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A Remark on the Coincidence of Hecke-eigenforms

by

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1. Introduction

This is a continuation of our previous paper [K]. In that paper, we gave a certain condition on the Fourier coefficients under which two Hecke-eigenforms coincide with each other. In this paper, we give a slightly stronger result than the result in [K] (cf. Theorem 1.1). Furthermore, we propose a conjecture on the coincidence of Hecke-eigenforms, and verify it under a certain assumption on the nonvanishing of the Koecher-Maaß Dirichlet series (cf. Theorem 2.2).

NOTATION. For an (m,n)-matrix X and an (m,m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. For an integral domain R of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R. In particular, we put $\mathcal{H}_n = \mathcal{H}_n(\mathbf{Z})$. We define the set $\mathcal{E}_n(R)$ of even-integral matrices over R by $\mathcal{E}_n(R) = 2\mathcal{H}_n(R)$. For a subset S of the set of matrices of degree n with entries in R, we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of the symmetric matrices of degree n with entries in real numbers, we denote by S^+ the subset of S consisting of positive definite matrices. Let R' be a subring of R. Two symmetric matrices A and A' with entries in R are called equivalent over R' with each other and write $A \underset{R'}{\sim} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2. Main result

In this section and the next, we freely use the terminologies about Siegel modular forms and Hecke operators following [K]. (See also, see [A] or [F].) Let

$$GSp_n(\mathbf{Q})^+ = \{M \in GL_{2n}(\mathbf{Q}) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0\},$$

where $J_n = \begin{pmatrix} O_n & E_n \\ -E_n & O_n \end{pmatrix}$. For a positive integer N let

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbf{Z}) \mid C \equiv 0 \bmod N \right\},$$

and for a Dirichlet character χ modulo N, we denote by $M_k(\Gamma_0^{(n)}(N), \chi)$ the space of modular forms of weight k and character χ belonging to $\Gamma_0^{(n)}(N)$. We also denote by $S_k(\Gamma_0^{(n)}(N), \chi)$ the subspace of $M_k(\Gamma_0^{(n)}(N), \chi)$ consisting of all cusp forms. We simply write $\Gamma_0^{(n)}(1)$ as $\Gamma^{(n)}$, and $M_k(\Gamma_0^{(n)}(N), \chi)$ as $M_k(\Gamma_0^{(n)}(N))$, and $S_k(\Gamma_0^{(n)}(N), \chi)$ as $S_k(\Gamma_0^{(n)}(N))$ if χ is the trivial character. Now let f be an element of $M_k(\Gamma^{(n)})$. Then f has the following Fourier expansion:

$$f(z) = \sum_A a(A) \exp(2\pi i tr(Az)),$$

where A runs over all semi-positive half-integral matrices of degree n. Let $\mathbf{L}_n = \mathbf{L}_{\mathbf{Q}}(\Gamma^{(n)}, GSp_n(\mathbf{Q})^+)$ denote the Hecke ring over $\mathbf{Q}$ associated with the Hecke pair ($\Gamma^{(n)}$, $GSp_n(\mathbf{Q})^+$). For each integer m define an element $T(m)$ of $\mathbf{L}_n$ by

$$T(m) = \sum_{d_1,\cdots,d_n,e_1,\cdots,e_n} \Gamma^{(n)}(d_1 \perp \cdots \perp d_n \perp e_1 \perp \cdots \perp e_n)\Gamma^{(n)},$$

where $d_1, \cdots, d_n, e_1, \cdots, e_n$ run over all positive integers satisfying

$$d_i | d_{i+1},\ e_{i+1} | e_i\ (i = 1, \cdots, n-1), d_n | e_n, \quad d_i e_i = m\ (i = 1, \cdots, n).$$

Furthermore, for $i = 1, \cdots, n$ and a prime number p not dividing N, put

$$T_i(p^2) = \Gamma^{(n)}(E_{n-i} \perp pE_i \perp p^2 E_{n-i} \perp pE_i)\Gamma^{(n)}.$$

As is well known, $\mathbf{L}_n$ is generated by all $T(p)$ and $T_i(p^2)$ $(i = 1, \cdots, n)$. For a $T \in \mathbf{L}_n$ let $|_k T$ denote *the Hecke operator* on $M_k(\Gamma^{(n)})$ associated to T. For a matrix $A = (a_{ij}) \in \mathcal{H}_n$, put

$$c(A) = \text{g.c.d}(a_{ii}\ (i = 1, \cdots, n), \quad 2a_{ij}\ (1 \le i < j \le n)).$$

We call A primitive if $c(A) = 1$. We denote by $\mathcal{C}_n$ the subset of $\mathcal{H}_n$ consisting of primitive matrices. Let $\mathcal{F}'_n$ be the subset of $\mathcal{H}_n^+$ consisting of all matrices of the form mA_0 with $A_0 \in \mathcal{C}_n$ and m a prime number not dividing $2^{2[n/2]} \det A_0$ or 1.

THEOREM 1.1. *Let*

$$f_1(z) = \sum_A a_1(A) \exp(2\pi i tr(Az))$$

and

$$f_2(z) = \sum_A a_2(A) \exp(2\pi i tr(Az))$$

be Hecke-eigenforms in $S_k(\Gamma^{(n)})$. Assume that

$$a_1(A) = a_2(A)$$

for any $A \in \mathcal{F}'_n$. Then we have

$$f_1(z) = f_2(z).$$

REMARK 1. Let

$$\mathcal{F}_n = \{mA_0 \mid A_0 \in \mathcal{C}_n^+, m \text{ a squarefree positive integer}\}.$$

In [K] we got a stronger result than the result in [B-K2], namely, we proved that two cuspidal Hecke eigenforms coincide with each other if their A-th Fourier coefficients coincide with each other for any $A \in \mathcal{F}_n$. Clearly $\mathcal{F}'_n$ is strictly contained in $\mathcal{F}_n$. Thus Theorem 1.1 is a refinement of the above result.

REMARK 2. In this paper, we treat only the case of modular forms of level 1. However, the result can be, to some extent, to the higher level case.

For a Hecke eigenform $f \in M_k(\Gamma^{(n)})$ and an element $T \in \mathbf{L}_n$ let $\lambda_f(T)$ denote the eigenvalue of T with respect to f. For a matrix $A \in \mathcal{H}_n$, we denote by $cl(A)$ the $GL_n(\mathbf{Z})$-equivalence class of A. Now to prove the theorem, on the set $\mathcal{H}_n^+/GL_n(\mathbf{Z})$ of $GL_n(\mathbf{Z})$-equivalence classes of $\mathcal{H}_n^+$ we define the following two orders; let $cl(A_1)$ and $cl(A_2)$ be elements of $\mathcal{H}_n^+/GL_n(\mathbf{Z})$. First we define $cl(A_1) \geq cl(A_2)$ if there exists an integral square matrix D of degree n such that $A_1 = A_2[D]$. In particular, we write $cl(A_1) > cl(A_2)$ if $cl(A_1) \geq cl(A_2)$ and $cl(A_1) \neq cl(A_2)$. Next we define $cl(A_1) \succeq cl(A_2)$ if $c(A_1) = mc(A_2)$ with some integer $m > 1$, or if $c(A_1) = c(A_2)$ and $cl(A_1) \geq cl(A_2)$. Similarly to above we write $cl(A_1) \succ cl(A_2)$ if $cl(A_1) \succeq cl(A_2)$ and $cl(A_1) \neq cl(A_2)$. These definitions $'\geq'$ and $'\succeq'$ do not depend on the choice of representatives of $\mathcal{H}_n^+/GL_n(\mathbf{Z})$, and define the orders on $\mathcal{H}_n^+/GL_n(\mathbf{Z})$. From now on, we simply write as $A_1 \succeq A_2$ instead of $cl(A_1) \succeq cl(A_2)$, and others, if there is no fear of confusion. To prove Theorem 1.1, we consider the problem in more general setting. For a positive integer N, let $\mathcal{F}_{n,N}$ be the subset of $\mathcal{H}_n^+$ consisting of all matrices of the form mA_0 with A_0 a primitive matrix and m a squarefree positive integer prime to N. Furthermore, let $\mathcal{H}_{n,N}$ denote the subset of $\mathcal{H}_n$ consisting of all matrices of the form mA_0 with A_0 a primitive matrix and m a positive integer prime to N. Furthermore, for a subset $\mathcal{S}$ of $\mathcal{H}_n^+$ we put $\mathcal{S}_N = \mathcal{S} \cap \mathcal{H}_{n,N}$.

PROPOSITION 1.2. *Let*

$$g(z) = \sum_A a(A) \exp(2\pi i tr(Az))$$

be an element of $M_k(\Gamma^{(n)})$. For any prime number p put

$$g|T(p)(z) = \sum_A a|T(p)(A) \exp(2\pi i tr(Az)).$$

Then for any $A \in \mathcal{H}_n^+$ we have

$$a|T(p)(A) = a(pA) + \sum_{A' \in \mathcal{H}_n^+, A' \prec pA} c(n,k;A')a(A'),$$

where $c(n, k; A')$ is a rational number determined by n, k and A'. In particular, for any element $A \in \mathcal{C}_n^+$ we have

$$a|T(p)(A) = a(pA) + \sum_{A' \in \mathcal{C}_n^+, A' \prec pA} c(n, k; A')a(A') .$$

PROPOSITION 1.3. *Let N be a positive integer. Let*

$$f_1(z) = \sum_A a_1(A) \exp(2\pi i tr(Az))$$

and

$$f_2(z) = \sum_A a_2(A) \exp(2\pi i tr(Az))$$

be Hecke-eigenforms in $S_k(\Gamma^{(n)})$. Assume that

$$a_1(A) = a_2(A)$$

for any $A \in \mathcal{C}_n^+$, and that $\lambda_1(p) = \lambda_2(p)$ for any prime number $p \not| N$, where $\lambda_i(p) = \lambda_{f_i}(T(p))$. Then we have

$$a_1(A) = a_2(A)$$

for any $A \in \mathcal{H}_{n,N}^+$.

Proof. We prove the assertion by induction with respect to the order $\prec$. The assertion holds for a minimal element in $\mathcal{H}_{n,N}^+$ because a minimal element is a primitive matrix. Assume that A is not minimal, and that the assertion holds for A' such that $A' \prec A$. We may assume that A is not primitive and that $p^{-1}A \in \mathcal{H}_{n,N}^+$ for some prime number p not dividing N. By Proposition 1.2, we have

$$a_i(A) = -\lambda_i(T(p))a_i(p^{-1}A) + \sum_{A' \in \mathcal{H}_n^+, A' \prec A} c(n, k; A')a_i(A')$$

for $i = 1, 2$. Clearly, $A' \in \mathcal{H}_{n,N}^+$ for $A' \prec A$. Thus, by induction hypothesis, we have

$$a_1(A) = a_2(A) .$$

This completes the induction.

PROPOSITION 1.4. *Let f be a Hecke eigenform in $S_k(\Gamma^{(n)})$. There exists an element $A \in \mathcal{C}_n^+$ such that $a_f(A) \neq 0$.*

Proof. Assume that $a_f(A) = 0$ for any $A \in \mathcal{C}_n^+$. Then, in the same manner as in the proof Proposition 1.3, we can show that

$$a_f(A) = 0$$

for any $A \in \mathcal{H}_n^+$. This is a contradiction.

PROPOSITION 1.5. *Let M and N be positive integers, and χ a Dirichlet character modulo M with conductor m_χ. Let*

$$g(z) = \sum_A a(A) \exp(2\pi i tr(Az))$$

be an element of $M_k(\Gamma_0^{(n)}(M), \chi)$. *Assume that* $a(A) = 0$ *for any* $A \in \mathcal{H}_{n,N}^+$, *and that* N *is prime to* M/m_χ. *Then* $g(z) = 0$.

Proof. The assertion can be proved in the same manner as in [Mi, Theorem 4.6.8 (1)].

Proof of Theorem 1.1. By Proposition 1.4, we can take an element $A_0 \in \mathcal{C}_n^+$ such that $a_1(A_0) = a_2(A_0) \neq 0$. Put $N = 2^{2[n/2]} \det A_0$ and let p be a prime number not dividing N. By Proposition 1.2, we have

$$\lambda_i(p) a_i(A_0) = a_i(pA_0) + \sum_{A' \in \mathcal{C}_n^+, A' \prec pA_0} c(n, k; A') a_i(A')$$

for $i = 1, 2$. By assumption we have

$$a_1(pA_0) = a_2(pA_0)$$

and

$$a_1(A') = a_2(A')$$

for any $A' \prec pA_0$. Thus we have

$$\lambda_1(p) = \lambda_2(p)$$

for any prime number $p \not| N$. Thus by Proposition 1.3, we have

$$a_1(A) = a_2(A)$$

for any $A \in \mathcal{H}_{n,N}^+$. Thus by Proposition 1.5, the assertion holds.

3. Conjecture

In this section we propose a conjecture on the coincidence of two Hecke-eigenforms, which is far stronger than Theorem 1.1. Before formulating our conjecture, we review a result of [I-K] on the expression of Koecher-Maaß Dirichlet series.

A half-integral matrix A over $\mathbf{Z}_p$ is called non-degenerate modulo p if the reduction $M_A \otimes_{\mathbf{Z}_p} \mathbf{Z}_p/p\mathbf{Z}_p$ of the quadratic space M_A is non-degenerate. We should remark that A is non-degenerate modulo p if and only if A is unimodular in the case of $p \neq 2$, where as it is non-degenerate modulo 2 if and only if $A = \frac{1}{2}U$ or $A \sim \frac{1}{2}U \perp c_0$ over $\mathbf{Z}_2$ with U an even-integral unimodular matrix and $c_0 \in \mathbf{Z}_2^*$ in the case of $p = 2$. Now define a subset $\mathcal{K}_n'(\mathbf{Z}_p)$ of $\mathcal{H}_n(\mathbf{Z}_p)$ by

$$\mathcal{K}_n'(\mathbf{Z}_p) = \{A \in \mathcal{H}_n(\mathbf{Z}_p) \mid A \sim V_0 \perp pV_1 \text{ with } V_0, V_1 \text{ non-degenerate modulo } p\}.$$

Further define a subset $\mathcal{K}_n''(\mathbf{Z}_2)$ of $\mathcal{H}_n(\mathbf{Z}_2)$ by

$$\mathcal{K}_n''(\mathbf{Z}_2) = \{A \in \mathcal{H}_n(\mathbf{Z}_2) \mid A \sim \frac{1}{2}V_0 \perp V \perp V_1 \text{ with } V_0, V_1 \text{ even-integral unimodular}$$

and V a diagonal unimodular matrix of degree 2 such that $\det V \equiv 1 \bmod 4\}$.

Put $\mathcal{K}_n(\mathbf{Z}_p) = \mathcal{K}'_n(\mathbf{Z}_2) \cup \mathcal{K}''_n(\mathbf{Z}_2)$ or $\mathcal{K}'_n(\mathbf{Z}_p)$ according as $p = 2$ or not. Put $\mathcal{K}_n = \mathcal{H}_n^{\times} \cap \prod_p \mathcal{K}_n(\mathbf{Z}_p)$. We note that $\mathcal{K}_2 \cap \mathcal{C}_2^+$ is the set of positive definite matrices A of degree 2 such that $-4\det A$ are fundamental discriminants. Let

$$f(z) = \sum_{A \in \mathcal{H}_n^+} a_f(A) \exp(2\pi i tr(Az))$$

be a cuspidal Hecke eigenform, and for each prime p let $\alpha_{0,p}, \alpha_{1,p}, \cdots, \alpha_{n,p}$ denote the p-Satake parameters determined by f (cf. [I-K]). We then define the standard zeta function $\zeta^{st}(f, s)$ of f by

$$\zeta^{st}(f, s) = \prod_p \{(1 - p^{-s}) \prod_{i=1}^{n} (1 - \alpha_{i,p} p^{-s})(1 - \alpha_{i,p}^{-1} p^{-s})\}^{-1} .$$

Furthermore, we define the Koecher-Maaß Dirichlet series $L(f, s)$ of f by

$$L(f, s) = \sum_A \frac{a_f(A)}{(\det A)^s e(A)} ,$$

where A runs over a complete set of representatives of $\mathcal{H}_n^+/GL_n(\mathbf{Z})$, and $e(A)$ denotes the order of the orthogonal group of A. For $A \in \mathcal{H}_n^+$ put

$$G_f(A) = \sum_{C \in \mathcal{G}(A)} \frac{a_f(C)}{e(C)} ,$$

where $\mathcal{G}(A)$ is the set of equivalence classes belonging to the genus of A. Then rewriting [I-K, Theorem 3.3], we have

THEOREM 2.1. *Under the above notation and the assumption we have*

$$L(f, s) = \frac{\zeta^{st}(f, 2s - k + 1)}{\zeta(2s - k + 1)} \sum_{\mathcal{G}(A)} G_f(A) S(A, s) ,$$

where $\mathcal{G}(A)$ runs over all genera of positive definite half-integral matrices in $\mathcal{K}_n$, and $S(A, s)$ is a certain Dirichlet series depending only on $\mathcal{G}(A)$.

Let

$$\mathcal{L}_n = \{A = p^i A_0 \mid p \text{ a prime number}, i = 0, 1, A_0 \in \mathcal{K}_n \cap \mathcal{C}_n^+\} .$$

Now we propose the following conjecture:

CONJECTURE. *Let*

$$f_1(z) = \sum_A a_1(A) \exp(2\pi i tr(Az))$$

and

$$f_2(z) = \sum_A a_2(A) \exp(2\pi i tr(Az))$$

be cuspidal Hecke-eigenforms of weight k with respect to $\Gamma^{(n)}$. *Assume that*

$$a_1(A) = a_2(A)$$

for any $A \in \mathcal{L}_n$. *Then we have*

$$f_1(z) = f_2(z)\,.$$

As for this conjecture, we have

THEOREM 2.2. *Assume the following*:
(*) *there is no Hecke eigenform* f *in* $S_k(\Gamma^{(n)})$ *such that* $L(f, s) = 0$.
Then for two Hecke eigenforms f_1 *and* f_2 *in* $S_k(\Gamma^{(n)})$, *the above conjecture is true.*

To prove this, we give a lemma.

LEMMA 2.3. *Let* f *be a Hecke-eigenform in* $S_k(\Gamma^{(n)})$ *such that* $L(f, s) \neq 0$. *Then there exists an element* $A \in \mathcal{K}_n \cap \mathcal{C}_n^+$ *such that* $a_f(A) \neq 0$.

Proof. Assume that $a_f(A) = 0$ for any $A \in \mathcal{K}_n \cap \mathcal{C}_n^+$. Then by Proposition 1.2, we have $a_f(A) = 0$ for any $A \in \mathcal{K}_n^+$. Thus by Theorem 2.1 we have $L(f, s) = 0$.

Proof of Theorem 2.2. Assume that

$$a_1(A) = a_2(A)$$

for any $A \in \mathcal{L}_n$. Then by the same argument as in the proof of Theorem 1.1 we have

$$(**) \qquad \lambda_{f_1}(T(p)) = \lambda_{f_2}(T(p))$$

for any prime number p. Furthermore, we have

$$a_1(A) = a_2(A)$$

for any $A \in \mathcal{K}_n^+$. Hence by Theorem 2.1 we have

$$\frac{\zeta^{st}(f_1, 2s - k + 1)}{\zeta^{st}(f_2, 2s - k + 1)} = \frac{L(f_1, s)}{L(f_2, s)}\,.$$

As is well known, the left-hand side of the above equality is invariant under the transformation $s \to k - 1/2 - s$ (cf. [B, Satz 4]), while the right-hand side is invariant under the transformation $s \to k - s$ (cf. [Ma, Section 15]). We note that the left-hand side of the above equality has neither zeros nor poles in some half plane $\mathrm{Re}s >> 0$. Thus by an argument similar to the proof of [B-K1,Theorem], we can prove

$$\zeta^{st}(f_1, 2s - k + 1) = \zeta^{st}(f_2, 2s - k + 1)$$

and

$$L(f_1, s) = L(f_2, s)\,.$$

In particular, the first equality implies that

$$\lambda_{f_1}(T_i(p^2)) = \lambda_{f_2}(T_i(p^2))$$

for $i = 0, \cdots, n$ and any prime number p, and this combined with (**) shows that f_1 and f_2 have the same eigenvalue for any Hecke operator. Thus, $f_1 - f_2$ is also a Hecke eigenform if it is not identically zero. Thus by assumption (*) f_1 is identical with f_2.

REMARK 1. In the above proof, we do not assume so called "multiplicity one conjecture" for Hecke eigenforms.

REMARK 2. The assumption (*) does not hold for lower weight case. In fact, there is an example of a Hecke eigenform f of weight 4 belonging to the congruence group $\Gamma_0^{(2)}(N)$ such that $L(f, s) = 0$ (cf. [B-S]). However, we do not know a counter example to (*) for higher weight modular forms. Furthermore, although the assumption (*) does not hold, we would still expect that the conjecture is true. Because a cusp form with a vanishing Koecher-Maaß Dirichlet series is thought to have a distinguished property, by which the conjecture is expected to be proved.

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