On the Distribution of the Zeros of the Riemann Zeta Function in the Neighborhood of its Zeros

by

Akio FUJII

(Received September 27, 2004)

§1. Introduction

We are concerned with the distribution of the zeros of the Riemann zeta function $\zeta(s)$ in the neighborhood of its zeros.

One of the main open problems in this area is Montgomery’s conjecture which states that for any $\alpha > 0$,

$$
\sum_{0 < \gamma, \gamma' \leq T \atop 0 < \gamma - \gamma' \leq \frac{2\pi\alpha}{\log T}} 1 = \frac{T}{2\pi} \log T \cdot \left\{ \int_0^a \left( 1 - \left( \frac{\sin \pi t}{\pi t} \right)^2 \right) \, dt + o(1) \right\}
$$

as $T \to \infty$, where $\gamma$ and $\gamma'$ run over the imaginary parts of the zeros of $\zeta(s)$ (cf. Montgomery [24]). As we have noticed on some occasions (cf. Fujii [6] [9] [11]), to study Montgomery’s Conjecture is equivalent to get an asymptotic formula for the sum

$$
\sum_{0 < \gamma \leq T} \left( S\left( \gamma + \frac{2\pi\alpha}{\log T} \right) - S(\gamma) \right)
$$

for any positive $\alpha$ as $T \to \infty$, where we put

$$
S(T) = \frac{1}{\pi \arg \zeta \left( \frac{1}{2} + iT \right)} \quad \text{for} \quad T \neq \gamma,
$$

and $S(T) = S(T + 0)$ if $T$ is an ordinate of the zeros of $\zeta(s)$. To see this we recall that

$$
\sum_{0 < \gamma \leq T} \left( S\left( \gamma + \frac{2\pi\alpha}{\log T} \right) - S(\gamma) \right) = \sum_{0 < \gamma, \gamma' \leq T \atop 0 < \gamma' - \gamma \leq \frac{2\pi\alpha}{\log T}} 1 - \frac{T}{2\pi \log T} \cdot \int_0^a \, dt + O(T).
$$

Although we have not been able to get such an asymptotic formula, we could give an upper bound for this. In fact, in [11], we have shown, without assuming any unproved hypothesis, that for $0 < \alpha \ll T \log T$,

$$
\sum_{0 < \gamma \leq T} \left( S\left( \gamma + \frac{2\pi\alpha}{\log T} \right) - S(\gamma) \right) = O(T \log T),
$$

169
where we always suppose in this article that \( T > T_0 \). More recently, we [17] have shown, under the Riemann Hypothesis (R.H.), that

\[
\left| \sum_{0 < \gamma \leq T} \left( S \left( \gamma + \frac{2\pi \alpha}{\log 2\pi} \right) - S(\gamma) \right) \right| \leq 4.5 \cdot \frac{T}{2\pi} \log \frac{T}{2\pi}
\]

uniformly for \( \alpha \) which satisfies \( 0 < \alpha \ll T \log T \).

Here we are concerned with an extension of the problem. More concretely, we are concerned with the following problem.

**Problem.** For any integer \( m \geq 1 \) and for any positive \( \alpha \) satisfying \( 0 < \alpha \ll \log T \), to get an asymptotic formula for the sum

\[
\sum_{0 < \gamma \leq T} \left( S_m \left( \gamma + \frac{2\pi \alpha}{\log 2\pi} \right) - S_m(\gamma) \right)
\]

and clarify its properties from a viewpoint of GUE statistics, where \( S_m(T) \) will be defined below.

For \( m = 1 \) and 2, we shall settle this problem without assuming any unproved hypothesis (Cf. Theorems 1 and 2 below.) In fact, our result for \( m = 1 \) reveals a GUE aspect as it should be expected.

To proceed further, we shall first recall the definitions and some of the known results on \( S(T) \) and \( S_m(T) \). The importance of the extension of \( S(T) \) to \( S_m(T) \) for \( m \geq 1 \) can be seen, for example, in the connection with the Lindelöf hypothesis or the Riemann Hypothesis as we shall also recall below.

We denote the non-trivial zeros of \( \zeta(s) \) by \( \rho = \beta + i\gamma \) with real numbers \( \beta \) and \( \gamma \). Let \( N(T) \) denote the number of the zeros \( \beta + i\gamma \) of \( \zeta(s) \) in \( 0 < \gamma < T, 0 < \beta < 1 \), when \( T \neq \gamma \) for any \( \gamma \). When \( T = \gamma \) for some \( \gamma \), then we put

\[
N(T) = \frac{1}{2} (N(T + 0) + N(T - 0)).
\]

Let

\[
S(T) = \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + iT \right) \quad \text{for} \quad T \neq \gamma,
\]

where the argument is obtained by the continuous variation along the straight lines joining \( 2, 2 + iT \), and \( \frac{1}{2} + iT \), starting with the value zero. When \( T = \gamma \), then we put, throughout the rest of this article,

\[
S(T) = \frac{1}{2} (S(T + 0) + S(T - 0)).
\]

Now, the well known Riemann-von Mangoldt formula (cf. p. 212 of Titchmarsh [27]) states that

\[
N(T) = \frac{1}{\pi} \vartheta(T) + 1 + S(T),
\]

where \( \vartheta(T) \) is the continuous function defined by

\[
\vartheta(T) = 3 \left( \log T \left( \frac{1}{4} + \frac{iT}{2} \right) \right) - \frac{1}{2} T \log \pi
\]
with $\vartheta(0) = 0$, $\Gamma(s)$ being the gamma-function. It is well known that

$$\vartheta(T) = \frac{T}{2} \log \frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + \frac{1}{48T} + \frac{7}{5760T^3} + \cdots$$

and that

$$S(T) \ll \log T.$$

The last estimate was refined under the Riemann Hypothesis (R.H.) by Littlewood [22] and later by Selberg [26] in different ways as follows:

$$S(T) = O\left(\frac{\log T}{\log \log T}\right).$$

Concerning the average behavior of $S(T)$, it is a well known result of Littlewood [22] and Selberg [26] that

$$\int_0^T S(t) dt = O(\log T).$$

It is noticed further on p. 335 of Titchmarsh [27] that *the Lindelöf hypothesis is equivalent to the statement that*

$$\int_0^T S(t) dt = o(\log T) \quad (T \to \infty).$$

If we assume the Riemann Hypothesis, then we have, due to Littlewood [22] and Selberg [26],

$$\int_0^T S(t) dt = O\left(\frac{\log T}{(\log \log T)^2}\right).$$

Concerning the mean value theorem of the higher order, Selberg [26] showed that for any integer $k \geq 1$,

$$\int_0^T S(t)^2 dt = 2k! \left(\frac{2\pi}{(2\pi)^{2k+1}}\right) T(\log \log T)^k + O(T(\log \log T)^{k-\frac{1}{2}}).$$

This enables us to determine the distribution function of $S(t)$. Under R.H., Selberg also showed that for each integer $k \geq 1$,

$$\int_0^T S(t)^2 dt = \frac{2k!}{(2\pi)^{2k+1}} T(\log \log T)^k + O(T(\log \log T)^{k-1}).$$

For $k = 1$, Goldston [21] has refined this by showing, under R.H., that

$$\int_0^T S(t)^2 dt = \frac{T}{2\pi^2} \log \log T + \frac{T}{2\pi^2} \left\{ \int_1^\infty \frac{F(a)}{a^2} da + C_o + \sum_{m=2}^\infty \sum_p \left( -\frac{1}{m} + \frac{1}{m^2} \right) \frac{1}{p^m} \right\} + o(T).$$
where \( C_o \) is the Euler constant, \( p \) runs over the prime numbers and \( F(a) \) is the Montgomery’s sum defined by

\[
F(a) \equiv F(a, T) \equiv \frac{1}{2\pi} \log T \sum_{0 < \gamma, \gamma' \leq T} \left( \frac{T}{2\pi} \right)^{ia(\gamma - \gamma')} w(\gamma - \gamma')
\]

with

\[
w(t) = \frac{4}{4 + t^2}.
\]

Concerning the mean value theorems for short intervals, we [8] [9] have shown that for \( 0 < \alpha \ll T \log T \)

\[
\int_0^T \left( S \left( t + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S(t) \right)^2 dt
\]

\[
= \begin{cases}
2k! (2\pi)^{2k} T (\log(2\pi \alpha) - \text{Ci}(2\pi \alpha) + C_o)^k \\
\quad + O(T(Ak)^k((\log(2\pi \alpha) - \text{Ci}(2\pi \alpha) + C_o)^{k - \frac{1}{2}} + k^k)) \\
\quad \text{if } 0 < \alpha \ll \log T
\end{cases}
\]

\[
= \begin{cases}
2k! (2\pi)^{2k} T \left( \log \log T - \log \left| \zeta \left( 1 + i \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) \right| \right)^k \\
\quad + O\left( T(Ak)^k \left( \log \log T - \log \left| \zeta \left( 1 + i \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) \right| \right)^{k - \frac{1}{2}} + k^k \right) \\
\quad \text{if } \log T \ll \alpha \ll T \log T
\end{cases}
\]

uniformly for an integer \( k \geq 1 \), where we put

\[
\text{Ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt.
\]

When \( 0 < \alpha \ll 1 \), this formula does not give an asymptotic formula. However, we [8] [9] can recover it, applying Goldston, for the case of \( k = 1 \), under the Riemann Hypothesis, as follows. For \( 0 < \alpha \ll T \log T \), we have

\[
\int_0^T \left( S \left( t + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S(t) \right)^2 dt
\]

\[
= \begin{cases}
\frac{T}{\pi^2} \left\{ \int_0^{2\pi a} \frac{1 - \cos a}{a} da + \int_1^\infty \frac{F(a)}{a^2} (1 - \cos(2\pi a)) da + o(1) \right\} \\
\quad \text{if } 0 < \alpha = o(\log T)
\end{cases}
\]

\[
= \begin{cases}
\frac{T}{\pi^2} \left\{ \log \log T - \log \left| \zeta \left( 1 + i \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) \right| + O(1) \right\} \\
\quad \text{if } \log T \ll \alpha \ll T \log T
\end{cases}
\]
Now we turn our attentions to the extension of $S_m(T)$ for $m \geq 1$ of $S(T)$, which will be defined below. When $T \neq \gamma$, then we put

$$S_0(T) = S(T)$$

and

$$S_m(T) = \int_0^T S_{m-1}(t) dt + C_m$$

for any integer $m \geq 1$, where $C_m$'s are some specified constants (cf. p. 2 of Fujii [13]). When $T = \gamma$, then we put

$$S_m(T) = \frac{1}{2} (S_m(T + 0) + S_m(T - 0)).$$

Concerning $S_m(T)$ for $m \geq 2$, Littlewood [22] and Selberg [26] have shown under the Riemann Hypothesis that

$$S_m(T) \ll \log T \left( \log \log T \right)^m + 1.$$

We recall our studies on $S_m(T)$ for $m \geq 2$ without assuming any unproved hypothesis. We shall first recall the relation between $S_m(T)$ and the integral $I_m(T)$ which will be defined as follows. When $T \neq \gamma$, then we put for $k \geq 1$

$$I_{2k-1}(T) = \frac{1}{\pi} (-1)^{k-1} \int_{\frac{1}{2}}^\infty \int_{\sigma}^\infty \cdots \int_{\sigma}^\infty \log \zeta(\sigma + iT) (d\sigma)^{2k-1}$$

and

$$I_{2k}(T) = \frac{1}{\pi} (-1)^k \int_{\frac{1}{2}}^\infty \int_{\sigma}^\infty \cdots \int_{\sigma}^\infty \log \zeta(\sigma + iT) (d\sigma)^{2k}.$$ 

When $T = \gamma$, then we put for $m \geq 1$

$$I_m(T) = \frac{1}{2} (I_m(T + 0) + I_m(T - 0)).$$

We now describe a relation between $S_m(T)$ and $I_m(T)$ as follows (cf. Lemma 1 on p. 3 and Theorem 1 on p. 4 of Fujii [13]).

**Lemma 1.** Let $m$ be an integer $\geq 1$. Then we have

$$S_m(T) = I_m(T) + W_m(T),$$

where we put

$$W_1(T) = 0,$$

for $m \geq 2$ and for $T \neq \gamma$,

$$W_m(T) = \sum_{\beta+\epsilon=\gamma} \frac{(-1)^{r-1}}{(2\pi)!} \cdot h! \sum_{\beta+i\gamma} \left( \beta - \frac{1}{2} \right)^{2r} (T - \gamma)^h$$

for $\epsilon > 0$ and $\beta > \frac{1}{2} \epsilon < T$. 

**Proof:** (details omitted)
for $m \geq 2$ and for $T = \gamma$

\[ W_m(T) = \frac{1}{2} (W_m(T + 0) + W_m(T - 0)) \]

and we have the estimate

\[ I_m(T) \ll \log T. \]

We notice that the estimate of $I_m(T)$ for $m = 1$ was obtained by Littlewood [22] and Selberg [26]. We [13, 15] have obtained the above estimate for $m \geq 2$ as a consequence of the following expression of $I_m(T)$:

\[ I_m(T) = -\frac{1}{\pi} \Im \left\{ \frac{i^m}{m!} \int_\frac{1}{2}^\infty \left( \sigma - \frac{1}{2} \right)^m \frac{\zeta'}{\zeta} (\sigma + iT) \, d\sigma \right\}. \]

For $\sigma \geq \frac{1}{2}$, let $N(\sigma, T)$ be the number of the zeros $\beta + i\gamma$ of $\zeta(s)$ such that $\beta > \sigma$ and $0 < \gamma < T$ when $T \neq \gamma$. When $T = \gamma$, then we put

\[ N(\sigma, T) = \frac{1}{2} (N(\sigma, T + 0) + N(\sigma, T - 0)). \]

Using these notations, we recall that

\[ W_m(T) = \sum_{\substack{h \geq 1, r \geq 0 \atop h + 2r = m}} (-1)^r N_{h, 2r}(T), \]

where we put for $h \geq 1$ and $r \geq 1$

\[ N_{h, 2r}(T) = \int_0^T \cdots \int_0^T \int_{\frac{1}{2}}^1 \cdots \int_{\frac{1}{2}}^1 N(\sigma, t) (d\sigma)^{2r} (dt)^h, \]

and for $h = 0$ and $r \geq 1$

\[ N_{0, 2r}(T) = \int_{\frac{1}{2}}^1 \cdots \int_{\frac{1}{2}}^1 N(\sigma, T) (d\sigma)^{2r}. \]

To get an upper bound for $W_m(T)$, we apply Selberg’s density theorem (cf. p. 232 of Selberg [26]): for some positive constant $C$,

\[ N(\sigma, T) \ll T \log T \cdot e^{-C(\sigma - \frac{1}{2}) \log T} \]

uniformly for $\sigma \geq \frac{1}{2}$. Thus, we have first

\[ \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \cdots \int_{\frac{1}{2}}^1 N(\sigma, T) (d\sigma)^{2r} \ll \frac{T}{(\log T)^{2r-1}} \]

and consequently,

\[ N_{h, 2r}(T) \ll \frac{T^{h+1}}{(\log T)^{2r-1}}. \]
On the Distribution of the Zeros of the Riemann Zeta Function in the Neighborhood of its Zeros

\[ W_m(T) \ll \frac{T^{m-1}}{\log T}, \]

and

\[ S_m(T) \ll \frac{T^{m-1}}{\log T} \quad \text{for any integer } m \geq 2. \]

This looks weak at first sight if we compare this with the conditional results mentioned above. However as we have noticed on p. 6 of Fujii [13], we have shown that the Riemann Hypothesis is equivalent to the statement that for any integer \( m \geq 3, \)

\[ S_m(T) = o(T^{m-2}). \]

Concerning the higher moments of \( S_m(T), \) a study has been given in Fujii [15] for \( m \geq 2, \) while the case for \( m = 1 \) was treated in Selberg [26]. On the other hand, Littlewood [23] gave its study for \( m \geq 1 \) under the Riemann Hypothesis.

We now describe our results concerning the problem which has been proposed above. It is worthwhile to notice that for \( m = 1 \) and \( m = 2, \) we can get an asymptotic formula without assuming any unproved hypothesis. We shall describe our result for \( m = 1 \) first.

**Theorem 1.** Suppose that \( 0 < \alpha = O\left( \frac{T}{\log T} \right). \) Then we have

\[
\sum_{0 < \gamma \leq T} \left( S_1\left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_1(\gamma) \right)
\]

\[
= \begin{cases}
\frac{T}{2\pi^2} \left[ \log(2\pi \alpha) - Ci(2\pi \alpha) + C_\alpha \right] + O\left( T \sqrt{\log(2\pi \alpha) - Ci(2\pi \alpha) + C_\alpha} \right) \\
\quad \quad \quad \text{if } 0 < \alpha \ll \log T
\end{cases}
\]

\[
+ \frac{T}{2\pi^2} \left\{ \log \log T - \log \left| \zeta\left( 1 + i \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) \right| \right\}
\]

\[
+ O\left( T \left| \log \log T - \log \left| \zeta\left( 1 + i \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) \right| \right|^{1/2} \right)
\]

\[
\quad \quad \quad \text{if } \log T \ll \alpha = O\left( \frac{T}{\log T} \right).
\]

If we assume the Riemann Hypothesis, then the result can be stated as follows.

**Theorem 2.** (On R.H.) Suppose that \( 0 < \alpha = O\left( \frac{T}{\log T} \right). \) Then we have

\[
\sum_{0 < \gamma \leq T} \left( S_1\left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_1(\gamma) \right)
\]
\[
\left\{ \begin{array}{ll}
\frac{T}{\pi^2} \left\{ \int_0^{2\pi} \frac{1 - \cos a}{a} \, da + \int_1^\infty \frac{F(a)}{a^2} (1 - \cos(2\pi \alpha a)) \, da + o(1) \right\} & \\
= \frac{T}{\pi^2} \left\{ \log \log T - \log \left| \zeta \left( 1 + i \frac{2\alpha}{\log T} \right) \right| + O(1) \right\} \\
& \text{if } \log T \ll \alpha = O \left( \frac{T}{\log T} \right),
\end{array} \right.
\]

where \( F(a) \) is defined above.

The right hand side for the range \( 0 < \alpha = o(\log T) \) is nothing but the GUE part of the number variance of \( N(T) \), which has been mentioned above (cf. also Berry [1] and Fujii [8]). In fact, we observe that

\[
\sum_{0 < \gamma \leq T} \left( S_1 \left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_1(\gamma) \right)
= \frac{1}{2} \int_0^T \left( S \left( t + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S(t) \right)^2 \, dt + O(a(\log T)^2).
\]

Thus we see that Theorems 1 and 2 are the immediate consequences of the mean value theorem of \( S(t) \) for short intervals as mentioned above.

For \( m = 2 \), we can show the following theorem.

**Theorem 3.** Suppose that \( 0 < \alpha \ll \log T \). Then we have

\[
\sum_{0 < \gamma \leq T} \left( S_2 \left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_2(\gamma) \right)
= -\frac{T}{2\pi^2} \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n(\log n)^3} \sin \left( \frac{2\pi \alpha \log n}{\log \frac{T}{2\pi}} \right)
+ \alpha \sum_{\beta > 1/2 \atop 0 < \gamma \leq T} \left( \beta - \frac{1}{2} \right)^2 + O \left( \frac{T}{\log T} \log \log T \right),
\]

where \( \Lambda(n) \) is the von-Mangoldt function defined by

\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

and the dash indicates the halving convention as above.

Since

\[
\sum_{\beta > 1/2 \atop 0 < \gamma \leq T} \left( \beta - \frac{1}{2} \right)^2 \ll \frac{T}{\log T},
\]

...
we get the following asymptotic formula for \( 0 < \alpha = o(\log T) \).

**Corollary 1.** Suppose that \( 0 < \alpha \ll \log T \). Then we have

\[
\sum_{0 < \gamma \leq T} \left( S_2 \left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_2(\gamma) \right)
\]

\[
= - \frac{T}{2\pi^2} \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n \log n} \sin \left( \frac{2\pi \alpha \log n}{\log \frac{T}{2\pi}} \right) + O \left( \frac{T}{\log T} (\alpha + \log \log T) \right).
\]

For \( m \geq 3 \), we get the following more general theorem.

**Theorem 4.** Suppose that \( 0 < \alpha \ll \log T \). Then we have for any integer \( m \geq 3 \)

\[
\sum_{0 < \gamma \leq T} \left( S_m \left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_m(\gamma) \right)
\]

\[
= - \frac{T}{2\pi^2} \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n \log n} \sin \left( \frac{2\pi \alpha \log n}{\log \frac{T}{2\pi}} \right) + O \left( \frac{T}{\log T} (\alpha + \log \log T) \right).
\]

where \( \left\lfloor \frac{m-1}{2} \right\rfloor \) is the integer part of \( \frac{m-1}{2} \), \( \left\{ \frac{m-1}{2} \right\} \) is the fractional part of \( \frac{m-1}{2} \), \( \vartheta(T) \) and \( W_m(T) \) are introduced above, \( \vartheta^{(j)}(t) \) is the \( j \)-th derivative of \( \vartheta(t) \) and we put

\[
\delta(1, m) = \begin{cases} 
1 & \text{if } m = 1 \\
0 & \text{otherwise}.
\end{cases}
\]

An immediate consequence is the following.

**Corollary 2.** Suppose that \( 0 < \alpha \ll \log T \). Then we have for any integer \( m \geq 3 \)

\[
\sum_{0 < \gamma \leq T} \left( S_m \left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_m(\gamma) \right) \ll T^{m-1}.
\]

If we assume the Riemann Hypothesis, we get an asymptotic formula for \( m \geq 2 \) as follows.
COROLLARY 3. (On R.H.) Suppose that $0 < \alpha \ll \log T$. Then we have for any integer $m \geq 2$

$$
\sum_{0 < \gamma \leq T} \left( S_m \left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_m(\gamma) \right)
= -\frac{T}{2\pi^2} (-1)^{\lceil \frac{m-1}{2} \rceil} \sum_{n=2}^{\infty} \frac{A(n)^2}{n(\log n)^{m+1}} \cdot \left( \cos \left( \frac{2\pi \alpha \log n}{\log \frac{T}{2\pi}} - \left\{ \frac{m-1}{2} \right\} \pi \right) - \cos \left( -\left\{ \frac{m-1}{2} \right\} \pi \right) \right)
+ O \left( \frac{T}{\log T} (\log \log T)^{\frac{1}{2} \left( 1 + \delta(1,m) - 1 \right)} \right).
$$

We shall prove our theorems by applying the following theorem.

THEOREM 5. Suppose that $0 \leq \alpha \ll T \log T$. Then we have for any integer $m \geq 1$

$$
\int_C S(t) S_m \left( t + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) dt = \frac{T}{2\pi^2} (-1)^{\lceil \frac{m-1}{2} \rceil} \sum_{n=2}^{\infty} \frac{A(n)^2}{n(\log n)^{m+1}} \cos \left( \frac{2\pi \alpha \log n}{\log \frac{T}{2\pi}} - \left\{ \frac{m}{2} \right\} \pi \right)
+ \int_C S(t) W_m \left( t + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) dt + O \left( \frac{T}{\log T} (\log \log T)^{\frac{1}{2} \left( 1 + \delta(1,m) \right)} \right).
$$

Finally, we remark that we can extend our analysis to more general sums like

$$
\sum_{0 < \gamma \leq T} \left( S_m \left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_m(\gamma) \right)^k
$$

for any integer $m \geq 1$ and for any integer $k \geq 1$. For simplicity, we shall state and prove the simplest case, namely, for $m = 1$ and $k \geq 2$ (cf. 6-3 in the section 6 below for the statement of the results under the Riemann Hypothesis for $m \geq 2$ and for $k \geq 2$). For $k = 2$, we have the following result.

THEOREM 6. Suppose that $0 < \alpha \ll \log T$. Then we have

$$
\sum_{0 < \gamma \leq T} \left( S_1 \left( \gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}} \right) - S_1(\gamma) \right)^2
= \frac{T \log T}{2\pi^3} \sum_{n=2}^{\infty} \frac{A(n)^2}{n(\log n)^2} \left( 1 - \cos \left( \frac{2\pi \alpha \log n}{\log \frac{T}{2\pi}} \right) \right) + O(T \sqrt{\log \log T}).
$$

For $k \geq 3$, we can prove the following theorem.
THEOREM 7. Suppose that $0 < \alpha \ll \log T$. Then we have for any integer $k \geq 3$

$$
\sum_{0 < \gamma \leq T} \left( S_1 \left( \gamma + \frac{2\pi \alpha}{\log T} \right) - S_1(\gamma) \right)^k
= \frac{T \log T}{(2\pi)^{k+1}} \sum_{\mu=0}^{k} \binom{k}{\mu} \sum_{m_1 - m_k = m_{\mu+1} - m_{\mu}}^{\infty} \frac{\Lambda(m_1) \cdots \Lambda(m_k)}{m_1 \cdots m_{\mu} (\log m_1)^2 \cdots (\log m_k)^2}
\cdot \left( \frac{1}{m_1^{ia_1}} - 1 \right) \cdots \left( \frac{1}{m_{\mu}^{ia_\mu}} - 1 \right) \left( \frac{1}{m_{\mu+1}} - 1 \right) \cdots \left( \frac{1}{m_k} - 1 \right)
+ \frac{2kT}{(2\pi)^{k+1}} \int_0^{2\pi a} \frac{1 - \cos(y)}{y} dy \sum_{\mu=0}^{k-1} \binom{k-1}{\mu}
\cdot \sum_{m_1 - m_{k-1} = 2}^{\infty} \frac{\Lambda(m_1) \cdots \Lambda(m_{k-1})}{m_1 \cdots m_{\mu} (\log m_1)^2 \cdots (\log m_{k-1})^2}
\cdot \left( \frac{1}{m_1^{ia_1}} - 1 \right) \cdots \left( \frac{1}{m_{\mu}^{ia_\mu}} - 1 \right) \left( \frac{1}{m_{\mu+1}} - 1 \right) \cdots \left( \frac{1}{m_{k-1}} - 1 \right)
+ O(T^{\sqrt{\log \log T}}),
$$

where we put

$$a = \frac{2\pi \alpha}{\log T}.$$

We shall prove our theorems in the following sections. In the section 2, we shall state some lemmas which we shall use below. In the section 3, we shall prove Theorem 5. In the section 4, we shall prove Theorems 1, 2, 3 and 4 using Theorem 5. In the section 5, we shall prove Theorems 6 and 7. In the section 6, we shall give some supplemental remarks. We shall see below that our proof will be mainly focused on the treatment of the main terms, since we have already estimated some of the error terms (cf. Lemma 4 below). We shall give the details of the proofs as much as possible for completeness.

§2. Some lemmas

We shall list up some lemmas which we shall use below. Let $s = \sigma + it$. We suppose that $\sigma \geq \frac{1}{4}$ and $t \geq 2$. Let $X$ be a positive number satisfying $4 \leq X \leq t^2$. As on pp. 233–235 of Selberg [26], we put

$$\sigma_{X,t} = \frac{1}{2} + 2 \max_{\rho} \left( \beta - \frac{1}{2} \frac{2}{\log X} \right),$$
where \( \rho \) runs through all zeros \( \beta + i \gamma \) of \( \zeta(s) \) for which

\[
|t - \gamma| \leq \frac{X^{\beta - \frac{1}{2}}}{\log X}.
\]

We put

\[
A_X(n) = \begin{cases} 
A(n) & \text{for } 1 \leq n \leq X \\
A(n) \frac{\log^2 X^3}{2 \log^2 X} & \text{for } X \leq n \leq X^2 \\
A(n) \frac{\log^2 X^3}{2 \log^2 X} & \text{for } X^2 \leq n \leq X^3.
\end{cases}
\]

We recall first the following lemma (cf. p. 17 of Fujii [13] and p. 185 and p. 246 of Selberg [26]).

**Lemma 2.** Suppose that \( t \geq 2 \) and \( 2 \leq X \leq t^2 \). Then we have for any integer \( m \geq 1 \)

\[
I_m(t) = \frac{1}{\pi^3} \left\{ \left( \sum_{n > X^3} \frac{A_X(n)}{n^{2 + it} (\log n)^{m+1}} \right)^m \right\}
\]

\[
+ O \left( \left( \sigma_{X,t} - \frac{1}{2} \right)^{m+1} \left| \sum_{n < X^3} \frac{A_X(n)}{n^{2 + it} (\log n)^{m+1-v}} \right| \right)
\]

\[
+ O \left( \left( \sigma_{X,t} - \frac{1}{2} \right)^{m+1} \int_{2}^{X^3} X^{\sigma_{X,t} - \frac{1}{2}} \left( \sum_{p > X^3} \frac{\Lambda_X(p) \log(Xp)}{p^{\sigma+t} \log p} \right) \, d\sigma \right)
\]

\[
= M_m(t) + R_m(t), \quad \text{say}.
\]

As on pp. 180–186 of Fujii [15], we have the following mean value theorem.
LEMMA 4. Suppose that $2 \leq X \leq T^b$ for some positive constant $b$. Then we have for any integer $m \geq 1$ and for any integer $k \geq 1$,
\[ \int_0^T \left| R_m(t) \right|^2 dt \ll T \left( \frac{1}{(\log X)^k} + \frac{(\log \log X)^k}{(\log X)^{2mk}} \right). \]

§3. Proof of Theorem 5

Let $m$ be an integer $\geq 1$. In this section we suppose that $a$ is a positive number $\ll T$ and we put $X = T^b$ and $T_1 = \sqrt{X}$ with an arbitrarily small positive number $\delta$.

By the partial integration, we get
\[ \int_{T_1}^T S(t)I_m(t + a)dt = S(T)I_m(T + a) \quad \text{and} \quad \int_{T_1}^T S(t)W_m(t + a)dt \ll T \log^2 T \ll \sqrt{T}. \]

Using Lemma 4, we get
\[ \int_{T_1}^T R_1(t)M'_m(t + a)dt \ll T \left( \int_{T_1}^T |R_1(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_{T_1}^T |M'_m(t + a)|^2 dt \right)^{\frac{1}{2}} \ll T \sqrt{\frac{\log \log T}{(\log T)^2}} \cdot \left( \int_{0}^{T + a} |M'_m(t)|^2 dt \right)^{\frac{1}{2}} \ll T \frac{(\log \log T)^{\frac{1}{2}(1 + \delta(1, m))}}{(\log T)^{\frac{1}{2}}}, \]

$\delta(1, m)$ being introduced in the statement of Theorem 4, since
\[ \int_0^{T+a} |M'_m(t)|^2 \, dt \ll \int_0^T \left| \sum_{n < X^3} \frac{A_X(n)}{n^{1/2} (\log n)^m} \right|^2 \, dt \]
\[ \ll \sum_{n < X^3} \frac{A^2(n)}{n (\log n)^{2m}} (T + n) \ll T (\log \log X)^{\delta(1,m)}. \]

Using Lemma 4 and Selberg’s mean value theorem on \( S(t) \) mentioned above, we get
\[ \int_{T_1}^T S(t) R_m(t + a) \, dt \ll \left( \int_{T_1}^T |S(t)|^2 \, dt \right)^{1/2} \cdot \left( \int_{T_1}^T |R_m(t + a)|^2 \, dt \right)^{1/2} \]
\[ \ll \sqrt{T \log \log T} \cdot \sqrt{T \left( \frac{1}{(\log T)^2} + \frac{\log \log T}{(\log T)^{2m}} \right)} \ll T \left( \log \log T \right)^{\frac{1}{2}(1+\delta(1,m))}. \]

Finally, we shall evaluate the integral
\[ \int_{T_1}^T M_1(t) M'_m(t + a) \, dt. \]
For \( m = 2\nu \), we have
\[ M_m(t) = \frac{1}{\pi} (-1)^{\nu} \sum_{n < X^3} \frac{A_X(n)}{\sqrt{n (\log n)^{m+1}}} \left( \frac{1}{n^{1/2} - \frac{1}{n-i t}} \right) \]
and for \( m = 2\nu + 1 \), we have
\[ M_m(t) = \frac{1}{\pi} (-1)^{\nu} \sum_{n < X^3} \frac{A_X(n)}{\sqrt{n (\log n)^{m+1}}} \left( \frac{1}{n^{1/2} + \frac{1}{n-i t}} \right). \]
Thus we have for \( m = 2\nu \)
\[ \int_{T_1}^T M_1(t) M'_m(t + a) \, dt = \frac{(-1)^{\nu+1}}{4\pi^2} \int_{T_1}^T \sum_{n < X^3} \frac{A_X(n)}{\sqrt{n (\log n)^{m+1}}} \left( \frac{1}{n^{1/2} + \frac{1}{n-i t}} \right) \]
\[ \cdot \sum_{g < X^3} \frac{\Lambda^2(g)}{\sqrt{g (\log g)^m}} \left( \frac{1}{g^{1/2} + \frac{1}{g^{1/2} (t+a)}} \right) \, dt \]
\[ = \frac{T}{2\pi^2} (-1)^{\nu+1} \sum_{n = 2}^{\infty} \frac{\Lambda^2(n)}{n (\log n)^{m+2}} \cos(a \log n) + O \left( \frac{T}{\log^m T} \right) \]
and for \( m = 2\nu + 1 \)
\[ \int_{T_1}^T M_1(t) M'_m(t + a) \, dt = \frac{(-1)^{\nu+1}}{4\pi^2} \int_{T_1}^T \sum_{n < X^3} \frac{A_X(n)}{\sqrt{n (\log n)^{m+1}}} \left( \frac{1}{n^{1/2} + \frac{1}{n-i t}} \right) \]
\[ \cdot \sum_{g < X^3} \frac{\Lambda^2(g)}{\sqrt{g (\log g)^m}} \left( \frac{1}{g^{1/2} + \frac{1}{g^{1/2} (t+a)}} \right) \, dt \]
\[ \frac{T}{2\pi^2} (-1)^{\nu+1} + \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n(\log n)^{m+2}} \sin(\alpha \log n) + O\left( \frac{T}{\log^m T} \right). \]

Consequently, we get

\[ \int_{C} S(t) S_{m}(t + a) dt = \frac{T}{2\pi^2} (-1)^{\frac{\nu}{2}} \sum_{n=2}^{\infty} \frac{\Lambda(n)^2}{n(\log n)^{m+2}} \cos \left( \alpha \log n - \left\{ \frac{m}{2} \right\} \pi \right) \]

\[ + \int_{C} S(t) W_{m}(t + a) dt + O\left( \frac{T}{\log T} \log \log T \right) \]

This proves our Theorem 5.

§4. Proof of Theorems 1, 2, 3 and 4

We put \( a = \frac{2\pi \alpha}{\log T} \) for any positive \( \alpha \). We suppose that \( m \) is an integer \( \geq 1 \) and that if \( m = 1 \),

\[ 0 < \alpha = O\left( \frac{T}{\log T} \right) \]

and if \( m \geq 2 \),

\[ 0 < \alpha \ll \log T. \]

Now, by the definition of \( S_{m}(T) \) and the Riemann-von Mangoldt formula, we have

\[ \sum_{0 < \gamma \leq T} (S_{m}(\gamma + a) - S_{m}(\gamma)) = \sum_{0 < \gamma \leq T} \int_{\gamma}^{\gamma + a} S_{m-1}(t) dt \]

\[ = \int_{C} (N(t) - N(t - a)) S_{m-1}(t) dt \]

\[ = \int_{C} \left( \frac{1}{\pi} \vartheta(t) - \frac{1}{\pi} \vartheta(t - a) \right) S_{m-1}(t) dt + \int_{C} (S(t) - S(t - a)) S_{m-1}(t) dt \]

\[ = U_1 + U_2, \quad \text{say}. \]

By the partial integration, we get

\[ U_1 = \left[ \left( \frac{1}{\pi} \vartheta(t) - \frac{1}{\pi} \vartheta(t - a) \right) S_{m}(t) \right]^{T+a}_{C} - \int_{C} \left( \frac{1}{\pi} \vartheta'(t) - \frac{1}{\pi} \vartheta'(t - a) \right) S_{m}(t) dt. \]

When \( m = 1 \),

\[ U_1 = O(a(\log T)^2) + O\left( a \int_{C} \frac{|S_{1}(t)|}{t} dt \right) = O(a(\log T)^2), \]

where we have used the estimate

\[ S_{1}(T) \ll \log T. \]
When \( m = 2 \), then
\[
U_1 = \frac{1}{\pi} (\vartheta(T + a) - \vartheta(T)) W_2(T + a) + O\left(a \int_{C}^{T+a} \frac{|S_2(t)|}{t} dt\right) + O(a \log^2 T)
\]
\[
= a \cdot W_2(T + a) + O\left(\frac{T}{\log T}\right) = O\left(\frac{T}{\log T}\right),
\]

where we have used the estimate
\[
I_2(T), \quad S_2(T) \ll \frac{T}{\log T}.
\]

When \( m \geq 3 \), we have
\[
U_1 = \frac{1}{\pi} (\vartheta(T + a) - \vartheta(T)) W_m(T + a)
\]
\[
+ \frac{1}{\pi} \sum_{j=1}^{m-2} \frac{(-a)^j}{j!} \int_{C}^{T+a} \vartheta(j+1)(t) W_m(t) dt + O\left(a \int_{C}^{T+a} \frac{T}{\log T} dt\right) + O(a \log^2 T)
\]
\[
\ll T^{m-1}.
\]

We turn to treat \( U_2 \). We see first that
\[
U_2 = \int_{C}^{T} S(t) S_{m-1}(t) dt - \int_{C}^{T} S(t) S_m(t + a) dt + O(1).
\]

When \( m = 1 \), we have
\[
U_2 = \int_{C}^{T} S^2(t) dt - \int_{C}^{T} S(t) S(t + a) dt + O(1)
\]
\[
= \frac{1}{2} \int_{0}^{T} (S(t + a) - S(t))^2 dt + O(\alpha T^2).
\]

Thus when \( m = 1 \), we get
\[
\sum_{0 < \gamma \leq T} (S_{1}(\gamma + a) - S_{1}(\gamma)) = \frac{1}{2} \int_{0}^{T} (S(t + a) - S(t))^2 dt + O(\alpha T^2).
\]

Combining this with our mean value theorem of \( S(t) \) for short interval which has been mentioned in the introduction, we get our conclusion as stated in Theorems 1 and 2.

When \( m = 2 \), we apply Theorem 5 and get
\[
U_2 = \left[\frac{1}{2} S^2(t)\right]_{C}^{T} - \int_{C}^{T} S(t) S_1(t + a) dt + O(1)
\]
\[
= -\frac{T}{2\pi^2} \sum_{n=2}^{\infty} \frac{\Lambda(n)^2}{n(\log n)^3} \sin(\alpha \log n) + O\left(\frac{T}{\log T} \log \log T\right),
\]

since we have
\[
W_1(T) = 0.
\]
Thus we get for $m = 2$

$$
\sum_{0<\gamma\leq T} (S_2(\gamma + a) - S_2(\gamma)) = -\frac{T}{2\pi^2} \sum_{n=2}^{\infty} \frac{\Lambda(n)^2}{n(n\log n)^3} \sin(a \log n) + \alpha \cdot W_2(T + a) + O\left(\frac{T}{\log T} \log \log T\right)
$$

This proves our Theorem 3 and Corollary 1.

Finally, for $m \geq 3$, by applying Theorem 5, we get

$$
U_2 = -\frac{T}{2\pi^2} (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{n=2}^{\infty} \frac{\Lambda(n)^2}{n(n\log n)^{m+1}} \left( \cos \left( a \log n - \left\{ \frac{m-1}{2} \right\} \pi \right) 
- \cos \left( -\left\{ \frac{m-1}{2} \right\} \pi \right) \right) 
+ \int_C^{T+a} S(t) W_{m-1}(t) dt - \int_C^{T} S(t) W_{m-1}(t+a) dt 
+ O\left(\frac{T}{\log T} \log \log T \right)^{\frac{3}{2}} \left(1 + \delta(1, m-1)\right).
$$

Thus we get for $m \geq 3$,

$$
\sum_{0<\gamma\leq T} (S_m(\gamma + a) - S_m(\gamma)) = -\frac{T}{2\pi^2} (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{n=2}^{\infty} \frac{\Lambda(n)^2}{n(n\log n)^{m+1}} \left( \cos \left( a \log n - \left\{ \frac{m-1}{2} \right\} \pi \right) 
- \cos \left( -\left\{ \frac{m-1}{2} \right\} \pi \right) \right) 
+ \frac{1}{\pi} \sum_{j=1}^{m-2} \frac{(-a)^j}{j!} \int_C^{T+a} \vartheta(j+1)(t) W_m(t) dt 
+ \int_C^{T+a} S(t) W_{m-1}(t) dt - \int_C^{T} S(t) W_{m-1}(t+a) dt 
+ O\left(\frac{T}{\log T} \log \log T \right)^{\frac{3}{2}} \left(1 + \delta(1, m-1)\right).
$$

This is our Theorem 4.
§5. Proof of Theorems 6 and 7

We suppose that \( k \geq 2 \) and \( a \) is positive and bounded, namely, \( 0 < \alpha \ll \log T \). In this section we put \( X = T^\delta \) with an arbitrarily small positive \( \delta \) and \( T_1 = \sqrt{X} \).

By the Riemann-von Mangoldt formula, we get first

\[
\sum_{0 < \gamma \leq T} (S_1(\gamma + a) - S_1(\gamma))^k = \int_C (S_1(t + a) - S_1(t))^k \frac{1}{\pi} \partial'(t) dt + [(S_1(t + a) - S_1(t))^k S(t)]_C^T \\
- k \int_C (S_1(t + a) - S_1(t))^{k-1} (S(t + a) - S(t)) S(t) dt
\]

\[= V_1 + V_2 - kV_3, \quad \text{say.} \]

We get, simply,

\[ V_2 \ll \log^{k+1} T. \]

We shall evaluate first \( V_1 \). Using Lemma 3, we get

\[
V_1 = \int_{T_1}^T (M_1(t + a) - M_1(t))^k \frac{1}{2\pi} \log \frac{t}{2\pi} dt \\
+ \sum_{r=0}^{k-1} \binom{k}{r} \int_{T_1}^T (M_1(t + a) - M_1(t))^r (R_1(t + a) - R_1(t))^{k-r} \frac{1}{2\pi} \log \frac{t}{2\pi} dt \\
+ O(T_1 \log^{k+1} T) \\
= V_4 + V_5 + O(T_1 \log^{k+1} T), \quad \text{say.}
\]

By Schwarz inequality, we get

\[ V_5 \ll \log T \cdot \sum_{r=0}^{k-1} \binom{k}{r} \left( \int_{T_1}^T (M_1(t + a) - M_1(t))^{2r} dt \right)^{\frac{1}{2}} \\
\cdot \left( \int_{T_1}^T (R_1(t + a) - R_1(t))^{2(k-r)} dt \right)^{\frac{1}{2}}. \]

By Lemma 4, we get

\[ \int_{T_1}^T (R_1(t + a) - R_1(t))^{2(k-r)} dt \ll T (\log X)^{k-r} \]

Next, we get for any integer \( 2 \leq R \leq 2k \)

\[ \int_0^T (M_1(t + a) - M_1(t))^R dt \]

\[= \frac{1}{(2\pi)^R} \int_0^T \left( \frac{A_X(m)}{\sqrt{m (\log m)^2}} \left\{ \frac{1}{m^it} \left( \frac{1}{m^{i\alpha}} - 1 \right) + \frac{1}{m^{-i\pi}} \left( \frac{1}{m^{-i\alpha}} - 1 \right) \right\} \right)^R dt \]
Thus we get

\[
\int_0^T \frac{1}{m_1^ia_1} \cdot \left( \frac{1}{m_1^ia_1} - 1 \right) \cdot \frac{1}{m_{\mu+1}^ia_{\mu+1}} \cdot \left( \frac{1}{m_{\mu+1}^ia_{\mu+1}} - 1 \right) \cdot \frac{1}{m_R^ia_R} \cdot \left( \frac{1}{m_R^ia_R} - 1 \right) \, dt
\]

\[
= \frac{T}{(2\pi)^k} \sum_{\mu=0}^R \frac{\Lambda_X(m_1) \cdots \Lambda_X(m_R)}{m_1 \cdots m_R (\log m_1)^2 \cdots (\log m_R)^2}
\]

\[
= \frac{T}{(2\pi)^k} \sum_{\mu=0}^R \frac{\Lambda_X(m_1) \cdots \Lambda_X(m_R)}{m_1 \cdots m_R (\log m_1)^2 \cdots (\log m_R)^2}
\]

\[
\cdot \left( \frac{1}{m_1^ia_1} - 1 \right) \cdot \left( \frac{1}{m_{\mu+1}^ia_{\mu+1}} - 1 \right) \cdot \left( \frac{1}{m_R^ia_R} - 1 \right) + O \left( \frac{T}{\log^2 T} \right)
\]

Thus we get

\[
V_5 \ll \log T \sum_{k=0}^{k-1} \binom{k}{r} T \frac{\log \log X}{(\log X)^{k-r}} \ll T \sqrt{\log \log T}.
\]

In a similar manner, we get

\[
V_4 = \frac{1}{(2\pi)^k} \int_{T_1} \left( \sum_{m \in X^3} \frac{\Lambda_X(m)}{\sqrt{m (\log m)^2}} \left( \frac{1}{m_{\mu+1}^ia_{\mu+1}} - 1 \right) \right)^k
\]

\[
= \frac{1}{(2\pi)^k} \sum_{\mu=0}^k \binom{k}{\mu} \sum_{m_1, \cdots, m_k \in X^3} \frac{\Lambda_X(m_1) \cdots \Lambda_X(m_k)}{m_1 \cdots m_k (\log m_1)^2 \cdots (\log m_k)^2}
\]

\[
\cdot \int_{T_1} \frac{1}{m_1^ia_1} \cdot \left( \frac{1}{m_1^ia_1} - 1 \right) \cdots \frac{1}{m_R^ia_R} \cdot \left( \frac{1}{m_R^ia_R} - 1 \right)
\]
Using these notations, we decompose

\[ V_1 = \frac{1}{m_1^{\nu_1}} \left( \frac{1}{m_1^{\nu_1} - 1} \right) \cdots \frac{1}{m_k^{\nu_k}} \left( \frac{1}{m_k^{\nu_k} - 1} \right) \frac{1}{2\pi} \log \frac{t}{2\pi} dt \]

\[ = \frac{\partial (T)}{2^{k+1} \pi^{k+1}} \sum_{\mu=0}^{k} \left( k \right) \sum_{m_1 \cdots m_k \in \mathbb{Z}^3} \frac{A_{\chi} (m_1) \cdots A_{\chi} (m_k)}{m_1 \cdots m_k (\log m_1)^2 \cdots (\log m_k)^2} \cdot \left( \frac{1}{m_1^{\nu_1}} - 1 \right) \cdots \left( \frac{1}{m_k^{\nu_k}} - 1 \right) + O(1) \]

\[ = \frac{\partial (T)}{2^{k+1} \pi^{k+1}} \sum_{\mu=0}^{k} \left( k \right) \sum_{m_1 \cdots m_k \in \mathbb{Z}^3} \frac{A_{\chi} (m_1) \cdots A_{\chi} (m_k)}{m_1 \cdots m_k (\log m_1)^2 \cdots (\log m_k)^2} \cdot \left( \frac{1}{m_1^{\nu_1}} - 1 \right) \cdots \left( \frac{1}{m_k^{\nu_k}} - 1 \right) + O(1) \]

Hence we get

\[ V_1 = \frac{\partial (T)}{2^{k+1} \pi^{k+1}} \sum_{\mu=0}^{k} \left( k \right) \sum_{m_1 \cdots m_k \in \mathbb{Z}^3} \frac{A_{\chi} (m_1) \cdots A_{\chi} (m_k)}{m_1 \cdots m_k (\log m_1)^2 \cdots (\log m_k)^2} \cdot \left( \frac{1}{m_1^{\nu_1}} - 1 \right) \cdots \left( \frac{1}{m_k^{\nu_k}} - 1 \right) + O(1) \]

We shall next evaluate \( V_3 \). For this purpose, we put

\[ M(t) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}^3} \frac{A_{\chi} (n)}{n^{2+it} \log n} \]

and

\[ R(t) = S(t) - M(t). \]

Using these notations, we decompose \( V_3 \) further as follows.

\[ V_3 = \int_{T_1}^{T} \sum_{r=0}^{k-1} \binom{k-1}{r} (M_1(t+a) - M_1(t))^r \cdot (R_1(t+a) - R_1(t))^{k+1-r} \]

\[ \cdot (S(t+a) - S(t)) S(t) dt + O(T_1 \log^{k+1} T) \]

\[ = V_6 + V_7 + V_8 + O(T_1 \log^{k+1} T), \quad \text{say}, \]
where we put

\[
V_6 = \int_{T_1}^T (M_1(t + a) - M_1(t))^{k-1}(M(t + a) - M(t))M(t)dt,
\]

\[
V_7 = \int_{T_1}^T (M_1(t + a) - M_1(t))^{k-1}(M(t + a) - M(t))R(t)dt
\]

+ \int_{T_1}^T (M_1(t + a) - M_1(t))^{k-1}(R(t + a) - R(t))M(t)dt

+ \int_{T_1}^T (M_1(t + a) - M_1(t))^{k-1}(R(t + a) - R(t))R(t)dt = \sum_{l=1}^3 V_{7,l}, \quad \text{say},
\]

and

\[
V_8 = \sum_{r=0}^{k-2} \binom{k-1}{r} \int_{T_1}^T (M_1(t + a) - M_1(t))^r (R_1(t + a) - R_1(t))^{k-1-r}
\]

\[
\cdot (M(t + a) - M(t))M(t)dt
\]

+ \sum_{r=0}^{k-2} \binom{k-1}{r} \int_{T_1}^T (M_1(t + a) - M_1(t))^r (R_1(t + a) - R_1(t))^{k-1-r}
\]

\[
\cdot (M(t + a) - M(t))R(t)dt
\]

+ \sum_{r=0}^{k-2} \binom{k-1}{r} \int_{T_1}^T (M_1(t + a) - M_1(t))^r (R_1(t + a) - R_1(t))^{k-1-r}
\]

\[
\cdot (R(t + a) - R(t))M(t)dt
\]

+ \sum_{r=0}^{k-2} \binom{k-1}{r} \int_{T_1}^T (M_1(t + a) - M_1(t))^r (R_1(t + a) - R_1(t))^{k-1-r}
\]

\[
\cdot (R(t + a) - R(t))R(t)dt = \sum_{l=1}^4 V_{8,l}, \quad \text{say}.
\]

By Theorem 4 on pp. 252-254 of Selberg [26], we get

\[
\int_{T_1}^T |R(t)|^4 dt \ll T
\]

and

\[
\int_{T_1}^T |M(t)|^4 dt \ll T(\log \log T)^2.
\]

By p. 233 of Fujii [3], we get

\[
\int_{T_1}^T (M(t + a) - M(t))^4 dt \ll T(\log(3 + a \log T))^2.
\]
Hence, we get

\[ V_{7,1} \ll \left( \int_{T_1}^T (M_1(t + a) - M_1(t))^{2(k-1)} dt \right)^{\frac{1}{2}} \]

\[ \cdot \left( \int_{T_1}^T (M(t + a) - M(t))^4 dt \right)^{\frac{1}{4}} \cdot \left( \int_{T_1}^T R(t)^4 dt \right)^{\frac{1}{4}} \]

\[ \ll T \sqrt{\frac{\log(3 + a \log T)}{\log \log T}} \]

\[ V_{7,2} \ll \left( \int_{T_1}^T (M_1(t + a) - M_1(t))^{2(k-1)} dt \right)^{\frac{1}{2}} \]

\[ \cdot \left( \int_{T_1}^T (R(t + a) - R(t))^4 dt \right)^{\frac{1}{4}} \cdot \left( \int_{T_1}^T M(t)^4 dt \right)^{\frac{1}{4}} \]

\[ \ll T \sqrt{\frac{\log \log T}{\log \log T}} \]

and

\[ V_{7,3} \ll \left( \int_{T_1}^T (M_1(t + a) - M_1(t))^{2(k-1)} dt \right)^{\frac{1}{2}} \]

\[ \cdot \left( \int_{T_1}^T (R(t + a) - R(t))^4 dt \right)^{\frac{1}{4}} \cdot \left( \int_{T_1}^T R(t)^4 dt \right)^{\frac{1}{4}} \]

\[ \ll T \sqrt{\frac{\log \log T}{\log \log T}} \]

Thus we get

\[ V_7 \ll T \sqrt{\frac{\log \log T}{\log \log T}} \]

Similarly, we get

\[ V_{8,1} \ll \sum_{r=0}^{k-2} \binom{k-1}{r} \left( \int_{T_1}^T |(M_1(t + a) - M_1(t))|^{4r} dt \right)^{\frac{1}{4}} \]

\[ \cdot \left( \int_{T_1}^T |(R_1(t + a) - R_1(t))|^{4(k-1-r)} dt \right)^{\frac{1}{4}} \]

\[ \cdot \left( \int_{T_1}^T |(M(t + a) - M(t))|^{4} dt \right)^{\frac{1}{4}} \cdot \left( \int_{T_1}^T |M(t)|^{4} dt \right)^{\frac{1}{4}} \]

\[ \ll \sum_{r=0}^{k-2} \binom{k-1}{r} T \left( \frac{(\log \log X)^{2(k-1-r)}}{(\log X)^{4(k-1-r)}} \cdot (\log(3 + a \log T))^2 (\log \log T)^2 \right)^{\frac{1}{4}} \]

\[ \ll T \frac{(\log \log T)^{\frac{3}{2}}}{\log T} . \]
In the same manner, we get
\[ V_{8,2} \ll \sum_{r=0}^{k-2} \left( \frac{k-1}{r} \right) \left( \int_{T_1}^{T} \left( |M_1(t + a) - M_1(t)|^{4r} \right) dt \right)^{\frac{1}{4}} \]
\[ \cdot \left( \int_{T_1}^{T} |(R_1(t + a) - R_1(t))|^{4(1-r)} dt \right)^{\frac{1}{4}} \]
\[ \cdot \left( \int_{T_1}^{T} |(M(t + a) - M(t))|^{4} dt \right)^{\frac{1}{4}} \cdot \left( \int_{T_1}^{T} |R(t)|^{4} dt \right)^{\frac{1}{4}} \]
\[ \ll \sum_{r=0}^{k-2} \left( \frac{k-1}{r} \right) T \left( \frac{\left( \log \log X \right)^{2(k-1-r)}}{\left( \log X \right)^{4(k-1-r)}} \cdot \left( \log(3 + a \log T) \right)^{2} \right)^{\frac{1}{4}} \ll T \log \log T / \log T . \]

And
\[ V_{8,3} \ll \sum_{r=0}^{k-2} \left( \frac{k-1}{r} \right) T \left( \frac{\left( \log \log X \right)^{2(k-1-r)}}{\left( \log X \right)^{4(k-1-r)}} \cdot \left( \log \log T \right)^{2} \right)^{\frac{1}{4}} \ll T \log \log T / \log T . \]

Consequently, we get
\[ V_{8} \ll T \left( \log \log T \right)^{\frac{3}{2}} / \log T . \]

Thus we are left to evaluate \( V_{6} \). We suppose first that \( k \geq 3 \). We decompose \( V_{6} \) further as follows.
\[ V_{6} = -\frac{1}{4\pi^{2}} \int_{0}^{T} (M_1(t + a) - M_1(t))^{k-1} \]
\[ \cdot \left( \sum_{n \leq X^3, n \leq p_q, 1 \leq j \leq 2} \frac{A_X(n)}{n \log n} \left\{ \frac{1}{n^{it}} \left( \frac{1}{n^{ia}} - 1 \right) - \frac{1}{n^{-it}} \left( \frac{1}{n^{-ia}} - 1 \right) \right\} + O(1) \right) \]
\[ \cdot \left( \sum_{g \leq X^3, g \leq p_q, 1 \leq j \leq 2} \frac{A_X(g)}{g \log g} \left\{ \frac{1}{g^{it}} - \frac{1}{g^{-it}} \right\} + O(1) \right) dt = \sum_{l=1}^{6} V_{6,l} , \]

where \( p \) and \( q \) run over the prime numbers and we put
\[ V_{6,1} = -\frac{1}{2\pi i} \int_{0}^{T} \left( \sum_{m \leq X^3} \frac{A_X(m)}{\sqrt{m} \log m} \left\{ \frac{1}{m^{it}} \left( \frac{1}{m^{ia}} - 1 \right) + \frac{1}{m^{-it}} \left( \frac{1}{m^{-ia}} - 1 \right) \right\} \right)^{k-1} \]
\[ \cdot \left( \sum_{p \leq X^3} \frac{A_X(p)}{p \sqrt{p} \log p} \left\{ \frac{1}{p^{ia}} - 1 \right\} \right) \]
\[
\left( \sum_{q < X^3} \frac{A_X(q)}{q \log q} \left\{ \frac{1}{q^{it}} - \frac{1}{q^{-it}} \right\} \right) dt, \\
V_{6.2} = -\frac{1}{4\pi^2} \int_0^T (M_1(t + a) - M_1(t))^{k-1} \\
\cdot \left( \sum_{p < X^3} \frac{A_X(p)}{p \log p} \left\{ \frac{1}{p^{it}} \left( \frac{1}{p^{ia}} \right) - \frac{1}{p^{it}} \left( \frac{1}{p^{-ia}} \right) \right\} \right) \\
\cdot \left( \sum_{q^2 < X^3} \frac{A_X(q^2)}{q \log q^2} \left\{ \frac{1}{q^{it}} - \frac{1}{q^{-it}} \right\} \right) dt, \\
V_{6.3} = -\frac{1}{4\pi^2} \int_0^T (M_1(t + a) - M_1(t))^{k-1} \\
\cdot \left( \sum_{p^2 < X^3} \frac{A_X(p^2)}{p \log p^2} \left\{ \frac{1}{p^{it}} \left( \frac{1}{p^{2ia}} \right) - \frac{1}{p^{it}} \left( \frac{1}{p^{-2ia}} \right) \right\} \right) \\
\cdot \left( \sum_{q < X^3} \frac{A_X(q)}{q \log q} \left\{ \frac{1}{q^{it}} - \frac{1}{q^{-it}} \right\} \right) dt, \\
V_{6.4} = -\frac{1}{4\pi^2} \int_0^T (M_1(t + a) - M_1(t))^{k-1} \\
\cdot \left( \sum_{p^2 < X^3} \frac{A_X(p^2)}{p \log p^2} \left\{ \frac{1}{p^{it}} \left( \frac{1}{p^{2ia}} \right) - \frac{1}{p^{it}} \left( \frac{1}{p^{-2ia}} \right) \right\} \right) \\
\cdot \left( \sum_{q^2 < X^3} \frac{A_X(q^2)}{q \log q^2} \left\{ \frac{1}{q^{it}} - \frac{1}{q^{-it}} \right\} \right) dt, \\
V_{6.5} = O \left( \int_C |(M_1(t + a) - M_1(t))^{k-1}|M(t)|dt \right)
\] and

\[
V_{6.6} = O \left( \int_C |(M_1(t + a) - M_1(t))^{k-1}||M(t + a) - M(t)||dt \right).
\]

We get first

\[
V_{6.2} \ll \left( \int_0^T |M_1(t + a) - M_1(t)|^{4(k-1)} dt \right)^{\frac{1}{4}} \\
\cdot \left( \int_0^T \left| \sum_{p < X^3} \frac{A_X(p)}{p \log p} \left\{ \frac{1}{p^{it}} \left( \frac{1}{p^{ia}} \right) - \frac{1}{p^{it}} \left( \frac{1}{p^{-ia}} \right) \right\} \right|^{4} dt \right)^{\frac{1}{4}}
\]
Similarly, we have
\[ V_{6,3} + V_{6,4} \ll T \sqrt{\log \log T} \]

Similarly, we have
\[ V_{6,5} \ll \int_C^T |(M_1(t + a) - M_1(t))|^{2(k-1)} dt \] \[ \ll T \sqrt{\log \log T} \]

Finally, we shall evaluate \( V_{6,1} \).

\[ V_{6,1} = \frac{1}{(2\pi)^k+1} \int_0^T \left( \sum_{m < X^3} \frac{A_X(m)}{\sqrt{m(\log m)}} \left[ \frac{1}{m} \left( \frac{1}{m^{ia}} - 1 \right) + \frac{1}{m^{-it}} \left( \frac{1}{m^{-ia}} - 1 \right) \right] \right)^{k-1} \]

\[ \cdot \left( \sum_{p < X^3} \frac{A_X(p)}{\sqrt{p(\log p)}} \left[ \frac{1}{p} \left( \frac{1}{p^{it}} - 1 \right) - \frac{1}{p^{-it}} \left( \frac{1}{p^{-ia}} - 1 \right) \right] \right)^{k-1} \]

\[ + \frac{1}{(2\pi)^k+1} \int_0^T \left( \sum_{m < X^3} \frac{A_X(m)}{\sqrt{m(\log m)}} \left[ \frac{1}{m} \left( \frac{1}{m^{ia}} - 1 \right) + \frac{1}{m^{-it}} \left( \frac{1}{m^{-ia}} - 1 \right) \right] \right)^{k-1} \]

\[ \cdot \left( \sum_{p < X^3} \frac{A_X(p)}{\sqrt{p(\log p)}} \left[ \frac{1}{p} \left( \frac{1}{p^{it}} - 1 \right) - \frac{1}{p^{-it}} \left( \frac{1}{p^{-ia}} - 1 \right) \right] \right)^{k-1} \]

\[ = V_0 + V_{10} + V_{11} + V_{12} \] say.
We see first that

\[ V_0 = \frac{T}{(2\pi)^{k+1}} \sum_{\mu=0}^{k-1} \binom{k-1}{\mu} \sum_{p,q < X^3} \frac{A_X(p)}{\sqrt{\log p}} \frac{A_X(q)}{\sqrt{\log q}} \left( \sum_{m_1 \cdots m_{k-1} < X^3} \frac{A_X(m_1) \cdots A_X(m_{k-1})}{\sqrt{m_1 \cdots m_{k-1} \log m_1^2 \cdots \log m_{k-1}^2}} \right) \]

\[ \cdot \left( \frac{1}{p^{1/a}} - 1 \right) \left( \frac{1}{m_1^{1/a}} - 1 \right) \cdots \left( \frac{1}{m_\mu^{1/a}} - 1 \right) \left( \frac{1}{m_{ \mu+1}^{1/a}} - 1 \right) \cdots \left( \frac{1}{m_{k-1}^{1/a}} - 1 \right) \]

\[ + O \left( \sum_{\mu=0}^{k-1} \binom{k-1}{\mu} \sum_{p,q < X^3} \frac{A_X(p)}{\sqrt{\log p}} \frac{A_X(q)}{\sqrt{\log q}} \left( \sum_{m_1 \cdots m_{k-1} < X^3} \frac{A_X(m_1) \cdots A_X(m_{k-1})}{\sqrt{m_1 \cdots m_{k-1} \log m_1^2 \cdots \log m_{k-1}^2}} \right) \right) \]

\[ \cdot \left( \frac{1}{p^{1/a}} - 1 \right) \left( \frac{1}{m_1^{1/a}} - 1 \right) \cdots \left( \frac{1}{m_\mu^{1/a}} - 1 \right) \left( \frac{1}{m_{ \mu+1}^{1/a}} - 1 \right) \cdots \left( \frac{1}{m_{k-1}^{1/a}} - 1 \right) \]

\[ + O(T) \]

\[ = V_{13} + V_{14} + O(T), \quad \text{say}. \]
We see next that

\[ V_{10} = \frac{T}{(2\pi)^{k+1}} \sum_{\mu=0}^{k-1} \binom{k-1}{\mu} \sum_{p, q < X^3} A_X(p) \frac{A_X(q)}{\sqrt{p} \log p \sqrt{q} \log q} \]

\[ \cdot \sum_{q^m \cdots m_{k-1} < X^3, q^m \cdots m_{k-1} \neq m_{k-1}} \frac{A_X(m_1) \cdots A_X(m_{k-1})}{\sqrt{m_1} \cdots \sqrt{m_{k-1}} (\log m_1)^2 \cdots (\log m_{k-1})^2} \]

\[ \cdot \left( \frac{1}{p^{-ia}} - 1 \right) \left( \frac{1}{m_1^{\mu}} - 1 \right) \cdots \left( \frac{1}{m_{k}^{\mu}} - 1 \right) \left( \frac{1}{m_{k+1}^{\mu}} - 1 \right) \cdots \left( \frac{1}{m_{k+1}^{\mu}} - 1 \right) \]

\[ + O(T) \]

\[ = \frac{T}{(2\pi)^{k+1}} \sum_{\mu=0}^{k-1} \binom{k-1}{\mu} \sum_{p < X^3} \frac{A_X^2(p)}{p \log^2 p} \]

\[ \cdot \sum_{q^m \cdots m_{k-1} < X^3, q^m \cdots m_{k-1} \neq m_{k-1}} \frac{A_X(m_1) \cdots A_X(m_{k-1})}{\sqrt{m_1} \cdots \sqrt{m_{k-1}} (\log m_1)^2 \cdots (\log m_{k-1})^2} \]

\[ \cdot \left( \frac{1}{p^{-ia}} - 1 \right) \left( \frac{1}{m_1^{\mu}} - 1 \right) \cdots \left( \frac{1}{m_{k}^{\mu}} - 1 \right) \left( \frac{1}{m_{k+1}^{\mu}} - 1 \right) \cdots \left( \frac{1}{m_{k+1}^{\mu}} - 1 \right) \]

\[ + T \sum_{\mu=0}^{k-1} \binom{k-1}{\mu} \sum_{p, q < X^3, p \neq q} \frac{A_X(p) A_X(q)}{\sqrt{p} \log p \sqrt{q} \log q} \]

\[ \cdot \sum_{q^m \cdots m_{k-1} < X^3, q^m \cdots m_{k-1} \neq m_{k-1}} \frac{A_X(m_1) \cdots A_X(m_{k-1})}{\sqrt{m_1} \cdots \sqrt{m_{k-1}} (\log m_1)^2 \cdots (\log m_{k-1})^2} \]

\[ \cdot \left( \frac{1}{p^{-ia}} - 1 \right) \left( \frac{1}{m_1^{\mu}} - 1 \right) \cdots \left( \frac{1}{m_{k}^{\mu}} - 1 \right) \left( \frac{1}{m_{k+1}^{\mu}} - 1 \right) \cdots \left( \frac{1}{m_{k+1}^{\mu}} - 1 \right) \]

\[ + O(T) \]

\[ = V_{15} + V_{16} + O(T), \quad \text{say.} \]

V_{14} and V_{16} can be treated as follows.

\[ V_{14} \ll T \sum_{\mu=0}^{k-1} \binom{k-1}{\mu} \sum_{p < X^3} \frac{1}{p} \sum_{q < X^3} \sum_{l=1}^{\mu} \frac{A_X(m_l) m_l^\mu}{(\log m_l)^2} \cdot \prod_{l=1}^{k-1} \sum_{m < X^3} \frac{A_X(m_h) m_h^\mu}{(\log m_h)^2} \]

\[ \cdot \sum_{j=\mu+1}^{k-1} \sum_{m_j < X^3} \frac{A_X(m_j)}{(\log m_j)^2} \cdot \sum_{p | m_j} \sum_{m_j \neq m_l} \frac{1}{p | m_j} \frac{A_X(m_j) m_j^\mu}{(\log m_j)^2} \cdot \prod_{l=1}^{k-1} \sum_{m < X^3} \frac{A_X(m_d) m_d^\mu}{(\log m_d)^2} \]
Similarly, we get

\[
V_{16} \ll T.
\]

\[V_{13} + V_{15} \text{ can be written in a simpler form as follows.}
\]

\[
V_{13} + V_{15} = \frac{2T}{(2\pi)^{k+1}} \sum_{\mu=0}^{k-1} \left( \frac{k-1}{\mu} \right) \sum_{p} \frac{1}{p \log p} \cdot \sum_{1 < q < X^3} \frac{1}{q^2 \log q} \cdot \sum_{i=1}^{\mu} \prod_{h=1}^{k_i} \sum_{m_h < X} A_X(m_h) m_h \log(m_h)^2
\]

\[
\cdot \sum_{j=0}^{k-1 \mu+1} \sum_{m_j < X} \frac{A_X(m_j)}{(\log m_j)^2}
\]

\[\ll T.
\]

\[V_{16} \ll T.
\]

\[V_{13} + V_{15} \text{ can be written in a simpler form as follows.}
\]

\[
V_{13} + V_{15} = \frac{2T}{(2\pi)^{k+1}} \sum_{\mu=0}^{k-1} \left( \frac{k-1}{\mu} \right) \sum_{p} \frac{1}{p \log p} \cdot \sum_{1 < q < X^3} \frac{1}{q^2 \log q} \cdot \sum_{i=1}^{\mu} \prod_{h=1}^{k_i} \sum_{m_h < X} A_X(m_h) m_h \log(m_h)^2
\]

\[
\cdot \sum_{j=0}^{k-1 \mu+1} \sum_{m_j < X} \frac{A_X(m_j)}{(\log m_j)^2}
\]

\[\ll T.
\]
\[ \sum_{m_1 \cdots m_{k-1} \leq X^\delta} \frac{\Lambda_X(m_1) \cdots \Lambda_X(m_{k-1})}{m_1 \cdots m_{\mu} (\log m_1)^2 \cdots (\log m_{k-1})^2} \]

\[ \cdot \left( \frac{1}{p^{ia}} - 1 \right) \left( \frac{1}{m_1^{ia}} - 1 \right) \cdots \left( \frac{1}{m_{\mu}^{ia}} - 1 \right) \left( \frac{1}{m_1^{-ia}} - 1 \right) \cdots \left( \frac{1}{m_{k-1}^{-ia}} - 1 \right) + O(T) \]

\[ \ll T \sum_{\mu=0}^{k-1} \sum_{p < X^\delta} \sum_{j=\mu+1}^{k-1} \frac{1}{p^j} \log p \sum_{m_j = p^j \cdot X^\delta} \sum_{r \geq 1} \frac{1}{p^r} \log p \sum_{m_{\mu} = m_1 \cdots m_{\mu} \leq m_{\mu}+1 \cdots m_{k-1}} \prod_{d \neq j} \Lambda(m_d) \left( \log m_d \right)^2 \]

\[ \cdot \frac{\Lambda(m_d)}{(\log m_d)^2} \cdot \frac{\Lambda(m_1) \cdots \Lambda(m_{\mu})}{(\log m_1)^2 \cdots (\log m_{\mu})^2} \]

\[ = O(T). \]

In the same manner, we get

\[ V_{12} = \frac{T}{(2\pi)^{k+1}} \sum_{\mu=0}^{k-1} \left( \frac{k-1}{\mu} \right) \sum_{p, q < X^\delta} \frac{\sqrt{\Lambda_X(p)}}{p} \frac{\sqrt{\Lambda_X(q)}}{q} \log \frac{\sqrt{\Lambda_X(m_1) \cdots \Lambda_X(m_{k-1})}}{\sqrt{m_1 \cdots m_{k-1} (\log m_1)^2 \cdots (\log m_{k-1})^2}} \]

\[ \cdot \left( \frac{1}{p^{ia}} - 1 \right) \left( \frac{1}{m_1^{ia}} - 1 \right) \cdots \left( \frac{1}{m_{\mu}^{ia}} - 1 \right) \left( \frac{1}{m_1^{-ia}} - 1 \right) \cdots \left( \frac{1}{m_{k-1}^{-ia}} - 1 \right) + O(T) \]

\[ = O(T). \]

Consequently, we get

\[ V_{b,1} = \frac{-2T}{(2\pi)^{k+1}} \int_0^{2\pi} \frac{1 - \cos(y)}{y} dy \]

\[ \cdot \sum_{\mu=0}^{k-1} \left( \frac{k-1}{\mu} \right) \sum_{m_1 \cdots m_{\mu} \leq \mu+1 \cdots m_{k-1}} \frac{\Lambda(m_1) \cdots \Lambda(m_{k-1})}{m_1 \cdots m_{\mu} (\log m_1)^2 \cdots (\log m_{k-1})^2} \]

\[ \cdot \left( \frac{1}{m_1^{ia}} - 1 \right) \cdots \left( \frac{1}{m_1^{-ia}} - 1 \right) \left( \frac{1}{m_{\mu}^{-ia}} - 1 \right) \cdots \left( \frac{1}{m_{k-1}^{-ia}} - 1 \right) + O(T). \]

Combining all of these evaluations, we get, for \( k \geq 3, \)

\[ \sum_{0 < \gamma \leq T} (S_1(\gamma + a) - S_1(\gamma))^k \]
This is our Theorem 7.

We suppose next that $k = 2$ and evaluate $V_h$. In this case, we have

$$
\int_C (M(t + a) - M_1(t))(M(t + a) - M(t))M(t)dt
$$

$$
= \frac{1}{8\pi^3} \int_0^T \left( \sum_{m < X^3} \frac{A_X(m)}{\sqrt{m} \log n} \sum_{g < X} \frac{A_X(n)}{\sqrt{g} \log g} \right) dt
$$

$$
= \frac{T}{8\pi^3} \sum_{m < X^3} \frac{A_X(m)}{\sqrt{m} \log m} \sum_{n < X^3} \frac{A_X(n)}{\sqrt{n} \log n} \sum_{g < X, g = mn} \frac{A_X(g)}{\sqrt{g} \log g}
$$

$$
\times \left\{ \left( \frac{1}{m^t} - 1 \right) \cdot \left( \frac{1}{n^t} - 1 \right) + \left( \frac{1}{m^t} - 1 \right) \cdot \left( \frac{1}{n^t} - 1 \right) \right\} \cdot \left\{ \frac{1}{g^t} - \frac{1}{g^{-t}} \right\} dt
$$

This is our Theorem 7.
\[ \cdot \left\{ \left( \frac{1}{m^{ia}} - 1 \right) \cdot \left( \frac{1}{n^{ia}} - 1 \right) + \left( \frac{1}{m^{-ia}} - 1 \right) \cdot \left( \frac{1}{n^{-ia}} - 1 \right) \right\} \]

\[ - \frac{T}{8\pi^3} \sum_{m < X^3, m = ng} \frac{\Lambda_X(m)}{\sqrt{m} \log m} \sum_{n < X^3} \frac{\Lambda_X(n)}{\sqrt{n} \log n} \sum_{g < X^3} \frac{\Lambda_X(g)}{\sqrt{g} \log g} \]

\[ \cdot \left\{ \left( \frac{1}{m^{ia}} - 1 \right) \cdot \left( \frac{1}{n^{ia}} - 1 \right) + \left( \frac{1}{m^{-ia}} - 1 \right) \cdot \left( \frac{1}{n^{-ia}} - 1 \right) \right\} + O(T) = O(T), \]

where we notice, for example, that

\[ \sum_{m < X^3} \frac{\Lambda_X(m)}{\sqrt{m} \log m} \sum_{n < X^3} \frac{\Lambda_X(n)}{\sqrt{n} \log n} \sum_{g < X^3, g = mn} \frac{\Lambda_X(g)}{\sqrt{g} \log g} \]

\[ \cdot \left\{ \left( \frac{1}{m^{ia}} - 1 \right) \cdot \left( \frac{1}{n^{ia}} - 1 \right) + \left( \frac{1}{m^{-ia}} - 1 \right) \cdot \left( \frac{1}{n^{-ia}} - 1 \right) \right\} \]

\[ = \sum_{p^j < X^3, j \geq 2} \frac{\Lambda_X(p^j)}{p^j \log p} \sum_{p^j = mn, m, n < X^3} \frac{\Lambda_X(m)}{\log m^2} \frac{\Lambda_X(n)}{\log n} \left( \frac{1}{m^{ia}} - 1 \right) \]

\[ + \frac{1}{n^{ia}} - 1 \right) \cdot \left( \frac{1}{n^{-ia}} - 1 \right) \right\} = O(1). \]

Moreover \( V_4 \) can be written simply as follows.

\[ V_4 = \vartheta(T) \frac{2}{2\pi^3} \left( \sum_{m=2}^{\infty} \frac{\Lambda^2(m)}{m (\log m)^4} \frac{1}{m-ia} - 1 \right)^2 + O(T) \]

\[ = \vartheta(T) \frac{2}{2\pi^3} \sum_{m=2}^{\infty} \frac{\Lambda^2(m)}{m (\log m)^4} (1 - \cos(a \log n)) + O(T). \]

Consequently, we get

\[ \int_0^T (S_1(t + a) - S_1(t))^2 \frac{1}{\pi} \vartheta(t) dt \]

\[ = \vartheta(T) \frac{2}{2\pi^3} \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n (\log n)^4} (1 - \cos(a \log n)) + O(T \sqrt{\log \log T}). \]

Hence, we get

\[ \sum_{0 < \gamma < T} (S_1(\gamma + a) - S_1(\gamma))^2 \]

\[ = \frac{T \log T}{2\pi^3} \sum_{n=2}^{\infty} \frac{\Lambda(n)^2}{n (\log n)^4} (1 - \cos(a \log n)) + O(T \sqrt{\log \log T}). \]

This is our Theorem 6.
§6. Concluding remarks

6-1. We can extend the present argument to the zeros of Dirichlet L-functions. This will appear elsewhere.

6-2. In the proof of Theorems 1, 2, 3 and 4, we can evaluate \( \sum_{0 < \gamma \leq T} S_m(\gamma + a) \) and \( \sum_{0 < \gamma \leq T} S_m(\gamma) \) separately as follows. We have, for any integer \( m \geq 1 \),

\[
\sum_{0 < \gamma \leq T} S_m(\gamma) = \frac{1}{\pi} \int_{C} \vartheta'(t) S_m(t) dt + [S_m(t) S(t)]_{C}^{T} - \int_{C} S_{m-1}(t) S(t) dt
\]

and

\[
\frac{1}{\pi} \int_{C} \vartheta'(t) S_m(t) dt = \frac{1}{\pi} \int_{C} \vartheta'(t) W_m(t) dt + O(T (\log \log T)^{\frac{1}{2}(1,m)}),
\]

where we take the same \( T_1 \) as in the section 3.

For \( m = 1 \), we get

\[
\sum_{0 < \gamma \leq T} S_1(\gamma) = -\int_{0}^{T} S^2(t) dt + O(T \sqrt{\log \log T}),
\]

since \( S_1(T) S(T) \ll \log^2 T \) and \( W_1(T) = 0 \). By Selberg’s mean value theorem on \( S(t) \), which is stated in the introduction, we get the following.

**Corollary 4.**

\[
\sum_{0 < \gamma \leq T} S_1(\gamma) = -\frac{T}{2\pi^2} \log \log T + O(T \sqrt{\log \log T}).
\]

For \( m \geq 2 \), we have

\[
\sum_{0 < \gamma \leq T} S_m(\gamma) = \frac{1}{\pi} \int_{C} \vartheta'(t) W_m(t) dt + O(T^{m-1}),
\]

since \( S_m(T) S(T) \ll T^{m-1} \) and

\[
\int_{C} S_{m-1}(t) S(t) dt \ll T^{m-1}.
\]

By the expression

\[
W_m(T) = 2 \sum_{h=2, \text{even}} \sum_{r \geq 1, h \geq 0} \frac{(-1)^{T-1}}{(2r)!} \cdot h! \sum_{\beta + iy \in \gamma} (\beta - \frac{1}{2})^{2r} (T - \gamma)^h,
\]

we get the following consequence.

**Corollary 5.** Let \( m \) be an integer \( \geq 2 \). Then the Riemann Hypothesis is equivalent to the statement that
\[ \sum_{0 \leq \gamma \leq T} S_m(\gamma) = o(T^{m-1} \log T) \quad \text{as} \quad T \to \infty. \]

If we assume the Riemann Hypothesis, then we have for any integer \( m \geq 1 \),
\[ \frac{1}{\pi} \int_C \theta'(t) S_m(t) dt = \left[ \frac{1}{\pi} \theta'(t) S_{m+1}(t) \right]_C - \frac{1}{\pi} \int_C \theta''(t) S_{m+1}(t) dt \ll \frac{\log^2 T}{(\log \log T)^{m+\frac{2}{2}}}. \]

Using this estimate, we get for \( m = 1 \) the following result as a consequence of Goldston’s mean value theorem on \( S(t) \), which is stated in the introduction.

**Corollary 6.** (On R.H.)
\[ \sum_{0 \leq \gamma \leq T} S_1(\gamma) = -\frac{T}{2\pi^2} \log \log T - \frac{T}{2\pi^2} \sum_{a=1}^{\infty} \frac{F(a)}{a^2} \, da + O(1). \]

For \( m \geq 2 \), applying Theorem 5, we get, under R.H.,
\[ \sum_{0 \leq \gamma \leq T} S_m(\gamma) = \frac{T}{2\pi^2} (-1)^{\frac{m-1}{2}} \sum_{n=1}^{\infty} \Lambda(n)^2 \, \cos \left( -\left\{ \frac{m-1}{2} \right\} \pi \right) \frac{1}{n^{m+1}} + O\left( \frac{T}{\log T} \sqrt{\log \log T} \right). \]

Alternatively, we can use the partial integration and observe that
\[ \int_0^T S_{m-1}(t) S(t) \, dt \]
\[ = \begin{cases} (-1)^{\frac{m-1}{2}} \int_0^T S_{\frac{m}{2}}^2(t) \, dt + O\left( \frac{\log^2 T}{(\log \log T)^{m+\frac{2}{2}}} \right) & \text{if } m \text{ is an odd integer} \\ O\left( \frac{\log^2 T}{(\log \log T)^{m+\frac{2}{2}}} \right) & \text{if } m \text{ is an even integer}. \end{cases} \]

Then using Selberg’s mean value theorem on \( S_1(T) \) (cf. p. 255 of Selberg [26]) and the author’s mean value theorem on \( S_m(T) \) for any integer \( m \geq 2 \) (cf. Theorem 3 on p. 173 of Fujii [15]), we get an asymptotic formula or an upper bound for \( \sum_{0 \leq \gamma \leq T} S_m(\gamma) \) under the Riemann Hypothesis in the following form.

**Corollary 7.** (On R.H.)
(i) If \( m \) is an odd integer \( \geq 3 \), then we have
\[ \sum_{0 \leq \gamma \leq T} S_m(\gamma) = \frac{T}{2\pi^2} (-1)^{\frac{m-1}{2}} \sum_{n=2}^{\infty} \frac{\Lambda(n)^2}{n^{m+1}} + O\left( \frac{T}{\log T} \right). \]
(ii) If \( m \) is an even integer \( \geq 2 \), then we have
\[
\sum_{0 < \gamma \leq T} S_m(\gamma) = O\left(\frac{\log^2 T}{(\log \log T)^{m+2}}\right).
\]

**6-3.** If we assume the Riemann Hypothesis, then our argument in the section 5 gives us the following theorem. The proof will appear elsewhere.

**THEOREM 8.** (On R.H.) Suppose that \( 0 < a \ll 1 \). Then we have for any integer \( m \geq 2 \) and any integer \( k \geq 2 \),
\[
\sum_{0 < \gamma \leq T} (S_m(\gamma + a) - S_m(\gamma))^k = T \log T \left(\frac{T}{2\pi}\right)^k \sum_{\mu=0}^{k} \binom{k}{\mu} (-1)^{(m+1)(k-\mu)}
\]
\[
\cdot \sum_{m_1, \ldots, m_n \geq 2} \frac{A(m_1) \cdots A(m_k)}{m_1 \cdots m_k (\log m_1)^{m+1} \cdots (\log m_k)^{m+1}}
\]
\[
\cdot \left(1 - \frac{1}{m_1} \right) \cdots \left(1 - \frac{1}{m_k} \right) \left(\frac{1}{m_1^{\mu_1}} - 1 \right) \cdots \left(\frac{1}{m_k^{\mu_k}} - 1 \right) + O\left(T \sqrt{\log \log T}\right).
\]
This implies, in particular, the following consequence.

**COROLLARY 8.** (On R.H.) Suppose that \( 0 < \alpha \ll \log T \). Then we have for any integer \( m \geq 2 \),
\[
\sum_{0 < \gamma \leq T} \left( S_m\left(\gamma + \frac{2\pi \alpha}{\log \frac{T}{2\pi}}\right) - S_m(\gamma) \right)^2
\]
\[
= T \log T \cdot \sum_{n=2}^{\infty} \frac{\alpha^2}{n (\log n)^{2m+2}} \left(1 - \cos\left(\frac{2\pi \alpha \log n}{\log \frac{T}{2\pi}}\right)\right) + O\left(T \sqrt{\log \log T}\right).
\]

**References**

On the Distribution of the Zeros of the Riemann Zeta Function in the Neighborhood of its Zeros


Department of mathematics
Rikkyo University
Nishi-ikebukuro, Toshimaku
Tokyo, 171–8501 Japan