On Regularity of Solutions to Overdetermined Non Linear Partial Differential Equations

by

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Introduction

This paper is concerned with regularity of solutions to an overdetermined system of nonlinear partial differential equations. For a single nonlinear equation, regularity and propagation of singularity theorems were obtained first by Bony [2], and then by Beals-Reed [1] using a different method. We show in this paper that these theorems may be extended to an overdetermined system of nonlinear partial differential equations with several unknown functions (Theorems 3.2, 3.4, 3.5 in Sect. 3). We also apply the obtained results to a certain class of overdetermined systems, and clarify some properties of their solutions (Theorem 4.1 in Sect. 4).

Let us make here a preliminary remark. Contrary to a determined equation, in investigating an overdetermined system, making use of the equations in it is insufficient in general. One needs to use some other equations derived from them by differentiations and algebraic operations. According to the prolongation theorem due to Cartan [3], Kuranishi [16], Matsuda [18], one may prolong it to what is called an involutive system. Such a system has good properties enough to deal with various problems. In particular the involutiveness enables us to introduce the characteristics and bicharacteristics in a natural way. Thus we are led to treating an involutive nonlinear system.

In Sect. 1, we recall some fundamental properties of an involutive nonlinear system. Sect. 2 is concerned with the characteristics. Our main results are stated and proved in Sect. 3. In the final Sect. 4, we apply them to a certain class of involutive systems.

Throughout the paper, manifolds, vector bundles, etc. are assumed to be smooth, that is, differentiable of class $C^\infty$ unless expressly stated otherwise.

1. Involutive nonlinear systems

Let us recall some facts about involutive nonlinear systems of partial differential equations (For details, see Pommaret [20]).

Let $E$ be a fibered manifold over a manifold $X$ with projection $\pi : E \to X$, and $J_k(E)$ be the bundle of $k$-jets of sections of $E$ ($k \geq 0$). The natural projection from $J_k(E)$ onto $J_j(E)$ will be denoted by $\pi^k_j$ ($k \geq j \geq 0$).
A nonlinear system of partial differential equations of order \( l \) on \( E \) is, by definition, a fibered submanifold \( \mathcal{R}_l \) of the fibered manifold \( J_l(E) \) over \( X \) with projection \( \pi_{l-l} = \pi \circ \pi_l' \). The \( k \)-prolongation \( \mathcal{R}_{l+k} \) of \( \mathcal{R}_l \) is defined to be the subset \( \mathcal{R}_{l+k} = J_k(\mathcal{R}_l) \cap J_{l+k}(E), J_{l+k}(E) \) being regarded canonically as a submanifold of \( J_k(J_l(E)) \). Clearly \( \pi_{l+k}^{l+k} \) maps \( \mathcal{R}_{l+k} \) into \( \mathcal{R}_{l+j}(k \geq j \geq 0) \). By a solution of \( \mathcal{R}_l \), we shall mean a not necessarily smooth section \( u \) of \( E \) over an open set \( U \subset X \) such that \( u \) is differentiable of class \( C^l \) and that its \( l \)-jet \( j^l_l(u) \) at \( x \) belongs to \( \mathcal{R}_l \) for each \( x \in U \).

Let \( T = TX \) denote the tangent bundle to \( X \), and \( T^* = T^*X \) the cotangent bundle. Let \( V(E) \) be the kernel of the differential \( \pi_0 : TE \to TX \) of \( \pi \) (the vertical bundle of a fibered manifold \( E \)). For each \( P \in J_l(E) \), one has the injection \( \epsilon \) from \( S'l^l(T^*E) \to V_\epsilon(E) \) to \( V_P(J_l)(\epsilon = \pi^{l'}_0(P), x = \pi(e)) \), and hence a bundle monomorphism \( \epsilon : S'^lT^* \to V(E) \to V(J_l) \) over \( J_l(E) \). Here \( S'^lT^* \) denotes the symmetric product of \( l \) copies of \( T^* \). The symbol \( G_l \) of \( \mathcal{R}_l \) is defined to be the bundle \( G_l = V(\mathcal{R}_l) \cap S'^lT^* \otimes V(E) \) over \( \mathcal{R}_l \), and the \( k \)-prolongation \( G_{l+k} \) of \( G_l \) to be \( S'^lT^* \otimes G_l \cap S'^{l+k}T^* \otimes V(E)(k = 1, 2, \ldots) \).

Given a point \( P \in \mathcal{R}_l \), we denote the fibers of the above bundles over the corresponding points by the same symbols as the bundles. \( G_l \) is considered as a vector subspace of \( \text{Hom}(T, S'^{l-1}T^* \otimes V(E)) \). For a set of vectors \( \{v_1, \ldots, v_l\} \) in \( T \), let \( G_l(v_1, \ldots, v_l) \) be the subspace of \( G_l \) consisting of those vectors \( \xi \) which annihilate all of \( v_1, \ldots, v_l \). Let \( g_k \) be the minimum of \( \dim G_l(v_1, \ldots, v_l) \) where vectors \( v_1, \ldots, v_l \) range over \( T \) (\( 1 \leq l \leq n \)), and \( g_0 \) be \( \dim G_l \). The inequality \( \dim G_{l+1} \leq \sum_{i=0}^n g_i \) holds (\( n = \dim X \)). The subspace \( G_l \) is said to be involutive if the equality \( \dim G_{l+1} = \sum_{i=0}^n g_i \) holds.

The symbol \( G_l \) is said to be involutive if its fiber over each point of \( \mathcal{R}_l \) is involutive.

I. Criterion of involutiveness (Kuranishi [16], cf. Pommaret[20]). A nonlinear system \( \mathcal{R}_l \) of order \( l \) on \( E \) is involutive if and only if the following three conditions are fulfilled:

(i) \( G_{l+1} \) is a vector bundle over \( \mathcal{R}_l \).

(ii) the symbol \( G_l \) is involutive,

(iii) the projection \( \pi^{l+1}_l : \mathcal{R}_{l+1} \to \mathcal{R}_l \) is surjective.

If \( \mathcal{R}_l \) is involutive, the integers \( g_i(0 \leq i \leq n) \) are (locally) constant on \( P \in \mathcal{R}_l \), and the inequalities \( g_0 \geq g_1 \geq \cdots \geq g_n = 0 \) hold. The non-negative integers \( s_i = g_i - g_{i-1}(1 \leq i \leq n) \) are called the Cartan characters of \( \mathcal{R}_l \).

II. If \( \mathcal{R}_l \) is involutive, then the prolongations \( \mathcal{R}_{l+k} \) are involutive; In particular, for any integer \( k \geq 0 \),

(a) \( \pi^{l+k+1}_l : \mathcal{R}_{l+k+1} \to \mathcal{R}_{l+k} \) is surjective, and

(b) \( G_{l+k} \) is a vector bundle over \( \mathcal{R}_l \).

(cf. Kuranishi [16], Matsuda [19], Goldschmidt [6])

For a given point \( P \in J_l(E) \), we write \( e = \pi^{l'}_0(P), x = \pi(e) \). Let \( R_P = \sum_{k=0}^\infty S^k T_x \) (the symmetric algebra), \( L_P = \sum_{k=0}^\infty L_{k+1} p \) where \( L_{k+1} = S^k T_x \otimes V_x^*(E) \). Here \( V_x^*(E) \) is the dual space to \( V_x(E) \). For each point \( P \in \mathcal{R}_l \), let \( N_P \) be the homogeneous submodule of the \( R_P \)-module \( L_P \) generated by the annihilator of \( G_l \) in \( L_{l+1} \). The characteristic
module $M_{P}$ of $R_{l}$ at $P$ is defined to be the smallest (homogeneous) submodule $M_{P}$ which contains $N_{P}$ and which has the property: $X_{P}z \subset M_{P}(z \in L_{P})$ implies $z \in M_{P}$, where $X_{P}$ is the ideal in $R_{P}$ generated by the homogeneous part $T_{x}$ of degree 1 (Kakié [10], cf. Goldschmidt [5]).

III. If $R_{l}$ is involutive, then for any $k \geq 1$,

(i) $M_{k, P} = N_{k, P}$, where $M_{k, P}$ and $N_{k, P}$ denote the homogeneous parts of degree $k$ of $M_{P}$ and $N_{P}$, respectively, and

(ii) the family $M_{k} = \{ M_{k, P} \; ; \; P \in R_{l} \}$ forms a vector bundle over $R_{l}$.

(For the proof of (i), see Kakié [10]. (ii) follows from (i), (II)-(b) and the fact that $N_{k, P}$ is the annihilator of $G_{1+\langle k-1 \rangle, P}$ in $L_{k, P}$.)

In later discussions, we also use local descriptions in terms of local coordinates. Let $(x, y) = (x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m})$ be a fibered chart of $E$, where the coordinates $(x_{i})$ are the pull-backs by $\pi$ of some coordinates (which we denote also by $(x_{i})$) of $X$, and $n = \dim X, n + m = \dim E$. Let $\nu = (\nu_{1}, \ldots, \nu_{n})$ be an ordered $n$-tuple of non-negative integers. We shall use the standard multi-index notations such as $(\partial/\partial x)^{\nu} = (\partial/\partial x_{1})^{\nu_{1}} \cdots (\partial/\partial x_{n})^{\nu_{n}}$, $|\nu| = \nu_{1} + \cdots + \nu_{n}$. Let $p_{j}^{\nu}$ be the function on $J_{k}(E)$ defined by $p_{j}^{\nu}(f) = (\partial/\partial x)\nu f_{j}(x)$, $f_{j}$ being $j$-coordinate of a section $f$ of $E$, $|\nu| \leq k$.

Associated to a fibered chart $(x, y)$, one has a local coordinate system of $J_{k}(E)$: 

$$(x_{i}, y_{j}, p_{j}^{\nu}; 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq |\nu| \leq k).$$

Here $x_{i}, y_{j}$ denote their pull-backs to $J_{k}(E)$ under the map $\pi_{0}^{k}$.

A nonlinear system $R_{l}$ may be described locally by the equations

$$\mathcal{R}_{l} : F_{\alpha} = 0 \quad (\alpha = 1, \ldots, r = [\text{codim} R_{l} \text{ in } J_{l}(E)])$$

where the Jacobian matrix $\partial(F_{1}, \ldots, F_{r})/\partial(y, p)$ is of rank $r$ everywhere. Let $\delta_{\alpha}^{\beta}$ denote the total differentiation with respect to $x_{i}$; $\delta_{\alpha}^{\beta} = \partial/\partial x_{i} + \sum_{j=1}^{m} \sum_{|\nu| = 0}^{\infty} \delta_{\nu}^{\nu_{1}} \partial/\partial p_{j}^{\nu_{1}}$, where $1_{i}$ is the multi-index with $i$-th component being 1 and the other ones being 0. Clearly $\delta_{\alpha}^{\beta}$ and $\delta_{\beta}^{\gamma}$ commute, and hence any total differentiation of higher order may be written as $(\partial^{\mu}) = (\delta_{\alpha}^{\mu})^{\nu_{1}} \cdots (\delta_{\beta}^{\mu})^{\nu_{m}}$. The $k$-prolongation of $R_{l}$ can be described by $\mathcal{R}_{l+k} : (\partial^{\mu}) F_{\alpha} = 0 \quad (1 \leq \alpha \leq r, 0 \leq |\nu| \leq k)$. The symbol $G_{l}$ and its prolongations are described as follows:

$$G_{l+k, P} = \left\{ \sum_{1 \leq j \leq m \mid |\nu| = |\mu| = k} \chi_{\nu j}^{\mu} (dx)^{\nu} \otimes \frac{\partial}{\partial y_{j}} \in S^{l+k} T^{\ast}_{x} \otimes V_{k}(E) ; \right.$$  

$$\sum_{1 \leq j \leq m \mid |\nu| = |\mu| = k} \frac{\partial F_{\alpha}(P)}{\partial p_{j}^{\nu}} \chi_{\nu j}^{\mu} = 0 \quad (1 \leq \alpha \leq r, |\nu| = k) \right\} .$$

For a function $\varphi$ on $J_{l+k}(E)$, we shall use the following notation:

$$\epsilon_{l+k}^{\ast}(d\varphi(P)) = \sum_{1 \leq j \leq m \mid |\nu| = |\mu| = k} \frac{\partial \varphi(P)}{\partial p_{j}^{\nu}} (\partial/\partial x)^{\nu} \otimes dy_{j}.$$  

Bearing in mind the above facts, we easily find the following.
We shall say that a function (locally) defined on $J_k(E)$ is a function of $q$-jets ($q \leq k$) if it is obtained by pulling back a function on $J_q(E)$ by the projection $\pi_d^k$.

**Lemma 1.1.** Assume that $\mathcal{R}_l$ is involutive. Let $k \geq 2$, and $P' \in \mathcal{R}_{l+k}$. For any smooth section $z$ of the bundle $M_{l+k}$ around $P_0 = \pi_{l+k}^l(P'_0)$, we can find a function $\varphi$ defined on a neighborhood $\mathcal{U}$ of $P'$ in $J_{l+k}(E)$ such that the following (i)–(iii) hold:

(i) $\varphi$ vanishes everywhere on $\mathcal{R}_{l+k} \cap \mathcal{U}$.

(ii) In terms of an (equivalently, any) local coordinate system $(x, y, p)$ of $J_{l+k}(E)$, $\varphi$ has the expression of the form

$$\varphi = \sum_{1 \leq j \leq m} \sum_{|\alpha| = l+k} a^j_{\alpha} p^\alpha_j + \sum_{1 \leq j \leq m} \sum_{|\mu| = l+k-1} b^j_{\mu} p^\mu_j + \psi,$$

where $a$'s are functions of $l$-jets, $b$'s are functions of $(l+1)$-jets, and $\psi$ is a function of $(l+k-2)$-jets.

(iii) $\epsilon_{l+k}^r(d\varphi(P')) = z(P'), \ P' \in \mathcal{R}_{l+k} \cap \mathcal{U}$.

**Proof.** We write $F^\nu_\mu = (\partial_\nu)^\mu F_\alpha$. Since $k \geq 2$, the following hold:

(1) In terms of local coordinates $(x,y,p)$, each function $F^\nu_\mu (|\mu| = k)$ of $(l+k)$-jets admits an expression satisfying the same conditions required for $\varphi$ in (ii).

(2) The module of local sections of $M_{l+k}$ is generated by

$$z^\mu_a = \sum_{1 \leq j \leq m} \sum_{|\alpha| = l+k} \frac{\partial F^\mu_\alpha}{\partial p^\alpha_j} \left( \frac{\partial}{\partial x_j} \right)^\nu \otimes dy_j \ (1 \leq \alpha \leq r, |\mu| = k).$$

Since $M_{l+k}$ is a vector bundle, we can find smooth functions $\epsilon_{\mu}^\alpha$ defined on a neighborhood of $P_0$ in $J_l(E)$ such that $z(P) = \sum_{1 \leq \alpha \leq r} \sum_{|\mu| = l+k} \epsilon_{\mu}^\alpha(P) x^\mu_\alpha$. Then it is easy to see that $\varphi$ fulfills the conditions (i)–(iii).

Q.E.D.

2. The characteristic variety and bicharacteristics

Let $\mathcal{R}_l$ be an involutive system of order $l$, and $G_l$ its symbol. The complexification of a real vector space $W$ will be denoted by $W_C; W_C = W \otimes_R C$. Let $P \in \mathcal{R}_l, e = \pi_l^0(P), x = \pi_1^0(e)$. We write $V_x = V_x(E)$. The characteristic variety $\mathcal{V}_P$ of $\mathcal{R}_l$ at $P$ is defined by

$$\mathcal{V}_P = \{ \xi \in (T^*_x) \cap [0]; (G_l)_* e \cap (\xi^* \otimes (V_x)_e) \neq \{0\} \}.$$ 

Let $d$ be the largest integer for which $s_d \neq 0, s_1, \ldots, s_n$ being the Cartan characters of $\mathcal{R}_l$. Then $\mathcal{V}_P$ is a projective algebraic variety of projective dimension $d - 1$ (cf. Kakié [10]). Incidentally we note that $\mathcal{R}_l$ is overdetermined (not underdetermined) if and only
if \( s_n = 0 \), equivalently, \( \mathcal{E}_P \neq (T^*_c)e \setminus \{0\} \). The characteristic variety \( \mathcal{E} \) of \( \mathcal{R}_l \) is the family \( \mathcal{E} = \{ \mathcal{E}_P; P \in \mathcal{R}_l \} \). This is a smooth family of projective algebraic variety. To see this, bearing in mind the description of the symbol \( \mathcal{G}_l \) in Sect. 1, let us introduce the \( r \times m \) matrix \( \Delta(P, \partial / \partial x) \) of which \((\alpha, j)\) entry is \( \sum_{|v|=0} \partial F_{\alpha}(P)/\partial \partial_j \partial x^v \). Let \( \{ \Delta_{\beta}; \beta = 1, 2, \cdots , r_0 \} \) be the set of \( m \times m \) submatrices of \( \Delta \). The determinants \( |\Delta_{\beta}| \) are homogeneous polynomials in \( n \) variables \( \partial / \partial x_i \) of degree \( ml \) with coefficients being smooth functions of \( P \).

**Lemma 2.1.** A (non-zero) complex covector \( \xi \in (T^*_c)e \) belongs to \( \mathcal{E}_P \) if and only if \( |\Delta_{\beta}|(P, \xi) = 0 \) \((\beta = 1, 2, \cdots , r_0)\).

**Proof.** In this proof, vector spaces are assumed to denote their complexifications. Let \( \sigma_{\xi} : V_{\xi} \to S^i T^*_c \otimes V_{\xi}/G_{l,P} \) be the linear mapping sending \( v \) to the canonical projection of \( \xi^l \otimes v \). Writing \( D_{\xi} = \text{Ann}(G_{l,P}) \subset S^i T^*_c \otimes V_{\xi}^* \), we have the dual mapping \( \sigma_{\xi}^*: D_{\xi} \to V_{\xi}^* \).

It is easily seen that

\[
\sigma_{\xi}^*((\partial / \partial x)^e) \otimes v^* = \xi^e v^* \quad (v^* \in V_{\xi}^*, |v| = l),
\]

where \( \xi = \sum_{1 \leq i \leq n} \xi_i dx^i \).

A covector \( \xi \) belongs to \( \mathcal{E}_P \) if and only if \( \sigma_{\xi} \) is not injective, equivalently, \( \sigma_{\xi}^* \) is not surjective. The latter condition may be stated as rank \( \Delta(P, \xi) < m \). Hence we have the required result.

The characteristic variety is more accurately described by the characteristic ideal (cf. Guillemin-Quillen-Sternberg [7]). The characteristic ideal \( m \) of \( \mathcal{R}_l \) is the family \( m = \{ m_P; P \in \mathcal{R}_l \} \) of ideals defined by

\[
m_P = \text{Ann}(L_P/M_P) = \{ f \in R_P; fL_P \subset M_P \}
\]

where \( M_P \) is the characteristic module of \( \mathcal{R}_l \). Since \( M_P \) used here is slightly different from that in [7], so is the ideal \( m_P \). Our ideal possesses the following convenient property which follows from the same property of \( M_P \): \( \mathcal{X}_P f \subset m_P (f \in R_P) \) implies \( f \in m_P \).

**Lemma 2.2** (Guillemin, Quillen and Sternberg [7]). (i) The characteristic variety \( \mathcal{E}_P \) is the set of zeros of the ideal \( m_P \) \((P \in \mathcal{R}_l)\).

(ii) The functions \( |\Delta_{\beta}|(P, \partial / \partial x) \) take values in the characteristic ideal \( m \).

**Proof.** For completeness we give a proof. To show (ii), let \( \Delta_{\beta}^* \) be the adjugate matrix of \( \Delta_{\beta} \). Then \( \Delta_{\beta}^* \Delta_{\beta} = |\Delta_{\beta}|(\text{unit matrix}) \). Using the basis \( \{ dy_1, \cdots , dy_m \} \) of \( V_{\xi}^* \), we identify \( L_P \) the set of column vectors with \( m \) components in \( R_P \). Then we may consider each \( \Delta_{\beta} \) as an \( R_P \)-linear mapping in \( L_P \). Recalling the definition of the characteristic module \( M_P \), we find that \( \Delta_{\beta}(L_0,P) \subset M_P \). Accordingly \( |\Delta_{\beta}|(L_0,P) = \Delta_{\beta}^* \Delta_{\beta}(L_0,P) \subset M_P \). Since \( L_0,P \) generates \( L_P \), we have \( |\Delta_{\beta}|(P) \in m_P \). Thus (ii) is proved. We next prove (i). Lemma 2.1 and (ii) just proved imply that the set of zeros of \( m_P \) is contained in \( \mathcal{E}_P \). Conversely, let \( \xi \in \mathcal{E}_P \). Take any homogeneous element \( f \in m_P \). Assume that \( k = \deg f \geq l \). Let us choose a non-zero vector \( v \in V_{\xi} \) such that \( \xi^l \otimes v \in G_{l,P} \). (The
vector spaces are assumed to denote their complexifications.) Then \( \xi^k \otimes v \in G_{i+(k-l),p} \) take \( v^* \in V^*_v \) such that \( v, v^*> \neq 0 \). The element \( f \otimes v^* \) belongs to \( M_k \), which coincides with \( \text{Ann}(G_{i+(k-l)}) \) by (III)-(i) in Sect. 1. Taking the dual pairing between \( \xi^k \otimes v \) and \( f \otimes v^* \), we have \( \langle v, v^* \rangle = f(\xi) = 0 \). Hence \( f(\xi) = 0 \). Assume \( k < l \). The result just proved indicates that \( \xi^v, f(\xi) \in m_P \) vanishes for any multi-index \( v \) with \( |v| = l \). Consequently we have \( f(\xi) = 0 \). Q.E.D.

Let \( R \) denote the bundle \( \{ R_{p}; p \in \mathcal{R}_l \} \) of rings over \( \mathcal{R}_l \). By a smooth homogeneous section of \( R \), we shall mean a smooth section of the vector bundle \( R_k = \{ R_{k,p}; p \in \mathcal{R}_l \} \) for some integer \( k \geq 0 \), where \( R_{k,p} = S^k T^*_x \). Using the local coordinates \( (x_1, \cdots, x_n) \) of \( X \) and writing \( \xi_i = \partial/\partial x_i \), a smooth homogeneous section \( f \) of \( R \) may be expressed as a homogeneous polynomial in \( \xi \) with coefficients being smooth functions on \( P \in \mathcal{R}_l \): \( f = \sum_{|v|=k} a_v(p)(\xi)^v \). For a function \( a \) on \( \mathcal{R}_l \), let \( da/dx_i \) denote the restriction to \( \mathcal{R}_{l+1} \) of the total derivative \( \partial a/\partial x_i \), where \( a^* \) is a smooth extension of \( a \) to a neighborhood of \( \mathcal{R}_l \) in \( J_1(E) \). Observe that \( da^*/dx_i \) is uniquely defined independently of the choice of an extension \( a^* \). For a smooth homogeneous section \( f \), we define \( df/dx_i = \sum_{|v|=k} da_v/dx_i (\xi)^v \).

Let \( f_1, f_2 \) be smooth homogeneous sections of \( R \). We denote by \( \{ f_1, f_2 \} \) the Poisson bracket \( \{ f_1, f_2 \} = \sum_{1\leq i\leq n}(\partial f_1/\partial \xi_i)df_2/dx_i - \partial f_2/\partial \xi_i df_1/dx_i \). Note that \( \{ f_1, f_2 \} \) is a smooth homogeneous section of \( (\rho^{l+1}_l)^* R \), where \( \rho^{l+1}_l : \mathcal{R}_{l+1} \to \mathcal{R}_l \) is the projection \( \pi^{l+1}_l \) restricted to \( \mathcal{R}_{l+1} \).

**Proposition 2.3.** Assume that \( \mathcal{R}_l \) is involutive. If \( f_1, f_2 \) be smooth homogeneous sections of the characteristic ideal \( m \), then the Poisson bracket \( \{ f_1, f_2 \} \) is a smooth homogeneous section of \( (\rho^{l+1}_l)^* m \subset (\rho^{l+1}_l)^* R \).

**Proof.** Let \( q_i = \deg f_i \). We first consider the case when \( q_i \geq l + 2 \). Let \( f_i = \sum_{|v|=q_i} a_v^{(i)}(\xi)^v \) be the expressions of \( f_i \). Using the local bases \( \{ dy_j; 1 \leq j \leq m \} \) of the module of sections of the dual bundle \( V^*(E) \) to \( V(E) \), we have smooth sections \( \zeta_j(\xi) = f_i \otimes dy_j \) of \( M_{q_i} \). By Lemma 1.1, we can find functions \( \varphi_j^{(i)} (j = 1, \cdots, m) \) defined locally on \( J_{q_j}(E) \) such that the \( \varphi_j^{(i)} \) vanish everywhere on \( \mathcal{R}_{l+(q_i-1)} \), and that, for each \( i, j \)

\[
\varphi_j^{(i)} = \sum_{|v|=q_j} a_v^{(i)} p_j^v + \sum_{1 \leq k \leq m} \sum_{|v|=q_i-1} b_v^{(i)} p_k^v + \psi
\]

where the functions \( a_v^{(i)} \) are assumed to be extended smoothly to an open set in \( J_i(E) \), \( b \)'s are functions of \((l+1)\)-jets, and \( \psi \) is a function of \((q_i-2)\)-jets. Using the differential operators \( X_i = \sum_{|v|=q_i} a_v^{(i)} (\partial_v)^v \), we put \( F_j = X_1(\varphi_j^{(2)}) - X_2(\varphi_j^{(1)}) \) \((1 \leq j \leq m)\). These are functions of \( q \)-jets, where \( q = q_1 + q_2 \). Applying the Leibniz rule, we easily find that each \( F_j \) is expressed as the sum of the function of \((q-1)\)-jets

\[
\sum_{1 \leq i \leq m} \sum_{|v|=q_1} \sum_{|v|=q_2} \{ a_v^{(1)}(\partial_v a_v^{(2)}) \mu_i - a_v^{(2)}(\partial_v a_v^{(1)}) \nu_j \} p_{j,v}^{n+1}
\]
and a function of \((q - 2)\)-jets. Clearly the \(F_j\)'s vanish on \(\mathcal{R}_{l+(q-l)}\). Since \(\rho_{q-l}^q\) is surjective by (II) in Sect. 1, we know that the \(F_j\)'s considered as functions of \((q-1)\)-jets vanish on \(\mathcal{R}_{l+(q-l-1)}\). On account of (IV) in Sect. 1, it follows that

\[
\epsilon_{q-l-1}(dF_j)(P') = \{f_1, f_2\}(P') \otimes dy_j \in M_P (j = 1, \ldots, m),
\]

where \(P'' \in \mathcal{R}_{l+(q-l-1)}\), \(P' = \pi_{l+1}^{q-1}(P'')\), \(P = \pi_l^{q-1}(P'')\). Consequently we have \(\{f_1, f_2\}(P') \in m_P\) for any \(P' \in \mathcal{R}_{l+1}\).

Consider the general case. Let \(\nu, \mu\) be multi-indices with \(|\nu| = |\mu| = l + 2\). By what we have just proved, the bracket \([\xi^\nu f_1, \xi^\mu f_2]\) is a section of \((\rho_{l+1}^{l+1})^* m\). On the other hand \([\xi^\nu f_1, \xi^\mu f_2] = \xi^{\nu+\mu}\{f_1, f_2\} (\text{mod } f_1, f_2)\). Hence we conclude that \(\xi^{\nu}\{f_1, f_2\}\) is a section of \((\rho_{l+1}^{l+1})^* m\) for any multi-index \(\nu'\) with \(|\nu'| = 2(l + 2)\). On account of the property of \(m\) explained just before Lemma 2.2, it follows that \(\{f_1, f_2\}\) is a section of \((\rho_{l+1}^{l+1})^* m\). Q.E.D.

We now introduce the bicharacteristics of an involutive overdetermined nonlinear system \(\mathcal{R}_l\). Let \(\mathfrak{S}\) be the real characteristic variety of \(\mathcal{R}_l\), which is the set of real characteristic covectors; \(\mathfrak{S} = \{\xi \in (\rho_{l+1}^{l+1})^* T^* \setminus \{0\}; \xi \in \mathcal{S}\}\), where \(\rho_{l+1}^l : \mathcal{R}_l \to X\) is the projection. Assume that we are given a solution \(u\) of \(\mathcal{R}_l\) differentiable of class \(C^{l+2}\). Let \(J[u] : T^* \to (\rho_{l+1}^{l+1})^* T^*\) be the bundle mapping sending each fiber \(T^*_x\) canonically to the fiber of \((\rho_{l+1}^{l+1})^* T^*\) over \(J^l(u)\). We shall denote by \(\mathfrak{S}[u]\) the inverse image of \(\mathfrak{S}\) under the injective mapping \(J^l[u] ; \mathfrak{S}[u] \subset T^*\). We define the field \(H[u]\) to be the one defined on \(\mathfrak{S}[u]\) which assigns to each point \(x^* \in \mathfrak{S}[u]\) the vector subspace of \(T^*_x (T^*)\) spanned by the Hamilton vector fields

\[
H_p = \sum_{1 \leq i \leq n} (\partial p/\partial \xi_i) \partial/\partial x_i - \sum_{1 \leq i \leq n} (\partial p/\partial x_i) \partial/\partial \xi_i,
\]

in which \(p\) varies in the set of differentiable functions of class \(C^2\) defined around \(x^*\) and vanishing identically on \(\mathfrak{S}[u]\). Here \((x_i, \xi_i)\) are the local coordinates of \(T^*_x\) defined by \(\xi_i(x^*) = a_i\) when \(x^* = \sum_{1 \leq i \leq n} a_i d x_i\). If \(\mathfrak{S}[u] \subset T^*\) is a regular submanifold differentiable of class \(C^2\) and of codimension \(d\) around \(x^*\), then the field \(H[u]\) has the constant rank \(d\) around \(x^*\).

We shall say that a point \(P_0^* \in \mathfrak{S}\) is a \(\rho\)-simple characteristic point with a regular local equation \(f_i(P, \xi) = 0 (1 \leq i \leq d)\) if \(\mathfrak{S}\) is a regular submanifold of codimension \(d\) in \((\rho_{l+1}^{l+1})^* T^*\) around \(P_0^*\) and if \(f_i(P, \xi) (1 \leq i \leq d)\) are smooth homogeneous sections of the characteristic ideal \(m\) around \(P_0\) such that the Jacobian matrix of \(f_1, \ldots, f_d\) with respect to \(\xi_1, \ldots, \xi_n\) is of rank \(d\).

A subset \(V \subset T^*\) is said to be involutive around \(x^* \in V\) if, for any functions \(p_1(x, \xi), p_2(x, \xi)\) defined around \(x^*, \) differentiable of class \(C^2\), and vanishing identically on \(V\), the Poisson bracket \(\{p_1, p_2\}\) vanishes identically on \(V\).
LEMMA 2.4. Let \( \mathcal{R}_t \) be an involutive system, and \( u \) be its solution differentiable of class \( C^{l+2} \). Let \( x_0^* \) be a point of \( \mathfrak{E}[u] \). Assume that the point \( P_0^* = J^1[u](x_0^*) \) is a \( \rho \)-simple characteristic point. Then the following are valid:

(i) The variety \( \mathfrak{E}[u] \subset T^* \) is involutive around \( x_0^* \).

(ii) For any point \( x^* \in \mathfrak{E}[u] \) near \( x_0^* \), \( H[u](x^*) \) is contained in the tangent space to \( \mathfrak{E}[u] \) at \( x^* \). Moreover the field \( H[u] \) is involutive around \( x_0^* \), that is, any vector fields \( v_1, v_2 \) on \( \mathfrak{E}[u] \) taking values in \( H[u] \), the commutator \( [v_1, v_2] \) also takes values in \( H[u] \) around \( x_0^* \).

Proof. Let \( f_i(P, \xi) = 0 \) \( (1 \leq i \leq d) \) be a regular local equation of \( \mathfrak{E}[u] \) around the \( \rho \)-simple point \( P_0^* \). Let \( p_i(x, \xi) = f_i(j_1^1(u), \xi) \). We see that \( p_i(x, \xi) = 0 \) \( (i = 1, \ldots, d) \) gives a regular local equation of \( \mathfrak{E}[u] \), and hence the vector fields \( H_{p_i} (1 \leq i \leq d) \) generate \( H[u] \) around \( x_0^* \). From Lemma 2.2 and Proposition 2.3, it follows that the Poisson bracket \( \{p_i, p_j\} \) vanishes everywhere on \( \mathfrak{E}[u] \). This implies (i). Since \( H_{p_i}(p_j) = \{p_i, p_j\} \), we have that \( H_{p_i}(x^*) \in T_{x^*}(\mathfrak{E}[u]) \). Thus the first part of (ii) is valid. To show the remaining assertion, we prove

\[
v_{ij} = [H_{p_i}, H_{p_j}] \equiv 0 \quad (\text{mod} \quad H_{p_1}, \ldots, H_{p_d}) \quad \text{on} \quad \mathfrak{E}[u].
\]

It suffices to show that, for any function \( f \) on \( T^* \), \( g_{ij} = v_{ij}(f) \) are expressed as a linear combination of \( H_{p_i}(f) \) and \( p_k \) \( (1 \leq k \leq d) \) with function coefficients. As a consequence of Jacobi’s identity, we have \( g_{ij} = \{\{p_i, p_j\}, f\} \). Since \( \{p_i, p_j\} \) can be expressed as a linear combination of \( p_k \) \( (1 \leq k \leq d) \), we have the required assertion. Q.E.D.

DEFINITION. Under the circumstances of Lemma 2.4, we call integral manifolds of \( H[u] \) of maximum dimension bicharacteristic manifolds of \( (\mathcal{R}_t, u) \).

The classical theory (cf. Chevalley [4]) indicates that, through any point \( x^* \in \mathfrak{E}[u] \), there passes a unique (local) bicharacteristic manifold of \( (\mathcal{R}_t, u) \) of which dimension equals to the codimension of \( \mathfrak{E}[u] \) in \( T^* \).

3. Regularity of solutions

Let \( s \) be a real number. A distribution \( g \) on the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) is said to be of class \( H^s \) at \( x_0 \in \mathbb{R}^n \) if there is a function \( \phi(x) \in C_0^n(\mathbb{R}^n) \) with \( \phi(x_0) \neq 0 \) such that \( (1 + |\xi|^2)^{\frac{s}{2}}(\hat{\phi}g)(\xi) \in L^2(\mathbb{R}^n) \), \( \hat{\phi}g)(\xi) \) being the Fourier transform of \( (\phi g)(x) \).

Let \( x_0^* \) be a point of the cotangent bundle \( T^* \mathbb{R}^n \setminus \{0\} \) excluded the zero section. A distribution \( g \) is said to be of class \( H^s_{ml} \) at \( x_0^* \) if there exists a classical pseudodifferential operator \( A \) of order 0 whose principal symbol does not vanish at \( x_0^* \) such that \( Ag \) is of class \( H^s \) at the origin of \( x_0^* \).

LEMMA 3.1. Let \( g_j(x)(1 \leq j \leq k) \) be functions on a neighborhood of \( x_0 \) in \( \mathbb{R}^n \) and of class \( H^s \) at \( x_0 \), and \( F(x, y_1, \ldots, y_k) \) be a smooth function defined on a neighborhood of the point \( (x_0, g_1(x_0), \ldots, g_k(x_0)) \) in \( \mathbb{R}^{n+k} \). Let \( x_0^* \) be a point of \( T^* \mathbb{R}^n \setminus \{0\} \) with origin \( x_0 \). If \( s > n/2 \), then the following are valid:
(i) (Schauder) The composite function \( F(x, g_1(x), \cdots, g_k(x)) \) is of class \( H^s \) at \( x_0 \).

(ii) (Bony) If furthermore \( g_j(x) \) are of class \( H^s_{ml} \) at \( x_0^s \), then the function \( F(x, g_1(x), \cdots, g_k(x)) \) is of class \( H^s_{ml} \) at \( x_0^s \) provided \( t < 2s - n/2 \).

(For the proof, see e.g. Bony [2])

This lemma allows us, in particular, to define the local and microlocal Sobolev classes of sections of a fibered manifold \( E \) as well.

**DEFINITION.** Let \( n/2 < s < t < 2s - n/2 \). Let \( \varphi \) be a section of \( E \) if the functions \( u_j(x) \) are of class \( H^s \) at \( x_0 \), the coordinate neighborhood being considered as a subset of \( R^n \). We say that \( u \) is of class \( H^s \) at \( x_0 \) if the functions \( u_j(x) \) are of class \( H^s \) at \( x_0 \) and of class \( H^s_{ml} \) at \( x_0^s \), \( x_0 \) being the origin of \( E \).

Let \( \Gamma \) be a subset of \( T^*X \setminus \{0\} \). By \( u \in H^s \cap H^s_{ml}(\Gamma) \), we mean that \( u \) is of class \( H^s \cap H^s_{ml} \) at each point of \( \Gamma \).

**THEOREM 3.2.** Let \( \mathcal{R}_i \) be an involutive system of order \( l \), and \( P_0^s \) be a point of its real characteristic variety \( \mathfrak{R} \). Assume that \( P_0^s \) is a \( \rho \)-simple characteristic point with a regular local equation \( f_i(P, \xi) = 0 \) \((i = 1, \cdots, d)\). Put \( N_0 = \max(l + 2, \deg f_i - 2; 1 \leq i \leq d) + n/2 \), where \( n = \dim X \). Then, for any real numbers \( s, t \) with \( N_0 < s \leq t < 2s - n/2 \), the following is valid: Let \( u \) be a solution of \( \mathcal{R}_i \) defined on an open set \( U \subset X \) and belonging to \( H^l_{loc}(U) \) such that the section \( j^l(u) \subset J^l(E) \) is located sufficiently near the origin of \( P_0^s \). If \( u \) is of class \( H^s \cap H^s_{ml} \) at \( x^* \in \mathfrak{R}[u] \) near \( J^l[u]^{-1}(P_0^s) \), then \( u \) is of class \( H^s \cap H^s_{ml} \) at any point on the bicharacteristic manifold of \( (\mathcal{R}_i, u) \) passing through \( x^* \).

The proof is carried out by invoking to a version of a theorem due to Bony [2] or Beals-Reed [1] for a single linear differential operator with non-smooth coefficients. Let \( A(x, D) \) be a linear differential operator of order \( q \) acting on vector-valued functions \( v(x) = (v_1(x), \cdots, v_m(x)) \) with \( v_j(x) \) being functions on \( R^n \) and taking the form

\[
A(x, D) = \sum_{|\nu| = q} a_{\nu}(x) \left( \frac{\partial}{\partial x} \right)^\nu E_m + \left( \sum_{|\mu| = q - 1} b_{jk}(x) \left( \frac{\partial}{\partial x} \right)^\mu \right)_{1 \leq j, k \leq m},
\]

where \( E_m \) is the \( m \times m \) unit matrix, and \( a_{\nu}(x) \) are real-valued functions. Set \( p(x, \xi) = \sum_{|\nu| = q} a_{\nu}(x) \xi^\nu \). Assuming that \( a_{\nu}(x) \) are of class \( C^2 \), let \( \gamma \) be a connected integral curve of the Hamilton vector field \( H_\gamma \) on which \( p(x, \xi) \) vanishes.

**LEMMA 3.3 (Beals-Reed [1], cf. Bony [2]).** Let \( n/2 < s < t < 2s - n/2 \). Suppose that \( a_{\nu}(x) \in H^{s+2} \cap H^{s+2}_{ml}(\gamma) \), \( b_{jk}(x) \in H^{s+1} \cap H^{s+1}_{ml}(\gamma) \). Let \( x_0^s \) be a point on \( \gamma \). If \( v \in H^{s+q-2} \cap H^{s+q-2}_{ml}(\gamma) \), \( A(x, D)v \in H^s \cap H^j_{ml}(\gamma) \), and \( v \in H^{s+q-2} \cap H^{s+q-2+\epsilon}_{ml}(x_0^s) \) for some \( 0 < \epsilon \leq 1 \), then \( v \in H^{s+q-2} \cap H^{s+q-2+\epsilon}_{ml}(\gamma) \).
Proof. This is a special case of Theorem 3.2 in Beals-Reed [1] except that we treat operators acting on vector-valued functions. Since the principal symbol of $A(x, D)$ is the unit matrix multiplied by $p(x, \xi)$, we find it easy to see that the proof in [1] can be used with little modification to verify the assertion. Q.E.D.

Proof of Theorem 3.2: We may assume that each section $f_i$ is of degree $\geq l + 2$ by multiplying, if necessary, a suitable section of $R$. Let

$$f_i(P, \xi) = \sum_{|v|=q} a^{(i)}_v(P) \xi^v \quad (P \in \mathcal{R}_i, \xi_k = \frac{\partial}{\partial x_k})$$

be the descriptions of the $f_i$. We set $p_i(x, \xi) = \sum_{|v|=q} a^{(i)}_v(j^{(i)}_v(u))\xi^v$. Then a bicharacteristic manifold of $(\mathcal{R}_i, u)$ is a $d$-dimensional integral manifold $\Gamma$ of the field $H[\mathcal{R}]$ generated by the Hamilton vector fields $H_p$. By Lemma 2.4, $H[\mathcal{R}]$ is involutive, and hence any point on $\Gamma$ can be joined by a curve composed of a finite number of integral curves of $H_{p_1}, \ldots, H_{p_d}$ to a given point on it. Therefore the proof is complete if we verify the following:

For each fixed $i \in \{1, \ldots, d\}$, if $u$ is of class $H^s \cap H^l_{ml}$ at $x^*_0 \in \mathcal{R}_s[\mathcal{R}]$, then $u$ is of class $H^s \cap H^l_{ml}$ at any point on the integral curve $\gamma_i$ of $H_{p_i}$ passing through $x^*_0$.

Fixing $i$, we will omit the sub- and super-script $i$ in the symbols in the course of the proof. Applying Lemma 1.1 to the smooth sections $f(P, \xi) \otimes d\gamma_j$, we can find (locally defined) functions $b^{jk}_\mu$ of $(l + 1)$-jets and functions $\psi_j$ of $(q - 2)$-jets such that

$$\sum_{|v|=q} a_v p_j^v + \sum_{1 \leq k \leq m} \sum_{|\mu|=q-1} b_{\mu}^{jk} p_k^\mu + \psi_j = 0$$

everywhere on $\mathcal{R}_{t+(q-1)}$ (j = 1, \ldots, m).

Let $y_j = u_j(x)$ (j = 1, \ldots, m) be the expression of $u$ in terms of local coordinates. Then the functions $u_j = u_j(x)$ satisfies the linear differential equations

$$\sum_{|v|=q} [a_v] \left(\frac{\partial}{\partial x}\right)^v u_j + \sum_{1 \leq k \leq m} \sum_{|\mu|=q-1} [b^{jk}_\mu] \left(\frac{\partial}{\partial x}\right)^\mu u_k + [\psi_j] = 0 \quad (j = 1, \ldots, m),$$

where $[a_v] = [a_v](x) = a_v(j^1_v(u))$, and the same for $[b^{jk}_\mu]$s, $[\psi]$s. By Lemma 3.1, we see that if $u \in H^{s \cap H^l_{ml}}(\gamma)$ for $q - 2 + n/2 < s \leq 2s - n/2$, then $[a_v] \in H^{t-\gamma} \cap H^l_{ml}(-\gamma)$, $[b^{jk}_\mu] \in H^{s-1+\gamma} \cap H^l_{ml}(-1+\gamma)$, $[\psi_j] \in H^{s+q+2} \cap H^l_{ml}(-q+2)(\gamma)$. Hence we can apply Lemma 3.3 to conclude that $u \in H^s \cap H^l_{ml}(\gamma)$ provided $u$ is of class $H^s \cap H^l_{ml}(\gamma)$ at the point $x^*_0 \in \gamma$, where $0 < \epsilon \leq 1$. Since $u \in H^s \cap H^l_{ml}(\gamma)$ with $t = s$, applying repeatedly this result, we conclude that for any $s, t$ with $N_0 < s < t < 2s - n/2$, $u \in H^s \cap H^l_{ml}(\gamma)$ provided $u \in H^s \cap H^l_{ml}(x^*_0)$.

Outside of the characteristic variety, we have the following

Q.E.D.
THEOREM 3.4. Let $\mathcal{R}_l$ be an involutive system of order $l$ on $E$, and $N_1$ be $\max\{ml - 1, l + 1\} + n/2$ ($n = \dim X, m = \dim E - n$). Let $u$ be a solution of $\mathcal{R}_l$, and $x_0^s$ be a point of $T^*\{0\} - \mathcal{H}[u]$. Assume $s > N_1$. If $u \in H^2 \cap H^s_{ml}(x_0^s)$, then $u \in H^2 \cap H^s_{ml}(x_0^s)$ for any $t < 2s - n/2$.

Proof. Let $P_0^s = J^i[u](x_0^s) \in (\rho_{c-})^*T^*$. On account of Lemma 2.2, we can find a smooth homogeneous section $f(P, \xi)$ of $m$ of degree $q = \max\{ml, l + 2\}$ such that $f(P_0^s) \neq 0$. As in the proof of Theorem 3.2, using Lemma 1.1, we can choose functions $a$’s of $l$-jets, functions $b$’s of $(l + 1)$-jets, and functions $\psi$’s of $(q - 2)$-jets in such a way that equations (3.2) hold true. Hence $u$ satisfies the differential equations (3.3). Consequently we may apply the following theorem to deduce the result:

Theorem. Let $A(x, D)$ be a linear differential operator defined by (3.1), and $x_0^s = (x_0, \xi_0)$ be a point of $T^*R^n$ such that $\sum_{|v| = q} a_v(x_0)\xi_0^v \neq 0$. Let $n/2 < s \leq t < 2s - n/2$. Assume that $a_v \in H^{s+1} \cap H^t_{ml}((x_0^s))$, $b^j_k \in H^1 \cap H^t_{ml}((x_0^s))$. If $v \in H^{s+q-1} \cap H^{t+q-1}((x_0^s))$ and $A(x, D)v \in H^s \cap H^t_{ml}(x_0^s)$, then $v \in H^{s+q-1} \cap H^{t+q}(x_0^s)$. (This theorem is merely a version of Theorem 2.1 in Beals-Reed [1].) Q.E.D.

Repeated applications of Theorem 3.4 and Sobolev’s lemma yield the following result.

THEOREM 3.5 (Elliptic regularity theorem). Let $\mathcal{R}_l$ and $N_1$ be as in Theorem 3.4. Let $u$ be a solution of $\mathcal{R}_l$ such that $\mathcal{H}[u]$ is empty. If $u \in H^s_{loc}(X)$ with $s > N_1$, then $u$ is differentiable of class $C^\infty$.

Remark. Assume that $\mathcal{R}_l$ is a linear involutive system (cf. e.g. Spencer [21]). Then one may consider its distribution solutions, and its characteristic variety and bicharacteristics are defined independently of solutions. On account of the theorems due to Hörmander [8, 9] on a single differential operator, we can deduce more easily the following results:

1. Let $x_0^s$ be a simple real non-zero characteristic covector, and $\Gamma$ be the bicharacteristic manifold passing through $x_0^s$. Let $u \in \mathcal{D}'(X)$ be a solution of $\mathcal{R}_l$. If $u$ is of class $H^s_{ml}$ at $x_0^s$ for a real number $s$, then $u$ is of class $H^s_{ml}$ at any point on $\Gamma$. In particular $WF(u) \supset \Gamma$ if and only if $WF(u) \ni x_0^s$, $WF(u)$ being the wave front set of $u$.

2. If $u \in \mathcal{D}'(X)$ is a solution of $\mathcal{R}_l$, then any real non-characteristic point is outside $WF(u)$.

4. Application

The theorems above may be meaningfully applied to solutions of Sobolev class. As a matter of fact, it is much more difficult to show that a general smooth system admits solutions of Sobolev class or of class $C^\infty$, even for linear equations, and few results has been obtained as yet (cf. Yang [22], Kakié [11, 12, 14, 15], MacKichan [17]). We take up here a special class of involutive systems to which the theorems are meaningfully applied.

Let $\mathcal{R}_1$ be an involutive system of order 1. Let $M_P = \bigcap_{j=1}^l Q_j$ be an irredundant primary decomposition of the characteristic module $M_P$ of $\mathcal{R}_1$ in the module $L_P$, and
$Q_j,P$ be $\mathcal{P}_j,P$-primary. We denote by $c(\mathcal{P}_j,P)$ the homogeneous part of degree 1 of $\mathcal{P}_j,P$; $c(\mathcal{P}_j,P) \subset T_x(x = \pi^1(T_x)(P))$. We shall make the following assumptions:

(A-1) The Cartan characters satisfy: $s_1 = \cdots = s_d > 0, s_{d+1} = \cdots = s_n = 0$ with $1 \leq d < n$.

(A-2) (i) $r = r(P)$ is constant on $P \in \mathcal{R}_1$; (ii) For each $j = 1, \cdots , r$, the family $c(\mathcal{P}_j) = \{c(\mathcal{P}_j,P); P \in \mathcal{R}_1\}$ forms a smooth vector bundle of rank $n - d$ over $\mathcal{R}_1$; (iii) $\mathcal{P}_j,P \subset Q_j,P (1 \leq j \leq r, P \in \mathcal{R}_1)$.

In [12, 14] we obtained the existence theorem that $\mathcal{R}_1$ admits local smooth solutions. The following is a version of it: $\mathcal{R}_1$ admits local (non-smooth) solutions of class $H^s$ for any sufficiently large $s$, and the solution can be parametrized by some arbitrary constants and $s_j$ functions of $d$ variables.

The assumption (A-2)(iii) implies that the characteristic ideal $m_P$ of $\mathcal{R}_1$ coincides with $
abla_{i=1}^r \mathcal{P}_{i,P}$. Hence the characteristic variety $\mathcal{E}_P$ is the union of $r$ irreducible varieties $W_{j,P} = \{\text{the set of zeros of } \mathcal{P}_{j,P}\}$. (A-1) implies that all $W_{j,P}$ are of projective dimension $d - 1$ (see Theorem 1.5 in Kakié [13]). Let $V_j,P$ denote the set of real points in $W_{j,P}$. From (A-2)(ii), we see that each $V_j,P$ is a $d$-dimensional plane in $T^*_x$. We set $V_j = \bigcup_{P \in \mathcal{R}_1} V_j,P; \Re \mathcal{E}$ is the union of $V_j (1 \leq j \leq r)$ excluded the zero-section. Let $\sum_{1 \leq i \leq n} c_{j,i}^P(P) \partial/\partial x_i (1 \leq \beta \leq n - d)$ be linearly independent sections of $c(\mathcal{P}_j)$. Then $V_j$ is a submanifold of $(\rho^1_{-1})^*T^*$ locally described by

$$f_{j}^{(\beta)}(P, \xi) = \sum_{1 \leq i \leq n} c_{j,i}^P(P) \xi_i = 0 \quad (\beta = 1, \cdots , n - d),$$

where $\xi = \sum_{1 \leq i \leq n} \xi_i dx_i \in T^*_x$.

Let $u$ be a given solution of $\mathcal{R}_1$ differentiable of class $C^3$. Define $V_j[u]$ be the union of $V_j,P$ where $P = j^1_x(u), x \in X$. Then $\Re \mathcal{E}[u]$ is the union $\bigcup_{1 \leq j \leq r} V_j[u]$. Set $p_{j}^{(\beta)}(x, \xi) = f_{j}^{(\beta)}(j^1_x(u), \xi)$. Then $V_j[u]$ is the submanifold of $T^*$ described by $p_{j}^{(\beta)}(x, \xi) = 0 \quad (\beta = 1, \cdots , n - d)$. We assert that each $V_j[u]$ is involutive. In fact, observing that the products $\prod_{1 \leq j \leq r} f_{j}^{(\beta)}$ are smooth homogeneous sections of $m$, we see that Lemma 2.4 may be used to conclude that $\Re \mathcal{E}[u]$ is involutive outside its singular points. Therefore $V_j[u]$ is involutive on an open dense subset. Since $V_j[u]$ has no singular point, it follows that $V_j[u]$ is involutive as required. This fact enables us to define the field $H_j[u]$ on $V_j[u]$ which assigns to each $x^* \in V_j[u]$ the subspace of $T_{x^*}(V_j[u])$ generated by $H_{j^1_x(u)}(1 \leq \beta \leq n - d); H_j[u]$ is involutive.

Let $\Lambda_j[u]$ be the field on $X$ assigning to each point $x \in X$ the subspace $c(\mathcal{P}_j,P)$ of $T_x$ where $P = j^1_x(u)$. The fields $\Lambda_j[u]$ are differentiable of class $C^2$. Denote by $\rho$ the mapping from $V_j[u]$ to $X$ induced by the projection $T^* \rightarrow X$. We find that $H_j[u]$ is $\rho$-related to $\Lambda_j[u]$, and hence that $\Lambda_j[u]$ is also involutive (cf. Chevalley [4, Chap.III]). Accordingly, through any given point $x \in X$, there passes a unique $(n - d)$-dimensional
integral manifold $\Sigma_j(x)$ of $A_j[u]$. We call it a characteristic manifold of $(R_1, u)$. It is to be observed that any bicharacteristic manifold lies over a characteristic manifold.

We now state a theorem which clarify some property of solutions.

**Theorem 4.1.** Assume that $u$ is a solution of $R_1$ of class $H^t$ at every point $x \in X$ with $s$ being greater than $\max\{r - 2, 3, m - 1\} + n/2$, and that $\mathfrak{R}[u]$ has no singular point, equivalently each $V_j[u]$ does not meet any other $V_k[u]$ outside the zero section. Let $I \subset X$ be a $d$-dimensional submanifold such that each non-zero vector tangent to $I$ does not belong to any $A_j[u]$ (a non-characteristic submanifold). Then, for any number $t < 2s - n/2$, the following are valid:

(i) Let $x_0$ be a point not belonging to $I$. Assume that it is sufficiently near $I$ so that each characteristic manifold $\Sigma_j(x_0)$ meets $I$ at a single point $x_j$. Then $u$ is of class $H^t$ at $x_0$ if $u$ is of class $H^t$ at all the points $x_j (1 \leq j \leq r)$.

(ii) If $u$ is of class $H^t$ at each point $x \in I$ except one point $x_0 \in I$, then $u$ is of class $H^t$ at any point of $X$ (near $x_0$) outside the characteristic manifolds $\Sigma_j(x_0)$ ($1 \leq i \leq r$).

To prove this, let $x_0^*$ be a real characteristic point of $R_1$ over $x_0$. It is on some $V_j[u]$. The bicharacteristic manifold $\Gamma_j(x_0^*)$ of $(R_1, u)$ has the base manifold $\Sigma_j(x_0)$. In particular $\Gamma_j(x_0^*)$ contains a characteristic point $x_j^* \in R_1$. Using the assumption, we can apply Theorem 3.2 to deduce that $u$ is of class $H^t \cap H^t_{ml}$ at $x_0^*$. On the other hand, let $x^* \in T_{x_0}^*$ be a non-characteristic point over $x_0$. Theorem 3.4 indicates that $u$ is of class $H^t \cap H^t_{ml}$ at $x^*$. Combining these results, we see that $u$ is of class $H^t$ at $x_0$. This completes the proof of (i). We can derive (ii) from (i) if we observe that if $x \in X$ is not on $\Sigma_j(x_0)$, $\Sigma_j(x)$ meets $I$ at a point distinct from $x_0$.

References


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