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Certain Vector Valued Siegel Modular Forms of Half Integral Weight and Degree Two

by

Ryuji TSUSHIMA

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To the memory of Professor Tsuneo Arakawa

1. Introduction

Let $\mathfrak{S}_g = \{Z \in M_g(\mathbf{C}) \mid {}^tZ = Z,\ \mathrm{Im}\, Z > 0\}$ be the Siegel upper half plane of degree g, $\Gamma_g = Sp\,(g, \mathbf{Z})$ the Siegel modular group of degree g and

$$\Gamma_g^* = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; \text{diagonal elements of } A\,{}^tB,\ C\,{}^tD \text{ are even} \right\} .$$

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we denote $(AZ + B)(CZ + D)^{-1}$ by $M\langle Z\rangle$. Let $\mathbf{e}(z) = \exp(2\pi i z)$ and for $Z \in \mathfrak{S}_g$ put

$$\theta(Z) = \sum_{\eta \in \mathbf{Z}^g} \mathbf{e}\left(\frac{1}{2}{}^t\eta Z \eta\right) .$$

If M belongs to Γ_g^*, $\theta(M\langle Z\rangle)/\theta(Z)$ is holomorphic on $\mathfrak{S}_g$. Let $\alpha = \begin{pmatrix} 2\cdot 1_g & O \\ O & 1_g \end{pmatrix}$ and let $\Theta(Z) = \theta(2Z) = \theta(\alpha\langle Z\rangle)$. Let

$$\Gamma_0^g(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; C \equiv O \pmod 4 \right\} .$$

Then $\Gamma_g^\alpha := \alpha^{-1}\Gamma_g^*\alpha \cap \Gamma_g$ contains $\Gamma_0^g(4)$. Hence if M belongs to $\Gamma_0^g(4)$ or more generally if M belongs to Γ_g^α, then $J(M, Z) := \Theta(M\langle Z\rangle)/\Theta(Z)$ is holomorphic on $\mathfrak{S}_g$ and satisfies the equality:

$$J(M, Z)^2 = \det(CZ + D)\psi(\det D) ,$$

where $\psi : 1 + 2\mathbf{Z} \to \{\pm 1\}$ is the non-trivial Dirichlet character modulo 4. $J(M, Z)$ is *the automorphy factor of weight* $1/2$.

DEFINITION 1.1. Let 1_g be the unit matrix of degree g and $J_g = \begin{pmatrix} O & 1_g \\ -1_g & O \end{pmatrix}$. Let

$$G_g = \left\{ M \in GL(2g, \mathbf{R}) \;\middle|\; {}^tMJ_gM = \nu(M)J_g \text{ with some } \nu(M) > 0 \right\}$$

be the symplectic group of degree g with similitudes. Let $\mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$. We define a group $\widetilde{G}_g$ which consists of the pairs $\xi = (M, \phi(Z))$, where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_g$ and $\phi(Z)$ is a non-zero holomorphic function on $\mathfrak{S}_g$ such that

$$\phi(Z)^2 = t(\xi)\nu(M)^{-1/2}\det(CZ + D)$$

for any $Z \in \mathfrak{S}_g$ with some $t(\xi) \in \mathbf{T}$. The multiplicative law is defined as follows:

$$(M_1, \phi_1(Z))(M_2, \phi_2(Z)) = (M_1M_2, \phi_1(M_2\langle Z\rangle)\phi_2(Z)) .$$

We denote the natural projection of $\widetilde{G}_g$ to G_g by p.

If we define $\iota(M) = (M, J(M, Z))$, then this is an injective homomorphism of Γ_g^α to $\widetilde{G}_g$. Let $\mu : GL(g, \mathbf{C}) \to GL(r, \mathbf{C})$ be an irreducible holomorphic representation. Then $\mu(CZ + D)$ is also an automorphy factor (with respect to Γ_g) and so is $J(M, Z)^{2k+1}\mu(CZ + D)$ (with respect to $\Gamma_0^g(4)$). We denote $\mu(CZ + D)$ by $\mu(M, Z)$.

For any holomorphic mapping $f : \mathfrak{S}_g \to \mathbf{C}^r$ and $\xi = (M, \phi(Z)) \in \widetilde{G}_g$, we put

$$f \mid [\xi]_{\mu,k+1/2}(Z) = \phi(Z)^{-(2k+1)}\mu(CZ + D)^{-1}f(M\langle Z\rangle) .$$

Then we have

$$f \mid [\xi\,\eta]_{\mu,k+1/2}(Z) = (f \mid [\xi]_{\mu,k+1/2}) \mid [\eta]_{\mu,k+1/2}(Z)$$

for any ξ and $\eta \in \widetilde{G}_g$.

In the following we assume that $g = 2$. Let $\mathrm{Sym}^j : GL(2, \mathbf{C}) \to GL(j + 1, \mathbf{C})$ be the symmetric tensor representation of degree j. When $\mu = \mathrm{Sym}^j$, we denote $f \mid [\xi]_{\mu,k+1/2}(Z)$ by $f \mid [\xi]_{j,k+1/2}(Z)$. A holomorphic mapping $f : \mathfrak{S}_2 \to \mathbf{C}^{j+1}$ is called *a Siegel modular form of half integral weight* with respect to $\Gamma_0^2(4)$, if f satisfies the following equality for any $M \in \Gamma_0^2(4)$ and $Z \in \mathfrak{S}_2$:

$$f \mid [\iota(M)]_{j,k+1/2}(Z) = f(Z) .$$

We denote by $M_{j,k+1/2}(\Gamma_0^2(4))$ the $\mathbf{C}$-vector space of all such mappings.

A form $f \in M_{j,k+1/2}(\Gamma_0^2(4))$ is called *a cusp form* if f belongs to the kernel of the Φ-operator. We denote the space of cusp forms by $S_{j,k+1/2}(\Gamma_0^2(4))$. Namely, f belongs to $S_{j,k+1/2}(\Gamma_0^2(4))$ if and only if

$$\lim_{\mathrm{Im}\, Z_2\to\infty} f \mid [\xi]_{j,k+1/2}(Z) = 0 ,$$

for any $\xi \in \widetilde{G}_2$ such that $p(\xi) \in \Gamma_2$, where $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$.

Let ψ be as before. We denote by $M_{j,k+1/2}(\Gamma_0^2(4), \psi)$ the $\mathbf{C}$-vector space of holomorphic mappings of $\mathfrak{S}_2$ to $\mathbf{C}^{j+1}$ which satisfy

$$f \mid [\iota(M)]_{j,k+1/2}(Z) = \psi(\det D)\, f(Z) ,$$

for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^2(4)$ and $Z \in \mathfrak{S}_2$. If j is odd, $M_{j,k+1/2}(\Gamma_0^2(4))$ and $M_{j,k+1/2}(\Gamma_0^2(4), \psi)$ are 0-spaces, because $-1_4 \in \Gamma_0^2(4)$ and $\mathrm{Sym}^j(-1_2) = -1_{j+1}$. So we assume that j is even and replace j with $2j$. Any $f \in M_{2j,k+1/2}(\Gamma_0^2(4), \psi)$ is a cusp form in the above sense ([T2], Theorem 4.6). The author calculated the dimensions of $S_{2j,k+1/2}(\Gamma_0^2(4))$ and $M_{2j,k+1/2}(\Gamma_0^2(4), \psi)$ for $k \geq 4$ in [T2].

Let $M_k(\Gamma_0^2(4))$ and $M_k(\Gamma_0^2(4), \psi)$ be the **C**-vector spaces of scalar valued Siegel modular forms with respect to $\Gamma_0^2(4)$ and the automorphy factor $\det(CZ + D)^k$ and $\det(CZ + D)^k \psi(\det D)$, respectively. Then $A(\Gamma_0^2(4), \psi) := \bigoplus_{k=0}^{\infty} M_k(\Gamma_0^2(4), \psi^k)$ is a graded ring.

$\bigoplus_{k=0}^{\infty} M_{2j,k+1/2}(\Gamma_0^2(4))$ and $\bigoplus_{k=0}^{\infty} M_{2j,k+1/2}(\Gamma_0^2(4), \psi)$ are $A(\Gamma_0^2(4), \psi)$-modules. Let $M_{2j,k}(\Gamma_2)$ be the space of vector valued Siegel modular forms with respect to Γ_2 and the automorphy factor $\det(CZ + D)^k \operatorname{Sym}^{2j}(CZ + D)$. T. Satoh determined $\bigoplus_{k=0}^{\infty} M_{2,2k}(\Gamma_2)$ explicitly in [Sa] by using the dimension formula ([T1]). The purpose of this paper is to determine the module structure of $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4))$ explicitly by using a similar method. Recently T. Ibukiyama determined $\bigoplus_{k=0}^{\infty} M_{2,2k+1}(\Gamma_2)$ and $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4), \psi)$ explicitly ([Ib1], [Ib2]).

2. Dimension of $M_{2,k+1/2}(\Gamma_0^2(4))$

For two formal power series $\sum a_k t^k$ and $\sum b_k t^k$ we write

$$\sum a_k t^k \equiv \sum b_k t^k \quad (k \geq m),$$

if $a_k = b_k$ for any $k \geq m$. We calculated the dimension of $S_{2,k+1/2}(\Gamma_0^2(4))$ $(k \geq 4)$ by using the holomorphic Lefschetz fixed point formula ([T2]). The restriction to k arises from the fact that the vanishing theorem holds when $k \geq 4$. We have

THEOREM 2.1. *The generating function of* $\dim S_{2,k+1/2}(\Gamma_0^2(4))$ *satisfies*

$$\sum_{k=0}^{\infty} \dim S_{2,k+1/2}(\Gamma_0^2(4))\, t^k \equiv \frac{-t^2 + t^3 + 3t^4 + 3t^5 - 3t^7}{(1-t)(1-t^2)^2(1-t^3)} \quad (k \geq 4).$$

Using this theorem and the surjectivity of Φ-operators we compute the dimension of $M_{2,k+1/2}(\Gamma_0^2(4))$. First we describe one-dimensional cusps of the Satake compactification $\overline{\Gamma_0^2(4)\backslash\mathfrak{S}_2}$ of $\Gamma_0^2(4)\backslash\mathfrak{S}_2$. We refer to [T2] for the complete configuration of the cusps. $\overline{\Gamma_0^2(4)\backslash\mathfrak{S}_2}$ has four one-dimensional cusps. Each one-dimensional cusp is biholomorphic to $\Gamma_0^1(4)\backslash\mathfrak{S}_1$. Let G_1 be the stabilizer of the one-dimensional rational boundary component (at infinity) of $\mathfrak{S}_2$ defined by $\operatorname{Im} Z_2 = \infty$ where Z_2 is the lower-right coefficient of Z. One-dimensional cusps correspond bijectively to the double cosets in $\Gamma_0^2(4)\backslash\Gamma_2/G_1$.

Let

$$P_1 = 1_4, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

P_1, P_2, P_3 and P_4 are the representatives of $\Gamma_0^2(4)\backslash\Gamma_2/G_1$ ([T2], Proposition 2.5). Let C_i be the one-dimensional cusp corresponding to the double coset $\Gamma_0^2(4)P_iG_1$ $(i = 1, 2, 3, 4)$, respectively.

Let $\xi = (P_3, \phi(Z)) \in \widetilde{G}_2$ and let

$$M_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (n \in \mathbf{Z}) \quad \text{and} \quad Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}.$$

Then we have

$$\lim_{\mathrm{Im}\, Z_2 \to \infty} J(P_3 M_n P_3^{-1}, P_3\langle Z\rangle) = (i)^n ,$$

where $i = \sqrt{-1}$ (cf. [T2], Theorem 3.9 (15) Φ_{15c}). Hence if $f \in M_{2j,k+1/2}(\Gamma_0^2(4))$, we have

$$\begin{aligned}
&\lim_{\mathrm{Im}\, Z_2 \to \infty} f|[\xi]_{2j,k+1/2}(Z) \\
&= \lim_{\mathrm{Im}\, Z_2 \to \infty} f|[\xi]_{2j,k+1/2}(M_n\langle Z\rangle) \\
&= \lim_{\mathrm{Im}\, Z_2 \to \infty} \phi(M_n\langle Z\rangle)^{-2k-1}\mu(P_3, M_n\langle Z\rangle)^{-1} f((P_3M_nP_3^{-1})P_3\langle Z\rangle) \\
&= \lim_{\mathrm{Im}\, Z_2 \to \infty} J(P_3M_nP_3^{-1}, P_3\langle Z\rangle)^{2k+1}\left(\frac{\phi(Z)}{\phi(M_n\langle Z\rangle)}\right)^{2k+1} f|[\xi]_{2j,k+1/2}(Z) \\
&= i^{n(2k+1)} \lim_{\mathrm{Im}\, Z_2 \to \infty} f|[\xi]_{2j,k+1/2}(Z) .
\end{aligned}$$

Therefore $\lim_{\mathrm{Im}\, Z_2\to\infty} f|[\xi]_{2j,k+1/2}(Z)$ is identically 0. Namely, the Φ-operator to the one-dimensional cusp C_3 is a 0-map.

Similarly as in the case of integral weight the Φ-operators to the zero-dimensional cusps are 0-maps, if $2j > 0$. We prove surjectivity of the Φ-operators to the one-dimensional cusps C_1, C_2 and C_4 by constructing Eisenstein series of Klingen type. Let V be the representation space of Sym^{2j}. There exists a non-zero vector $v_0 \in V$ such that

$$\mathrm{Sym}^{2j}(M)v_0 = a^{2j}v_0 \quad \text{for all} \quad M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in GL(2, \mathbf{C}) .$$

Such a vector v_0 is uniquely determined up to a constant multiple. Let $f \in S_{2j+k+1/2}(\Gamma_0^1(4))$ and $\begin{pmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2 \end{pmatrix} \in \mathfrak{S}_2$. We denote $f(Z_1)v_0$ by $\widetilde{f}(Z)$. Note that $P_4 \in \Gamma_2^{\alpha}$. Let $\xi_i = \iota(P_i)$ $(i = 1$ or $4)$ and let

$$E_{i,f}(Z) = \sum_{\gamma} \widetilde{f} \mid [\xi_i^{-1}\iota(\gamma)]_{2j,k+1/2}(Z) ,$$

where γ is over $(P_iG_1P_i^{-1}\cap\Gamma_0^2(4))\backslash\Gamma_0^2(4)$. It is easily verified that $\widetilde{f} \mid [\xi_i^{-1}\iota(\gamma)]_{2j,k+1/2}(Z)$ is independent on the choice of γ.

Now we return to the case of general degree and let

$$\Gamma^{g,0}(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, B \equiv O \pmod 4 \right\}.$$

Then $\alpha \Gamma_g^* \alpha^{-1} \cap \Gamma_g$ contains $\Gamma^{g,0}(4)$. Let $\Theta^0(Z) = \theta(Z/2)$. If M belongs to $\Gamma^{g,0}(4)$, then

$$J^0(M, Z) := \Theta^0(M\langle Z\rangle)/\Theta^0(Z)$$

is holomorphic on $\mathfrak{S}_g$. By using $J^0(M, Z)$ we define the space $S_{k+1/2}(\Gamma^{g,0}(4))$ similarly as before.

Let

$$M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $P_2 M P_2^{-1}$ belongs to $\Gamma_0^2(4)$ if and only if b is divisible by 4. Hence the cusp C_2 is $\Gamma^{1,0}(4)\backslash\mathfrak{S}_1$. Let $f \in S_{2j+k+1/2}(\Gamma_0^1(4))$. Then $f^0(Z) := f(Z/4)$ belongs to $S_{2j+k+1/2}(\Gamma^{1,0}(4))$. Let $\widetilde{f^0}(Z) = f^0(Z_1)v_0$ for $Z \in \mathfrak{S}_2$. Let $\xi_2 = (P_2, \phi_2(Z)) \in \widetilde{G}_2$ and

$$E_{2,f}(Z) = \sum_{\gamma} \widetilde{f^0} \mid [\xi_2^{-1}\iota(\gamma)]_{2j,k+1/2}(Z)$$

where γ is over $(P_2 G_1 P_2^{-1} \cap \Gamma_0^2(4))\backslash\Gamma_0^2(4)$. $\widetilde{f^0} \mid [\xi_2^{-1}\iota(\gamma)]_{2j,k+1/2}(Z)$ is independent of the choice of γ. This is similarly proved as in [T2], Theorem 5.2. The following proposition is essentially due to [K], [G] and [A].

THEOREM 2.2. *If $k \geq 4$, then the sum in $E_{i,f}(Z)$ $(i = 1, 2, 4)$ converges absolutely and uniformly on any compact subset of $\mathfrak{S}_2$.*

$E_{i,f}(Z)$ belongs to $M_{2j,k+1/2}(\Gamma_0^2(4))$. Similarly as in the case of integral weight we can prove that $\lim_{\mathrm{Im} Z_2\to\infty} E_{i,f} \mid [\xi_i]_{2j,k+1/2}(Z) = f(Z_1)v_0$ $(i = 1, 4)$, $\lim_{\mathrm{Im} Z_2\to\infty} E_{2,f} \mid [\xi_2]_{2,k+1/2}(Z) = f^0(Z_1)v_0$ and $\lim_{\mathrm{Im} Z_2\to\infty} E_{i,f} \mid [\xi_{i'}]_{2j,k+1/2}(Z) = 0$ $(i \neq i')$. Hence we have

COROLLARY 2.3. *The generating function of* $\dim M_{2,k+1/2}(\Gamma_0^2(4))$ *satisfies*

$$\begin{aligned} \sum_{k=0}^{\infty} \dim M_{2,k+1/2}(\Gamma_0^2(4))\, t^k &\equiv \sum_{k=0}^{\infty} \dim S_{2,k+1/2}(\Gamma_0^2(4))\, t^k \\ &\quad + 3\sum_{k=0}^{\infty} \dim S_{k+5/2}(\Gamma_0^1(4))\, t^k \quad (k \geq 4) \\ &= \frac{-t^2 + t^3 + 3t^4 + 3t^5 - 3t^7}{(1-t)(1-t^2)^2(1-t^3)} + 3\frac{(t^2+t^3)}{(1-t^2)^2} \\ &= \frac{2t^2 + t^3}{(1-t)(1-t^2)^2(1-t^3)}. \end{aligned}$$

As we see in Theorem 3.4 below, the generating function of $\dim M_{2,k+1/2}(\Gamma_0^2(4))$ is exactly equal to $\dfrac{2t^2+t^3}{(1-t)(1-t^2)^2(1-t^3)}$.

3. The structure of the module $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4))$

From the result of J.-I. Igusa ([Ig]), we have the following

THEOREM 3.1.

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4), \psi^k)\, t^k = \frac{1}{(1-t)(1-t^2)^2(1-t^3)}\,.$$

T. Ibukiyama explicitly represented the generators of $A(\Gamma_0^2(4), \psi)$ by theta constants ([HI]). We recall the definition of theta constants.

DEFINITION 3.2. Let $m \in \mathbf{Z}^{2g}$ and let m' and m'' be the vectors in $\mathbf{Z}^g$ determined by the first and the last g coefficients of m. For $Z \in \mathfrak{S}_g$ and $W \in \mathbf{C}^g$, we define

$$\theta_m(Z, W) = \sum_{p\in\mathbf{Z}^g} \mathbf{e}\left(\frac{1}{2}\cdot {}^t(p+m'/2)Z(p+m'/2) + {}^t(p+m'/2)(W+m''/2)\right)\,.$$

Since we have $\theta_{m+2n}(Z, W) = (-1)^{{}^t m' n''}\theta_m(Z, W)$, $\theta_m(Z, W)$ essentially depends only on m mod 2. We put $\theta_m(Z) = \theta_m(Z, 0)$. If ${}^t m' m''$ is odd, then $\theta_m(Z)$ is identically zero. So we assume that ${}^t m' m''$ is even and call $\theta_m(Z)$ *theta constant*. If $g = 2$, there are 10 theta constants. If ${}^t m = (m_1, m_2, m_3, m_4)$, we denote $\theta_m(Z)$ by $\theta_{m_1m_2m_3m_4}(Z)$. Hence we have $\theta(Z) = \theta_{0000}(Z)$.

Let

$$f_1(Z) = (\theta_{0000}(Z))^2\,,$$
$$X(Z) = ((\theta_{0000}(Z))^4 + (\theta_{0001}(Z))^4 + (\theta_{0010}(Z))^4 + (\theta_{0011}(Z))^4)/4\,,$$
$$g_2(Z) = (\theta_{0000}(Z))^4 + (\theta_{0100}(Z))^4 + (\theta_{1000}(Z))^4 + (\theta_{1100}(Z))^4\,,$$
$$f_3(Z) = (\theta_{0001}(Z)\theta_{0010}(Z)\theta_{0011}(Z))^2\,.$$

Then we have

THEOREM 3.3 ([HI]). *The graded ring $A(\Gamma_0^2(4), \psi)$ is generated by algebraically independent modular forms $f_1(2Z)$, $X(2Z)$, $g_2(2Z)$ and $f_3(2Z)$ whose weights are* 1, 2, 2 *and* 3, *respectively.*

Let V be $\{S \in M_2(\mathbf{C}) \mid {}^tS = S\}$. We define the action of $M \in GL(2, \mathbf{C})$ on V by $S \mapsto MS\,{}^tM$. This action defines a representation of $GL(2, \mathbf{C})$ which is equivalent to

Sym^2. Let F be a differentiable function on $\mathfrak{S}_2$ and let

$$\Delta F = \begin{pmatrix} \dfrac{\partial F}{\partial Z_1} & \dfrac{1}{2}\dfrac{\partial F}{\partial Z_{12}} \\ \dfrac{1}{2}\dfrac{\partial F}{\partial Z_{12}} & \dfrac{\partial F}{\partial Z_2} \end{pmatrix}.$$

If $M \in \Gamma_2$, it holds that

$$\Delta(F(M\langle Z\rangle)) = (CZ+D)^{-1}(\Delta F)(M\langle Z\rangle)\,{}^t(CZ+D)^{-1}$$

(cf. [M] and [Sh2], (4.2)). Hence if F satisfies $F(M\langle Z\rangle) = F(Z)$, we have

$$(\Delta F)(M\langle Z\rangle) = (CZ+D)\Delta(F(Z))\,{}^t(CZ+D).$$

Let $f \in M_k(\Gamma_0^2(4), \psi^k)$ and $g \in M_{\ell+1/2}(\Gamma_0^2(4))$. Then $g^{2k}/f^{2\ell+1}$ is a (meromorphic) modular form of weight 0. Therefore $\Delta(g^{2k}/f^{2\ell+1})$ is a (meromorphic) modular form with respect to $\mathrm{Sym}^2(CZ+D)$. $f^{2\ell+2}/g^{2k-1}$ is a (meromorphic) modular form of with respect to $J(M,Z)^{2k+2\ell+1}$. Hence

$$\begin{aligned}[f,g] &:= \frac{1}{k(2\ell+1)}(f^{2\ell+2}/g^{2k-1})\Delta(g^{2k}/f^{2\ell+1}) \\ &= \frac{1}{\ell+1/2} f\Delta g - \frac{1}{k} g\Delta f\end{aligned}$$

becomes a holomorphic modular form and belongs to $M_{2,k+\ell+1/2}(\Gamma_0^2(4))$. T. Satoh used the differential operator of this type first in the case of integral weight in [Sa].

In the following we denote $\theta(2Z)$, $f_1(2Z)$, $X(2Z)$, $g_2(2Z)$ and $f_3(2Z)$ by θ, f_1, X, g_2 and f_3, respectively. Our main theorem in this paper is the following

THEOREM 3.4. *$\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4))$ is a free $A(\Gamma_0^2(4), \psi)$-module of rank three and the generators are $[X,\theta]$, $[g_2,\theta]$ and $[f_3,\theta]$.*

Proof. Let $h_1, h_2 \in M_{k-2}(\Gamma_0^2(4), \psi^{k-2})$ and $h_3 \in M_{k-3}(\Gamma_0^2(4), \psi^{k-3})$. Assume that

$$h_1[X,\theta] + h_2[g_2,\theta] + h_3[f_3,\theta]$$

is identically zero. We have

$$\begin{aligned}[X,\theta] &= \frac{1}{2}(X^2/\theta^3)\Delta(\theta^4/X), \\ [g_2,\theta] &= \frac{1}{2}(g_2^2/\theta^3)\Delta(\theta^4/g_2), \\ [f_3,\theta] &= \frac{1}{3}(f_3^2/\theta^5)\Delta(\theta^6/f_3).\end{aligned}$$

Hence we have

$$\begin{pmatrix} \dfrac{\partial(\theta^4/X)}{\partial Z_1} & \dfrac{\partial(\theta^4/g_2)}{\partial Z_1} & \dfrac{\partial(\theta^6/f_3)}{\partial Z_1} \\ \dfrac{\partial(\theta^4/X)}{\partial Z_{12}} & \dfrac{\partial(\theta^4/g_2)}{\partial Z_{12}} & \dfrac{\partial(\theta^6/f_3)}{\partial Z_{12}} \\ \dfrac{\partial(\theta^4/X)}{\partial Z_2} & \dfrac{\partial(\theta^4/g_2)}{\partial Z_2} & \dfrac{\partial(\theta^6/f_3)}{\partial Z_2} \end{pmatrix} \begin{pmatrix} \dfrac{1}{2}(X^2/\theta^3)h_1 \\ \dfrac{1}{2}(g_2^2/\theta^3)h_2 \\ \dfrac{1}{3}(f_3^2/\theta^5)h_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Assume that h_1, h_2 or h_3 is not identically zero. Then the determinant of the matrix in the left hand side is identically zero. This contradicts the fact that $f_1(=\theta^2)$, X, g_2 and f_3 are algebraically independent. Therefore h_1, h_2 and h_3 are identically zero. Thus the assertion was proved for $k \geq 4$ from Corollary 2.3.

We prove the assertion for $0 \leq k \leq 3$. Let $f \in M_{2,5/2}(\Gamma_0^2(4))$. Then Xf, $g_2 f \in M_{2,9/2}(\Gamma_0^2(4))$ and we have

$$Xf = h_1[X,\theta] + h_2[g_2,\theta] + h_3[f_3,\theta]\,,$$
$$g_2 f = h_1'[X,\theta] + h_2'[g_2,\theta] + h_3'[f_3,\theta]\,,$$

where $h_1, h_2, h_1', h_2' \in M_2(\Gamma_0^2(4))$ and $h_3, h_3' \in M_1(\Gamma_0^2(4), \psi)$. Hence it follows that

$$(g_2h_1 - Xh_1')[X,\theta] + (g_2h_2 - Xh_2')[g_2,\theta] + (g_2h_3 - Xh_3')[f_3,\theta] = 0\,.$$

So from the result for $k \geq 4$ we have

$$g_2h_1 - Xh_1' = g_2h_2 - Xh_2' = g_2h_3 - Xh_3' = 0\,.$$

Therefore h_1, h_2 and h_3 are divisible by X. This means that h_1 and h_2 are multiples of X by constants and h_3 is identically 0. Hence $M_{2,5/2}(\Gamma_0^2(4))$ is spanned by $[X,\theta]$ and $[g_2,\theta]$. Similarly we can prove that $M_{2,1/2}(\Gamma_0^2(4)) \simeq M_{2,3/2}(\Gamma_0^2(4)) \simeq \{0\}$ and $M_{2,7/2}(\Gamma_0^2(4))$ is spanned by $f_1[X,\theta]$, $f_1[g_2,\theta]$ and $[f_3,\theta]$. □

Next we mention the case of $M_{2,k+1/2}(\Gamma_0^2(4),\psi)$. We have

THEOREM 3.5.

$$\sum_{k=0}^{\infty} \dim M_{2,k+1/2}(\Gamma_0^2(4),\psi)\, t^k = \frac{t^5 + 2t^6}{(1-t)(1-t^2)^2(1-t^3)}\,.$$

Proof. The assertion for $k \geq 4$ follows from dimension formula ([T2], Theorem 4.5). If $f \in M_{2,k+1/2}(\Gamma_0^2(4),\psi)$, then $\Theta^2 f$ belongs to $M_{2,k+3/2}(\Gamma_0^2(4),\psi)$. Since $M_{2,9/2}(\Gamma_0^2(4),\psi) \simeq \{0\}$, it follows $M_{2,7/2}(\Gamma_0^2(4),\psi) \simeq M_{2,5/2}(\Gamma_0^2(4),\psi) \simeq M_{2,3/2}(\Gamma_0^2(4),\psi) \simeq M_{2,1/2}(\Gamma_0^2(4),\psi) \simeq \{0\}$. □

REMARK 3.6. $\bigoplus_{k=0}^{\infty} M_k(\Gamma_0^2(4),\psi^{k+1})$ is a free $A(\Gamma_0^2(4),\psi)$-module of rank 1. Its generator is of weight 11 which we denote by f_{11}. The form of type $[f,g]$ in $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4),\psi)$ of the lowest weight is $[f_{11},\Theta]$. Its weight is 23/2. Hence $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4),\psi)$ is not spanned by forms of this type. The structure of the module $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4), \psi)$ was explicitly determined by T. Ibukiyama. Especially $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4), \psi)$ is a free $A(\Gamma_0^2(4), \psi)$-module of rank 3 ([Ib2]).

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Department of Mathematics
Meiji University
Tama-ku, Kawasaki-shi
214–8571 Japan
e-mail: tsushima@math.meiji.ac.jp