

## Paramodular Forms of Degree 2 and Level 3

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**Abstract.** Let  $\Gamma_3 \subset \mathrm{Sp}_2(\mathbb{Q})$  be the paramodular group of level 3 and  $\Gamma_3^* \subset \mathrm{Sp}_2(\mathbb{R})$  the maximal normal discrete extension of  $\Gamma_3$  of index 2. Denote by  $D\Gamma_3^*$  the commutator subgroup of  $\Gamma_3^*$ . The main goal of the present note is to determine the structure of the graded ring of paramodular forms for  $D\Gamma_3^*$ . Since all generators constructed here are actually modular forms for  $\Gamma_3^*$  with certain multiplier-systems, we can derive generators for the graded rings of paramodular forms for all groups  $\Gamma$  with  $D\Gamma_3^* \subset \Gamma \subset \Gamma_3^*$ , especially  $\Gamma_3$ .

### 1. Introduction

In [16], Igusa determined generators of the graded ring of modular forms for  $\Gamma_1 = \mathrm{Sp}_2(\mathbb{Z})$ , the paramodular group of degree 2 and level 1. This was the first example, where generators of the graded ring of paramodular forms of degree 2 are known. Later, Freitag [9] gave another proof of Igusa's result, using a distinguished Siegel modular form  $\Theta_5$  with (uniquely determined) nontrivial multiplier-system and known zero-divisor. Then, using similar techniques, Freitag [10] determined generators of the graded ring of modular forms of even weight for  $\Gamma_2^*$ , the maximal normal discrete extension of  $\Gamma_2$  of index 2. Only recently, these results were extended to  $\Gamma_2$  by Ibukiyama and Onodera [19]. As far as we know, these are the only results, where generators of the graded ring of modular forms for the paramodular group of degree 2 and level  $t$  are known explicitly. Apart from that, in degree 2, there is Ibukiyama's formula for the dimensions of the spaces of cusp-forms [17], which can be used to deduce some information about generators of the graded ring of modular forms for  $\Gamma_t$  (for small level  $t$  at least, see e.g. [18], [19]). The general result of Runge [27, Theorem 2.3], which describes the even part of the graded ring of modular forms for paramodular groups of arbitrary degree as invariants of a certain space of theta-constants, hardly gives any explicit information on generators.

As another example, in this note we solve the case of degree 2 and level 3. More precisely we determine generators of the graded ring of modular forms for the commutator-subgroup  $D\Gamma_3^*$  where  $\Gamma_3^*$  is the maximal normal discrete extension of  $\Gamma_3$  (which is of index 2). Our method is in some sense the same as Freitag's [9]. By results of Borchers [1], one can nowadays construct paramodular forms of degree 2 with known zero-divisor, so-called Borchers-products.  $\Theta_5$  is an example of such a Borchers-product. More examples were given in [15]. We construct Borchers-products for  $\Gamma_3^*$  (following [1] or [3] more closely than [15]). These paramodular forms have non-trivial multiplier-systems in general and we have to consider modular forms for  $D\Gamma_3^*$  quite naturally. The crucial point is, that we can find Borchers-products with "minimal" zero-divisor. As in [9], [10] the only thing we have to do then, is to lift those modular forms on the divisor, that are restrictions of paramodular forms, to paramodular forms for  $\Gamma_3^*$ . Here we use "arithmetical liftings" (generalizations of Maaß's construction [24], [25]), introduced by Gritsenko [13], [14] and Gritsenko-Nikulin [15].

The Borchers-products constructed in section 4, already appeared in [15] and it was noted there that these modular forms can be used "to construct all generators of the graded rings of modular forms for  $\Gamma_2$  and  $\Gamma_3$ ". In fact, for level 2 one can find all forms for the commutator-subgroup  $D\Gamma_2^*$  (which has index 4 in  $\Gamma_2$ ) by the very same method used here. There are good reasons to believe, that  $t = 2$  and  $t = 3$  are the only cases, where the problem is as easy as in the Siegel-case (see remark 4.4).

Now we give a short description of the following sections:

In section 2, we fix our notation concerning the paramodular group  $\Gamma_t$  and the extension  $\Gamma_t^*$ . From [20] we cite a special case of a general result on generators of paramodular groups, which we did not find anywhere else in the literature. Moreover we give a description of the character-groups of  $\Gamma_3$  and  $\Gamma_3^*$  following [6].

In section 3 arithmetical liftings from vector-valued modular forms of half-integral weight for the metaplectic group  $\text{Mp}_2(\mathbb{Z})$  to paramodular forms for  $\Gamma_3^*$  with multiplier-systems are defined. The main result, proposition 3.6, is essentially a reformulation of results from [14] and [15]. We explicitly calculate the dimensions of the associated Maaß-spaces, using a dimension-formula of Skoruppa [29], [7].

In section 4 we apply Borchers theory in order to find paramodular forms for  $\Gamma_t^*$ . Borchers theory is formulated in terms of orthogonal groups. The (well-known) connection with paramodular forms is cited from [15] and [3] mainly. The input for Borchers lift are vector-valued modular forms of weight  $-\frac{1}{2}$  for  $\text{Mp}_2(\mathbb{Z})$  with poles of small order at the cusp. Since we are looking for forms with "minimal" zero-divisor, we have to find vector-valued modular forms with poles of small order at the cusp. To this end, an obstruction-problem from [2] is solved.

In section 5 we use (some of) the forms, constructed in the preceding sections in order to prove our main result, theorem 5.2. It states, that the ring of paramodular forms for  $D\Gamma_3^*$  is generated by tree Borchers-products of weight 1, 6 and 12, together with four Maaß-lifts, needed to generate all the modular forms on the product of two upper half-planes with multiplier-systems of order 3. Moreover, we find all relations among the generators. In the same way, we find generators of the rings of paramodular forms for

$\Gamma_3^*$  and  $\Gamma_3$  and all relations among them. The formula for the dimensions of the spaces of paramodular forms for  $\Gamma_3$ , following from these results, coincides with Ibukiyama's formula [18].

In the remaining part of this section we set up basic notations:

For a ring  $R$  (always assumed to be commutative and with unity), we denote by  $R^{n \times n}$  the set of  $n$  by  $n$  matrices with entries in  $R$ . Given  $A, B$  in  $R^{n \times n}$ , we write  $A^t$  for the transpose of  $A$  and define  $B[A] = A^t B A$ . Let  $I_n \in R^{n \times n}$  be the identity-matrix of dimension  $n$  (if  $n$  is obvious, we just write  $I$  instead of  $I_n$ ).  $\text{Sp}_n(R)$ , the symplectic group of degree  $n$  with entries in the ring  $R$ , is given by  $\text{Sp}_n(R) = \{M \in R^{2n \times 2n} \mid \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\}$ .

For  $m \in \mathbb{N}, n \in \mathbb{Z}$  we write  $m|n$ , if  $m$  divides  $n$ .

For a group  $G$  the group of abelian characters of  $G$  is denoted by  $G^{\text{ab}}$  and the commutator-group is denoted by  $DG$ . If  $G^{\text{ab}}$  is finite (which will always be the case later), one has  $G^{\text{ab}} \cong G/DG$ . For  $n \in \mathbb{N}$  we set  $C_n = \mathbb{Z}/n\mathbb{Z}$ .

We use  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  as generators of  $\text{SL}_2(\mathbb{Z})$ . The principal congruence subgroup (of level  $n$ ) is  $\text{SL}_2(\mathbb{Z})[n] := \{M \in \text{SL}_2(\mathbb{Z}) \mid M \equiv I \pmod{n}\mathbb{Z}\}$ . The space of (elliptic) modular forms of weight  $k$  for  $\text{SL}_2(\mathbb{Z})$  is denoted by  $[\text{SL}_2(\mathbb{Z}), k, 1]$ . For  $4 \leq k \in 2\mathbb{N}$  denote by  $g_k \in [\text{SL}_2(\mathbb{Z}), k, 1]$  the normalized elliptic Eisenstein-series of weight  $k$ . Explicitly,  $g_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \in \mathbb{N}} \sigma_{k-1}(n) e^{2\pi i n \tau}$ , where  $B_k$  is the  $k^{\text{th}}$  Bernoulli-number and  $\sigma_k(n) = \sum_{d \in \mathbb{N}, d|n} d^k$ . Let  $\eta(\tau) = e^{2\pi i \tau/24} \prod_{n \in \mathbb{N}} (1 - e^{2\pi i n \tau})$  be the Dedekind-eta-function and  $v_\eta$  be the multiplier-system of  $\eta$ .  $v_\eta^2$  is a generator of  $\text{SL}_2(\mathbb{Z})^{\text{ab}} \cong C_{12}$ .  $\Delta_{12} = \eta^{24} = \frac{1}{12^3} (g_4^3 - g_6^2)$  is the first non-trivial cusp-form for  $\text{SL}_2(\mathbb{Z})$  (up to normalization).

## 2. Paramodular groups of degree 2

We think of paramodular groups (of degree 2) as subgroups of the rational symplectic group  $\text{Sp}_2(\mathbb{Q})$ . For  $t \in \mathbb{N}$  define  $P_t := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$  (a polarization, chosen in normal form without restriction) and  $D_t = \begin{pmatrix} t & 0 \\ 0 & P_t \end{pmatrix}$ . Later on we will specialize  $t = 3$ , but if possible, we give results for general  $t$ .

DEFINITION 2.1. *The paramodular group  $\Gamma_t$  (of level  $t$ ) is given by*

$$\Gamma_t := \{M \in \text{Sp}_2(\mathbb{Q}) \mid D_t^{-1} M D_t \in \mathbb{Z}^{4 \times 4}\}.$$

The conjugated group  $\widehat{\Gamma}_t := D_t^{-1} \Gamma_t D_t \subset \mathbb{Z}^{4 \times 4}$  is the integral paramodular group (of level  $t$ ). Note that  $\widehat{\Gamma}_t$  leaves the form  $\begin{pmatrix} 0 & -P_t \\ P_t & 0 \end{pmatrix}$  invariant.

As is well known [22], [14], paramodular groups have non-trivial discrete extensions in  $\text{Sp}_2(\mathbb{R})$  for  $t > 1$ . In our special case (were  $t$  will be prime later on), we define a distinguished extension of index 2 of  $\Gamma_t$ . Set

$$V_t := \begin{pmatrix} U_t & 0 \\ 0 & U_t^t \end{pmatrix} \in \text{Sp}_2(\mathbb{R}) \quad \text{with} \quad U_t := \begin{pmatrix} 0 & \sqrt{t} \\ 1/\sqrt{t} & 0 \end{pmatrix}.$$

Then  $V_t^2 = I$  and  $\gamma_t : M \mapsto V_t M V_t^{-1}$  is an involution in  $\text{aut}(\Gamma_t)$ .

DEFINITION 2.2. *The extended paramodular group  $\Gamma_t^*$  (of level  $t$ ) is the group, generated by  $V_t$  over  $\Gamma_t$ , i.e.*

$$\Gamma_t^* := \langle \Gamma_t \cup \{V_t\} \rangle = \Gamma_t \cup \Gamma_t V_t \subset \mathrm{Sp}_2(\mathbb{R}).$$

$\Gamma_t^*$  is an extension of index 2 of  $\Gamma_t$  for  $t > 1$ . In general, there is an even bigger maximal normal discrete extension  $\Gamma_t^{\max} \supset \Gamma_t^*$ , which is generated by (suitably defined elements)  $V_d \in \mathrm{Sp}_2(\mathbb{R})$  for all  $d \parallel t$  (see [15, 1.3] for details). If  $t$  is square-free, then  $\Gamma_t^{\max}$  is maximal discrete, and if  $t$  is prime, then  $\Gamma_t^* = \Gamma_t^{\max}$  is maximal discrete too (though in general, it is not). Typical elements of  $\Gamma_t$  are

$$\begin{aligned} J_t &= \begin{pmatrix} 0 & -P_t^{-1} \\ P_t & 0 \end{pmatrix}, \\ \mathrm{rot}(U) &= \begin{pmatrix} U & 0 \\ 0 & U^{-\mathrm{tr}} \end{pmatrix} \quad \text{for } U \in \Omega_t = \{M \in \mathrm{GL}_2(\mathbb{Z}) \mid P_t M P_t^{-1} \in \mathbb{Z}^{2 \times 2}\}, \\ \mathrm{trans}(S) &= \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \quad \text{for } S \in \Sigma_t = \{M \in \mathbb{Q}^{2 \times 2} \mid M = M^{\mathrm{tr}}, M P_t \in \mathbb{Z}^{2 \times 2}\}, \\ M_1 \times_t M_2 &= \begin{pmatrix} a_1 & 0 & b_2 & 0 \\ 0 & a_2 & 0 & b_2/t \\ c_1 & 0 & d_1 & 0 \\ 0 & t c_2 & 0 & d_2 \end{pmatrix} \quad \text{for } M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \end{aligned}$$

We need generators of  $\Gamma_t$ . From [20, Satz 1.12] we cite

LEMMA 2.3.  *$\Gamma_t$  is generated by  $J_t$  and  $\mathrm{trans}(S)$  for  $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1/t \end{pmatrix}$ .*

In the sequel, modular forms for  $\Gamma_3^*$  with arbitrary multiplier-systems will be considered. For the rest of this section we specialize to  $t = 3$ . Since multiplier-systems for  $\Gamma_3$  and  $\Gamma_3^*$  have integral weight [5, Satz 14], multiplier-systems are just (abelian) characters of  $\Gamma_3$  resp.  $\Gamma_3^*$ .

The groups  $\Gamma_t^{\mathrm{ab}}$  and  $\Gamma_t^{*\mathrm{ab}}$  are known by [6]. Characters of  $\Gamma_3$  arise in the following way: There are surjective homomorphisms ( $\mathbb{F}_p$  is the field with  $p$  elements)

$$\begin{aligned} \alpha_2 : \Gamma_3 &\rightarrow \mathrm{Sp}_2(\mathbb{F}_2), \quad M \mapsto D_3^{-1} M D_3 \bmod 2\mathbb{Z}, \\ \beta_3 : \Gamma_3 &\rightarrow \mathrm{SL}_2(\mathbb{F}_3)^2, \quad M \mapsto \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_4 & 3b_4 \\ c_4/3 & d_4 \end{pmatrix} \right) \bmod 3\mathbb{Z}. \end{aligned}$$

Recall, that  $D_3^{-1} \Gamma_3 D_3$  is the integral paramodular group and that  $\begin{pmatrix} 0 & -P_3 \\ P_3 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \bmod 2\mathbb{Z}$ . Since  $\mathrm{Sp}_2(\mathbb{F}_2)$  is isomorphic to the symmetric group on six elements (e.g. via the action on the six odd theta-characteristics in  $\mathbb{F}_2^4$ ), there is a character  $\widehat{\kappa}$  of  $\mathrm{Sp}_2(\mathbb{F}_2)$  of order 2. The pull-back of  $\widehat{\kappa}$  gives a character  $\kappa := \widehat{\kappa} \circ \alpha_2 \in \Gamma_3^{\mathrm{ab}}$  of order 2 (in the same way, the nontrivial character of the Siegel modular group  $\Gamma_1 = \mathrm{Sp}_2(\mathbb{Z})$  arises [23]).

As is well known,  $\mathrm{SL}_2(\mathbb{F}_3)^{\mathrm{ab}}$  is (isomorphic to) a cyclic group of order 3, generated by a character  $\widehat{\mu}$ , which is uniquely determined by  $\widehat{\mu} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{2\pi i/3}$ .

The corresponding characters of  $\mathrm{SL}_2(\mathbb{F}_3)^2$ , which arise by first projecting on the  $j^{\mathrm{th}}$  component, are denoted by  $\widehat{\mu}_j$ . The pull-back of the characters  $\widehat{\mu}_j$  gives two (independent) characters  $\mu_j := \widehat{\mu}_j \circ \beta_3 \in \Gamma_3^{\mathrm{ab}}$  of order 3.

The following lemma is a special case of [6, Theorem 4.2].

LEMMA 2.4.  $\Gamma_3^{\mathrm{ab}} \cong \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_3$  is generated by  $\kappa$ ,  $\mu_1$  and  $\mu_2$ .

Explicit values of  $\kappa$  and  $\mu_1$  on  $J_3$ , the subgroups of rotations and the subgroups of translations are given by

$$(2.1) \quad \mu_1(M) = \begin{cases} e^{2\pi i s_1/3} & \text{for } M = \mathrm{trans} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_4/3 \end{pmatrix}, \\ 1 & \text{for } M = \mathrm{rot} \begin{pmatrix} u_1 & 3u_2 \\ u_3 & u_4 \end{pmatrix} \text{ and } M = J_3, \end{cases}$$

$$(2.2) \quad \kappa(M) = \begin{cases} (-1)^{s_1+s_2+s_4} & \text{for } M = \mathrm{trans} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_4/3 \end{pmatrix}, \\ (-1)^{(1+u_1+u_4)(1+u_2+u_3)+u_1u_4} & \text{for } M = \mathrm{rot} \begin{pmatrix} u_1 & 3u_2 \\ u_3 & u_4 \end{pmatrix}, \\ 1 & \text{for } M = J_3. \end{cases}$$

(For more explicit formulas see e.g. [23], [15, Lemma 1.2], [6]). Note that  $\mu_2 = \mu_1 \circ \gamma_3$ , so explicit values of  $\mu_2$  can easily be read of (2.1) too.

The involution  $\gamma_3 \in \mathrm{aut}(\Gamma_3)$  acts on  $\Gamma_3^{\mathrm{ab}}$  by  $\nu \mapsto \nu \circ \gamma_3$ . If  $\nu \circ \gamma_3 = \nu$ , we say that  $\nu$  is symmetric.  $\nu \in \Gamma_3^{\mathrm{ab}}$  can be extended to a character of  $\Gamma_3^*$  if (and only if)  $\nu$  is symmetric. In this case,  $\nu$  is extended to a character of  $\Gamma_3^*$  by  $\nu(V_3) = 1$ . It was shown in [6, Sec. 5], that  $\kappa$  is symmetric. Since  $\mu_1 \circ \gamma_3 = \mu_2$ , the characters  $\mu_1$  and  $\mu_2$  are not symmetric, but on the other hand,  $\mu := \mu_1\mu_2$  is symmetric. We extend  $\kappa$  and  $\mu$  to characters of  $\Gamma_3^*$  as above (and denote this extended characters by the same symbols again), i.e. as characters of  $\Gamma_3^*$  we have  $\kappa(V_3) = \mu(V_3) = 1$ . Since  $\Gamma_3^*$  is an extension of index 2 of  $\Gamma_3$ , generated by  $V_3$ , another character  $\chi$  of  $\Gamma_3^*$  is defined by  $\chi(V_3) = -1$  and  $\chi(\Gamma_3) = \{1\}$ .

The following lemma is a special case of [6, Cor. 5.5].

LEMMA 2.5.  $\Gamma_3^{*\mathrm{ab}} \cong \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_3$  is generated by  $\chi$ ,  $\kappa$  and  $\mu$ .

Explicit values of  $\mu$  on  $J_3$ , rotations  $\mathrm{rot}(U)$ ,  $U \in \Omega_t$ , and translations  $\mathrm{trans}(S)$ ,  $S \in \Sigma_t$ , are given by

$$(2.3) \quad \mu(M) = \begin{cases} e^{2\pi i (s_1+s_4)/3} & \text{for } M = \mathrm{trans} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_4/3 \end{pmatrix}, \\ 1 & \text{for } M = \mathrm{rot} \begin{pmatrix} u_1 & 3u_2 \\ u_3 & u_4 \end{pmatrix} \text{ and } M = J_3. \end{cases}$$

Since  $V_3 \notin D\Gamma_3^*$ , we have  $D\Gamma_3^* \subset \Gamma_3$ . In fact, lemma 2.5 implies, that  $[\Gamma_3 : D\Gamma_3^*] = 6$  and  $\Gamma_3/D\Gamma_3^*$  is generated by the coset trans  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D\Gamma_3^*$ .

For  $\nu \in \Gamma_3^{\text{ab}}$  or  $\nu \in \Gamma_3^{*\text{ab}}$  we define  $\tilde{\nu} \in \text{SL}_2(\mathbb{Z})^{\text{ab}}$  via  $\tilde{\nu}(M) := \nu(M \times_t I)$ .

### 3. Jacobi-forms with characters and Maaß-Lifts

In this section we use “arithmetical liftings”, defined by Gritsenko [13, 14] and Gritsenko-Nikulín [15] to construct paramodular forms with certain multiplier-systems. This is the first of two fundamental methods, used to construct generators of the graded ring of modular forms for  $D\Gamma_3^*$ . The other one is Borcherds-products, being presented in the following section.

First we fix our notation concerning paramodular forms. Let  $\mathbb{H}_n$  be the Siegel upper half-plane of degree  $n$  and  $(M, Z) \mapsto M \cdot Z$  the usual action of  $\text{Sp}_n(\mathbb{R})$  on  $\mathbb{H}_n$  (as biholomorphic transformations). The standard factor of automorphy is  $j_n(M, Z) = \det(CZ + D)$ , if  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathbb{R})$ . The corresponding action of weight  $k \in \mathbb{Z}$  on functions  $f : \mathbb{H}_n \rightarrow \mathbb{C}$  is given by  $f|_k M(Z) := j_n(M, Z)^{-k} f(M \cdot Z)$ .

DEFINITION 3.1. *Assume that  $\Gamma \subset \Gamma_t^*$  with finite index. Let  $\nu$  be a character of  $\Gamma$ . A holomorphic function  $f : \mathbb{H}_2 \rightarrow \mathbb{C}$  is a paramodular form of weight  $k \in \mathbb{Z}$  with character  $\nu$  for  $\Gamma$  if*

$$f|_k M(Z) = \nu(M) f \text{ for all } M \in \Gamma.$$

*$f$  is a cusp-form, if additionally  $\lim_{y \rightarrow \infty} f|_k M \begin{pmatrix} z & 0 \\ 0 & iy \end{pmatrix} = 0$  for all  $M \in \text{Sp}_2(\mathbb{Q})$  and  $z \in \mathbb{H}_1$ . The space of paramodular forms of weight  $k$  with character  $\nu$  for  $\Gamma$  is denoted by  $[\Gamma, k, \nu]$ . The subspace of cusp-forms is denoted by  $[\Gamma, k, \nu]_{\text{cusp}}$ .*

Since in the following we will have to consider (elliptic and Jacobi) modular forms of half-integral weight too, we need the metaplectic group  $\text{Mp}_2(\mathbb{Z})$ . This is two-fold cover of  $\text{SL}_2(\mathbb{Z})$ , consisting of pairs  $(M, \omega)$ , where  $M \in \text{SL}_2(\mathbb{Z})$  and  $\omega : \mathbb{H}_1 \rightarrow \mathbb{C}$  is a holomorphic square-root of  $j_1(M, \tau)$ , i.e. we have  $\omega(\tau)^2 = j_1(M, \tau)$  (see [7, Sec. 4.2] for some more details on  $\text{Mp}_2(\mathbb{Z})$ ). Standard generators of  $\text{Mp}_2(\mathbb{Z})$  are

$$\widehat{T} := \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad \widehat{J} := \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

By  $\sqrt{\tau}$  we always denote the principal value of the square root of  $\tau$ , determined by  $\Re(\sqrt{\tau}) > 0$  or  $\Re(\sqrt{\tau}) = 0$  and  $\Im(\sqrt{\tau}) \geq 0$ . The subgroup  $\text{Mp}_2(\mathbb{Z})[n] := \{(M, \omega) \mid M \equiv I \pmod{n\mathbb{Z}}\}$  is the principal congruence subgroup (of level  $n$ ) of the metaplectic group.  $\text{Mp}_2(\mathbb{Z})$  acts with weight  $k \in \frac{1}{2}\mathbb{Z}$  on functions  $f : \mathbb{H}_1 \rightarrow V$  (where  $V$  is a  $\mathbb{C}$ -vector-space) by

$$(3.1) \quad f|_k(M, \omega)(\tau) := \omega(\tau)^{-2k} f(M \cdot \tau).$$

Let  $H(\mathbb{Z})$  be the integral Heisenberg-group as in [15, Sec. 1]. We define the metaplectic Jacobi-group  $\text{MJ}_2(\mathbb{Z})$  to be

$$\text{MJ}_2(\mathbb{Z}) := \text{Mp}_2(\mathbb{Z}) \ltimes H(\mathbb{Z}),$$

where the action of  $\text{Mp}_2(\mathbb{Z})$  on  $H(\mathbb{Z})$  is given by the action of the first component. The parabolic subgroup  $\Gamma_{t,\infty} \subset \Gamma_t$  is defined by

$$\Gamma_{t,\infty} := \{M \in \Gamma_t \mid M \text{ has last row } (0, 0, 0, 1)\}.$$

Note that  $\Gamma_{1,\infty} \cong \text{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$ . Thus we can think of  $\text{MJ}_2(\mathbb{Z})$  as a two-fold cover of  $\Gamma_{1,\infty}$  and  $\text{MJ}_2(\mathbb{Z})$  acts with weight  $k \in \frac{1}{2}\mathbb{Z}$  on functions  $f : \mathbb{H}_2 \rightarrow V$  (where  $V$  is again a  $\mathbb{C}$ -vector-space) by

$$\begin{aligned} & f|_k((M, \omega), [u, v; w])(Z) \\ & := \omega(z_1)^{-2k} f\left((M \times_1 I) \text{rot} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \text{trans} \begin{pmatrix} 0 & v \\ v & w - uv \end{pmatrix} \cdot Z\right), \\ & Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}. \end{aligned}$$

We also write  $f|_k(M, \omega)$  for  $f|_k((M, \omega), [0, 0; 0])$  and  $f|_k[u, v; w]$  for  $f|_k((I, 1), [u, v; w])$ . Let  $v_H$  be the character of  $H(\mathbb{Z})$ , defined by

$$(3.2) \quad v_H([u, v; w]) := (-1)^{u+v+uv+w}.$$

Following [15, Lemma 3.1], all characters of  $\text{MJ}_2(\mathbb{Z})$  are of the form  $v_{a,b} := v_\eta^a \times v_H^b$  with  $a \in \mathbb{Z}/24\mathbb{Z}$  and  $b \in \mathbb{Z}/2\mathbb{Z}$ .  $v_{a,b}$  factors over  $\text{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$ , if and only if  $a$  is even (or equivalently if  $v_{a,b}$  has order  $\leq 12$ ). In this case we see, that  $v_{4,1}$  is the restriction of  $\kappa\mu^2 \in \Gamma_3^{*ab}$  to  $\Gamma_{1,\infty} \cong \text{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$ . Therefore, precisely the characters  $v_{4j,j}$ ,  $j \in \mathbb{Z}/6\mathbb{Z}$ , can be lifted (from  $\Gamma_{1,\infty}$ ) into  $\Gamma_3^{*ab}$ . This will be used frequently later.

For a function  $\Phi : \mathbb{H}_1 \times \mathbb{C} \rightarrow \mathbb{C}$  we define  $\tilde{\Phi}_m$  on  $\mathbb{H}_2$  by  $\tilde{\Phi}_m \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \Phi(z_1, z_2)e^{2\pi i m z_3}$ . We give a definition of Jacobi-forms with character which is suitable for our needs (compare [15, Def. 1.4]).

**DEFINITION 3.2.** *Let  $v_{a,b}$  be a character of  $\text{MJ}_2(\mathbb{Z})$ . A holomorphic function  $\Phi : \mathbb{H}_1 \times \mathbb{C} \rightarrow \mathbb{C}$  is a Jacobi-form of weight  $k \in \frac{1}{2}\mathbb{Z}$ , index  $m \in \frac{1}{2}\mathbb{Z}$ , with character  $v_{a,b}$ , if  $\tilde{\Phi}_m$  satisfies*

$$(3.3) \quad \tilde{\Phi}_m|_k M = v_{a,b}(M)\tilde{\Phi}_m \text{ for all } M \in \text{MJ}_2(\mathbb{Z})$$

and  $\Phi$  admits a Fourier-expansion

$$\Phi(z_1, z_2) = \sum_{\substack{n,l \in \mathbb{Q}, n \geq 0 \\ 4mn - l^2 \geq 0}} \alpha(n, l)e^{2\pi i(nz_1 + lz_2)}$$

(where  $n, l$  have bounded denominators, depending on  $v_{a,b}$ ). Moreover, if  $\alpha(n, l) \neq 0$  implies  $4mn - l^2 > 0$ , then  $\Phi$  is a cusp-form. The space of Jacobi-forms of weight  $k$  and index  $m$  with character  $v_{a,b}$  is denoted by  $[\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$ . The subspace of cusp-forms is denoted by  $[\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]_{\text{cusp}}$ .

A first simple observation is  $2m \equiv b \pmod{2\mathbb{Z}}$  or  $[\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}] = \{0\}$ . This holds because  $\Phi \in [\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$  implies

$$e^{2\pi imw} \tilde{\Phi}_m(Z) = \tilde{\Phi}_m \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 + w \end{pmatrix} = \tilde{\Phi}_m|_k((I, 1), [0, 0; w])(Z) = (-1)^{bw} \tilde{\Phi}_m(Z).$$

Therefore we will frequently assume  $2m \equiv b \pmod{2\mathbb{Z}}$ . Another consequence of the transformation (3.3) is that  $f \in [\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$  has non-zero Fourier-coefficients  $\alpha(n, l)$  for  $n \equiv \frac{a}{24} \pmod{\mathbb{Z}}$  and  $l \equiv \frac{b}{2} \pmod{\mathbb{Z}}$  only.

If weight and index are integral and the character is trivial, our Jacobi-forms are just the Jacobi-forms from [8], such as the Eisenstein-series  $E_{k,m}$  for even  $k \geq 4$  and the first cusp-forms of index 1,  $\phi_{10,1}$  and  $\phi_{12,1}$  (in the notation of [8]). Examples of Jacobi-forms with non-trivial character are the Dedekind-function  $\eta \in [\text{MJ}_2(\mathbb{Z}), \frac{1}{2}, 0, v_\eta \times 1]$  (which does not depend on the second variable of course) and the theta-series

$$\begin{aligned} \vartheta_{1/2}(\tau, z) &:= \sum_{m \in \mathbb{Z}} \binom{-4}{m} e^{2\pi i(m^2\tau/8 + mz/2)} \in [\text{MJ}_2(\mathbb{Z}), \frac{1}{2}, \frac{1}{2}, v_\eta^3 \times v_H], \\ \vartheta_{3/2}(\tau, z) &:= \sum_{m \in \mathbb{Z}} \binom{12}{m} e^{2\pi i(m^2\tau/24 + mz/2)} \in [\text{MJ}_2(\mathbb{Z}), \frac{1}{2}, \frac{3}{2}, v_\eta \times v_H]. \end{aligned}$$

Note that  $\vartheta_{1/2}(\tau, 0) = 0$  for all  $\tau \in \mathbb{H}_1$ , whereas  $\vartheta_{3/2}(\tau, 0) = 2\eta(\tau) \neq 0$  for all  $\tau \in \mathbb{H}_1$  (as can be seen from the well-known product-expansions [15]). More examples of Jacobi-forms with non-trivial character can be constructed in the following way: If  $\varphi \in [\text{MJ}_2(\mathbb{Z}), k, m, 1]_{\text{cusp}}$  is a cusp-form with trivial character, then (we may assume  $m \in \mathbb{Z}$  without restriction and)

$$(3.4) \quad \varphi \eta^{-j} \in [\text{MJ}_2(\mathbb{Z}), k - j/2, m, v_\eta^{-j} \times 1], \quad \text{if } j \in \mathbb{N} \text{ with } jm \leq 18.$$

This is because  $\eta^{-j}(\tau) = e^{-2\pi i j\tau/24}(1 + \mathcal{O}(e^{2\pi i\tau}))$  and for a cusp-form in  $[\text{MJ}_2(\mathbb{Z}), k, m, 1]_{\text{cusp}}$ , non-trivial Fourier-coefficients  $\alpha(n, l)$  exist for  $4nm - l^2 \geq 3$  only. Then  $4(n - j/24)m - l^2 \geq 3 - jm/6 \geq 0$  for  $18 \geq jm$ .

As is well known, Jacobi-forms appear as Fourier-Jacobi-coefficients of paramodular forms. In the reverse, Jacobi-forms can be lifted to paramodular forms (by so called ‘‘arithmetical liftings’’, i.e. generalizations of Maaß’s construction [25]), as described in [13, 14] (for trivial character) and [15] (for nontrivial character).

We take a slightly different point of view so far, as we use the correspondence of Jacobi-forms to vector-valued modular forms for  $\text{Mp}_2(\mathbb{Z})$  in order to lift such vector-valued forms. In this way, ‘‘arithmetical liftings’’ (for trivial character) were described in [1, Th. 14.3] in a more general context.

As in [8], it is easily seen, that the Fourier-coefficients  $\alpha(n, l)$  of a Jacobi-form of weight  $m$  depend on  $4mn - l^2$  and  $l \pmod{2m\mathbb{Z}}$  only, essentially. If non-trivial characters are to be taken into account, this has to be changed slightly.

LEMMA 3.3. *Assume  $\Phi \in [\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$ . Then  $e^{-\pi ibl/2m} \alpha(n, l)$  depends on  $4mn - l^2$  and  $l \pmod{2m\mathbb{Z}}$  only.*



*Proof.* We may assume  $m \equiv b/2 \pmod{\mathbb{Z}}$  (or  $\Phi = 0$ ). It follows from the definition, using the transformation-formula (3.3) for  $\text{rot} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$  and  $\text{trans} \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$ , that

$$e^{2\pi i m(u^2\tau + 2uz)} \Phi(\tau, z + u\tau + v) = (-1)^{b(u+v)} \Phi(\tau, z).$$

This implies for  $n, l \in \mathbb{Q}$

$$(3.5) \quad \alpha(n + mu^2 + lu, l + 2mu) = e^{-2\pi i lv} (-1)^{b(u+v)} \alpha(n, l) = (-1)^{bu} \alpha(n, l),$$

since  $l \equiv b/2 \pmod{\mathbb{Z}}$  (or  $\alpha(n, l) = 0$ ). With  $(n', l') = (n + mu^2 + lu, l + 2mu)$ , we have  $u = (l' - l)/2m$  and  $4mn' - l'^2 = 4mn - l^2$ . Now (3.5) can be formulated as

$$e^{-\pi i bl'/2m} \alpha(n', l') = e^{-\pi i bl'/2m} (-1)^{bu} \alpha(n, l) = e^{-\pi i bl'/2m} \alpha(n, l).$$

■

For  $m, x \in \frac{1}{2}\mathbb{Z}$  and  $b \in \mathbb{Z}$  with  $m \equiv x \equiv \frac{b}{2} \pmod{\mathbb{Z}}$  define a theta-series  $\theta_{m,x,b}$  on  $\mathbb{H}_1 \times \mathbb{C}$  by

$$\theta_{m,x,b}(\tau, z) = \sum_{l \equiv x \pmod{2m\mathbb{Z}}} e^{\pi i bl/2m} e^{2\pi i(l^2\tau/4m + lz)}.$$

Obviously,  $\theta_{m,x,b}$  depends on  $x \pmod{2m\mathbb{Z}}$  only. For  $b = 0$  (and  $m$  integral), these are just the theta-series  $\theta_{m,x}$  from [8, §5, (4)].  $\theta_{m,x,b}$  can be reduced to  $\theta_{m,x}$  as follows: If  $m$  is integral, then  $\theta_{m,x,b}(\tau, z) = \theta_{m,x}(\tau, z + b/4m)$ . In the case  $m \in \frac{1}{2} + \mathbb{Z}$ , we have

$$\begin{aligned} \theta_{m,x,b}(\tau, z) &= \sum_{l \equiv x \pmod{2m\mathbb{Z}}} e^{2\pi i(l^2\tau/4m + lz + lb/4m)} = \sum_{2l \equiv 2x \pmod{4m\mathbb{Z}}} e^{2\pi i((2l)^2\tau/8m + 2l(z/2 + b/8m))} \\ &= \theta_{2m,2x}(\tau/2, z/2 + b/8m). \end{aligned}$$

Especially we find the following transformation law for  $\theta_{m,x,b}$  under the generators of  $\text{Mp}_2(\mathbb{Z})$  (compare [8, §5], [1, Sec. 4]).

$$(3.6) \quad (\theta_{m,x,b})_m^\sim |_{1/2} \widehat{T} = e^{2\pi i x^2/4m} (\theta_{m,x,b})_m^\sim,$$

$$(3.7) \quad \begin{aligned} (\theta_{m,x,b})_m^\sim |_{1/2} \widehat{J} &= e^{2\pi i bx/4m} \frac{1}{\sqrt{2mi}} \sum_{x' : (m+\mathbb{Z})/2m\mathbb{Z}} \\ &\times e^{-2\pi i x'x/2m} e^{-2\pi i bx'/4m} (\theta_{m,x',b})_m^\sim. \end{aligned}$$

Let  $\mathbb{V}_m := \{f : (m + \mathbb{Z})/2m\mathbb{Z} \rightarrow \mathbb{C}\}$  be the vector-space of complex functions on  $(m + \mathbb{Z})/2m\mathbb{Z}$ . The characteristic functions  $f_x^c \in \mathbb{V}_m, x \in (m + \mathbb{Z})/2m\mathbb{Z}$ , form a basis of  $\mathbb{V}_m$ .

We define the  $\mathbb{V}_m$ -valued theta-series  $\Theta_{m,b} : \mathbb{H}_1 \times \mathbb{C} \rightarrow \mathbb{V}_m$  by  $\Theta_{m,b}(\tau, z)(x) = \theta_{m,x,b}(\tau, z)$ . Now the formulas (3.6) and (3.7) imply, that  $\Theta_{m,b}$  is a  $\mathbb{V}_m$ -valued modular form for  $\text{Mp}_2(\mathbb{Z})$  with a multiplier-system  $\rho_{m,b}$  (i.e. a representation  $\text{Mp}_2(\mathbb{Z}) \rightarrow \text{GL}(\mathbb{V}_m)$ ), determined by (3.6) and (3.7) (more precisely,  $\Theta_{m,b}$  is a  $\mathbb{V}_m$ -valued metaplectic Jacobi-form; see the following definition 3.4). Explicitly, for  $f \in \mathbb{V}_m$  we have

$$\begin{aligned}
 (\rho_{m,b}(\widehat{T})f)(x) &= e^{2\pi i x^2/4m} f(x), \\
 (\rho_{m,b}(\widehat{J})f)(x) &= e^{2\pi i b x/4m} \frac{1}{\sqrt{2mi}} \sum_{x':(m+\mathbb{Z})/2m\mathbb{Z}} e^{-2\pi i x'x/2m} e^{-2\pi i b x'/4m} f(x').
 \end{aligned}$$

Up to isomorphism,  $\rho_{m,b}$  does not depend on  $b$ . Given  $n \in 2\mathbb{Z}$  we define  $A_n \in \text{GL}(\mathbb{V}_m)$  by  $(A_n f)(x) = e^{2\pi i n x/4m} f(x)$  (note that  $e^{2\pi i n x/4m}$  depends on  $x$  modulo  $2m$  only, since  $n$  is even). Now if  $b, b' \in \mathbb{Z}$  satisfy  $b \equiv b' \pmod{2\mathbb{Z}}$ , then we see  $\rho_{m,b'}(M) = A_{b'-b} \rho_{m,b}(M) A_{b'-b}^{-1}$ , or, in other words,  $\rho_{m,b'}$  and  $\rho_{m,b}$  are conjugate by  $A_{b'-b}$ .

**DEFINITION 3.4.** *Let  $\rho : \text{Mp}_2(\mathbb{Z}) \rightarrow \text{GL}(V)$  be a finite-dimensional representation, such that  $\rho$  factors over a principal congruence group  $\text{Mp}_2(\mathbb{Z})[n]$ . A holomorphic function  $f : \mathbb{H}_1 \rightarrow V$  is a ( $V$ -valued) meromorphic modular form of weight  $k \in \frac{1}{2}\mathbb{Z}$  with multiplier-system  $\rho$ , if*

$$f|_k M = \rho(M) f \quad \text{for all } M \in \text{Mp}_2(\mathbb{Z})$$

(here the action of  $\text{Mp}_2(\mathbb{Z})$  is defined as in (3.1)) and  $f$  has at most a pole at the cusp  $i\infty$ . If in addition  $f$  is bounded in any region  $\Im(\tau) \geq y_0 > 0$  (i.e. if there is no pole at the cusp), then  $f$  is a (holomorphic) modular form. Moreover, if  $\lim_{\Im(\tau) \rightarrow \infty} f(\tau) = 0$ , then  $f$  is cusp-form. The space of meromorphic modular forms of weight  $k$  with multiplier-system  $\rho$  is denoted by  $[\text{Mp}_2(\mathbb{Z}), k, \rho]_{\text{mer}}$ , the subspace of (holomorphic) modular forms is denoted by  $[\text{Mp}_2(\mathbb{Z}), k, \rho]$  and the subspace of cusp-forms by  $[\text{Mp}_2(\mathbb{Z}), k, \rho]_{\text{cusp}}$ .

Of course, if  $k$  is integral, the action of  $\text{Mp}_2(\mathbb{Z})$  factors over  $\text{SL}_2(\mathbb{Z})$  and we can think of  $[\text{Mp}_2(\mathbb{Z}), k, \rho]$  as a space of ( $V$ -valued) elliptic modular forms. In our case, the vector-space  $V$  will almost always be  $\mathbb{V}_m$  for some  $m \in \frac{1}{2}\mathbb{Z}$ . For  $f : \mathbb{H}_1 \rightarrow \mathbb{V}_m$ , we define the components  $f_x : \mathbb{H}_1 \rightarrow \mathbb{C}$  for  $x \in (m + \mathbb{Z})/2m\mathbb{Z}$  by  $f(\tau) = \sum_{x:(m+\mathbb{Z})/2m\mathbb{Z}} f_x(\tau) f_x^c$ .

On  $\mathbb{V}_m$  there is a scalar product, defined by

$$(f, g) = \sum_{x:(m+\mathbb{Z})/2m\mathbb{Z}} f(x)g(x)$$

(this pairing is *not* hermitian, but respects holomorphy instead). For a representation  $\rho$  of  $\text{Mp}_2(\mathbb{Z})$  on  $\mathbb{V}_m$ , denote by  $\rho^*$  the dual representation of  $\rho$  with respect to the pairing above, i.e.  $\rho^*$  satisfies  $(\rho^*(M)f, \rho(M)g) = (f, g)$  for all  $f, g \in \mathbb{V}_m$  and  $M \in \text{Mp}_2(\mathbb{Z})$ .

**LEMMA 3.5.** *Let  $k \in \frac{1}{2}\mathbb{Z}$ ,  $v_{a,b} \in \text{MJ}_2(\mathbb{Z})^{\text{ab}}$  and  $m \in \frac{1}{2}\mathbb{Z}$  with  $2m \equiv b \pmod{2\mathbb{Z}}$ . Then*

$$\begin{aligned}
 &[\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, v_\eta^a \rho_{m,b}^*] \rightarrow [\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}], \\
 &f = (f_x)_{x:(m+\mathbb{Z})/2m\mathbb{Z}} \mapsto (f, \Theta_{m,b}) = \sum_{x:(m+\mathbb{Z})/2m\mathbb{Z}} f_x \theta_{m,x,b}
 \end{aligned}$$

is an isomorphism of the vector-spaces.

*Proof.* If  $f \in [\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, v_\eta^a \rho_{m,b}^*]$ , then  $F = (f, \Theta_{m,b})$  transforms as a Jacobi-form of weight  $k$ , index  $m$  with character  $v_{a,b}$ . For the Heisenberg-part of  $\text{MJ}_2(\mathbb{Z})$  this

follows from

$$\begin{aligned}\tilde{F}_m|_k[u, v; w](Z) &= \tilde{F}_m\left(\begin{array}{cc} z_1 & z_2 + uz_1 + v \\ z_2 + uz_1 + v & z_3 + u^2z_1 + 2uz_2 + uv + w \end{array}\right) \\ &= (f(z_1), \Theta_{m,b}(z_1, z_2 + uz_1 + v)) e^{2\pi im(z_3 + u^2z_1 + 2uz_2 + uv + w)} \\ &= (-1)^{b(u+v)} e^{2\pi im(uv+w)} (f(z_1), \Theta_{m,b}(z_1, z_2)) e^{2\pi imz_3} \\ &= v_H([u, v; w])^b \tilde{F}_m(Z)\end{aligned}$$

for  $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$  (note that  $\theta_{m,x,b}(\tau, z + u\tau + v) = (-1)^{b(u+v)} e^{-2\pi im(u^2\tau + 2uz)}$   $\times \theta_{m,x,b}(\tau, z)$  for  $u, v \in \mathbb{Z}$  since  $2m \equiv b \pmod{2\mathbb{Z}}$ ). Given  $M \in \text{Mp}_2(\mathbb{Z})$  we have

$$\begin{aligned}\tilde{F}_m|_k M(Z) &= (f|_{k-1/2} M(z_1), (\Theta_{m,b})_m \tilde{F}_m|_{1/2} M(Z)) \\ &= v_\eta^a(M) (\rho_{m,b}^*(M) f(z_1), \rho_{m,b}(M) (\Theta_{m,b})_m \tilde{F}_m(Z)) \\ &= v_\eta^a(M) (f(z_1), \Theta_{m,b}(z_1, z_2)) e^{2\pi imz_3} = v_\eta^a(M) \tilde{F}_m(Z),\end{aligned}$$

since  $\rho_{m,b}^*$  is the dual of  $\rho_{m,b}$  with respect to the given scalar product on  $\mathbb{V}_m$ . Moreover  $F$  satisfies the cusp-condition (since all  $f_x$  and  $\theta_{m,x,b}$  do). The mapping is injective, since for any fixed  $\tau \in \mathbb{H}_1$ , the theta-series  $\theta_{m,x,b}(\tau, \cdot)$ ,  $x \in (m + \mathbb{Z})/2m\mathbb{Z}$ , are linearly independent (as functions of the second variable). Finally we show that the mapping is surjective. Let  $\Phi \in [\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$  with Fourier-development

$$\Phi(\tau, z) = \sum_{\substack{n \equiv n_0, l \equiv l_0 \pmod{\mathbb{Z}}, \\ 4mn - l^2 \geq 0}} \alpha(n, l) e^{2\pi i(n\tau + lz)}$$

(where  $n_0 = \frac{a}{24}$  and  $l_0 = \frac{b}{2}$ , of course). By lemma 3.3 we know, that  $c_l(4mn - l^2) := e^{-\pi ibl/2m} \alpha(n, l)$  depends on  $4mn - l^2$  and  $l \pmod{2m\mathbb{Z}}$  only. Therefore we have

$$\begin{aligned}\Phi(\tau, z) &= \sum_{\substack{l \equiv l_0 \pmod{\mathbb{Z}} \\ 4mn - l^2 \geq 0}} \sum_{\substack{n \equiv n_0 \pmod{\mathbb{Z}} \\ 4mn - l^2 \geq 0}} c_l(4mn - l^2) e^{2\pi i(4mn - l^2)\tau/4m} e^{\pi ibl/2m} e^{2\pi i(l^2\tau/4m + lz)} \\ &= \sum_{l' : (l_0 + \mathbb{Z})/2m\mathbb{Z}} \sum_{\substack{N \equiv 4mn_0 - l'^2 \pmod{\mathbb{Z}}, \\ N \geq 0}} c_{l'}(N) e^{2\pi iN\tau/4m} \sum_{l \equiv l' \pmod{\mathbb{Z}}} e^{\pi ibl/2m} e^{2\pi i(l^2\tau/4m + lz)} \\ &= \sum_{l' : (l_0 + \mathbb{Z})/2m\mathbb{Z}} f_{l'}(\tau) \theta_{m,l',b}(\tau, z) = (f, \Theta_{l,b})(\tau, z),\end{aligned}$$

(note that  $l' \in \frac{1}{2}\mathbb{Z}$ , thus  $l \equiv l' \pmod{\mathbb{Z}}$  implies  $l^2 \equiv l'^2 \pmod{\mathbb{Z}}$  with  $f = (f_x)_{x:(m+\mathbb{Z})/2m\mathbb{Z}}$  where

$$f_x(\tau) = \sum_{0 \leq N \equiv 4mn_0 - x^2 \pmod{\mathbb{Z}}} c_{l'}(N) e^{2\pi iN\tau/4m}.$$

Now  $\Phi \in [\text{MJ}_2(\mathbb{Z}), k, m, v_{a,b}]$  implies  $f \in [\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, v_\eta^a \rho_{m,b}^*]$  (using the linear independence of the theta-series once more). ■

The arithmetical lifting uses Hecke-operators on Jacobi-forms of integral weight with characters (since paramodular forms have integral weight). In this case, the character factors over  $\text{SL}_2(\mathbb{Z}) \times H(\mathbb{Z})$ , i.e. it is of the form  $v_{a,b}$  with  $a$  even. Thus let  $\Phi \in [\text{MJ}_2(\mathbb{Z}), k, m, \xi \times v_H^\varepsilon]$  be a Jacobi-form of integral weight  $k$ , where  $\xi \in \text{SL}_2(\mathbb{Z})^{\text{ab}}$ . Let  $Q \in \mathbb{N}$  satisfy  $\text{SL}_2(\mathbb{Z})[Q] \subset \text{kern}(\xi)$ . Given  $l \in \mathbb{N}$  we define the Hecke-operator  $\mathcal{T}^{(Q)}(l)$  on  $\Phi$  as in [15, (1.12)] by

$$\tilde{\Phi}_m|_k \mathcal{T}^{(Q)}(l)(z_1, z_2, z_3) := l^{2k-3} \sum_{ad=l, b \bmod d} d^{-k} \xi(\sigma_a) \Phi\left(\frac{az_1+bQ}{d}, az_2\right) e^{2\pi i l m z_3}.$$

Here,  $\sigma_a \in \text{SL}_2(\mathbb{Z})$  has to satisfy  $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \pmod{Q}$ . As was proved in [15, Lemma 1.7], if  $\text{gcd}(l, 2^\varepsilon Q) = 1$ , one has  $\Phi|_k \mathcal{T}^{(Q)}(l) \in [\text{MJ}_2(\mathbb{Z}), k, lm, \xi_l \times v_H^\varepsilon]$ , where  $\xi_l$  is a twist of  $\xi$  (especially,  $\xi_l = \xi$ , if  $l \equiv 1 \pmod{Q}$ , the only case we will need later on).

Now we can formulate the main result of this section:

**PROPOSITION 3.6.** *Assume  $k \in \mathbb{Z}$ . Let  $d$  be a divisor of 6 and  $\nu = \mu^2 \kappa \in \Gamma_3^{*\text{ab}}$ . Set  $Q = \frac{6}{d}$ . Then there is an injective homomorphism*

$$\mathcal{M} : [\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \tilde{\nu}^d \rho_{d/2,d}^*] \rightarrow [\Gamma_3^*, k, \nu^d \chi^k],$$

defined by  $(c_l(n))$  is the  $n^{\text{th}}$  Fourier-coefficient of  $\tau \mapsto f(\tau)(l)$

$$\mathcal{M}(f)(Z) = c_0(0) \frac{-B_k}{2k} g_k(z_1) + \sum_{m \in \mathbb{N}, m \equiv 1 \pmod{QZ}} m^{2-k} (f, \Theta_{d/2,d})_{d/2} \tilde{\nu}^d|_k \mathcal{T}^{(Q)}(m)(Z)$$

Cusp-forms are mapped to cusp-forms by  $\mathcal{M}$ .

*Proof.* Recall that  $\nu = v_\eta^4 \times v_H$  as characters of the Jacobi-group  $\text{SL}_2(\mathbb{Z}) \times H(\mathbb{Z})$ , thus  $\tilde{\nu} = v_\eta^4$ . Therefore by lemma 3.5 we have  $\Phi := f \cdot \Theta_{d/2,d} \in [\text{MJ}_2(\mathbb{Z}), k, \frac{d}{2}, v_{4d,d}]$ .

First assume  $d < 6$ . Then  $\nu^d \neq 1$  is non-trivial and every  $f \in [\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \tilde{\nu}^d \rho_{d/2,d}^*]$  is a cusp-form, i.e.  $c_0(0) = 0$ . Thus in the notation of [15, Th. 1.12] we have  $\mathcal{M}(f) = \text{Lift}_1(\Phi)$ . Note that the series defining  $\mathcal{M}(f)$  converges for all  $k \geq 0$  in this case, since  $\Phi$  is a cusp-form. Now [15, Th. 1.12] implies the claim (because of  $Q = \frac{6}{d}$ , the lift has level  $Q d/2 = 3$ ).

Now let  $d = 6$ . In this case the character  $\nu^d$  is trivial and the Hecke-operators  $\mathcal{T}^{(1)}(m)$  are just the Hecke-operators  $T_-(m)$  from [14]. Thus in the notation of [14, Hauptsatz 2.1] we have  $\mathcal{M}(f) = F_\Phi$ . For a cusp-form  $\Phi$ , this series is  $\text{Lift}_1(\Phi)$  as in [15] again and converging for  $k \geq 0$ . In general,  $\Phi$  is cuspidal if and only if  $\Phi(\tau, 0) \in [\text{SL}_2(\mathbb{Z}), k, 1]$  is cuspidal. Thus if  $\Phi$  is not cuspidal, we may assume  $k \geq 4$  and the series defining  $\mathcal{M}(f)$  converges in this case too. Now [14, Hauptsatz 2.1, Satz 3.8] imply the claim (note that Jacobi-forms of prime index  $p$  are necessarily eigenforms of the operator  $W_p$  from [8]). ■

$\mathcal{M}$  is the Maaß-lift and the image  $\mathcal{M}_{k,d} := \mathcal{M}([\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \tilde{\nu}^d \rho_{d/2,d}^*])$  is the Maaß-space with character  $\nu^d \chi^k$ .

Using the arithmetical lifting  $\widehat{\text{Lift}}_{-1}$  from [15, Th. 1.12], one can define homomorphisms  $[\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \widetilde{v}^d \rho_{d/2,d}^*] \rightarrow [\Gamma_3^*, k, \overline{v}^d \chi^k]$  too, but these are not necessarily injective (and they are not injective in our case, as can be seen, combining results from section 5 with the dimension-formulas for the spaces  $[\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \widetilde{v}^d \rho_{d/2,d}^*]$ , given in this section).

Note that the representation  $\rho_{m,b}$  is reducible in almost all cases, i.e. except for  $m = \frac{1}{2}$  and  $m = 1$ . For our needs, the decomposition is given as follows:  $\widehat{\mathcal{J}}^2 = (-I, i)$  is a central element in  $\text{Mp}_2(\mathbb{Z})$ . On  $\mathbb{V}_m$ , the element  $\widehat{\mathcal{J}}^2$  acts via  $\rho_{m,b}$  as

$$(3.8) \quad (\rho_{m,b}(\widehat{\mathcal{J}}^2)f)(x) = -ie^{2\pi i b x / 2m} f(-x).$$

Define  $W(-1) := \rho_{m,b}(\widehat{\mathcal{J}}^2)$ . Since  $\widehat{\mathcal{J}}^2$  is central,  $W(-1)$  commutes with  $\rho_{m,b}$ . Thus all eigenspaces of  $W(-1)$  are invariant under  $\rho_{m,b}$ . Because  $W(-1)^2 = -\text{id}_{\mathbb{V}_m}$ ,  $W(-1)$  has order 4 and non-trivial eigenspaces for eigenvalues  $\pm i$  only. In this case denote by  $\mathbb{V}_{m,s} \subset \mathbb{V}_m$  the eigenspace with eigenvalue  $s$  of  $W(-1)$ . The restriction of  $\rho_{m,b}$  to  $\mathbb{V}_{m,s}$  is denoted by  $\rho_{m,b,s}$ . In our special cases (where  $2m|6$ ), it turns out, that  $\rho_{m,b,s}$  is always irreducible (though it is not in general; compare [8] and [29]). Note that  $W(-1)$  acts as a scalar (that is,  $W(-1) = c \text{id}_{\mathbb{V}_m}$  for some  $c \in \mathbb{C}^\times$ ), if and only if  $m \leq 1$ . Thus  $\rho_{m,b}$  is reducible, if  $m > 1$  (and easily seen to be irreducible for  $m \leq 1$ ). The decomposition  $\rho_{m,b} = \rho_{m,b,i} \oplus \rho_{m,b,-i}$  induces a decomposition

$$[\text{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho_{i,b}^*] = [\text{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho_{i,b,i}^*] \oplus [\text{Mp}_2(\mathbb{Z}), k, v_\eta^a \rho_{i,b,-i}^*],$$

and via the isomorphism from lemma 3.5 a decomposition

$$[\text{MJ}_2(\mathbb{Z}), k, t, v_{a,b}] = [\text{MJ}_2(\mathbb{Z}), k, t, v_{a,b}, i] \oplus [\text{MJ}_2(\mathbb{Z}), k, t, v_{a,b}, -i].$$

Via the Maaß-lift  $\mathcal{M}$ , we also get a decomposition

$$\mathcal{M}_{k,d} = \mathcal{M}_{k,d,i} \oplus \mathcal{M}_{k,d,-i},$$

where  $\mathcal{M}_{k,d,s} := \mathcal{M}([\text{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \widetilde{v}^d \rho_{d/2,d,s}^*])$ .

The decomposition of  $\rho_{m,b}$  with respect to  $W(-1)$  has to be known, if we want to evaluate Skoruppa's dimension-formula [29, 7] for  $[\text{Mp}_2(\mathbb{Z}), k, \rho]$ , since for this formula,  $\rho(\widehat{\mathcal{J}}^2)$  has to act as a scalar.

LEMMA 3.7 ([7, Th. 4.2], [29, Satz 5.1]). *Let  $\rho : \text{Mp}_2(\mathbb{Z}) \rightarrow \text{GL}(V)$  be a representation of dimension  $n$  of  $\text{Mp}_2(\mathbb{Z})$  with  $\text{Mp}_2(\mathbb{Z})[N] \subset \text{kern}(\rho)$  for some  $N \in \mathbb{N}$  and  $\rho(\widehat{\mathcal{J}}^2) = \zeta \text{id}_V$  with a fourth root of unity  $\zeta$ . Let  $l_j \in \mathbb{R}$ ,  $j = 1, \dots, n$ , be such that  $e^{2\pi i l_j}$  runs through the eigenvalues of  $\rho(\widehat{\mathcal{T}})$ . Define*

$$A(\rho) := \#\{j \mid l_j \equiv 0 \pmod{\mathbb{Z}}\}, \quad B(\rho) := \sum_{j=1}^n \mathbb{B}_1(l_j)$$

(here  $\mathbb{B}_1$  is given by  $\mathbb{B}_1(x) = 0$  for  $x \in \mathbb{Z}$  and  $\mathbb{B}_1(x) = x - [x] - \frac{1}{2}$  for  $x \in \mathbb{R} \setminus \mathbb{Z}$ ). Then for  $k \in \frac{1}{2}\mathbb{Z}$  one has

$$\dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho] - \dim[\mathrm{Mp}_2(\mathbb{Z}), 2 - k, \rho^*]_{\mathrm{cusp}} = \begin{cases} 0, & \text{if } \rho(\widehat{-1}) \neq i^{-2k} \mathrm{id}_V, \\ \frac{k-1}{12} \dim(\rho) + \frac{1}{2} A(\rho) - B(\rho) + \frac{1}{4} \Re(e^{2\pi i k/4} \mathrm{trace} \rho(\widehat{J})) \\ + \frac{2}{3\sqrt{3}} \Re(e^{2\pi i(k+\frac{1}{2})/6} \mathrm{trace} \rho(\widehat{JT})), & \text{if } \rho(\widehat{-1}) = i^{-2k} \mathrm{id}_V. \end{cases}$$

Note that for  $k \geq 2$ , the formula gives an explicit expression for  $\dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$ , since  $\dim[\mathrm{Mp}_2(\mathbb{Z}), 2 - k, \rho^*]_{\mathrm{cusp}} = 0$  in this case. For  $k = \frac{1}{2}$  and  $k = \frac{3}{2}$  there is an explicit formula for  $\dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$  in [29] also.

Let  $\mathcal{R} = \mathbb{C}[g_4, g_6]$  be the graded ring of elliptic modular forms. Using the dimension-formula from lemma 3.7 as in [29, Satz 7.3], we see that

$$[\mathrm{Mp}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, \rho] := \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} [\mathrm{Mp}_2(\mathbb{Z}), k, \rho]$$

always is a free module of rank  $\dim(\rho)$  over  $\mathcal{R}$ . By lemma 3.5, the same is true for the analogously defined spaces  $[\mathrm{MJ}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, m, v_{a,b}]$  of Jacobi-forms.

A basis for the eigenspace  $\mathbb{V}_{m,s}$  is given as follows (recall  $f_x^c \in \mathbb{V}_m$  being the characteristic function of  $x \in (m + \mathbb{Z})/2m\mathbb{Z}$  and  $W(-1)^2 = -\mathrm{id}_{\mathbb{V}_m}$ ): For  $s \in \{\pm i\}$  let

$$f_{x,s}^c := f_x^c - s W(-1) f_x^c \in \mathbb{V}_{m,s}.$$

If  $2x \not\equiv 0 \pmod{2m}$  then  $f_{x,s}^c \neq 0$ . If on the other hand  $2x \equiv 0 \pmod{2m}$  then we have  $W(-1) f_x^c = -i(-1)^{2m} f_x^c$ , i.e.  $f_x^c \in \mathbb{V}_{m,-i(-1)^{2m}}$ . Now let

$$\mathcal{B}_{0,s} = \left\{ f_x^c \mid x \in (m + \mathbb{Z})/2m\mathbb{Z}, 2x \equiv 0 \pmod{2m}, s = -i(-1)^{2m} \right\},$$

$$\mathcal{B}_{1,s} = \left\{ f_{x,s}^c \mid x \in \pm(m + \mathbb{Z})/2m\mathbb{Z}, 2x \not\equiv 0 \pmod{2m} \right\}.$$

Then  $\mathcal{B}_s := \mathcal{B}_{0,s} \cup \mathcal{B}_{1,s}$  is a basis of  $\mathbb{V}_{m,s}$ . Using this basis, one can calculate all the parameters in the dimension-formula from lemma 3.7 for  $\rho_{m,b,s}$ . For example one has

$$\mathrm{trace}(\rho_{m,b,s}(\widehat{J})) = \sum_{\substack{x: \pm(m+\mathbb{Z})/2m\mathbb{Z}, \\ 2x \not\equiv 0 \pmod{2m}}} \frac{e^{-2\pi i x^2/2m} + i s e^{2\pi i x^2/2m}}{\sqrt{2mi}} + \sum_{\substack{x: (m+\mathbb{Z})/2m\mathbb{Z}, \\ 2x \equiv 0 \pmod{2m}}} \frac{e^{-2\pi i x^2/2m}}{\sqrt{2mi}},$$

$$\mathrm{trace}(\rho_{m,b,s}(\widehat{JT})) = \sum_{\substack{x: \pm(m+\mathbb{Z})/2m\mathbb{Z}, \\ 2x \not\equiv 0 \pmod{2m}}} \frac{e^{-2\pi i x^2/4m} + i s e^{2\pi i x^2 3/4m}}{\sqrt{2mi}} + \sum_{\substack{x: (m+\mathbb{Z})/2m\mathbb{Z}, \\ 2x \equiv 0 \pmod{2m}}} \frac{e^{-2\pi i x^2/4m}}{\sqrt{2mi}}.$$

Of course it is possible to find explicit formulas for the traces for all the irreducible constituents of  $\rho_{m,b}$ , as was shown for  $m \in \mathbb{Z}$  in [29].

The following table lists the parameters of the dimension formula for  $[\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \rho]$  with  $\rho = \widetilde{v}^d \rho_{d/2,d,s}^*$ , where  $d$  is a divisor of 6 and  $v = \mu^2 \kappa \in \Gamma_3^{*ab}$ :

$d$	$s$	$A(\rho)$	$B(\rho)$	$\dim(\rho)$	$\text{trace } \rho(\widehat{J})$	$\text{trace } \rho(\widehat{J}\widehat{T})$	$\rho$
1	$i$	0	$\frac{-11}{24}$	1	$\frac{1-i}{\sqrt{2}}$	$\frac{\sqrt{3}-i}{2}$	$\widetilde{\nu}\rho_{1/2,i}^*$
2	$-i$	0	$\frac{-7}{12}$	2	0	$\frac{-i-\sqrt{3}}{2}$	$\widetilde{\mu}\rho_{1,-i}^*$
3	$i$	0	$\frac{-5}{12}$	2	0	$\frac{-\sqrt{3}+i}{2}$	$\widetilde{\kappa}\rho_{3/2,i}^*$
3	$-i$	0	$\frac{-1}{24}$	1	$-\frac{1+i}{\sqrt{2}}$	$\frac{\sqrt{3}+i}{2}$	$\widetilde{\kappa}\rho_{3/2,-i}^*$
6	$i$	0	$\frac{7}{12}$	2	0	$\frac{-\sqrt{3}+i}{2}$	$\rho_{3,i}^*$
6	$-i$	1	$\frac{1}{3}$	4	0	$\frac{\sqrt{3}+i}{2}$	$\rho_{3,-i}^*$

Using these parameters, we can calculate the dimensions of the Maaß-spaces  $\mathcal{M}_{k,d,s}$ . The following table lists the dimensions of  $\mathcal{M}_{k,d,s}$  for  $k \leq 14$ . There are no nontrivial forms of weight  $k \leq 0$ . The last two columns list the character  $\nu$  of the forms in the Maaß-space  $\mathcal{M}_{k,d,s}$  and a basis of the module  $[\text{MJ}_2(\mathbb{Z}), \mathbb{Z}, d/2, (v_{4,1})^d, s]$  over  $\mathcal{R}$ . We use the following abbreviations:

$$\begin{aligned} \phi_{1,1/2} &:= \eta \vartheta_{1/2} \in [\text{MJ}_2(\mathbb{Z}), 1, \frac{1}{2}, v_{4,1}, i], \\ \phi_{4,1} &:= \frac{\phi_{12,1}}{\eta^{16}} \in [\text{MJ}_2(\mathbb{Z}), 4, 1, v_{8,0}, -i], \\ \phi_{6,3/2} &:= \eta^{11} \vartheta_{3/2} \in [\text{MJ}_2(\mathbb{Z}), 6, \frac{3}{2}, v_{12,1}, -i], \\ \phi_{8,3} &:= \frac{\phi_{12,1}}{\eta^{12}} \vartheta_{1/2}^4 \in [\text{MJ}_2(\mathbb{Z}), 8, 3, 1, -i], \\ \phi_{9,3} &:= \eta^{14} \vartheta_{1/2}^3 \vartheta_{3/2} \in [\text{MJ}_2(\mathbb{Z}), 9, 3, 1, i], \\ \phi_{11,3} &:= \frac{\phi_{12,1}}{\eta^4} \vartheta_{1/2} \vartheta_{3/2} \in [\text{MJ}_2(\mathbb{Z}), 11, 3, 1, i]. \end{aligned}$$

$d$	$s$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\nu$	
1	$i$	1	0	0	0	1	0	1	0	1	0	1	0	2	0	$\mu^2 \kappa$	$\phi_{1,1/2}$
2	$-i$	0	1	0	1	0	1	0	2	0	2	0	2	0	3	$\mu$	$\phi_{1,1/2}^2, \phi_{4,1}$
3	$i$	0	0	1	0	1	0	1	0	2	0	2	0	2	0	$\kappa$	$\phi_{1,1/2}^3, \phi_{1,1/2}\phi_{4,1}$
3	$-i$	0	0	0	0	1	0	0	0	1	0	1	0	1		$\kappa$	$\phi_{6,3/2}$
6	$i$	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	$\phi_{9,3}, \phi_{11,3}$
6	$-i$	0	0	0	1	0	2	0	2	0	3	0	4	0	4	1	$E_{4,3}, E_{6,3}, \phi_{1,1/2}^6, \phi_{8,3}$

Note that it is easy to deduce the dimensions of  $\mathcal{M}_{k,d,s}$  for  $k \geq 15$  from the values given in the table, since for  $k \geq 2$  we have by the dimension formula  $\dim[\text{Mp}_2(\mathbb{Z}), k + 12, \rho] = \dim(\rho) + \dim[\text{Mp}_2(\mathbb{Z}), k, \rho]$ .

For  $k = 1$  and  $k = 2$  lemma 3.7 does not give the dimensions of  $\mathcal{M}_{k,d,s}$  explicitly and the results from [29, Satz 5.2] have to be used. Alternatively, some ad hoc arguments can be given as follows: Since  $[\mathrm{Mp}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, \rho]$  is a free module of rank  $\dim(\rho)$  over  $\mathcal{R}$ , we find  $\dim[\mathrm{Mp}_2(\mathbb{Z}), k, \rho] \leq \dim[\mathrm{Mp}_2(\mathbb{Z}), k+l, \rho]$  for all  $4 \leq l \in 2\mathbb{N}$ . Especially  $\dim \mathcal{M}_{k,d,s} \leq r_{d,s}(k) := \min\{\dim \mathcal{M}_{k+4,d,s}, \dim \mathcal{M}_{k+6,d,s}\}$ . For  $k = 1$  or  $k = 2$  this minimum is 0 except for the following cases:

$\dim \mathcal{M}_{1,1,i} = 1$ , since  $\tilde{\mu}^2 \tilde{\kappa} \rho_{1/2,i}^*$  is the multiplier-system of the  $\eta$ -function and  $r_{1,i}(1) = 1$ .

$\dim \mathcal{M}_{2,2,-i} = 1$ , since  $r_{2,-i}(2) = 1$  and  $\phi_{1,1/2}^2 \in [\mathrm{MJ}_2(\mathbb{Z}), 2, 1, v_{8,0}, -i]$ .

$\dim \mathcal{M}_{2,6,-i} = 0$ , since  $\dim[\mathrm{MJ}_2(\mathbb{Z}), 2, 3, 1] = 0$  was proved in [8, Th. 9.1].

$\dim \mathcal{M}_{1,3,i} = 0$ , since  $\dim \mathcal{M}_{2,6,-i} = 0$  and  $0 \neq \varphi \in [\mathrm{MJ}_2(\mathbb{Z}), 1, \frac{3}{2}, v_{12,1}, i]$  would imply  $0 \neq \varphi^2 \in [\mathrm{MJ}_2(\mathbb{Z}), 2, 3, 1, -i]$ .

In section 5 we need the restriction of Maaß-lifts to the diagonal, which in the case of paramodular forms is properly defined by a certain Witt-operator (see (5.1)).

If  $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \tilde{v}^d \rho_{d/2,d}^*]$ , then we associate with  $f$  the “Nullwert” of the Jacobi-form  $(f, \Theta_{d/2,d})$ , which we denote by  $\Psi_0 f$ , i.e.

$$(\Psi_0 f)(z) = (f, \Theta_{d/2,d})(z, 0).$$

Obviously we have  $\Psi_0 f \in [\mathrm{SL}_2(\mathbb{Z}), k, \tilde{v}^d]$ . The mapping  $f \mapsto \Psi_0 f$  is a surjective homomorphism (as we will prove immediately). The importance of this surjectivity is, that we can show now, how to lift certain modular forms from the diagonal as Maaß-lifts to paramodular forms. Although the following lemma is needed only in a very few special cases later on, we give a slightly more general statement here.

LEMMA 3.8. 1) Let  $d$  be a divisor of 6 and  $v = \mu^2 \kappa \in \Gamma_3^{*ab}$ .

$$[\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \tilde{v}^d \rho_{d/2,d}^*] \rightarrow [\mathrm{SL}_2(\mathbb{Z}), k, \tilde{v}^d], \quad f \mapsto \Psi_0 f$$

is a surjective homomorphism.

2) Let  $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \rho_{3,6}^*]$  and assume that  $(\Psi_0 f)(\tau) = \sum_{n \in \mathbb{N}_0} \alpha(n) e^{2\pi i n \tau}$  is an simultaneous eigenform of all Hecke-operators  $\mathcal{T}^{(1)}(m)$ . Then

$$(3.9) \quad W_3(\mathcal{M}(f))(z_1, z_3) = \frac{1}{\alpha(1)} \Psi_0 f(z_1) \Psi_0 f(z_3).$$

*Proof.* 1)  $[\mathrm{SL}_2(\mathbb{Z}), k, \tilde{v}^d] = \{0\}$ , if  $k$  is odd, since  $\tilde{v}(-I) = 1$ . Thus we may assume  $k \equiv 0 \pmod{2\mathbb{Z}}$ . Then there are three cases left:

Case 1:  $d = 2$ . Since  $[\mathrm{SL}_2(\mathbb{Z}), k, \tilde{v}^2] = \eta^8[\mathrm{SL}_2(\mathbb{Z}), k - 4, 1]$  by lemma 5.1 and  $\phi_{4,1}(\tau, 0) = \phi_{12,1}(\tau, 0)/\eta^{16}(\tau) = 12\eta^8(\tau)$ , a pre-image of  $\eta^8 f$  in  $[\mathrm{MJ}_2(\mathbb{Z}), k, 1, (v_{4,1})^2]$  is given by  $\frac{1}{12}\phi_{4,1} f$ .

Case 2:  $d = 3$ . Since  $[\mathrm{SL}_2(\mathbb{Z}), k, \tilde{v}^3] = \eta^{12}[\mathrm{SL}_2(\mathbb{Z}), k - 6, 1]$  by lemma 5.1 and  $\phi_{6,3/2}(\tau, 0) = 2\eta^{12}(\tau)$ , a pre-image of  $\eta^{12} f$  in  $[\mathrm{MJ}_2(\mathbb{Z}), k, \frac{3}{2}, (v_{4,1})^3]$  is given by  $\frac{1}{2}\phi_{6,3/2} f$ .



Case 3:  $d = 6$ . Since a basis of  $[\mathrm{SL}_2(\mathbb{Z}), k, 1]$  is given by  $g_4^a g_6^b$  for all  $(a, b) \in \mathbb{N}_0^2$  with  $4a + 6b = k$ , it is sufficient to find pre-images of these forms. A pre-image of  $g_4^a g_6^b$  in  $[\mathrm{MJ}_2(\mathbb{Z}), k, 3, 1]$  is  $g_4^{a-1} g_6^b E_{4,3}$ , if  $a > 0$ , or  $g_4^a g_6^{b-1} E_{6,3}$ , if  $b > 0$ .

2) Assume  $f \in [\mathrm{Mp}_2(\mathbb{Z}), k - \frac{1}{2}, \rho_{3,6}^*]$  is such that  $\Psi_0 f$  is an simultaneous eigenform of all Hecke-operators  $\mathcal{T}^{(1)}(m)$  (which are the usual Hecke-operators on elliptic modular forms as in [26] or [21]), i.e.  $\Psi_0 f|_k \mathcal{T}^{(1)}(l) = l^{k-2} \frac{\alpha(l)}{\alpha(1)} \Psi_0 f$  for all  $l \in \mathbb{N}$  (because of the normalization of  $\mathcal{T}^{(1)}(l)$  chosen here). From the definition we derive for  $\Phi = (f, \Theta_{3,6})$

$$\begin{aligned} \tilde{\Phi}_3|_k \mathcal{T}^{(1)}(m)(z_1, 0, z_3) &= m^{2k-3} \sum_{ad=m, b \bmod d} d^{-k} \Phi\left(\frac{az_1+b}{d}, 0\right) e^{2\pi i 3m z_3} \\ &= (\Psi_0 f)|_k \mathcal{T}^{(1)}(m)(z_1) e^{2\pi i 3m z_3} = m^{k-2} \frac{\alpha(m)}{\alpha(1)} (\Psi_0 f)(z_1) e^{2\pi i 3m z_3}. \end{aligned}$$

Finally we find

$$\begin{aligned} W_3(\mathcal{M}(f))(z_1, z_3) &= \mathcal{M}(f) \begin{pmatrix} z_1 & 0 \\ 0 & z_3/3 \end{pmatrix} = c_0(0) \frac{-B_k}{2k} g_k(z_1) \\ &\quad + \sum_{m \in \mathbb{N}} m^{2-k} \tilde{\Phi}_3|_k \mathcal{T}^{(1)}(m)(z_1, 0, z_3/3) \\ &= \alpha(0) \frac{-B_k}{2k} g_k(z_1) + \sum_{m \in \mathbb{N}} \frac{\alpha(m)}{\alpha(1)} (\Psi_0 f)(z_1) e^{2\pi i 3m z_3/3} \\ &= \alpha(0) \frac{-B_k}{2k} g_k(z_1) + (\Psi_0 f)(z_1) \frac{1}{\alpha(1)} \sum_{m \in \mathbb{N}} \alpha(m) e^{2\pi i m z_3} \\ &= \alpha(0) \left( \frac{-B_k}{2k} g_k(z_1) - \frac{1}{\alpha(1)} (\Psi_0 f)(z_1) \right) + \frac{1}{\alpha(1)} (\Psi_0 f)(z_1) (\Psi_0 f)(z_3) \\ &= \frac{1}{\alpha(1)} (\Psi_0 f)(z_1) (\Psi_0 f)(z_3), \end{aligned}$$

since  $\alpha(0) = 0$  or  $\Psi_0 f$  is a multiple of the Eisenstein-series  $g_k$  (in which case  $\frac{1}{\alpha(1)} \Psi_0 f(z_1) = \frac{-B_k}{2k} g_k(z_1)$ ). ■

If a suitable Hecke-theory for  $[\mathrm{SL}_2(\mathbb{Z}), k, \tilde{\nu}]$  with nontrivial characters  $\tilde{\nu}$  had been developed, one might prove formulas for the restrictions of the Maaß-lifts to the diagonal in the same way in general.

#### 4. Divisors and Borchers-products

Using results of R. Borchers [1], it is possible to find paramodular forms (of degree 2) with known (zero-)divisors, so called Borchers-products. Borchers theory is formulated in terms of orthogonal groups and paramodular forms arise from these using well-known

isomorphisms of the underlying groups. A description can be found e.g. in [15, Sec. 1.3] and, more general, in [12].

Throughout this section we assume  $t \in \mathbb{N}$ . Consider the lattice  $L := \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}$  equipped with the quadratic form  $q_t((l_1, l_2, l_3, l_4, \beta)) = l_1 l_2 + l_3 l_4 - t\beta^2$ .  $L$  has signature  $(2, 3)$ . Let

$$K = \{\lambda \in M \mid l_1 = l_2 = 0\},$$

$$\mathcal{K}_+ = \{Y = (y_1, y_3, y_2) \in K \otimes \mathbb{R} \mid q_t(Y), y_1 > 0\}.$$

Associated with  $L$  is the half-space  $\mathbb{H}_L = K \otimes \mathbb{R} + i\mathcal{K}_+$ , which is essentially the Siegel-half-space  $\mathbb{H}_2$  of degree 2 via the biholomorphic transformation

$$\omega_t : \mathbb{H}_2 \rightarrow \mathbb{H}_L, \quad \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mapsto (z_1, tz_3, z_2).$$

Let  $O(L)^+ = O(L) \cap SO(L \otimes \mathbb{R})^+$ , where  $SO(L \otimes \mathbb{R})^+$  is the connected component of the identity in the special orthogonal group of  $L \otimes \mathbb{R}$ .  $PO(L)^+ = O(L)^+ / \{\pm \text{id}\}$  acts on  $\mathbb{H}_L$  as a group of biholomorphic transformations.

As described in [15], the arithmetic structure  $PO(L)^+$  corresponds to the group  $P\Gamma_t^{\max} = \Gamma_t^{\max} / \{\pm I\}$  in the symplectic setting, i.e. there is an isomorphism  $\Omega_t : P\Gamma_t^{\max} \rightarrow PO(L)^+$ , which is compatible with the identification of the associated half-spaces via  $\omega_t$ . In other words, there is a commutative diagram

$$\begin{array}{ccc} P\Gamma_t^{\max} \times \mathbb{H}_2 & \xrightarrow{(\Omega_t, \omega_t)} & PO(L)^+ \times \mathbb{H}_L \\ \downarrow & & \downarrow \\ \mathbb{H}_2 & \xrightarrow{\omega_t} & \mathbb{H}_L \end{array}$$

where the vertical arrows indicate the action of  $\Gamma_t^{\max}$  resp.  $O(L)^+$  on the corresponding half-space. Explicit formulas for  $\Omega_t$  and the action of  $O(L)^+$  on  $\mathbb{H}_2$  can be found in [12, Prop. 2.6] and [15, Sec. 1.3].

Using the automorphic embedding  $(\Omega_t, \omega_t)$  we can think of modular forms for (subgroups of)  $O(L)^+$  as paramodular forms. Note that the weight of the forms is the same on both sides (see [3, Sec. 3.3] for details on the translation of factors of automorphy). We do not worry about how multiplier-systems correspond exactly, since they are left unspecified in the next theorem and we will determine them in the symplectic setting.

Let  $L'$  be the dual of  $L$  (with respect to the bilinear form  $b_t(x, y) = q_t(x + y) - q_t(x) - q_t(y)$  associated with  $q_t$ ). Explicitly  $L' = \mathbb{Z}^2 \times \mathbb{Z}^2 \times \frac{1}{2t}\mathbb{Z}$ , thus we can identify  $L'/L$  with  $\frac{1}{2t}\mathbb{Z}/\mathbb{Z}$  (in the obvious way). For  $\lambda = (l_1, l_2, l_3, l_4, \beta) \in L'$  with  $q_t(\lambda) < 0$  define

$$\lambda^\perp := \left\{ Z \in \mathbb{H}_2 \mid l_1 - tl_2 \det(Z) + \text{trace} \left( \begin{pmatrix} l_3 & -t\beta \\ -t\beta & tl_4 \end{pmatrix} Z \right) = 0 \right\}.$$

This is a rational quadratic divisor. The discriminant of  $\lambda^\perp$  is  $\delta(\lambda^\perp) = -4tq_t(\lambda)$ , if  $\lambda \in L'$  is primitive. As is easily seen,  $\Gamma_t^{\max}$  acts on the set of rational quadratic divisors of fixed discriminant: Given  $M \in \Gamma_t^{\max}$  and  $\lambda^\perp$  a rational quadratic divisor, the set  $M\lambda^\perp := \{M \cdot$

$Z \mid Z \in \lambda^\perp$  is again a rational quadratic divisor with  $\delta(\lambda^\perp) = \delta(M\lambda^\perp)$  (in fact this action realizes a homomorphism  $\Gamma_t^{\max} \rightarrow \text{PO}(L)^+$ , which is essentially the isomorphism  $\Omega_t$ ). Following Freitag and Hermann [12, Lemma 4.4], one can show:

LEMMA 4.1. *All rational quadratic divisors of fixed discriminant are equivalent under  $\Gamma_t^{\max}$ .*

Note that the  $\Gamma_t$ -orbits of rational quadratic divisors of fixed discriminant are distinguished by their image in the discriminant-group  $L'/L$ . The importance of lemma 4.1 is, that if  $t$  is prime, any modular form for  $\Gamma_t^*$ , that vanishes on a rational quadratic divisor  $\lambda^\perp$ , vanishes on the orbit  $\Gamma_t^*\lambda^\perp = \bigcup_{M \in \Gamma_t^*} M\lambda^\perp$ . This will be used extensively in section 5 to determine generators of graded rings of modular forms for  $\Gamma_3^*$  (with multiplier-systems).

Set  $\mathbb{V}_L := \{f : L'/L \rightarrow \mathbb{C}\}$  and let  $\rho_L : \text{Mp}_2(\mathbb{Z}) \rightarrow \text{GL}(\mathbb{V}_L)$  be the Weil-representation associated with the quadratic module  $(L'/L, q_t \bmod \mathbb{Z})$  as in [1, Sec. 4]. Then  $\rho_L \cong \rho_{i,0}^*$  (where  $\rho_{i,0}^*$  is the representation of  $\text{Mp}_2(\mathbb{Z})$ , which already appeared in the preceding section) via the identification  $\frac{x}{2t} \mapsto x$  of  $L'/L = \frac{1}{2t}\mathbb{Z}/\mathbb{Z}$  and  $\mathbb{Z}/2t\mathbb{Z}$ . Again we have a decomposition  $\rho_L = \rho_{L,i} \oplus \rho_{L,-i}$ , where  $\rho_{L,s}$  is the restriction of  $\rho_L$  to the eigenspace of  $\rho_L(\widehat{J}^2)$  with eigenvalue  $s$ . Note that  $\rho_{L,s} \cong \rho_{i,0,-s}^*$  because of the dual.  $\text{O}(L)^+$  acts on  $\mathbb{V}_L$  by  $(Mf)(l) = f(M^{-1}l)$ . The discriminant-kernel  $\text{O}(L)_d^+ \subset \text{O}(L)^+$  is the subgroup, fixing  $\mathbb{V}_L$  pointwise. Via the action on  $\mathbb{V}_L$ , there is an induced action of  $\text{O}(L)^+$  on the space  $[\text{Mp}_2(\mathbb{Z}), k, \rho_L]_{\text{mer}}$ . The crucial point is, that  $\text{O}(L)^+/\text{O}(L)_d^+ \cong \text{O}(L/L')$  is the orthogonal group of  $(L'/L, q_t \bmod \mathbb{Z})$ . Therefore, the action of  $\text{O}(L)^+$  commutes with the action of  $\text{Mp}_2(\mathbb{Z})$  via  $\rho_L$  on  $\mathbb{V}_L$  (and, for square-free  $t$  at least,  $\text{O}(L/L')$  decomposes  $\rho_L$  into irreducible constituents).

$f \in [\text{Mp}_2(\mathbb{Z}), k, \rho_L]_{\text{mer}}$  has a Fourier-expansion

$$f = \sum_{l \in L'/L} \sum_{-\infty \ll n \in q_t(l) + \mathbb{Z}} c(l, n) e^{2\pi i n \tau} f_l^c.$$

Note that  $c(l, n)$  is defined for  $l \in \frac{1}{2t}\mathbb{Z}$  also (via the identification of  $\frac{1}{2t}\mathbb{Z}/\mathbb{Z}$  with  $L'/L$ ).

The following fundamental theorem is a special case of [1, Th. 13.3] or [3, Th. 3.19] for lattices of signature  $(2, 3)$ .

THEOREM 4.2 *Let  $f \in [\text{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\text{mer}}$  with Fourier-coefficients  $c(l, n)$  for  $l \in L'/L$  and  $-\infty \ll n \in q_t(l) + \mathbb{Z}$ . Assume  $c(l, n) \in \mathbb{Z}$  for all  $n \leq 0$  and all  $l \in L'/L$ . Then there is a (meromorphic) modular form  $\mathcal{B}(f)$  of weight  $c(0, 0)/2$  for the subgroup of  $\text{O}(L)^+$  fixing  $f$  with some multiplier-system, such that all zeros and poles of  $\mathcal{B}(f)$  are along rational quadratic divisors  $\lambda^\perp$ ,  $\lambda \in L'$  primitive,  $q_t(\lambda) < 0$ , with multiplicity*

$$\sum_{r \in \mathbb{N}} c(r\lambda, r^2 q_t(\lambda)).$$

We will refer to the functions  $\mathcal{B}(f)$  as Borchers-products. As the name suggests, these functions have product-expansions, but these are converging only “near cusps” in general. As explained above, we think of Borchers-products  $\mathcal{B}(f)$  as paramodular forms

also. A description of the product-expansions of Borcherds-products in the symplectic setting was given in our special case in [15, Th. 2.1] (there the input are “weak” Jacobi-forms instead of meromorphic vector-valued modular forms, which is essentially the same by arguments analogous to lemma 3.5).

Every  $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$  is fixed by the discriminant-kernel  $\mathrm{O}(L)_d^+$  at least. But the irreducible constituents of  $\rho_L$  are fixed by nontrivial subgroups of  $\mathrm{O}(L)^+ / \mathrm{O}(L)_d^+$  in general. For weight  $-\frac{1}{2}$ , non-trivial contributions to  $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$  only come from  $\rho_{L,i}$  since  $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,-i}]_{\mathrm{mer}}$  is trivial. This implies (using the explicit formula for  $\rho_{i,0}^*(\widehat{\mathcal{J}}^2)$  given in (3.8)), that if  $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ , then  $f(\tau)(-l) = f(\tau)(l)$ , i.e.  $f$  is invariant under  $M \in \mathrm{O}(L)^+$ , if  $M$  acts as multiplication by  $-1$  on the discriminant-group (at least). Note that  $\Omega_t(V_t)$  acts as  $-1$  on the discriminant-group. Thus Borcherds-products are always paramodular forms for  $\Gamma_t^*$  at least. In general, there is a distinguished irreducible constituent of  $\rho_L$ , which is invariant under the full group  $\mathrm{O}(L)^+$ .

We summarize results from [1, Th. 13.3], [3, Th. 3.19] and [15, Th. 2.1] in our special case (in the symplectic setting):

**COROLLARY 4.3.** *Let  $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$  with Fourier-coefficients  $c(l, n)$  for  $l \in L'/L$  and  $-\infty \ll n \in q_t(l) + \mathbb{Z}$ . Assume  $c(l, n) \in \mathbb{Z}$  for all  $n \leq 0$  and all  $l \in L'/L$ .*

1) *There is a (meromorphic) modular form  $\mathcal{B}(f)$  of weight  $c(0, 0)/2$  for  $\Gamma_t^*$  with some multiplier-system, such that all zeros and poles of  $\mathcal{B}(f)$  are along rational quadratic divisors  $\lambda^\perp$ ,  $\lambda \in L'$  primitive,  $q_t(\lambda) < 0$ , with multiplicity*

$$\sum_{r \in \mathbb{N}} c(r\lambda, r^2 q_t(\lambda)).$$

2) *Let  $n_0 := -\min\{n \in \frac{1}{4t}\mathbb{Z} \mid c(l, n) \neq 0 \text{ for some } l \in L'/L\}$ , define*

$$A = \frac{1}{24} \sum_{l \in \mathbb{Z}} c(l/2t, -l^2/4t), \quad B = \frac{1}{2} \sum_{l \in \mathbb{N}} l c(l/2t, -l^2/4t),$$

$$C = \frac{1}{4} \sum_{l \in \mathbb{Z}} l^2 c(l/2t, -l^2/4t), \quad D = \sum_{n \in \mathbb{N}, l \in \mathbb{Z}} \sigma_1(n) c(l/2t, -n - l^2/4t)$$

and set  $\lambda_W = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$  (this is essentially the Weyl-vector from [1, Th. 13.3]). Then  $\mathcal{B}(f)$  has a product-expansion, converging for  $\Im(Z) > n_0$ , of the form

$$\mathcal{B}(f)(Z) = e^{2\pi i \operatorname{trace}(\lambda_W Z)} \prod_{(m,n,l) > 0} (1 - e^{2\pi i(nz_1 + lz_2 + tmz_3)})^{c(l/2t, mn - l^2/4t)}$$

(here  $(m, n, l) > 0$  means  $m, n \in \mathbb{N}_0$ ,  $l \in \mathbb{Z}$  and  $l < 0$  or  $m + n > 0$ ). Moreover, the Borcherds-product satisfies  $\mathcal{B}(f)(V_t \cdot Z) = (-1)^D \mathcal{B}(f)(Z)$ .

Borcherds lift is multiplicative: For  $f$  and  $g$  satisfying the assumptions from corollary 4.3, one finds  $\mathcal{B}(f + g) = \mathcal{B}(f)\mathcal{B}(g)$ .

In the remaining part of this section, we give some explicit examples of Borcherds-products for  $\Gamma_3^*$ . For applications in the following section, we are interested in forms

with “minimal” divisor (and “small” weight). From Borchers theorem it is heuristically clear, that we have to determine meromorphic forms in  $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,i}]_{\mathrm{mer}}$  with poles of minimal order. The following table lists representatives  $\lambda_j$  of all the orbits  $\Omega_3(\Gamma_3^*)\lambda_j$  with primitive  $\lambda_j \in L'$  such that  $-1 \leq q_3(\lambda_j) = \frac{-j}{12} < 0$ . The 4<sup>th</sup> column gives the defining equation of the rational quadratic divisor  $\lambda_j^\perp$  with discriminant  $\delta(\lambda_j^\perp) = j \leq 12$ . In the 5<sup>th</sup> column, the order  $\mathrm{ord}_\psi$  of the Borchers-product  $\psi = \mathcal{B}(f)$ , associated with a form  $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,i}]_{\mathrm{mer}}$  with Fourier-coefficients  $c(l, n)$ , along  $\lambda_j^\perp$  is given, if  $f$  has a pole of order  $\leq 1$  at the cusp  $i\infty$ .

$j$	$\lambda_j$	$q(\lambda_j)$	$Z$	$\mathrm{ord}_\psi(\lambda_j^\perp)$
1	$(0, 0, 0, 0, \frac{1}{6})$	$-\frac{1}{12}$	$z_2 = 0$	$c(\frac{1}{6}, -\frac{1}{12}) + c(\frac{2}{6}, -\frac{4}{12}) + c(\frac{3}{6}, -\frac{9}{12})$
4	$(1, 0, 0, 0, \frac{2}{6})$	$-\frac{1}{3}$	$z_2 = \frac{1}{2}$	$c(\frac{1}{3}, -\frac{1}{3})$
9	$(1, 0, 0, 0, \frac{3}{6})$	$-\frac{3}{4}$	$z_2 = \frac{1}{3}$	$c(\frac{1}{2}, -\frac{3}{4})$
12	$(0, 0, 1, -1, 0)$	$-1$	$z_3 = \frac{1}{3}z_1$	$c(0, -1)$

In order to construct Borchers-products explicitly, the only question that remains is: How to find modular forms  $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,i}]_{\mathrm{mer}}$  with explicitly given singularities? There are (at least) two possibilities:

1) The first method is to find holomorphic modular forms of weight  $12n - 1/2, n \in \mathbb{N}$  with multiplier-system  $\rho_{L,i}$  and divide by  $\Delta_{12}^n$ . In other words, we can use

$$[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_{L,i}]_{\mathrm{mer}} = \sum_{n \in \mathbb{N}} \Delta_{12}^{-n} [\mathrm{Mp}_2(\mathbb{Z}), 12n - \frac{1}{2}, \rho_{L,i}].$$

Since for  $t = 3$  we have  $\rho_{L,i} \cong \rho_{3,0,-i}^*$ , and we gave an explicit basis of the module  $[\mathrm{MJ}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, 3, 1, i] \cong [\mathrm{Mp}_2(\mathbb{Z}), \frac{1}{2}\mathbb{Z}, \rho_{3,0,-i}^*]$  over  $\mathcal{R}$  in section 3 on page 13, all forms in the spaces  $[\mathrm{Mp}_2(\mathbb{Z}), 12n - \frac{1}{2}, \rho_{L,i}]$  can be calculated explicitly in principle.

2) There is a second method given by Borchers in [2]. The result from [2, Th. 3.1] states, that there is a “simple” criterion for a given singularity of type  $(\sum_{\substack{n \in q_t(l) + \mathbb{Z} \\ -\infty \ll n \leq 0}} h(l, n)q^n)_{l \in L'/L}$  (at  $i\infty$ ) to be extensible to a meromorphic form in

$[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ : There exists  $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$  (with Fourier-coefficients  $c_f(l, n)$ ), such that  $h(l, n) = c_f(l, n)$  for all  $l$  and  $n \leq 0$ , if and only if every  $g \in [\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*]$  (with Fourier-coefficients  $c_g(l, n)$ ) satisfies

$$(4.1) \quad \sum_{l \in L'/L} \sum_{n \in q_t(l) + \mathbb{Z}, n \leq 0} h(l, n)c_g(l, -n) = 0.$$

Note that in general for  $f \in [\mathrm{Mp}_2(\mathbb{Z}), 2 - k, \rho]_{\mathrm{mer}}$  and  $g \in [\mathrm{Mp}_2(\mathbb{Z}), k, \rho^*]$ , we have  $(f, g) \in [\mathrm{SL}_2(\mathbb{Z}), 2, 1]_{\mathrm{mer}}$  and therefore  $\sum_{l, n \leq 0} c_f(l, n)c_g(l, -n) = 0$ .

Speaking informally, we can say that all obstructions for a given singularity to be extensible to a meromorphic form in  $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$  come from forms in  $[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*]$ . Therefore we call  $[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*]$  the obstruction-space for  $[\mathrm{Mp}_2(\mathbb{Z}), -\frac{1}{2}, \rho_L]_{\mathrm{mer}}$ .

Using the dimension-formula from lemma 3.7 (and the parameters from the table on page 13 as well as  $\rho_L^* \cong \rho_{3,0}$ ), we find  $\dim[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*] = 1$ . A generator of the obstruction-space can be realized as a vector-valued Eisenstein-series  $E_{\frac{5}{2}}$  as given in [4, Th. 4.8]. The Fourier-development of  $E_{\frac{5}{2}}$  is given by (here we set  $q = e^{2\pi i\tau}$ )

$$\begin{aligned} E_{\frac{5}{2}}(\tau)(0) &= 1 - 24q^1 - 72q^2 + \mathcal{O}(q^3) \\ E_{\frac{5}{2}}(\tau)(1/6) = E_{\frac{5}{2}}(\tau)(5/6) &= -1q^{1/12} - 12q^{13/12} + \mathcal{O}(q^{25/12}) \\ E_{\frac{5}{2}}(\tau)(2/6) = E_{\frac{5}{2}}(\tau)(4/6) &= -7q^{1/3} - 55q^{4/3} + \mathcal{O}(q^{7/3}) \\ E_{\frac{5}{2}}(\tau)(3/6) &= -34q^{3/4} - 48q^{7/4} + \mathcal{O}(q^{11/4}). \end{aligned}$$

Note that  $E_{\frac{5}{2}}$  actually lies in  $[\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_{L,i}^*]$  thus for  $l \in L'/L$  we find

$$-iE_{\frac{5}{2}}(\tau)(l) = \left(\rho_L^*(\widehat{J}^2)E_{\frac{5}{2}}\right)(\tau)(l) = -iE_{\frac{5}{2}}(\tau)(-l).$$

If we restrict the order of the pole at  $i\infty$  to be  $\leq 1$ , the obstruction-problem (4.1) admits the following singularities  $H_l = \sum_{n \in q_l(l) + \mathbb{Z}, -1 \leq n \leq 0} h(l, n)q^n$ ,  $l \in L'/L$ , as solutions:

$$\begin{array}{l} H_0 = \\ H_{1/6} = H_{5/6} = \\ H_{2/6} = H_{4/6} = \\ H_{3/6} = \end{array} \begin{array}{l} 0q^{-1} + 2q^0 \\ 1q^{-1/12} \\ 0q^{-1/3} \\ 0q^{-3/4} \end{array} \left| \begin{array}{l} 0q^{-1} + 12q^0 \\ -1q^{-1/12} \\ 1q^{-1/3} \\ q^{-3/4} \end{array} \right| \begin{array}{l} 0q^{-1} + 32q^0 \\ -1q^{-1/12} \\ 0q^{-1/3} \\ 1q^{-3/4} \end{array} \left| \begin{array}{l} 1q^{-1} + 24q^0 \\ 0q^{-1/12} \\ 0q^{-1/3} \\ 0q^{-3/4} \end{array} \right.$$

The corresponding Borcherds-products are denoted by  $\psi_1, \psi_6, \psi_{16}$  and  $\psi_{12}$  (the index always indicates the weight of the Borcherds-product). The following table lists the weight  $k$  and the order of the Borcherds-product  $\psi_k$  along the rational quadratic divisors  $\lambda_j^\perp$  of discriminant  $\leq 12$  (there are no zeros along rational quadratic divisors of discriminant  $> 12$ ). The 6<sup>th</sup> column gives the multiplier-system  $\nu$  of the Borcherds-products. In the last four columns, the parameters  $A, B, C$  and  $D$  from corollary 4.3 (originally [15, Theorem 2.1]) are listed.

$k$	$\lambda_1^\perp$	$\lambda_4^\perp$	$\lambda_9^\perp$	$\lambda_{12}^\perp$	$\nu$	$A$	$B$	$C$	$D$
1	1	0	0	0	$\chi\kappa\mu^2$	1/6	1/2	1/2	0
6	0	1	0	0	$\kappa$	1/2	1/2	3/2	0
16	0	0	1	0	$\mu$	4/3	1	4	0
12	0	0	0	1	$\chi$	1	0	0	1

Since the multiplier-system of Borcherds-products is left undetermined by theorem 4.2, we have to give additional arguments. Again, there are several possibilities:

1) Let  $\nu$  be the multiplier-system of the Borcherds-product  $\psi_k$ . From the parameter  $A$  and the product-expansion of  $\psi_k$  in corollary 4.3, one can easily deduce  $\psi_k(Z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = e^{2\pi i A} \psi_k(Z)$ , thus  $\nu(\text{trans} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = e^{2\pi i A} = (\mu^2 \kappa) (\text{trans} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})^{6A}$ . Furthermore,  $\psi_k|_k V_3 = (-1)^{k+D} \psi_k$  follows directly from corollary 4.3, thus  $\nu(V_3) = (-1)^{k+D}$ . Since each character  $\nu \in \Gamma_3^{*ab}$  is uniquely determined by the values  $\nu(\text{trans} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$  and  $\nu(V_3)$ , we have  $\nu = (\mu^2 \kappa)^{6A} \chi^{k+D}$ .

2) Alternatively, some ad hoc arguments can be given for some of the Borcherds-products at least. By the way, we get other useful information as a side-effect. For example, we can identify some Borcherds-products with certain Maaß-lifts.

Note that modular forms of weight  $k$  with multiplier-system  $\nu$  necessarily have zeros along certain rational quadratic divisors  $\lambda^\perp$ , if this divisor is fixed pointwise by a transformation  $M_\lambda \in \Gamma_3^*$ , such that  $j_2(M_\lambda, Z)^{-k} \neq \nu(M_\lambda)$  for all  $Z \in \lambda^\perp$ . Here are some examples of such transformations for (some of) the rational quadratic divisors of discriminant  $\leq 1$ :

$$\begin{aligned} Z \in \lambda_1^\perp &\implies Z = Z \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right], \\ Z \in \lambda_4^\perp &\implies Z = Z \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ Z \in \lambda_{12}^\perp &\implies Z = Z \left[ \begin{pmatrix} 0 & \sqrt{3} \\ 1/\sqrt{3} & 0 \end{pmatrix} \right]. \end{aligned}$$

In all cases we have  $j_2(M_\lambda, Z) = -1$ . Thus for  $f \in [\Gamma_3^*, k, \chi^j (\mu^2 \kappa)^l]$  we find

$$f(Z) = 0 \text{ on } \begin{cases} \lambda_1^\perp, & \text{if } (-1)^k \neq 1, \\ \lambda_4^\perp, & \text{if } (-1)^k \neq (-1)^l, \\ \lambda_{12}^\perp, & \text{if } (-1)^k \neq (-1)^j. \end{cases}$$

Some special cases frequently needed later are

$$(4.2) \quad f = 0 \text{ on } \lambda_1^\perp \text{ for } f \in [\Gamma_3^*, 2k+1, \nu] \quad (\text{all } k \in \mathbb{Z}, \nu \in \Gamma_3^{*ab}),$$

$$(4.3) \quad f = 0 \text{ on } \lambda_4^\perp \text{ for } f \in [\Gamma_3^*, 2k, \kappa(\chi\mu)^j] \quad (\text{all } k, j \in \mathbb{Z}),$$

$$(4.4) \quad f = 0 \text{ on } \lambda_{12}^\perp \text{ for } f \in [\Gamma_3^*, 2k, \chi(\kappa\mu)^j] \quad (\text{all } k, j \in \mathbb{Z}).$$

Now we can determine the multiplier-system of the Borcherds-products  $\psi_1$  and  $\psi_6$ .

Let  $0 \neq f_1 \in \mathcal{M}_{1,1,i}$  be a generator of the Maaß-space of weight 1 with character  $\chi\mu^2\kappa$ . It follows from (4.2), that  $f_1 = 0$  on  $\lambda_1^\perp$ . Then  $f_1/\psi_1$  is a non-trivial paramodular form in  $[D\Gamma_3^*, 0, 1] \cong \mathbb{C}$ . Thus we have  $\psi_1 = cf_1$  for some  $c \in \mathbb{C}^\times$ . Especially,  $\psi_1$  has multiplier-system  $\chi\mu^2\kappa$ .

Let  $0 \neq f_6 \in \mathcal{M}_{6,3,-i}$  be a generator of the Maaß-space of weight 6 with character  $\kappa$ . It follows from (4.3), that  $f_6 = 0$  on  $\lambda_4^\perp$ . Then  $f_6/\psi_6$  is a non-trivial paramodular form in  $[D\Gamma_3^*, 0, 1] \cong \mathbb{C}$ . Thus we have  $\psi_6 = cf_6$  for some  $c \in \mathbb{C}^\times$ . Especially,  $\psi_6$  has multiplier-system  $\kappa$ .

The other Borcherds-products can't be dealt with in the same way. But for  $\psi_{12}$  at least, we can find some other arguments (we do not need  $\psi_{16}$  in section 5). Note that the zeros of  $\psi_{12}$  imply, that the multiplier-system of  $\psi_{12}$  is of the form  $\nu = \chi^j \mu^l$  for some  $j, l \in \mathbb{Z}$  (if  $\kappa$  would appear in  $\nu$ , then we had  $\psi_{12} = 0$  on  $\lambda_4^\perp$  using (4.3)). We want it to be  $\chi$ .  $\psi_{12}$  cannot be a Maaß-lift in this case and we cannot give an argument analogous to  $\psi_1$  and  $\psi_6$  but have to find another realization of  $\psi_{12}$ . The idea is the following: If we find a non-trivial form  $f \in [\Gamma_3, k, \xi]$ , where  $\xi$  is a *non-symmetric* character in  $\Gamma_3^{\text{ab}}$  (such as  $\mu_1$ ) of order  $n$ , then  $f^n - f^n|_{nk}V_3$  is a non-trivial form with character  $\chi$  for  $\Gamma_3^* f^n - f^n|_kV_3$  is non-trivial, since otherwise we would have  $f^n(Z) = f^n|_{nk}V_3(Z) = (-1)^{-nk} f^n(V_3 \cdot Z)$  for all  $Z \in \mathbb{H}_2$ . Because  $\mathbb{H}_2$  is simply connected, this would imply  $f(Z) = (-1)^{-k} \zeta f(V_3 \cdot Z)$  for all  $Z \in \mathbb{H}_2$  with  $\zeta$  a fixed  $n^{\text{th}}$  root of unity. From this we could derive for all  $M \in \Gamma_3$

$$\begin{aligned} \xi(M)f(Z) &= f|_kM(Z) = \zeta (f|_kV_3M)(Z) = \zeta \xi(V_3MV_3^{-1})f|_kV_3(Z) \\ &= \xi(V_3MV_3^{-1})f(Z). \end{aligned}$$

Thus  $\xi$  would be symmetric, in contradiction to our assumption on  $\xi$ .

As an example of such a form for  $\Gamma_3$  with non-symmetric character, we want to define an Eisenstein-series  $E_4(\mu_1) \in [\Gamma_3, 4, \mu_1]$  of Klingen-type for  $\Gamma_3$  with (non-symmetric) character  $\mu_1$  by

$$E_4(\mu_1)(Z) := \sum_{M: \pm\Gamma_{3,\infty} \setminus \Gamma_3} \mu_1(M)^{-1} \eta^8((M \cdot Z)^*) 1|_4M(Z)$$

(here  $(M \cdot Z)^*$  is the upper left entry of  $M \cdot Z$ ). The summation is well-defined, since  $\tilde{\mu}_1$  is the multiplier-system of  $\eta^8$ . Moreover,  $\lim_{y \rightarrow \infty} E_4(\mu_1)\left(\begin{smallmatrix} \tau & 0 \\ 0 & iy \end{smallmatrix}\right) = \eta^8(\tau) \neq 0$ , thus  $E_4(\mu_1)$  is non-trivial. Since  $\mu_1$  has order 3, the argument given above, now would lead to  $\psi_{12} \in [\Gamma_3^*, 12, \chi]$ , if the sum defining  $E_4(\mu_1)$  would converge absolutely. But this is not the case. Probably the convergence can be fixed using some sort of Hecke-trick. We avoid this minor problem in the following way. We define an Eisenstein-series  $E_8(\mu_1^2) \in [\Gamma_3, 8, \mu_1^2]$  of Klingen-type for  $\Gamma_3$  with (non-symmetric) character  $\mu_1^2$  by

$$E_8(\mu_1^2)(Z) := \sum_{M: \pm\Gamma_{3,\infty} \setminus \Gamma_3} \mu_1(M)^{-2} \eta^{16}((M \cdot Z)^*) 1|_8M(Z).$$

Note that the sum converges absolutely now and is well-defined again, i.e. we have  $0 \neq E_8(\mu_1^2) \in [\Gamma_3, 8, \mu_1^2]$ . In this case, the argument given above leads to  $0 \neq f_{24} := E_8(\mu_1^2)^3 + E_8(\mu_1^2)^3|_{24}V_3 \in [\Gamma_3^*, 24, \chi]$ . From (4.4) we get  $f_{24} = 0$  on  $\lambda_{12}^\perp$ . Thus  $f_{12} := f_{24}/\psi_{12}$  is a non-trivial form of weight 12 for  $\Gamma_3^*$  (with some multiplier-system).  $f_{12}|_{12}V_3 = -f_{12}$  would imply  $f_{12}/\psi_{12} \in [D\Gamma_3^*, 0, 1]$  and  $f_{24} = c\psi_{12}^2$  with some constant  $c \in \mathbb{C}^\times$ . As a square,  $\psi_{12}^2$  can have multiplier-systems of 3-power-order only,



but the multiplier-system of  $f_{24}$  has order 2. Thus we must have  $f_{12}|_{12}V_3 = f_{12}$  and  $\psi_{12}|_{12}V_3 = -\psi_{12}$ , i.e. the multiplier-system of  $\psi_{12}$  is of the form  $\nu = \chi\mu^l$  for some  $l \in \mathbb{Z}$ . Now if  $\mu^l \neq 1$ , then  $W_3(\psi_{12})(z_1, z_3) = \psi_{12}\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_3/3 \end{smallmatrix}\right)$  would be a form of weight 12 on  $\mathbb{H}_1 \times \mathbb{H}_1$  with (non-trivial) multiplier-system  $v_\eta^{8l} \times v_\eta^{8l}$ . With  $l \in \{1, 2\}$ , lemma 5.1 leads to  $W_3(\psi_{12})(z_1, z_3) = c \eta^{8l}(z_1)\eta^{8l}(z_3)g_{12-4l}(z_1)g_{12-4l}(z_3)$  for some  $c \in \mathbb{C}^\times$ . On the other hand we have

$$\begin{aligned} W_3(f|_k V_3)(z_1, z_3) &= (-1)^k f\left(V_3 \cdot \begin{pmatrix} z_1 & 0 \\ 0 & z_3/3 \end{pmatrix}\right) = (-1)^k f\left(\begin{pmatrix} z_3 & 0 \\ 0 & z_1/3 \end{pmatrix}\right) \\ &= (-1)^k W_3(f)(z_3, z_1) \end{aligned}$$

in general. Since  $\psi_{12}|_{12}V_3 = -\psi_{12}$ , this implies  $W_3(\psi_{12})(z_1, z_3) = -W_3(\psi_{12})(z_3, z_1)$ . But obviously  $\eta^{8l}(z_1)\eta^{8l}(z_3)g_{12-4l}(z_1)g_{12-4l}(z_3)$  is invariant under interchanging  $z_1$  and  $z_3$ . Thus in this case  $W_3(\psi_{12}) = 0$  would follow, but  $\psi_{12}$  does not vanish on  $\lambda_1^\perp$  (identically). All together,  $\psi_{12}$  has multiplier-system  $\chi$ .

REMARK 4.4. 1) There is no  $M \in \Gamma_3^*$ , which fixes  $\lambda_9^\perp$  pointwise. Thus vanishing along  $\lambda_9^\perp$  can't be correlated to certain combinations of characters/weights as with the other rational-quadratic divisors (of norm  $\leq 12$ ) above.

2) The characters  $\mu^j$  cannot be used to derive zeros of forms along any rational-quadratic divisor in general. For example, both  $\psi_1^2$  and  $\psi_{16}$  have character  $\mu$ , but their zero-divisors are disjoint.

3) All four Borcherds-products  $\psi_k$  already appeared in [15] (Th. 2.6, (3.22), Exam. 1.17 and (4.8) in [15] resp.).

4) In some sense the problem of finding generators for  $[D\Gamma_3^*, \mathbb{Z}, 1]$  is as easy as in the Siegel case (that is  $[D\Gamma_1, \mathbb{Z}, 1]$ ), since in both cases the rational-quadratic divisors can be separated by Borcherds-products (i.e. for any rational-quadratic divisor  $\lambda^\perp$  there is a Borcherds-product vanishing exactly along the orbit  $\Gamma_t^* \lambda^\perp$  with order 1). The crucial point is, that there are no cusp-forms in the obstruction-space in both cases, since by results of Bruinier [3] the Eisenstein-series in the obstruction-space determines the weight of a Borcherds-product only. By the same reason it can be expected, that the analogous problem for  $\Gamma_2^*$  can be solved in the same way (though everything is known in this case by [19], for  $\Gamma_2$  and  $\Gamma_2^*$  at least). For  $t \geq 4$  the obstruction-space seems to have dimension  $\geq 2$  (the obstruction-space is in fact contained in the subspace associated with  $\rho_{L,i}^*$ ). The following table lists  $D = \dim[\text{Mp}_2(\mathbb{Z}), \frac{5}{2}, \rho_L^*]$  for some small  $t$ .

$t$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$D$	1	1	1	2	2	2	2	3	3	3	3	4	3	4	4	5	4	5	4	6

In general, finding generators for  $[D\Gamma_t^*, \mathbb{Z}, 1]$  gets more and more involved, as the dimension of the obstruction-space increases.

### 5. Graded rings of modular forms

The divisor  $\lambda_1^\perp$  can be used for a reduction process in the same way as it was used by Freitag in the case of Siegel modular forms of degree two. More precisely, if we can lift all (generators for the ring of) automorphic forms on  $\lambda_1^\perp$ , that arise from paramodular forms of level 3 by restriction to  $\lambda_1^\perp$ , then for any paramodular form a suitable linear combination with this lifts is divisible by the Borcherds-product  $\psi_1$  from the preceding section. It will turn out, that it is sufficient to lift generators of a certain subring only, if we make use of some of the other Borcherds-products too.

First we introduce some notation: For  $n \in \mathbb{N}$  let  $\mathbb{C}_n := \mathbb{C}[X_1, \dots, X_n]$  be the ring of polynomials in the  $n$  (independent) indeterminants  $X_1, \dots, X_n$ . If  $l \leq n$  we have a natural inclusion  $\mathbb{C}_l \subset \mathbb{C}_n$ . For  $P \in \mathbb{C}_n$  and  $j \in \{1, \dots, n\}$  let  $\deg_j(P)$  denote the degree of  $P$  with respect to  $X_j$ .

We define the Witt-Operator  $W_3$  on functions  $f : \mathbb{H}_2 \rightarrow \mathbb{C}$  by

$$(5.1) \quad W_3(f)(z_1, z_3) := f \begin{pmatrix} z_1 & 0 \\ 0 & \frac{1}{3}z_3 \end{pmatrix}.$$

From the embedding of  $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \rightarrow \Gamma_t$ , defined in section 2, we see, that  $f \in [\Gamma_3^*, k, \nu]$  implies  $W_3(f) \in [SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}), k, \tilde{\nu}]$ , i.e.  $W_3(f)$  is an elliptic modular form of weight  $k$  with character  $\tilde{\nu}$  in each of the two variables separately. Moreover from the transformation of  $f$  under  $V_3$  it follows that  $W_3(f)(z_3, z_1) = (-1)^k \nu(V_3) W_3(f)(z_1, z_3)$ .

First we need more information about  $[SL_2(\mathbb{Z}), k, \tilde{\nu}]$ . The following lemma 5.1 is well-known. On the other hand it is prototypical for analogous statements in higher dimensions, such as our main theorem 5.2 and the following lemmata. Therefore we sketch a proof of lemma 5.1 also.

For even  $n \in \mathbb{N}$  set  $\Gamma(n) := \text{kern}(v_\eta^n) \subset SL_2(\mathbb{Z})$  (this is the invariance group of  $\eta^n$ ).

LEMMA 5.1. *Let  $n \in \mathbb{N}$  be an even divisor of 24 and  $m = \frac{24}{n}$ .*

$$1) \quad [\Gamma(n), \mathbb{Z}, 1] := \bigoplus_{k \in \mathbb{Z}} [\Gamma(n), k, 1] = \mathbb{C}[\eta^n, g_4, g_6].$$

More precisely, for  $j \in \{0, \dots, m-1\}$  we have

$$[SL_2(\mathbb{Z}), k, v_\eta^{nj}] = \eta^{nj} [SL_2(\mathbb{Z}), k - nj/2, 1].$$

2) *The generators of  $[\Gamma(n), \mathbb{Z}, 1]$  satisfy the relation  $(\eta^n)^m = \frac{1}{12^3}(g_4^3 - g_6^2)$ .*

3) *Let  $P_n = X_1^m - \frac{1}{12^3}(X_2^3 - X_3^2) \in \mathbb{C}_3$  and  $I = (P_n) \subset \mathbb{C}_3$  be the ideal, generated by  $P_n$ . Then  $[\Gamma(n), \mathbb{Z}, 1] \cong \mathbb{C}_3/I$ .*

*Proof.* 1) If  $n \in \mathbb{N}$  is an even divisor of 24, then  $DSL_2(\mathbb{Z}) \subset \Gamma(n)$  and  $\Gamma(n)/DSL_2(\mathbb{Z})$  acts on  $[\Gamma(n), k, 1]$  as a group of commuting operators. Thus if  $f \in [\Gamma(n), k, 1]$ , then we can decompose  $f = \sum_{j: \mathbb{Z}/m\mathbb{Z}} f_j$  with  $f_j \in [SL_2(\mathbb{Z}), k, v_\eta^{nj}]$ . Therefore without restriction, we can assume  $f \in [SL_2(\mathbb{Z}), k, v_\eta^{nj}]$  with  $j \in \{0, \dots, m-1\}$ . Then

$$f(\tau + 1) = v_\eta^{nj}(T)f(\tau) = e^{2\pi ij/m} f(\tau)$$

and  $f$  has a zero at  $i\infty$  of order  $\frac{j}{m}$  at least. This implies  $f_j/\eta^{nj} \in [\mathrm{SL}_2(\mathbb{Z}), k - nj/2, 1]$ .

2) follows from  $(\eta^n)^m = \Delta_{12} = \frac{1}{12^3}(g_4^3 - g_6^2)$ , as is well-known.

3) The relation stated in 2) shows  $P_n(\eta^n, g_4, g_6) = 0$  on  $\mathbb{H}_1$ . Now assume that  $Q \in \mathbb{C}_3$  satisfies  $Q(\eta^n, g_4, g_6) = 0$  on  $\mathbb{H}_1$ . We have to show, that  $Q \in I$ . After reducing  $Q$  modulo  $I$ , we may assume  $\deg_1(Q) < m$ . Write  $Q = \sum_{j=0}^{m-1} X_1^j R_j$  with  $R_j \in \mathbb{C}[X_2, X_3]$  for all  $j$ . From our assumption  $\sum_{j=0}^{m-1} (\eta^n)^j R_j(g_4, g_6) = 0$  on  $\mathbb{H}_1$  follows. The summand  $(\eta^n)^j R_j(g_4, g_6)$  has character  $v_\eta^{nj}$ . Since these characters are all different, all summands have to vanish separately. Thus we get  $R_j(g_4, g_6) = 0$  on  $\mathbb{H}_1$  for all  $j$ . Because  $g_4$  and  $g_6$  are algebraically independent,  $R_j = 0$  follows. Thus we arrive at  $Q = 0 \in I$ . ■

Using lemma 3.8 we can choose Maaß-lifts  $E_k \in \mathcal{M}_{k,6,-i} \subset [\Gamma_3^*, k, 1]$  for  $k \in \{4, 6, 12\}$ , such that

$$W_3(E_k)(z_1, z_3) = g_k(z_1)g_k(z_3).$$

Especially,  $W_3(E_4)$ ,  $W_3(E_6)$  and  $W_3(E_{12})$  generate the ring  $[\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), 2\mathbb{Z}, 1]_{\mathrm{symm}}$  of symmetric modular forms of even weight on  $\mathbb{H}_1 \times \mathbb{H}_1$  (here symmetric means  $f(z_1, z_3) = f(z_3, z_1)$  for all  $(z_1, z_3) \in \mathbb{H}_1 \times \mathbb{H}_1$ ; see e.g. [11, III, Folg. 4.1]). Moreover these forms are algebraically independent on  $\mathbb{H}_1 \times \mathbb{H}_1$ . Note that, despite the notation,  $E_k$  is not necessarily an Eisenstein-series for  $\Gamma_3^*$  of Siegel-type. From dimension-formulas we will prove later, it can be seen, that in fact  $E_k$  is an Eisenstein-series (up to a non-zero factor) for  $k \in \{4, 6\}$ .

Now we choose a Maaß-lift  $f_4 \in \mathcal{M}_{4,2,-i} \subset [\Gamma_3^*, 4, \mu]$ , such that

$$W_3(f_4)(z_1, z_3) = \eta^8(z_1)\eta^8(z_3).$$

In fact,  $f_4$  is uniquely determined by this property, since  $\dim \mathcal{M}_{4,2,-i} = 1$ . Note, that if  $f \in \mathcal{M}_{4,2,-i}$ , then  $W_3(f) \in \mathbb{C}\eta^8(z_1)\eta^8(z_3)$  by lemma 5.1. Note too, that  $W_3(f) = 0$  implies  $f = 0$ : If  $W_3(f) = 0$ , we have  $f/\psi_1^2 \in [\Gamma_3^*, 2, 1]$  and  $W_3(f/\psi_1^2) = 0$  again, thus  $f/\psi_1^4 \in [\Gamma_3^*, 0, \mu^2] = \{0\}$  and  $f = 0$ .

Let  $\psi_6, \psi_{12}$  be the Borchers-products from section 4. Without loss of generality we can scale these products in such a way, that

$$W_3(\psi_6)(z_1, z_3) = \eta^{12}(z_1)\eta^{12}(z_2),$$

$$W_3(\psi_{12})(z_1, z_3) = \Delta_{12}(z_1)g_6^2(z_2) - g_6^2(z_1)\Delta_{12}(z_2).$$

Note that  $[\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), 12, 1]_{\mathrm{anti}}$  (the space of anti-symmetric modular forms of weight 12 on  $\mathbb{H}_1 \times \mathbb{H}_1$ ) has dimension 1 and is generated by  $\Delta_{12}(z_1)g_6(z_2)^2 - g_6(z_1)^2\Delta_{12}(z_2)$  (and 12 is the smallest weight  $k$  such that  $\dim[\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), k, 1]_{\mathrm{anti}} \neq 0$ ).

With the Maaß-lifts just chosen and the Borchers-products, constructed in section 4, we have found enough forms in order to generate the graded ring of paramodular forms for  $D\Gamma_3^*$ . We summarize the properties of these forms (the lower index always indicates the weight of the modular form):

$$E_k \in [\Gamma_3^*, k, 1] \quad \text{for } k = 4, 6, 12, \quad f_4 \in [\Gamma_3^*, 4, \mu],$$

$$\psi_1 \in [\Gamma_3^*, 1, \chi\kappa\mu^2], \quad \psi_6 \in [\Gamma_3^*, 6, \kappa], \quad \psi_{12} \in [\Gamma_3^*, 12, \chi].$$

For later use we define some forms (the lower index is the weight again):

$$h_6 = \psi_1^6 \in [\Gamma_3^*, 6, 1], \quad h_8 = \psi_1^4 f_4 \in [\Gamma_3^*, 8, 1], \quad h_{10} = \psi_1^2 f_4^2 \in [\Gamma_3^*, 10, 1],$$

$$h_9 = \psi_1^3 \psi_6 \in [\Gamma_3^*, 9, \chi], \quad h_{11} = \psi_1 \psi_6 f_4 \in [\Gamma_3^*, 11, \chi],$$

$$h_{21} = \psi_1^3 \psi_6 \psi_{12} \in [\Gamma_3^*, 21, 1], \quad h_{23} = \psi_1 f_4 \psi_6 \psi_{12} \in [\Gamma_3^*, 23, 1].$$

Now we state our main result.

**THEOREM 5.2.**

$$1) \quad [D\Gamma_3^*, \mathbb{Z}, 1] := \bigoplus_{k \in \mathbb{Z}} [D\Gamma_3^*, k, 1] = \mathbb{C}[f_4, \psi_6, \psi_{12}, \psi_1, E_4, E_6, E_{12}].$$

2)  $\psi_1, E_4, E_6$  and  $E_{12}$  are algebraically independent.

3) The generators of  $[D\Gamma_3^*, \mathbb{Z}, 1]$  satisfy relations of the form (with certain constants  $c_j \in \mathbb{C}$  and polynomials  $p_1, p_2 \in \mathbb{C}_4$  and  $p_3 \in \mathbb{C}_6$ )

$$f_4^3 - c_1 \psi_1^4 f_4 E_4 = p_1(\psi_1^6, E_4, E_6, E_{12}),$$

$$\psi_6^2 - c_2 \psi_1^4 f_4 E_4 = p_2(\psi_1^6, E_4, E_6, E_{12}),$$

$$\psi_{12}^2 = p_3(\psi_1^4 f_4, \psi_1^2 f_4^2, \psi_1^6, E_4, E_6, E_{12}).$$

4)  $[D\Gamma_3^*, \mathbb{Z}, 1] \cong \mathbb{C}_7/I$ , where  $I = (P_1, P_2, P_3) \subset \mathbb{C}_7$  is the ideal, generated by

$$P_1 = X_1^3 - c_1 X_4^4 X_1 X_5 - p_1(X_4^6, X_5, X_6, X_7),$$

$$P_2 = X_2^2 - c_2 X_4^4 X_1 X_5 - p_2(X_4^6, X_5, X_6, X_7) \quad \text{and}$$

$$P_3 = X_3^2 - p_3(X_4^4 X_1, X_4^2 X_1^2, X_4^6, X_5, X_6, X_7).$$

*Proof.* 1)  $\Gamma_3^*/D\Gamma_3^*$  acts on  $[D\Gamma_3^*, k, 1]$  as a group of commuting operators. The decomposition of  $[D\Gamma_3^*, k, 1]$  into the eigenspaces of these operators is

$$(5.2) \quad [D\Gamma_3^*, k, 1] = \bigoplus_{\nu \in \Gamma_3^{*ab}} [\Gamma_3^*, k, \nu].$$

Now let  $f \in [D\Gamma_3^*, k, 1]$ . Using the decomposition (5.2), we can write  $f = \sum_{\nu \in \Gamma_3^{*ab}} f_\nu$  with  $f_\nu \in [\Gamma_3^*, k, \nu]$ . Therefore we can assume  $f \in [\Gamma_3^*, k, \nu]$  without restriction.

Case 1:  $k \equiv 1 \pmod{2\mathbb{Z}}$ . Then  $W_3(f) = 0$  by (4.2) and  $f/\psi_1 \in [\Gamma_3^*, k-1, \nu\chi\kappa\mu]$ .

Case 2:  $k \equiv 0 \pmod{2\mathbb{Z}}$ ,  $\nu(V_3) = -1$ . Then  $f = 0$  on  $\lambda_{12}^\perp$  by (4.4), thus  $f/\psi_{12} \in [\Gamma_3^*, k-12, \nu\chi]$ .

Case 3:  $k \equiv 0 \pmod{2\mathbb{Z}}$ ,  $\nu = \kappa(\chi\mu)^j$  for some  $j \in \mathbb{Z}$ . Then  $f = 0$  on  $\lambda_{\frac{1}{4}}^\perp$  by (4.3), thus  $f/\psi_6 \in [\Gamma_3^*, k-6, \nu\kappa]$ .

Case 4:  $k \equiv 0 \pmod{2\mathbb{Z}}$ ,  $\nu = \mu^j$ ,  $j \in \{0, 1, 2\}$ . Now,  $W_3(f) \in [\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), k, v_\eta^{8j}]_{\mathrm{symm}}$ , i.e.  $W_3(f)$  is a modular form for  $\mathrm{SL}_2(\mathbb{Z})$  with multiplier-system  $v_\eta^{8j}$  in each of the two variables. Using lemma 5.1 we find  $W_3(f)(z_1, z_2) = \eta^{8j}(z_1)\eta^{8j}(z_2)h(z_1, z_2)$  with a symmetric modular form  $h \in [\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}), k - 4j, 1]_{\mathrm{symm}}$ . Thus there is a polynomial  $P$  such that

$$h(z_1, z_3) = P(g_4(z_1)g_4(z_3), g_6(z_1)g_6(z_3), g_{12}(z_1)g_{12}(z_3)).$$

This shows, that  $W_3(f)$  can be lifted as  $f_4^j P(E_4, E_6, E_{12})$  to a paramodular form on  $\mathbb{H}_2$ . Now we have  $f - f_4^j P(E_4, E_6, E_{12}) = 0$  on  $\lambda_1^\perp$  and  $(f - f_4^j P(E_4, E_6, E_{12}))/\psi_1 \in [\Gamma_3^*, k - 1, \chi\kappa\mu^{j+1}]$ .

2) Assume  $Q \in \mathbb{C}_4$  satisfies  $Q(\psi_1, E_4, E_6, E_{12}) = 0$  on  $\mathbb{H}_2$ . Write  $Q = \sum_{t \in \mathbb{N}_0} X_1^t R_t$ , where  $R_t \in \mathbb{C}[X_2, X_3, X_4]$  for all  $t$ . Restricting to  $\mathbb{H}_1 \times \mathbb{H}_1$ , we find

$$0 = W_3(Q(\psi_1, E_4, E_6, E_{12})) = R_0(W_3(E_4), W_3(E_6), W_3(E_{12})).$$

Since  $W_3(E_4)$ ,  $W_3(E_6)$  and  $W_3(E_{12})$  are algebraically independent (on  $\mathbb{H} \times \mathbb{H}$ ),  $R_0 = 0$  follows. Then  $Q = X_1(\sum_{t \in \mathbb{N}_0} X_1^t R_{t+1})$  is divisible by  $X_1$  and  $\tilde{Q} = \sum_{t \in \mathbb{N}_0} X_1^t R_{t+1}$  satisfies  $\tilde{Q}(\psi_1, E_4, E_6, E_{12}) = 0$  on  $\mathbb{H}_2$  again. Inductively  $R_t = 0$  for all  $t$ , and  $Q = 0$  altogether, follows. Therefore  $\psi_1, E_4, E_6$  and  $E_{12}$  are algebraically independent.

3) Obviously,  $f_4^3, \psi_6^2 \in [\Gamma_3^*, 12, 1]$  and  $\psi_{12}^2 \in [\Gamma_3^*, 24, 1]$ . From lemma 5.3, 1) it follows, that  $f_4^3, \psi_6^2$  and  $\psi_{12}^2$  are polynomials in  $h_8, h_{10}, h_6, E_4, E_6$  and  $E_{12}$ . Thus the claim is true for  $\psi_{12}^2$ . If  $P \in \mathbb{C}_6$  is a polynomial, such that  $P(h_8, h_{10}, h_6, E_4, E_6, E_{12}) \in [\Gamma_3^*, 12, 1]$ , then  $P$  is of the form  $P = cX_1X_4 + p$  with  $c \in \mathbb{C}$ ,  $p \in \mathbb{C}[X_3, X_4, X_5, X_6]$ , since all monomials of weight 12 containing  $h_8$  or  $h_{10}$  are given by  $h_8E_4$ . Therefore the polynomials for  $f_4^3$  and  $\psi_6^2$  are of the form given in the lemma.

4) The relations stated in 3) show  $P_j(f_4, \psi_6, \psi_{12}, \psi_1, E_4, E_6, E_{12}) = 0$  on  $\mathbb{H}_2$  for  $j = 1, 2, 3$ . We have to prove, that if  $Q \in \mathbb{C}_7$  is such that  $Q(f_4, \psi_6, \psi_{12}, \psi_1, E_4, E_6, E_{12}) = 0$  on  $\mathbb{H}_2$ , then  $Q \in I := (P_1, P_2, P_3)$ . Therefore assume  $Q(f_4, \dots, E_{12}) = 0$  on  $\mathbb{H}_2$ . Moreover, after reducing  $Q$  modulo  $I$ , we can assume  $\deg_3(Q) \leq 1$ ,  $\deg_2(Q) \leq 1$  and  $\deg_1(Q) \leq 2$ . Write  $Q = \sum_{t \in \mathbb{N}_0} X_4^t R_t$ , where  $R_t \in \mathbb{C}_7$  satisfies  $\deg_4(R_t) = 0$  for all  $t$  (i.e.  $X_4$  does not appear in  $R_t$ ). Now we show  $R_0 = 0$ . First write

$$R_0 = \sum_{\substack{0 \leq r_1 \leq 2, \\ 0 \leq r_2, r_3 \leq 1}} X_1^{r_1} X_2^{r_2} X_3^{r_3} U_{r_1 r_2 r_3} \quad \text{with} \quad U_{r_1 r_2 r_3} \in \mathbb{C}[X_5, X_6, X_7] \text{ for all } r_1, r_2, r_3.$$

Restricting to  $\lambda_1^\perp$ , we find (here  $f \otimes g$  is defined by  $(f \otimes g)(z_1, z_2) = f(z_1)g(z_2)$ )

$$\begin{aligned} 0 &= W_3(Q(f_4, \dots, E_{12})) \\ &= R_0(\eta^8 \otimes \eta^8, \eta^{12} \otimes \eta^{12}, W_3(\psi_{12}), 0, g_4 \otimes g_4, g_6 \otimes g_6, g_{12} \otimes g_{12}) \\ &= \sum_{0 \leq r_1 \leq 2, 0 \leq r_2, r_3 \leq 1} (\eta \otimes \eta)^{8r_1 + 12r_2} W_3(\psi_{12})^{r_3} U_{r_1 r_2 r_3} (g_4 \otimes g_4, g_6 \otimes g_6, g_{12} \otimes g_{12}) \end{aligned}$$

on  $\mathbb{H}_1 \times \mathbb{H}_1$ . Since the characters of the summands come from the factors  $(\eta \otimes \eta)^{8r_1+12r_2}$  and are therefore all different, and because each summand is  $(-1)^{r_3}$ -symmetric (under  $(z_1, z_4) \mapsto (z_4, z_1)$ ), the summands vanish one for one. Thus we have  $U_{r_1 r_2 r_3}(g_4 \otimes g_4, g_6 \otimes g_6, g_{12} \otimes g_{12}) = 0$  on  $\mathbb{H}_1 \times \mathbb{H}_1$  and, by the argument applied in the proof of 2),  $U_{r_1 r_2 r_3} = 0$  for all  $r_1, r_2, r_3$ . This proves  $R_0 = 0$ . Now  $Q = X_4(\sum_{t \in \mathbb{N}_0} X_4^t R_{t+1})$  follows, i.e.  $Q$  is divisible by  $X_4$ , and we can apply the same argument to  $Q/X_4$  again. Inductively we derive  $R_t = 0$  for all  $t$ . Therefore  $Q = 0 \in I$ , as had to be shown.

Since all generators for  $[D\Gamma_3^*, \mathbb{Z}, 1]$  are modular forms for  $\Gamma_3^*$  (with multiplier-systems), we can in principle find generators for the graded rings of paramodular forms for all groups  $\Gamma$  with  $D\Gamma_3^* \subset \Gamma \subset \Gamma_3^*$ . We give three examples. The following lemma will be useful for a reduction-process from forms on a 4-dimensional half-space, e.g. the hermitian half-space of degree 2. An analogous example was given by Freitag [11], where he used paramodular forms in  $[\Gamma_2^*, 2\mathbb{Z}, 1]$  to determine generators for the ring of symmetric Hermitian modular forms of degree 2 for  $\mathbb{Q}(\sqrt{-1})$ . Actually, this was our prime motivation for the present note and we hope to use the results presented here in order to find generators for the ring of Hermitian modular forms of degree 2 for  $\mathbb{Q}(\sqrt{-2})$  soon.

LEMMA 5.3.

$$1) \quad [\Gamma_3^*, 2\mathbb{Z}, 1] := \bigoplus_{k \in 2\mathbb{Z}} [\Gamma_3^*, k, 1] = \mathbb{C}[h_8, h_{10}, h_6, E_4, E_6, E_{12}],$$

2)  $h_6, E_4, E_6$  and  $E_{12}$  are algebraically independent.

3) The generators of  $[\Gamma_3^*, 2\mathbb{Z}, 1]$  satisfy the following relations (here  $c_1 \in \mathbb{C}$  and  $p_1 \in \mathbb{C}_4$  are the same as in theorem 5.2, 3))

$$\begin{aligned} h_8^2 &= h_6 h_{10}, \\ h_{10}^2 &= h_8(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})), \\ h_8 h_{10} &= h_6(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})). \end{aligned}$$

4)  $[\Gamma_3^*, 2\mathbb{Z}, 1] \cong \mathbb{C}_6/I$ , where  $I = (P_1, P_2, P_3) \subset \mathbb{C}_6$  is the ideal, generated by

$$\begin{aligned} P_1 &= X_1^2 - X_2 X_3, \\ P_2 &= X_2^2 - X_1(c_1 X_1 X_4 + p_1(X_3, X_4, X_5, X_6)) \quad \text{and} \\ P_3 &= X_1 X_2 - X_3(c_1 X_1 X_4 + p_1(X_3, X_4, X_5, X_6)). \end{aligned}$$

*Proof.* 1) Let  $f \in [\Gamma_3^*, k, 1]$  and  $k \equiv 0 \pmod{2\mathbb{Z}}$ . We apply the reduction process from lemma 5.2. First there is a polynomial  $Q_1 \in \mathbb{C}_3$ , such that  $f - Q_1(E_4, E_6, E_{12}) = 0$  along  $\lambda_1^\perp$  of order 2 (since  $k$  is even). Then we have  $g_1 = (f - Q_1(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-2, \mu_1^2]$ . Now there is a polynomial  $Q_2 \in \mathbb{C}_3$ , such that  $g_1 - f_4^2 Q_2(E_4, E_6, E_{12}) = 0$  along  $\lambda_1^\perp$  of order 2 again. Thus  $g_2 = (g_1 - Q_2(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-4, \mu_1]$  and once more, there is a polynomial  $Q_3 \in \mathbb{C}_3$ , such that  $g_2 - f_4 Q_3(E_4, E_6, E_{12}) = 0$  along  $\lambda_1^\perp$  of order 2. Finally  $g_3 = (g_2 - Q_3(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-6, 1]$  follows. Summarizing, we have

$$\begin{aligned}
f &= g_1 \psi_1^2 + P_1(E_4, E_6, E_{12}) = (g_2 \psi_1^2 + f_4^2 P_2(E_4, E_6, E_{12})) \psi_1^2 + P_1(E_4, E_6, E_{12}) \\
&= (g_3 \psi_1^2 + f_4 P_3(E_4, E_6, E_{12})) \psi_1^4 + \psi_1^2 f_4^2 P_2(E_4, E_6, E_{12}) + P_1(E_4, E_6, E_{12}) \\
&= g_3 \psi_1^6 + f_4 \psi_1^4 P_3(E_4, E_6, E_{12}) + f_4^2 \psi_1^2 P_2(E_4, E_6, E_{12}) + P_1(E_4, E_6, E_{12}).
\end{aligned}$$

Inductively  $f \in \mathbb{C}[\psi_1^6, \psi_1^4 f_4, \psi_1^2 f_4^2, E_4, E_6, E_{12}]$  follows.

2) follows from theorem 5.2, 2).

3) The first relation follows directly from the definition of the  $h_j$ 's. For the other two relations remember  $f_4^3 = c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})$  from theorem 5.2.

4) The relations stated in 3) show  $P_j(h_8, h_{10}, h_6, E_4, E_6, E_{12}) = 0$  for  $j = 1, 2, 3$ . Now assume that  $Q \in \mathbb{C}_6$  satisfies  $Q(h_8, h_{10}, h_6, E_4, E_6, E_{12}) = 0$  on  $\mathbb{H}_2$ . Reducing  $Q$  modulo  $I$ , we can assume that the degree of  $Q$  as a polynomial in  $X_1$  and  $X_2$  is 1 at most, i.e.  $Q$  is of the form

$$Q = U_0 + X_1 U_1 + X_2 U_2 \quad \text{with} \quad U_j \in \mathbb{C}[X_3, X_4, X_5, X_6] \text{ for } j = 0, 1, 2.$$

Write  $U_j = \sum_{k \geq 0} X_3^k R_{j,k}$  with  $R_{j,k} \in \mathbb{C}[X_4, X_5, X_6]$ ,  $j = 0, 1, 2, k \in \mathbb{N}_0$ . We show, that  $R_{0,0} = R_{1,0} = R_{2,0} = 0$ . Note that  $h_8$  and  $h_{10}$  vanish along  $\lambda_1^\perp$  of order 4 and 2 resp. Thus by restriction to  $\lambda_1^\perp$  we have

$$\begin{aligned}
0 &= W_3(Q(h_8, h_{10}, h_6, E_4, E_6, E_{12})) = W_3(U_0(h_6, E_4, E_6, E_{12})) \\
&= R_{0,0}(W_3(E_4), W_3(E_6), W_3(E_{12}))
\end{aligned}$$

on  $\mathbb{H}_1^2$ . This implies  $R_{0,0} = 0$ . Now, on  $\mathbb{H}_2$ , we have

$$\begin{aligned}
0 &= \sum_{k \geq 1} h_6^k R_{0,k}(E_4, E_6, E_{12}) + \sum_{k \geq 0} h_8 h_6^k R_{1,k}(E_4, E_6, E_{12}) \\
&\quad + \sum_{k \geq 0} h_{10} h_6^k R_{2,k}(E_4, E_6, E_{12}) \\
&= \psi_1^2 \left( \psi_1^4 \sum_{k \geq 0} h_6^k R_{0,k+1}(E_4, E_6, E_{12}) + \sum_{k \geq 0} \psi_1^2 f_4^2 h_6^k R_{1,k}(E_4, E_6, E_{12}) \right. \\
&\quad \left. + \sum_{k \geq 0} f_4^2 h_6^k R_{2,k}(E_4, E_6, E_{12}) \right).
\end{aligned}$$

Now the term in the brackets has to vanish on  $\mathbb{H}_2$  and by restriction to  $\lambda_1^\perp$  we get  $0 = W_3(f_4)^2 R_{2,0}(W_3(E_4), W_3(E_6), W_3(E_{12}))$  on  $\mathbb{H}_1^2$ . Since  $f_4$  does not vanish along  $\lambda_1^\perp$ , this implies  $R_{2,0} = 0$ . Now, on  $\mathbb{H}_2$ , we have

$$\begin{aligned}
0 &= \sum_{k \geq 1} h_6^k R_{0,k}(E_4, E_6, E_{12}) + \sum_{k \geq 0} h_8 h_6^k R_{1,k}(E_4, E_6, E_{12}) \\
&\quad + \sum_{k \geq 1} h_{10} h_6^k R_{2,k}(E_4, E_6, E_{12}) \\
&= \psi_1^4 \left( \psi_1^2 \sum_{k \geq 0} h_6^k R_{0,k+1}(E_4, E_6, E_{12}) + \sum_{k \geq 0} f_4^2 h_6^k R_{1,k}(E_4, E_6, E_{12}) \right. \\
&\quad \left. + \sum_{k \geq 0} f_4^2 \psi_1^4 h_6^k R_{2,k+1}(E_4, E_6, E_{12}) \right).
\end{aligned}$$

Again the term in the brackets has to vanish on  $\mathbb{H}_2$  and restriction to  $\lambda_1^\perp$  leads to  $0 = W_3(f_4)^2 R_{1,0}(W_3(E_4), W_3(E_6), W_3(E_{12}))$  this time. As before  $R_{1,0} = 0$  follows. Altogether we see, that

$$Q = X_3 \left( \sum_{k \geq 0} X_3^k R_{0,k+1} + X_1 \sum_{k \geq 0} X_3^k R_{1,k+1} + X_2 \sum_{k \geq 0} X_3^k R_{2,k+1} \right)$$

is divisible by  $X_3$ . Now the same argument can be applied to  $Q/X_3$  again. Inductively,  $R_{j,k} = 0$  for all  $j, k$  follows. Therefore  $Q = 0$  and  $Q \in I$  is proved.  $\blacksquare$

LEMMA 5.4.

$$1) \quad [\Gamma_3^*, \mathbb{Z}, 1] := \bigoplus_{k \in \mathbb{Z}} [\Gamma_3^*, k, 1] = \mathbb{C}[h_{21}, h_{23}, h_8, h_{10}, h_6, E_4, E_6, E_{12}],$$

2) *The generators of  $[\Gamma_3^*, \mathbb{Z}, 1]$  satisfy the following relations (here  $c_j \in \mathbb{C}$ ,  $p_1, p_2 \in \mathbb{C}_4$  and  $p_3 \in \mathbb{C}_6$  are as in theorem 5.2, 3))*

$$\begin{aligned}
h_8^2 &= h_6 h_{10}, \\
h_{10}^2 &= h_8(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})), \\
h_8 h_{10} &= h_6(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})), \\
h_{21}^2 &= h_6(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12})) p_3(h_8, h_{10}, h_6, E_4, E_6, E_{12}), \\
h_{23}^2 &= h_{10}(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12})) p_3(h_8, h_{10}, h_6, E_4, E_6, E_{12}), \\
h_{21} h_{23} &= h_8(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12})) p_3(h_8, h_{10}, h_6, E_4, E_6, E_{12}), \\
h_{21} h_8 &= h_{23} h_6, \\
h_{21} h_{10} &= h_{23} h_8, \\
h_{23} h_{10} &= h_{21}(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})).
\end{aligned}$$

3)  $[\Gamma_3^*, \mathbb{Z}, 1] \cong \mathbb{C}_8/I$ , where  $I = (P_j \mid j = 1, \dots, 9) \subset \mathbb{C}_8$  is the ideal, generated by



$$\begin{aligned}
P_1 &= X_3^2 - X_4X_5, \\
P_2 &= X_4^2 - X_3(c_1X_3X_6 + p_1(X_5, X_6, X_7, X_8)), \\
P_3 &= X_3X_4 - X_5(c_1X_3X_6 + p_1(X_5, X_6, X_7, X_8)), \\
P_4 &= X_1^2 - p_4(X_3, X_4, X_5, X_6, X_7, X_8)X_5, \\
P_5 &= X_2^2 - p(X_3, X_4, X_5, X_6, X_7, X_8)X_4, \\
P_6 &= X_1X_2 - p(X_3, X_4, X_5, X_6, X_7, X_8)X_3, \\
P_7 &= X_1X_3 - X_2X_5, \\
P_8 &= X_1X_4 - X_2X_3, \\
P_9 &= X_2X_4 - X_1(c_1X_3X_6 + p_1(X_5, X_6, X_7, X_8)),
\end{aligned}$$

with  $P_4 = (c_2X_3X_6 + p_2(X_5, X_6, X_7, X_8)) \cdot p_3(X_5, X_6, X_7, X_8)$ .

*Proof.* 1) Let  $f \in [\Gamma_3^*, k, 1]$ . If  $k \equiv 0 \pmod{2\mathbb{Z}}$ , then  $f \in \mathbb{C}[h_8, h_{10}, h_6, E_4, E_6, E_{12}]$  by lemma 5.3. Therefore assume  $k \equiv 1 \pmod{2\mathbb{Z}}$ . The reduction-process from lemma 5.2 now leads to  $f/\psi_1\psi_6\psi_{12} \in [\Gamma_3^*, k - 19, \mu]$  and the existence of  $P \in \mathbb{C}_3$ , such that  $f/\psi_1\psi_6\psi_{12} - f_4P(E_4, E_6, E_{12})$  vanishes along  $\lambda_1^\perp$ . Now  $\tilde{f} = (f/\psi_1\psi_6\psi_{12} - f_4P(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k - 23, 1]$  follows. We arrive at

$$f = \psi_1\psi_6\psi_{12}(\psi_1^2\tilde{f} + f_4P(E_4, E_6, E_{12})) \in h_{21}[\Gamma_3^*, k - 21, 1] + h_{23}[\Gamma_3^*, k - 23, 1].$$

Since  $k - 21$  and  $k - 23$  are even, lemma 5.3 again implies the claim.

2) The relations involving  $h_8^2, h_{10}^2$  and  $h_8h_{10}$  are the same as in lemma 5.3. The remaining relations follow from the definitions of the  $h_k$  together with the relations from theorem 5.2.

3) The relations stated in 2) show  $P_j(h_{21}, h_{23}, h_8, h_{10}, h_6, E_4, E_6, E_{12}) = 0$  for  $j = 1, \dots, 9$ . Now assume that  $Q \in \mathbb{C}_8$  satisfies  $Q(h_{21}, h_{23}, h_8, h_{10}, h_6, E_4, E_6, E_{12}) = 0$  on  $\mathbb{H}_2$ . Reducing  $Q$  modulo  $I$ , we can assume that the degree of  $Q$  as a polynomial in  $X_1, \dots, X_4$  is 2 at most. Furthermore all terms of degree 2 in  $X_1, \dots, X_4$  can be reduced to  $X_2X_3$ . Then  $Q$  is of the form (here we set  $X_0 = 1$ )

$$Q = \sum_{0 \leq j \leq 4} X_j U_j + X_2 X_3 U_{23} \quad \text{with} \quad U_j \in \mathbb{C}[X_5, X_6, X_7, X_8] \text{ for } j = 0, \dots, 4, 23.$$

Write  $U_j = \sum_{k \geq 0} X_5^k R_{j,k}$  with  $R_{j,k} \in \mathbb{C}[X_6, X_7, X_8]$ ,  $j = 0, \dots, 4, 23$ ,  $k \in \mathbb{N}_0$ . We show, that  $R_{j,0} = 0$  for  $j = 0, \dots, 4, 23$ . Note that  $1, h_{21} = \psi_1^3\psi_6\psi_{12}, h_{23} = \psi_1 f_4\psi_6\psi_{12}, h_8 = \psi_1^4 f_4, h_{10} = \psi_1^2 f_4^2$  and  $h_{23}h_8 = \psi_1^5 f_4^2\psi_6\psi_{12}$  vanish along  $\lambda_1^\perp$  of order 0, 3, 1, 4, 2, 5 resp. Since these orders are all different, we can proceed exactly as in the proof of lemma 5.3, i.e. by extracting the highest power of  $\psi_1$  and restriction to  $\lambda_1^\perp$  of  $Q(h_{21}, \dots, E_{12})$ , successively  $R_{0,0} = R_{2,0} = R_{4,0} = R_{1,0} = R_{3,0} = R_{5,0} = 0$  follows. Altogether we see, that  $Q$  is divisible by  $X_5$  and the same argument can be applied to  $Q/X_5$  again. Inductively,  $R_{j,k} = 0$  for all  $j, k$  follows. Therefore  $Q = 0$  and  $Q \in I$  is proved. ■

Since the dimensions  $\dim[\Gamma_3, k, 1]$  are known by results of Ibukiyama ([17], [18]), we determine generators for  $[\Gamma_3, \mathbb{Z}, 1]$  in order to compare the dimension formulas. Note that  $\Gamma_3 = \ker(\chi)$ .

LEMMA 5.5.

$$1) \quad [\Gamma_3, \mathbb{Z}, 1] := \bigoplus_{k \in \mathbb{Z}} [\Gamma_3, k, 1] = \mathbb{C}[h_9, h_{11}, h_8, h_{10}, \psi_{12}, h_6, E_4, E_6, E_{12}],$$

2) *The generators of  $[\Gamma_3, \mathbb{Z}, 1]$  satisfy the following relations (here  $c_j \in \mathbb{C}$ ,  $p_1, p_2 \in \mathbb{C}_4$  and  $p_3 \in \mathbb{C}_6$  are as in theorem 5.2, 3) again)*

$$\begin{aligned} h_8^2 &= h_6 h_{10}, \\ h_{10}^2 &= h_8(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})), \\ h_8 h_{10} &= h_6(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})), \\ h_9^2 &= h_6(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12})), \\ h_{11}^2 &= h_{10}(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12})), \\ h_9 h_{11} &= h_8(c_2 h_8 E_4 + p_2(h_6, E_4, E_6, E_{12})), \\ h_9 h_8 &= h_{11} h_6, \\ h_9 h_{10} &= h_{11} h_8, \\ h_{11} h_{10} &= h_9(c_1 h_8 E_4 + p_1(h_6, E_4, E_6, E_{12})), \\ \psi_{12}^2 &= p_3(h_8, h_{10}, h_6, E_4, E_6, E_{12}). \end{aligned}$$

3)  $[\Gamma_3, \mathbb{Z}, 1] \cong \mathbb{C}_9/I$ , where  $I = (P_j \mid j = 1, \dots, 10) \subset \mathbb{C}_9$  is the ideal, generated by

$$\begin{aligned} P_1 &= X_3^2 - X_4 X_6, \\ P_2 &= X_4^2 - X_3(c_1 X_3 X_7 + p_1(X_6, X_7, X_8, X_9)), \\ P_3 &= X_3 X_4 - X_6(c_1 X_3 X_7 + p_1(X_6, X_7, X_8, X_9)), \\ P_4 &= X_1^2 - X_6(c_2 X_3 X_7 + p_2(X_6, X_7, X_8, X_9)), \\ P_5 &= X_2^2 - X_4(c_2 X_3 X_7 + p_2(X_6, X_7, X_8, X_9)), \\ P_6 &= X_1 X_2 - X_3(c_2 X_3 X_7 + p_2(X_6, X_7, X_8, X_9)), \\ P_7 &= X_1 X_3 - X_2 X_6, \\ P_8 &= X_1 X_4 - X_2 X_3, \\ P_9 &= X_2 X_4 - X_1(c_1 X_3 X_7 + p_1(X_6, X_7, X_8, X_9)), \\ P_{10} &= X_5^2 - p_3(X_3, X_4, X_6, X_7, X_8, X_9). \end{aligned}$$

*Proof.* 1) Let  $f \in [\Gamma_3, k, 1]$ . We can decompose  $f$  into eigenfunctions of  $V_3$ , i.e.  $f = f_1 + f_\chi$  with  $f_\nu \in [\Gamma_3^*, k, \nu]$ . Thus without restriction assume  $f \in [\Gamma_3^*, k, \nu]$ ,

$v \in \{1, \chi\}$ . By lemma 5.3 it is sufficient to show, that  $f$  is a polynomial in  $h_9, h_{11}$  and  $\psi_{12}$  over  $[\Gamma_3^*, 2\mathbb{Z}, 1]$ . In order to prove this, we apply the reduction process from theorem 5.2.

Case 1:  $k \equiv 0 \pmod{2\mathbb{Z}}, v = 1$ . Then  $f \in [\Gamma_3^*, 2\mathbb{Z}, 1]$ .

Case 2:  $k \equiv 0 \pmod{2\mathbb{Z}}, v = \chi$ . Then  $f/\psi_{12} \in [\Gamma_3^*, k-12, 1]$  and  $f \in \psi_{12}[\Gamma_3^*, 2\mathbb{Z}, 1]$ .

Case 3:  $k \equiv 1 \pmod{2\mathbb{Z}}, v = 1$ . Then  $f/\psi_1\psi_6\psi_{12} \in [\Gamma_3^*, k-19, \mu]$  and there exists  $P \in \mathbb{C}_3$ , such that  $f/\psi_1\psi_6\psi_{12} - f_4P(E_4, E_6, E_{12})$  vanishes along  $\lambda_1^\perp$  of order 2. Now  $\tilde{f} = (f/\psi_1\psi_6\psi_{12} - f_4P(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-21, 1]$  and  $f = \psi_1^3\psi_6\psi_{12}\tilde{f} + \psi_1\psi_6\psi_{12}f_4P(E_4, E_6, E_{12}) \in h_9\psi_{12}[\Gamma_3^*, 2\mathbb{Z}, 1] + h_{11}\psi_{12}[\Gamma_3^*, 2\mathbb{Z}, 1]$  follows.

Case 4:  $k \equiv 1 \pmod{2\mathbb{Z}}, v = \chi$ . Then  $f/\psi_1\psi_6 \in [\Gamma_3^*, k-7, \mu]$  and there exists  $P \in \mathbb{C}_3$ , such that  $f/\psi_1\psi_6 - f_4P(E_4, E_6, E_{12})$  vanishes along  $\lambda_1^\perp$  of order 2. Now  $\tilde{f} = (f/\psi_1\psi_6 - f_4P(E_4, E_6, E_{12}))/\psi_1^2 \in [\Gamma_3^*, k-9, 1]$  and  $f = \psi_1^3\psi_6\tilde{f} + \psi_1\psi_6f_4P(E_4, E_6, E_{12}) \in h_9[\Gamma_3^*, 2\mathbb{Z}, 1] + h_{11}[\Gamma_3^*, 2\mathbb{Z}, 1]$  follows.

2) As in the previous cases (lemma 5.3, lemma 5.4), all relations follow from the definitions of the  $h_j$ 's and the relations, stated in theorem 5.2.

3) The relations stated in 2) show  $P_j(h_9, h_{11}, h_8, h_{10}, \psi_{12}, h_6, E_4, E_6, E_{12}) = 0$  for  $j = 1, \dots, 10$ . Now assume  $Q \in \mathbb{C}_9$  satisfies  $Q(h_9, h_{11}, h_8, h_{10}, \psi_{12}, h_6, E_4, E_6, E_{12}) = 0$  on  $\mathbb{H}_2$ . If we reduce  $Q$  modulo  $(P_{10})$ , we can assume that  $\deg_5(Q) \leq 1$ . Note that  $X_5$  does not appear in any of the other generators of  $I$ . If we reduce  $Q$  modulo  $(P_1, \dots, P_9)$ , we can assume, that the degree of  $Q$  as a polynomial in  $X_1, \dots, X_4$  is 2 at most. Furthermore all terms of degree 2 in  $X_1, \dots, X_4$  can be reduced to  $X_2X_3$ . Then  $Q$  is of the form (we set  $X_0 = 1$  as before)

$$Q = \sum_{0 \leq l \leq 1} X_5^l \left( \sum_{0 \leq j \leq 4} X_j U_{j,l} + X_2 X_3 U_{23,l} \right) \quad \text{with} \quad U_{j,l} \in \mathbb{C}[X_6, X_7, X_8, X_9]$$

for  $l = 1, 2$ , and  $j = 0, \dots, 4, 23$ . Write  $U_{j,l} = \sum_{k \geq 0} X_6^k R_{j,l,k}$  with  $R_{j,l,k} \in \mathbb{C}[X_7, X_8, X_9]$ , for  $l = 1, 2, j = 0, \dots, 4, 23, k \in \mathbb{N}_0$ . We show, that  $R_{j,l,0} = 0$  for  $l = 1, 2, j = 0, \dots, 4, 23$ . Principally, we can now proceed as in lemma 5.4, since  $1, h_9 = \psi_1^3\psi_6, h_{11} = \psi_1 f_4\psi_6, h_8 = \psi_1^4 f_4, h_{10} = \psi_1^2 f_4^2$  and  $h_{23}h_8 = \psi_1^5 f_4^2\psi_6\psi_{12}$  again vanish along  $\lambda_1^\perp$  of order 0, 3, 1, 4, 2, 5 resp.. The only difference is, that we now always get a sum of two terms (since  $Q$  is linear in  $X_5$  by assumption)

$$R_{j,0,0}(W_3(E_4), W_3(E_6), W_3(E_{12})) + W_3(\psi_{12})R_{j,1,0}(W_3(E_4), W_3(E_6), W_3(E_{12})),$$

which has to vanish on  $\mathbb{H}_1^2$ . But since  $W_3(\psi_{12})$  is antisymmetric (under  $(z_1, z_3) \mapsto (z_1, z_3)$ ), whereas all  $W_3(E_k)$  are symmetric, the summands have to vanish one for one. The proof now runs as before again, i.e. since  $R_{j,l,0} = 0$  for all  $j, l$ ,  $Q$  is divisible by  $X_6$  and, inductively,  $Q = 0$  as well as  $Q \in I$  follows.

From theorem 5.2, lemma 5.4 and lemma 5.5 we can deduce the generating functions for the dimensions  $\dim[\Gamma, k, 1]$  for  $\Gamma = D\Gamma_3^*, \Gamma_3^*$  and  $\Gamma_3$ . We find

COROLLARY 5.6.

$$\begin{aligned} \sum_{k \in \mathbb{N}_0} \dim [D\Gamma_3^*, k, 1] t^k &= \frac{(1+t^4+t^8)(1+t^6)(1+t^{12})}{(1-t)(1-t^4)(1-t^6)(1-t^{12})}, \\ \sum_{k \in \mathbb{N}_0} \dim [\Gamma_3^*, k, 1] t^k &= \frac{1+t^8+t^{10}+t^{21}+t^{23}+t^{31}}{(1-t^4)(1-t^6)^2(1-t^{12})}, \\ \sum_{k \in \mathbb{N}_0} \dim [\Gamma_3, k, 1] t^k &= \frac{(1+t^{12})(1+t^8+t^9+t^{10}+t^{11}+t^{19})}{(1-t^4)(1-t^6)^2(1-t^{12})}, \end{aligned}$$

In the case of  $\Gamma_3$ , the same generating function was already deduced by Ibukiyama [18] from his general formula for  $\dim [\Gamma_t, k, 1]_{\text{cusp}}$  for  $k \geq 5$ , given in [17].

Modular functions on  $\Gamma_3^* \backslash \mathbb{H}_2$  are quotients of modular forms. Thus lemma 5.4 allows us to determine the function-field of the Satake-compactification  $\overline{\Gamma_3^* \backslash \mathbb{H}_2}$  of  $\Gamma_3^* \backslash \mathbb{H}_2$ . Let

$$\mathcal{K} = \mathbb{C} \left[ \frac{h_8}{E_4^2}, \frac{h_6}{E_6}, \frac{h_6^2}{E_4^3} \right].$$

From the relations in lemma 5.4, 3) we deduce

$$\left( \frac{h_8}{E_4^2} \right)^2 = \frac{h_6 h_{10}}{E_4^4} = \frac{h_6^2 E_6}{E_4^3 h_6} \frac{h_{10}}{E_4 E_6} \in \mathcal{K}$$

and therefore  $\frac{h_{10}}{E_4 E_6} \in \mathcal{K}$ . The polynomial  $p_1$  in theorem 5.2, 3) satisfies  $p_1(h_6, E_4, E_6, E_{12}) \in [\Gamma_3^*, 12, 1]$ . This implies

$$p_1(X_1, X_2, X_3, X_4) = c_{11} X_1^2 + c_{12} X_1 X_3 + c_{13} X_3^2 + c_{14} X_2^3 + c_{15} X_4$$

for some constants  $c_j \in \mathbb{C}$ . Recall that  $p_1$  was chosen in such a way that  $f_4^3 - c_1 h_8 E_4 = p_1(h_6, E_4, E_6, E_{12})$ . Restricting to  $\lambda_1^\perp$  we find

$$\begin{aligned} W_3(f_4^3)(z_1, z_3) &= (\eta_8(z_1)\eta_8(z_3))^3 = \Delta_{12}(z_1)\Delta_{12}(z_3) \\ &= c_{13} (g_6(z_1)g_6(z_3))^2 + c_{14} (g_4(z_1)g_4(z_3))^3 + c_{15} g_{12}(z_1)g_{12}(z_3) \end{aligned}$$

(because of  $W_3(h_6) = W_3(h_8) = 0$ ). This implies  $c_{15} \neq 0$ , since  $\Delta_{12} \otimes \Delta_{12} \notin \mathbb{C}[g_4 \otimes g_4, g_6 \otimes g_6]$ . Using the relation for  $h_{10}^2$  in lemma 5.4, 3) we deduce

$$\begin{aligned} \left( \frac{h_{10}}{E_4 E_6} \right)^2 &= \frac{c_1 E_4 h_8^2 + h_8 (c_{11} h_6^2 + c_{12} h_6 E_6 + c_{13} E_6^2 + c_{14} E_4^3 + c_{15} E_{12})}{E_4^2 E_6^2} \\ &= c_1 \left( \frac{h_8}{E_4^2} \right)^2 \frac{E_4^3}{h_6^2} \left( \frac{h_6}{E_6} \right)^2 + c_{11} \frac{h_8}{E_4^2} \left( \frac{h_6}{E_6} \right)^2 + c_{12} \frac{h_8}{E_4^2} \frac{h_6}{E_6} + c_{13} \frac{h_8}{E_4^2} \\ &\quad + c_{14} \frac{h_8}{E_4^2} \frac{E_4^3}{h_6^2} \left( \frac{h_6}{E_6} \right)^2 + c_{15} \frac{h_8}{E_4^2} \frac{E_{12}}{E_6^2}. \end{aligned}$$

Therefore  $\frac{E_{12}}{E_6^2} \in \mathcal{K}$ . Now we can prove

COROLLARY 5.7.  $\overline{\Gamma_3^* \setminus \mathbb{H}_2}$  is a rational variety. The function-field is given by  $\mathbb{C} \left[ \frac{h_8}{E_4^2}, \frac{h_6}{E_6}, \frac{h_6^2}{E_4^3} \right]$ .

*Proof.* Let  $\mathcal{K} = \mathbb{C} \left[ \frac{h_8}{E_4^2}, \frac{h_6}{E_6}, \frac{h_6^2}{E_4^3} \right]$  as above. Assume  $f$  is a modular function on  $\Gamma_3^* \setminus \mathbb{H}_2$ . Then  $f = g/g'$  with  $g, g' \in [\Gamma_3^*, k, 1]$  is a quotient of modular forms of some weight  $k \in \mathbb{N}$ . We may assume  $k \in 4\mathbb{N}$  (since  $g/g' = g^4/g^3 g'$  as modular functions). In this case, every monomial  $h_6^{n_1} E_4^{n_2} E_6^{n_3} E_{12}^{n_4}$  of weight  $k = 6n_1 + 4n_2 + 6n_3 + 12n_4$  satisfies

$$E_4^{-k/4} h_6^{n_1} E_4^{n_2} E_6^{n_3} E_{12}^{n_4} = \left( \frac{h_6}{E_6} \right)^{n_1} \left( \frac{E_{12}}{E_6^2} \right)^{n_4} \left( \frac{E_6^2}{E_4^3} \right)^{(n_3+n_1)/2+n_4} \in \mathcal{K}.$$

Since  $k$  is even, lemma 5.4, 3) implies, that  $g$  is of the form  $g = U_0 + h_8 U_1 + h_{10} U_3$  with polynomials  $U_j \in \mathbb{C}[h_6, E_4, E_6, E_{12}]$ . Therefore  $E_4^{-k/4} U_0, E_4^{-k/4} (E_4^2 U_1), E_4^{-k/4} (E_4 E_6 U_3) \in \mathcal{K}$  and

$$E_4^{-k/4} g = E_4^{-k/4} U_0 + \frac{h_8}{E_4^2} E_4^{2-k/4} U_1 + \frac{h_{10}}{E_4 E_6} E_4^{1-k/4} E_6 U_3 \in \mathcal{K}$$

follows from the remarks above. Since the same argument applies to  $g'$ , we find  $f = g/g' = E_4^{-k/4} g / E_4^{-k/4} g' \in \mathcal{K}$ .

Since the field of modular functions on  $\overline{\Gamma_3^* \setminus \mathbb{H}_2}$  has transcendence-degree 3, the generators given in the corollary have to be algebraically independent. ■

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