}{|cz + d|} \right)^{1/2}, \quad \epsilon_d = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ \sqrt{-1} & d \equiv 3 \pmod{4}, \end{cases}
\]

where the Legendre symbol \( \left( \frac{d}{4} \right) \) has the same meaning as in [Shm]. Let

\[
\mathfrak{S}_{1/2}^+ (\Gamma_0(4) \backslash \mathfrak{H}, \mu) = \left\{ F \in L^2(\Gamma_0(4) \backslash \mathfrak{H}) \left| \begin{array} {c} F(\gamma \cdot z) = J(\gamma, z) F(z) \ (\forall \gamma \in \Gamma_0(4)) \\ \Delta_{1/2} F + \mu(1 - \mu) F = 0, \int_0^1 F(x + iy) \, dx = 0 \end{array} \right. \right\},
\]

where

\[
LF(z) = \frac{1}{4} e^{i \pi/4} \left( \frac{z}{|z|} \right)^{-1/2} \sum_{\nu \text{mod} 4} F \left( -1 + \frac{4\nu z}{16z^2} \right).
\]

We call an \( F \in \mathfrak{S}_{1/2}^+ (\Gamma_0(4) \backslash \mathfrak{H}, \mu) \) a Maass cusp form of weight 1/2. A Maass cusp form \( F \) of weight 1/2 has a Fourier expansion of the form

\[
F(z) = \sum_{n \neq 0} \rho(n) W_{\text{sgn}(n), \mu-1/4} (4\pi |n| y) e(nz),
\]

where \( W_{\text{sgn}(n), \mu-1/4} \) is the usual Whittaker function (see [E, 6.9]).
To describe the Maass correspondence, a kind of theta liftings, between \( S^+_{0}(I_0 \setminus \mathfrak{H}, \lambda) \) and \( S^+_{1/2}(I_0(4) \setminus \mathfrak{H}, \mu) \), we need the Siegel Theta series. Put

\[
Q = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

Let \( r : \text{SL}_2(\mathbb{R}) \to \text{GL}_3(\mathbb{R}) \) be the second symmetric tensor representation:

\[
r\left(\begin{array}{ccc} a & b \\ c & d \end{array}\right) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.
\]

The image of \( \text{SL}_2(\mathbb{R}) \) coincides with the identity component of \( \text{SO}(Q)(\mathbb{R}) \).

Let

\[
\Theta(z, g) = y^{3/4} \sum_{v \in \mathbb{Z}^3} e\{ (xQ + iyR)[r(g)^{-1}v] \} \quad (z = x + iy \in \mathfrak{H}, \ g \in \text{SL}_2(\mathbb{R}))
\]

be the Siegel Theta series. Then \( \Theta(z, g) \) has the following properties:

(i) \( \Theta(y \cdot z, g) = J(y, z)\Theta(z, g), \ y \in I_0(4); \)

(ii) \( \Theta(z, ygk) = \Theta(z, g), \ (y \in I_0, \ k \in \text{SO}(2)(\mathbb{R})); \)

(iii) \( \Theta(z, \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}) \) is an even function of \( \xi \). For a \( \Phi \in S^+_{0}(I_0 \setminus \mathfrak{H}, \lambda) \), put

\[
\Theta(\Phi)(z) = \int_{I_0(4) \setminus \text{SL}_2(\mathbb{R})} \Phi(g \cdot \sqrt{-1})\Theta(z, g) \, dg.
\]

Then, it is known that \( \Theta(\Phi) \) is in \( S^+_{1/2}(I_0(4) \setminus \mathfrak{H}, \mu) \) for \( \mu = \frac{2k+1}{4} \) (see [KS, Proposition 2.3]).

Let

\[
\Theta(\Phi)(z) = \sum_{n \neq 0} \rho(n)W_{\frac{1}{2}+\text{sgn}(n), \mu-\frac{1}{4}}(4\pi |n| y)e[nx]
\]

be the Fourier expansion. To describe the Fourier coefficients \( \rho(n) \), it is necessary to fix the normalization of Haar measures on \( \text{SO}(Y)(\mathbb{R}) \) for nondegenerate \( Y \in \text{Sym}_2(\mathbb{R}) \). Let \( h_Y \) be an element in \( \text{SL}_2(\mathbb{R}) \) such that

\[
Y = \begin{cases} 
\pm h_Y \begin{pmatrix} \sqrt{\det Y} & 0 \\ 0 & \sqrt{\det Y} \end{pmatrix}^{t} h_Y & (\text{Disc } Y < 0), \\
h_Y \begin{pmatrix} 0 & \sqrt{\det Y} \\ \sqrt{\det Y} & 0 \end{pmatrix}^{t} h_Y & (\text{Disc } Y > 0).
\end{cases}
\]

Then, putting

\[
k(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix},
\]

(2.1)
we have
\[ \text{SO}(Y)(\mathbb{R}) = \begin{cases} h_Y \{ k(\theta) \mid \theta \in \mathbb{R}/(2\pi \mathbb{Z}) \} h_Y^{-1} & (\text{Disc } Y < 0), \\ h_Y \{ \pm a(y) \mid y > 0 \} h_Y^{-1} & (\text{Disc } Y > 0) \end{cases} \]
and we normalize the Haar measure \( d\mu_Y \) on \( \text{SO}(Y)(\mathbb{R}) \) by
\[
\begin{align*}
\int \phi(h_Y k(\theta) h_Y^{-1}) d\mu_Y(\theta) &= \int \phi(\pm h_Y a(y) h_Y^{-1}) \frac{dy}{\sqrt{\pi}} \\
\int_{\text{Disc } Y < 0}^{\text{Disc } Y > 0} &\quad (\text{Disc } Y < 0), \\
\int_{\text{Disc } Y > 0}^{\text{Disc } Y < 0} &\quad (\text{Disc } Y > 0).
\end{align*}
\]
Then, under a suitable normalization of the Haar measure \( d\sigma \) on \( \text{SL}_2(\mathbb{R}) \), we have
\[ \rho(n) = |n|^{-3/4} \sum_{\substack{Y \in \Gamma_0 \setminus \text{Sym}^2(\mathbb{Z}) \\text{Disc } Y = n}} M_{\phi}(Y), \quad (2.2) \]
where
\[ M_{\phi}(Y) = \int_{\text{SO}(Y)(\mathbb{R})} \phi(h_Y h_Y^{-1} h) d\mu_Y(h) \times \begin{cases} 1 & (n > 0), \\ \frac{1}{\sqrt{2\pi}} & (n < 0). \end{cases} \]
Note that
\[ M_{\phi}(mY) = M_{\phi}(Y) \quad (m \in \mathbb{Z} \setminus \{0\}). \quad (2.3) \]
We need a little bit more general periods for later use. For \( Y, Y' \) with \( \text{sgn } Y = \text{sgn } Y' \), put
\[ M_{\phi}(Y, Y') = \int_{\text{SO}(Y)(\mathbb{R})} \phi(h_Y h_Y'^{-1} h) d\mu_Y(h). \]
As is proved in [S3, Lemma 6.3], the period \( M_{\phi}(Y, Y') \) is a spherical function as a function of \( Y' \). We define the spherical functions \( \Psi_{\lambda}(Y) (\lambda \in \mathbb{C}) \) on the space of nondegenerate real symmetric matrices of size 2 by
\[
\Psi_{\lambda}(Y) = \begin{cases} P_{-\lambda}(\cosh(\log |\alpha|)) & (\text{Disc } Y < 0), \\ P_{-\lambda/2}^{1/2}(\tanh(\log |\alpha|)) + P_{-\lambda/2}^{1/2}(-\tanh(\log |\alpha|)) & (\text{Disc } Y > 0), \\ \cosh(\log |\alpha|)^{1/2} \end{cases}
\]
where \( \alpha \) is an eigenvalue of \( |\det Y|^{-1/2}Y \), \( P_{-\lambda}(z) \) denotes the Legendre function, which is given by
\[ P_{-\lambda}(z) = F \left( \lambda, 1 - \lambda; 1, \frac{1 - z}{2} \right), \]
and \( P_{-\lambda/2}^{1/2}(z) \) denotes the associated Legendre function, which is given by
\[ P_{-\lambda/2}^{1/2}(z) = \frac{1}{\Gamma\left(\frac{1}{2} - \lambda\right)} \left( 1 + z \right)^{-\lambda-1/4} \cdot F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2} - \lambda, \frac{1 - z}{2} \right). \]
The functions $\Psi_{\lambda}(Y)$ does not depend on the choice of $\alpha$ and have the following integral representations:

$$\Psi_{\lambda}(Y) = |\det Y|^{\lambda/2} \int_{0}^{2\pi} |Y[k_0]|_{11}^{-\lambda} \, d\theta \times \left\{ \begin{array}{ll} \frac{1}{2\pi} & (\text{Disc} \, Y < 0) , \\ \frac{1}{\sqrt{2\pi} \Gamma(1-\lambda)} & (\text{Disc} \, Y > 0) . \end{array} \right.$$ \hspace{1cm} (2.4)

The integral representation of $\Psi_{\lambda}(Y)$ with $\text{Disc} \, Y < 0$ is absolutely convergent for all $\lambda \in \mathbb{C}$ and that of $\Psi_{\lambda}(Y)$ with $\text{Disc} \, Y > 0$ is absolutely convergent for $\text{Re}(\lambda) < 0$ and analytically continued to a meromorphic function of $\lambda$.

**Lemma 10 ([S3, Lemma 6.3]).** We have

$$\begin{cases} M_{\Phi}(Y, Y') = 4\sqrt{\pi} \cdot M_{\Phi}(Y)\Psi_{\lambda}(Y') & (\text{Disc} \, Y, \text{Disc} \, Y' < 0) , \\
M_{\Phi}(Y, Y') + M_{\Phi}(-Y, Y') \over 2 = \frac{\Gamma(1-\lambda/2)^2}{2^{\lambda+1/2}\sqrt{\pi}} \cdot M_{\Phi}(Y)\Psi_{\lambda}(Y') & (\text{Disc} \, Y, \text{Disc} \, Y' > 0) . \end{cases}$$

### 2.2. Zeta functions

For $\Phi_1 \in \mathcal{S}_0^+(I_0 \setminus \mathfrak{f}, \lambda_1)$, $\Phi_2 \in \mathcal{S}_0^+(I_0 \setminus \mathfrak{f}, \lambda_2)$ and $f \in \mathcal{S}(V(\mathbb{R}))$, we define the zeta integral $Z(f; \Phi_1, \Phi_2; s)$ by setting

$$Z(f; \Phi_1, \Phi_2; s) = \int_{0}^{\infty} t^{2s-1} \, dt \int_{\text{SL}_2(\mathbb{R})/I_0} \phi_1(g_1) \, d\mu_{g_1} \int_{\text{SL}_2(\mathbb{R})/I_0} \phi_2(g_2) \, d\mu_{g_2} \times \sum_{x \in \mathcal{C}} f(tp(g_1, g_2, h)x) \, dh ,$$

where $\phi_1(g_i) = \Phi_1(g_i^{-1} \cdot \sqrt{-1}) \, (i = 1, 2)$. Recall that $G_{\kappa}(\mathbb{R}) \ni k = (k_1, k_2, k_3) \mapsto (k_1, k_2) \in \text{SO}(\psi_1(x))(\mathbb{R}) \times \text{SO}(\psi_2(x))(\mathbb{R})$ is an isomorphism. We normalize the Haar measure $d\mu_x$ on $G_{\kappa}(\mathbb{R})$ by

$$d\mu_x(k) = d\mu_x(k_1, k_2, k_3) = d\mu_{\psi_1}(k_1) d\mu_{\psi_2}(k_2) .$$

We can normalize the Haar measures $d\mu_{g_1}, d\mu_{g_2}, d\mu_{h}$ appearing in the zeta integral above so that the following integral formula holds for any $x \in V_{ij}$ and any $F \in L^1(\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R}))$:

$$\int_{0}^{\infty} dt \int_{\text{SL}_2(\mathbb{R})} F(g_1, g_2, th) \, d\mu_{g_1} \, d\mu_{g_2} \, dh = \int_{V_{ij}} |P(y)|^{-2} \, dy \int_{G_{\kappa}(\mathbb{R})} F(g_k) \, d\mu_x(k) ,$$

where $g_k$ is an element in $G^+$ such that $\rho(g_k)x = y$.

**Lemma 11.** The zeta integral $Z(f; \Phi_1, \Phi_2; s)$ is absolutely convergent for $\text{Re}(s) > 2$.

**Proof.** Put

$$k(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} , \quad a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} , \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} .$$
Then, as a fundamental domain of $SL_2(\mathbb{R})$ for $\Gamma_0$, we may take

$$\mathcal{F} := \left\{ g = k(\theta)a(y^{-1})m(-x) \mid 0 < \theta < \pi, \ |x| \leq \frac{1}{2}, \ x^2 + y^2 \geq 1 \right\}.$$ 

Since $\Phi_1$ and $\Phi_2$ are assumed to be cuspidal, for any $\alpha > 0$ there exist positive constants $C_1, C_2$ satisfying

$$|\phi_i(g)| = |\Phi_i(x + y\sqrt{-1})| < C_i \ y^{-\alpha}, \quad (g = k(\theta)a(y^{-1})m(-x) \in \mathcal{F}, \ i = 1, 2).$$

Take an $f_0 \in \mathcal{S}(V(\mathbb{R}))$ such that the inequality

$$|f(\rho(k(\theta_1)n(x_1), k(\theta_2)n(x_2), k(\theta_3)n(x_3))v)| \leq f_0(v)$$

holds for any $\theta_1, \theta_2, \theta_3 \in \mathbb{R}, |x_1|, |x_2|, |x_3| \leq 1$ and $v \in V(\mathbb{R})$. Then, for any $g_1, g_2, g_3 \in \mathcal{F}$ and $t > 0$, we have

$$|f(\rho(g_1, g_2, g_3)v)| \leq f_0(\rho(a(y_1^{-1}), a(y_3^{-1}), a(y_3^{-1}))v).$$

For $\beta > 0$, there exists a positive constant $C_\beta$ satisfying

$$|f_0(v)| \leq C_\beta (\text{tr}^t vv)^{-\beta} \quad (v \neq 0).$$

Then we have

$$|f_0(\rho(a(y_1^{-1}), a(y_2^{-1}), a(y_3^{-1}))v)| \leq C_\beta t^{-2\beta} \sum_{i,j,k=1,2} y_1^{(i-1)y_2^{-y(i)}y_3^{(k-1)}} (v_k^{(i)})^2.$$ 

When $y_1, y_2 \geq 1/2$, we have $y_1^{\pm 1}, y_2^{\pm 1} \geq 2^{-4} y_1^{-1} y_2^{-1}$. Hence

$$|f_0(\rho(a(y_1^{-1}), a(y_2^{-1}), a(y_3^{-1}))v)| \leq 2^{4\beta} C_\beta t^{-2\beta} (y_1 y_2)^\beta \left( y_3^{-1} ||v_1||^2 + y_3 ||v_2||^2 \right)^{-\beta},$$

where we put $||v_k|| = \left( \sum_{i,j=1,2} (v_k^{(i)})^2 \right)^{1/2}$. If $v \not\in S$, then neither $v_1$ nor $v_2$ is equal to 0 and we obtain

$$|f_0(\rho(a(y_1^{-1}), a(y_2^{-1}), a(y_3^{-1}))v)| \leq 2^{3\beta} C_\beta t^{-2\beta} (y_1 y_2)^\beta ||v_1||^{-\beta} ||v_2||^{-\beta} \quad (v \in \mathcal{L} \setminus S).$$

Therefore we have the inequality

$$\sum_{v \in \mathcal{L} \setminus S} |f_0(\rho(a(y_1^{-1}), a(y_2^{-1}), a(y_3^{-1}))v)| \leq 2^{3\beta} C_\beta t^{-2\beta} (y_1 y_2)^\beta \left( \sum_{v_1 \in M_2(\mathbb{Z}) \setminus \{0\}} ||v_1||^{-\beta} \right)^2.$$ 

The infinite sum with respect to $v_1$ is the Epstein zeta function of the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ in 4 variables and is absolutely convergent for $\text{Re}(\beta) > 4$. Now assume that $s > 2$ and take positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfying

$$2s < \beta_1 < \alpha_1 + 1, \quad 2s > \beta_2 > 4, \quad \alpha_2 > \beta_2 - 1.$$
Then the integral \( Z(f; \Phi_1, \Phi_2; s) \) is dominated by

\[
C' \left( \sum_{n \in \mathbb{Z} \setminus \{0\} \atop \|v_1\|^2 = 1} \right)^2 \left( \int_1^\infty \frac{t^{4s-2\beta_1}}{t} \int_{1/2}^\infty \frac{y_1^{\beta_1-\alpha_1}}{y_1^2} \int_{1/2}^\infty \frac{d\alpha_1}{y_1^2} \int_{1/2}^\infty \frac{d\beta_1}{y_1^2} \right) + C'' \left( \sum_{n \in \mathbb{Z} \setminus \{0\} \atop \|v_1\|^2 = 1} \right)^2 \int_0^1 \frac{t^{4s-2\beta_2}}{t} \int_{1/2}^\infty \frac{y_1^{\beta_2-\alpha_2}}{y_1^2} \int_{1/2}^\infty \frac{d\alpha_2}{y_1^2} \int_{1/2}^\infty \frac{d\beta_2}{y_1^2}
\]

for some positive constants \( C', C'' \). From the choice of the constants \( \alpha_1, \alpha_2, \beta_1, \beta_2 \), this implies the convergence of the integral \( Z(f; \Phi_1, \Phi_2; s) \).

We define the local zeta functions with spherical functions as follows:

\[
\zeta_{ij}^{(\infty)}(f; \lambda_1, \lambda_2; s) = \int_{V_{ij}} |P(v)|^{s/2} \Psi_{\lambda_1}(\psi_1(v)) \Psi_{\lambda_2}(\psi_2(v)) f(v) \, dv
\]

\[
(f \in \mathcal{S}(V(\mathbb{R})), \, i, j = 0, 1, 2)
\]

(for the definition of \( V_{ij} \), see (1.5)). The integrals on the right hand side are absolutely convergent if \( \text{Re}(s) > 2 - \frac{\text{Re}(\lambda_1) + \text{Re}(\lambda_2)}{2} \) and \( \text{Re}(\lambda_1), \text{Re}(\lambda_2) < 0 \), and have analytic continuations to meromorphic functions of \( (\lambda_1, \lambda_2, s) \) in \( \mathbb{C}^3 \). We also put

\[
\zeta_+^{(\infty)}(f; \lambda_1, \lambda_2; s) = \zeta_{11}^{(\infty)}(f; \lambda_1, \lambda_2; s), \quad \zeta_-^{(\infty)}(f; \lambda_1, \lambda_2; s) = \sum_{i,j=0,2} \zeta_{ij}^{(\infty)}(f; \lambda_1, \lambda_2; s).
\]

This kind of zeta integrals and local zeta functions were examined in [S3]. By [S3, Proposition 5.2], there exist certain Dirichlet series of the form

\[
\zeta_{\pm}(\Phi_1, \Phi_2; s) = \sum_{x \in \rho(f) \setminus \mathcal{L}_{\pm}} M_{\Phi_1, \Phi_2}(x) \left| \frac{P(x)}{x} \right|^s,
\]

where the coefficients \( M_{\Phi_1, \Phi_2}(s) \) are period integrals of \( \Phi_1 \Phi_2 \) over \( \mathcal{G}_s^\phi(\mathbb{R})/\mathcal{G}_s^\phi(\mathbb{Z}) \), and they satisfy the identity

\[
Z(f; \Phi_1, \Phi_2; s) = \zeta_+(\Phi_1, \Phi_2; s) \zeta_{\pm}^{(\infty)}(f; \lambda_1, \lambda_2; s) + \zeta_-(\Phi_1, \Phi_2; s) \zeta_{\pm}^{(\infty)}(f; \lambda_1, \lambda_2; s).
\]

(2.5)

Our main result is a formula expressing the zeta functions \( \zeta_{\pm}(\Phi_1, \Phi_2; s) \) in terms of the Maass lifts \( \Theta(\Phi_1) \) and \( \Theta(\Phi_2) \).

Let \( F_1 \in \mathcal{S}_{1/2}^+(\Gamma_0(4) \setminus \mathcal{L}, \mu_1) \) (\( \mu_1 = \frac{3i+1}{4} \)) and \( F_2 \in \mathcal{S}_{1/2}^+(\Gamma_0(4) \setminus \mathcal{L}, \mu_2) \) (\( \mu_2 = \frac{3i+1}{4} \)) be Maass cusp forms of weight 1/2. The Maass forms \( F_1 \) and \( F_2 \) have the Fourier
expansion of the following form:

\[
F_1(z) = \sum_{n \neq 0} \rho_1(n) W_{\frac{1}{2} \text{sgn}(n), \mu_1 - \frac{1}{2}} (4\pi |n| y) e[nx].
\]

\[
F_2(z) = \sum_{n \neq 0} \rho_2(n) W_{\frac{1}{2} \text{sgn}(n), \mu_2 - \frac{1}{2}} (4\pi |n| y) e[nx].
\]

Using the Fourier coefficients of \(F_1\) and \(F_2\), we define Dirichlet series \(D_{\pm}(F_1, F_2; s)\) as follows:

\[
D_{\pm}(F_1, F_2; s) = \sum_{n=1}^{\infty} \frac{\rho_1(\pm n) \rho_2(\pm n)}{n^s}. 
\] (2.6)

Then our main theorem is the following:

**Theorem 1.** We have

\[
\zeta_+(\Phi_1, \Phi_2; s) = \frac{\Gamma(1 - \frac{2s}{\lambda_1 + 1}) \Gamma(1 - \frac{2s}{\lambda_2 + 1})}{2^s \pi^{\frac{1}{2}}} \cdot \zeta(2s - 1) D_+ \left( \Theta(\Phi_1), \Theta(\Phi_2); s - \frac{3}{2} \right),
\]

\[
\zeta_-(\Phi_1, \Phi_2; s) = 4\pi \cdot \zeta(2s - 1) D_- \left( \Theta(\Phi_1), \Theta(\Phi_2); s - \frac{3}{2} \right).
\]

For the proof of Theorem 1, we introduce the following auxiliary zeta integral, the contribution of the primitive part \(L^{\text{pr}}\):

\[
Z_{\text{pr}}(f; \Phi_1, \Phi_2; s) = \int_0^\infty t^{4s-1} dt \int_{\text{SL}_2(\mathbb{R})/\Gamma_0} \phi_1(g_1) dg_1 \int_{\text{SL}_2(\mathbb{R})/\Gamma_0} \phi_2(g_2) dg_2
\]

\[
\times \int_{\text{SL}_2(\mathbb{R})/\Gamma_0} \sum_{x \in \mathcal{L}_{\text{pr}}} f(t \rho(g_1, g_2, h)x) dh .
\]

**Lemma 12.** If \(\text{Re}(s) > 2\), we have

\[
Z(f; \Phi_1, \Phi_2; s) = \zeta(2s) \zeta(2s - 1) Z_{\text{pr}}(f; \Phi_1, \Phi_2; s).
\]

**Proof.** Put \(\Delta = \{ T \in \mathcal{M}_2(\mathbb{Z}) \mid \det T > 0 \} \). Then it is easy to see that the mapping

\[
\mathcal{L}^{\text{pr}} \times \Delta/\Gamma_0 \ni (x, T) \mapsto \rho(1, 1, T)x \in \mathcal{L} \setminus \mathbf{S}
\]

is a bijection. Hence the lemma is an immediate consequence of the following lemma. \(\square\)

**Lemma 13.** Let \(f\) be a right \(\text{GL}_2(\mathbb{Z})\)-invariant measurable function on \(\text{GL}_2(\mathbb{R})\). Then the identity

\[
\int_0^\infty t^{4s-1} dt \int_{\text{SL}_2(\mathbb{R})/\Gamma_0} \sum_{T \in \Delta/\Gamma_0} f(tT) dh
\]

\[
= \zeta(2s) \zeta(2s - 1) \int_0^\infty t^{4s-1} dt \int_{\text{SL}_2(\mathbb{R})/\Gamma_0} f(tT) dh
\]

holds, if \(\text{Re}(s) > 1\) and the integral on the left hand side of the identity is absolutely convergent.
Proof. For a positive integer \( n \), put \( \Delta_n = \{ T \in \Delta \mid \det T = n \} \). Let \( \Delta_n = \bigcup \delta_i \Gamma_0 \) be the left coset decomposition of \( \Delta_n \). We also put \( \Gamma_{\Delta, n} = \Gamma_0 \cap \left( \bigcap_i \delta_i \Gamma_0 \delta_i^{-1} \right) \). Then we have

\[
\int_0^\infty t^{4s-1} \frac{dt}{\text{SL}_2(\mathbb{R})/\Gamma_0} \sum_{T \in \Delta_n/\Gamma_0} f(\theta T) \, dh
\]

\[= \frac{1}{[\Gamma_0 : \Gamma_{\Delta, n}]} \int_0^\infty t^{4s-1} \frac{dt}{\text{SL}_2(\mathbb{R})/\Gamma_{\Delta, n}} \sum_i f(\theta \delta_i) \, dh \]

\[= \frac{1}{n^{2s}} [\Gamma_0 : \Gamma_{\Delta, n}] \sum_i \int_0^\infty t^{4s-1} \frac{dt}{\text{SL}_2(\mathbb{R})/\Gamma_{\Delta, n}} f(\theta) \, dh \]

\[= \frac{1}{n^{2s}} \left( \sum_i \left[ \frac{[\Gamma_0 : \delta_i^{-1} \Gamma_{\Delta, n} \delta_i]}{[\Gamma_0 : \Gamma_{\Delta, n}]} \right] \right) \int_0^\infty t^{4s-1} \frac{dt}{\text{SL}_2(\mathbb{R})/\Gamma_0} f(\theta) \, dh \]

Since \( [\Gamma_0 : \delta_i^{-1} \Gamma_{\Delta, n} \delta_i] = 1 \), we obtain

\[
\int_0^\infty t^{4s-1} \frac{dt}{\text{SL}_2(\mathbb{R})/\Gamma_0} \sum_{T \in \Delta_n/\Gamma_0} f(\theta T) \, dh
\]

\[= \frac{\nu(\Delta_n/\Gamma_0)}{n^{2s}} \int_0^\infty t^{4s-1} \frac{dt}{\text{SL}_2(\mathbb{R})/\Gamma_0} f(\theta) \, dh \]

Hence the lemma follows from the identity

\[
\sum_{n=1}^\infty \frac{\nu(\Delta_n/\Gamma_0)}{n^{2s}} = \sum_{n=1}^\infty \left( \sum_{d|n} d \right) n^{-2s} = \zeta(2s) \zeta(2s - 1) \quad \square
\]

Proof of Theorem 1. By Lemma 12, it is enough to calculate \( Z_{pf}(f; \Phi_1, \Phi_2; s) \). We have

\[
Z_{pf}(f; \Phi_1, \Phi_2; s) = \sum_{i, j} \sum_{x \in \Gamma \cap \mathcal{D}'(\mathbb{R}) \setminus \mathcal{V}_j} \frac{\text{SO}(\psi_1(x)(\mathbb{Z})) \times \text{SO}(\psi_2(x)(\mathbb{Z})) : G_2(\mathbb{Z})}{|P(x)|^s}
\]

\[\times \int_{\mathcal{V}_j} |P(y)|^{s-2} f(y) \, dy \int_{\text{SO}(\psi_1(x)(\mathbb{Z}))} \phi_1(g_1, y, k_1) \, d\mu_{\psi_1(x)}(k_1) \]

\[\times \int_{\text{SO}(\psi_2(x)(\mathbb{Z}))} \phi_2(g_2, y, k_2) \, d\mu_{\psi_2(x)}(k_2) \]

(2.7)

where \( g_1, y, g_2, y \) are elements in \( \text{SL}_2(\mathbb{R}) \) for which there exists an \( h_y \in \text{GL}_2(\mathbb{R}) \) satisfying \( \rho(g_1, y, g_2, y, h_y) x = y \). Note that we may take \( g_1, y = h_{\psi_1(y)} h_{\psi_1(x)}^{-1} \) and \( g_1, y = h_{\psi_2(y)} h_{\psi_2(x)}^{-1} \) (for the definition of \( h_y \), see (2.1)). Therefore the integral with respect to \( k_1 \) (resp. \( k_2 \)) is
equal to $M_{\Phi_1}(\psi_1(x), \psi_1(y))$ (resp. $M_{\Phi_2}(\psi_2(x), \psi_2(y))$). Put

$$X_{ij}^p(n) = \begin{cases} (Y_1, Y_2) \in X^p & \text{Disc } Y_1 = \text{Disc } Y_2 = (-1)^{-1}n, \\ \text{sgn } Y_1 = (i, 2 - i), \ \text{sgn } Y_2 = (j, 2 - j) \end{cases}.$$ \hfill (2.3)

Then, by Lemmas 9 and 10, the right hand side of (2.7) is equal to

$$\sum_{i,j} \sum_{n=1}^{\infty} n^{-s} \sum_{(Y_1, Y_2) \in I_G^0 \setminus X_{ij}^p(n)} \int_{V_{ij}} |P(y)|^{s-2} M_{\Phi_1}(Y_1, \psi_1(y)) M_{\Phi_2}(Y_2, \psi_2(y)) f(y) \, dy$$

$$= 2^4 \pi \sum_{i,j=0,2} \zeta_{ij}^{\text{pr}}(\Phi_1, \Phi_2; s) \zeta_{ij}^{(\infty)}(f; \lambda_1, \lambda_2; s)$$

$$+ \frac{\Gamma(1 - \frac{i}{4})^2 \Gamma(1 - \frac{j}{4})^2}{2^{i+1} \pi} \lambda_{ij}^{\text{pr}}(\Phi_1, \Phi_2; s) \lambda_{ij}^{(\infty)}(f; \lambda_1, \lambda_2; s),$$

where

$$\lambda_{ij}^{\text{pr}}(\Phi_1, \Phi_2; s) = \sum_{n=1}^{\infty} n^{-s} \sum_{(Y_1, Y_2) \in I_G^0 \setminus X_{ij}^p(n)} M_{\Phi_1}(Y_1) M_{\Phi_2}(Y_2).$$

By (2.2), (2.3) and (2.6), we have

$$\lambda_{ij}^{\text{pr}}(\Phi_1, \Phi_2; s) = \begin{cases} \frac{1}{\xi(2s)} \cdot D_+(\Theta(\Phi_1), \Theta(\Phi_2); s - \frac{3}{2}) & (i = j = 1), \\ \frac{1}{4\xi(2s)} \cdot D_-(\Theta(\Phi_1), \Theta(\Phi_2); s - \frac{3}{2}) & (i, j = 0, 2). \end{cases}$$

Therefore we obtain

$$Z_{\text{pr}}(f; \Phi_1, \Phi_2; s)$$

$$= \frac{(1 - \frac{i}{4})^2 \Gamma(1 - \frac{j}{4})^2}{2^{i+1} \pi} \cdot D_+(\Theta(\Phi_1), \Theta(\Phi_2); s - \frac{3}{2}) \cdot \zeta_{ij}^{(\infty)}(f; \lambda_1, \lambda_2; s)$$

$$+ 4\pi \cdot D_-(\Theta(\Phi_1), \Theta(\Phi_2); s - \frac{3}{2}) \cdot \zeta_{ij}^{(\infty)}(f; \lambda_1, \lambda_2; s).$$

This proves the theorem. \hfill \Box

### 2.3. Functional equation

We identify the vector space $V^*$ dual to $V$ with $V$ via the bilinear form

$$\langle x, y \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle.$$ 

Then the contragredient representation $\rho^*$ is given by

$$\rho^* (g_1, g_2, h)(x_1, x_2) = (g_1 x_1 g_2^{-1}, g_1 x_2 g_2^{-1}) h^{-1}$$

and $P(x)$ is the fundamental relative invariant of the dual prehomogeneous vector space $(G, \rho^*, V)$. Therefore the local zeta functions and the global zeta functions associated with $(G, \rho^*, V)$ coincide with $\zeta_{ij}^{(\infty)}(f; \lambda_1, \lambda_2; s)$ and $\zeta_{ij}(\Phi_1, \Phi_2; s)$, respectively. Moreover, if we replace $G$ with $G' = \text{SL}_2 \times \text{SL}_2 \times \text{GL}_2$, then $(G', \rho, V)$ is a prehomogeneous...
vector space equivalent to \((G, \rho, V)\) and \(G' = L \times U\) (\(L = SU_2 \times SU_2 \times GL_1, U = SU_2\)) determines a symmetric structure in the sense of [S3, §4]. Hence the general theory developed in [S3] can apply to our zeta functions attached to a pair of Maass cusp forms and the local zeta functions \(\xi^{(\infty)}_\pm (f; \lambda_1, \lambda_2; s)\) and the zeta functions \(\xi_\pm (\Phi_1, \Phi_2; s)\) satisfy certain functional equations ([S3, Theorems 5.3, 5.4]).

**Theorem 2.** The following functional equation holds for any \(f \in \mathcal{S}(V(\mathbb{R}))\):

\[
\begin{bmatrix}
\xi_\pm^{(\infty)}(f; \lambda_1, \lambda_2; s) \\
\xi_\pm^{(\infty)}(f; \lambda_1, \lambda_2; s)
\end{bmatrix}
= \Gamma(\lambda_1, \lambda_2; s)
\begin{bmatrix}
\xi_\pm^{(\infty)}(f; \lambda_1, \lambda_2; 2-s) \\
\xi_\pm^{(\infty)}(f; \lambda_1, \lambda_2; 2-s)
\end{bmatrix},
\]

where

\[
\Gamma(\lambda_1, \lambda_2; s) = 2^{-1} \pi^{-4s+2} \Gamma\left(s + \frac{\lambda_1 + \lambda_2}{2} - 1\right) \Gamma\left(s + \frac{\lambda_1 - \lambda_2 - 1}{2}\right) \Gamma\left(s + \frac{\lambda_2 - \lambda_1 - 1}{2}\right) \Gamma\left(s - \frac{\lambda_1 + \lambda_2}{2}\right)
\times
\left(\begin{array}{c}
2 \pi \Gamma(1-\lambda_2) \Gamma(1-\lambda_2) \\
\sin \pi \lambda_1 + \sin \pi \lambda_2
\end{array}\right),
\]

Proof. Let \(B_2\) be the subgroup of \(SL_2\) consisting of all upper triangular matrices in \(SL_2\) and consider the subgroup \(G' = B_2 \times B_2 \times GL_2\) of \(G\). Then, \((G', \rho, V)\) with the smaller group \(G'\) is still a prehomogeneous vector space and the fundamental relative invariants are given by

\[
P(x) = \det \pi(x), \quad P_1(x) = \psi_1(x)_{11}, \quad P_2(x) = \psi_2(x)_{11}.
\]

The local zeta functions of \((G', \rho, V)\) are given by the integrals

\[
\Xi_\pm(f; \mu_1, \mu_2; s) = \int_{V'_x} |P(x)|^{s/2} |P_1(x)|^{\mu_1} |P_2(x)|^{\mu_2} f(x) \, dx \quad (f \in \mathcal{S}(V(\mathbb{R}))),
\]

where \(V'_x = \{x \in V(\mathbb{R}) | \pm P(x) > 0, \ P_1(x) P_2(x) \neq 0\}\). By the integral representation (2.4) of the spherical function \(\Phi_\lambda\), we have

\[
\xi_\pm^{(\infty)}(f; \lambda_1, \lambda_2; s) = \Xi_\pm(\mathcal{K}(f); -\lambda_1, -\lambda_2; s + \frac{\lambda_1 + \lambda_2}{2}) \times \left\{\begin{array}{c}
\frac{2 \pi \Gamma(1-\lambda_2) \Gamma(1-\lambda_2)}{\sin \pi \lambda_1 + \sin \pi \lambda_2} \quad (+\text{case}), \\
\frac{1}{4\pi^2} \quad (-\text{case}).
\end{array}\right.
\]

Thus the theorem is reduced to the functional equation of the local zeta functions of \((G', \rho, V)\), which is a special case of the functional equations given in [S2, Theorem 4.1]. We omit further details of the explicit calculation. \(\Box\)

The following global functional equation is an immediate consequence of Theorem 2 and [S3, Theorem 5.4].
THEOREM 3. The zeta functions \( \zeta_{\pm} (\Phi_1, \Phi_2; s) \) have analytic continuations to meromorphic functions of \( s \). Moreover, if we put

\[
\xi_{\pm} (\Phi_1, \Phi_2; s) = \pi^{-2s} \Gamma (s + \frac{\lambda_1 - \lambda_2 - 1}{2}) \Gamma (s - \frac{\lambda_1 + \lambda_2}{2}) \zeta_{\pm} (\Phi_1, \Phi_2; s),
\]

then they satisfy the following functional equation:

\[
\left( \frac{\xi_+ (\Phi_1, \Phi_2; 2 - s)}{\xi_- (\Phi_1, \Phi_2; 2 - s)} \right) = C (\lambda_1, \lambda_2; s) \left( \frac{\xi_+ (\Phi_1, \Phi_2; s)}{\xi_- (\Phi_1, \Phi_2; s)} \right),
\]

where

\[
C (\lambda_1, \lambda_2; s) = \frac{1}{\sin \pi (s + \frac{\lambda_1 + \lambda_2}{2}) \cos \pi (s + \frac{\lambda_1 - \lambda_2}{2})}
\times
\left( \frac{\pi}{\sin \pi s \sin \frac{\pi \lambda_1}{2} \sin \frac{\pi \lambda_2}{2}} \frac{\Gamma (2 - \lambda_1) \Gamma (2 - \lambda_2)}{\sin 2\pi s + \sin \pi \lambda_1 + \sin \pi \lambda_2} \frac{\Gamma (1 - \lambda_1) \Gamma (1 - \lambda_2)}{\pi \sin \pi \lambda_1 \sin \pi \lambda_2} \right).
\]

References


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