Extensions of Herglotz’ Theorem

by

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1. Introduction

Since we are interested exclusively in abelian (i.e., commutative) *-semigroups then by a *-semigroup we understand an abelian semigroup S equipped with an involution, that is, a self-map \( s \mapsto s^* \) such that \((s^*)^* = s\) and \((st)^* = t^*s^* \) for all \( s, t \in S \). Suppose \( S \) is a *-semigroup. For subsets \( H \) and \( K \) of \( S \), define \( HK = \{ xy, x \in H, y \in K \} \), abbreviated

\[ aK \text{ (resp. } Ha) \text{ in case } H \text{ (resp. } K) \text{ is } \{ a \} \text{ for some } a. \]

For \( n \in \mathbb{N} \), define \( S \cdots S \) inductively by \( S \cdots S = S \) and \( S \cdots S = S \cdots S \) if \( n > 1 \). A function \( \psi : SS \to C \) is positive definite if \( \sum_{j,k=1}^n c_j \overline{c_k} \psi(s_j^* s_k) \geq 0 \) for all \( n \in \mathbb{N}, s_1, \ldots, s_n \in S \), and \( c_1, \ldots, c_n \in C \). Denote by \( \mathcal{P}(S) \) the set of all positive definite functions on \( S \) (i.e., defined on \( SS \)). If \( S \) is commutative and has an identity \( 1 \) (an element \( 1 \) such that \( 1s = s = s1 \) for all \( s \in S \), which implies \( 1^* = 1 \)) then a function \( \psi : SS \to C \) is called completely positive definite if for each \( r \in S \) the function \( s \mapsto \psi(rs) \) is in \( \mathcal{P}(S) \). Each completely positive definite function \( \psi \) on \( S \) is nonnegative since if \( r \in S \) then the function \( s \mapsto \psi(rs) \), being positive definite, is nonnegative at \( 1 \). The set of all completely positive definite functions on \( S \) is denoted by \( \mathcal{P}_c(S) \).

A character on \( S \) is a nonzero function \( \sigma : S \to C \) such that \( \sigma(s^*) = \overline{\sigma(s)} \) and \( \sigma(st) = \sigma(s)\sigma(t) \) for all \( s, t \in S \). Let \( S^* \) be the set of all characters on \( S \), and \( S^*_\text{c} \) the subset of nonnegative characters.

Consider a subset \( \Gamma \) of \( S^* \). Let \( \mathcal{A}(\Gamma) \) be the least \( \sigma \)-ring of subsets of \( \Gamma \) rendering measurable (in the sense of Halmos [15]) for each \( s \in S \) the function \( \hat{s} : \Gamma \to C \) defined by \( \hat{s}(\sigma) = \sigma(s) \) for \( \sigma \in \Gamma \). Clearly, if \( \Gamma \) and \( \Delta \) are subsets of \( S^* \) such that \( \Gamma \subset \Delta \) then \( \mathcal{A}(\Gamma) = \{ A \cap \Gamma, A \in \mathcal{A}(\Delta) \} \). Let \( F_+(\Gamma) \) be the set of those measures on \( \mathcal{A}(\Gamma) \) that integrate \( |\hat{s}|^2 \) for all \( s \in S \). Clearly, if \( \Gamma \) and \( \Delta \) are subsets of \( S^* \) such that \( \Gamma \subset \Delta \) and if \( \mu \in F_+(\Gamma) \) then the measure \( A \mapsto \mu(A \cap \Gamma) \) on \( \mathcal{A}(\Delta) \) is in \( F_+(\Delta) \). We denote by \( F(\Gamma) \) the complex linear hull of \( F_+(\Gamma) \). This is a set of complex measures defined on a suitable subring of \( \mathcal{A}(\Gamma) \). Each of them has the form \( \mu_1 - \mu_2 + i(\mu_3 - \mu_4) \) for some \( \mu_i \in F_+(\Gamma) \) \((i = 1, 2, 3, 4)\).
For \( \mu \in F(\Gamma) \) define a function \( \mathcal{L} \mu : SS \to C \) by

\[
\mathcal{L} \mu (s) = \int_\Gamma \sigma (s) \, d\mu (\sigma )
\]

for \( s \in SS \). The integral exists by Hölder’s inequality. A function \( \varphi : SS \to C \) is a \( \Gamma \)-moment function (called a moment function if \( \Gamma = S^* \), and a Stieltjes moment function if \( \Gamma = S^+_1 \)) if \( \varphi = \mathcal{L} \mu \) for some \( \mu \in F_+(\Gamma) \), and a \( \Gamma \)-moment function \( \varphi \) is \( \Gamma \)-determinate (called determinate if \( \Gamma = S^* \), and Stieltjes determinate if \( \Gamma = S^+_1 \)) if there is only one such \( \mu \). Let \( \mathcal{H}_S(\Gamma) \) (written \( \mathcal{H}(\mathcal{S}) \) if \( \Gamma = S^* \), and \( \mathcal{H}_S(S) \) if \( \Gamma = S^+_1 \)) be the set of all \( \Gamma \)-moment functions, and \( \mathcal{H}_{S, \det}(\Gamma) \) (written \( \mathcal{H}_{\det}(\mathcal{S}) \) if \( \Gamma = S^* \), and \( \mathcal{H}_{S, \det}(S) \) if \( \Gamma = S^+_1 \)) the subset of \( \Gamma \)-determinate ones. We have \( \mathcal{H}_{S, \det}(\Gamma) \subset \mathcal{H}(S) \subset \mathcal{P}(S) \) since if \( \mu \in F_+(\Gamma) \), \( n \in \mathbb{N}, s_1, \ldots, s_n \in S \), and \( c_1, \ldots, c_n \in C \) then

\[
\sum_{j,k=1}^n \mathcal{L} \mu (s_j^* s_j) = \int_\Gamma \left( \sum_{j=1}^n c_j \sigma (s_j) \right)^2 \, d\mu (\sigma ) \geq 0.
\]

If \( \Gamma \subset S^+_1 \), then a similar computation shows \( \sum_{j,k=1}^n \mathcal{L} \mu (s_j^* r s_j) \geq 0 \), so \( \mathcal{H}_S(S) \subset \mathcal{P}_S(S) \).

The \( * \)-semigroup \( S \) is called semiperfect if \( \mathcal{H}(S) = \mathcal{P}(S) \), and perfect if \( \mathcal{H}_{\det}(S) = \mathcal{P}(S) \). It is called Stieltjes semiperfect if \( \mathcal{H}_S(S) = \mathcal{P}_S(S) \), and Stieltjes perfect if \( \mathcal{H}_{S, \det}(S) = \mathcal{P}_S(S) \).

On every abelian semigroup, the mapping \( s \mapsto s \) is an involution, the identical involution. On every abelian group, the mapping \( s \mapsto s^{-1} \) (or \( s \mapsto -s \) if composition is written additively) is an involution, the inverse involution.

The group of integers with the inverse involution is perfect by Herglotz’ Theorem [17]. More generally, every abelian group with the inverse involution is perfect by the discrete version of the Bochner–Weil Theorem [23, 24, 32]. More generally yet, a \( * \)-semigroup \( S \) is perfect if it is \( * \)-divisible in the sense that for each \( s \in S \) there exist \( t \in S \) and \( m, n \in \mathbb{N} \) such that \( m + n \geq 2 \) and \( s = t^m t^n \). For the case of semigroups with identity, see the paper by Ressel and the author [11]; for the general case, the paper by Sakakibara and the author [12]. The latest result in this direction is that the equality can be replaced by \( s^* s = s^* t^m t^n \) (Sakakibara and the author [13]).

The oldest example of a non-perfect semiperfect semigroup is the additive semigroup of nonnegative integers (with its unique involution, the identity), which is semiperfect by Hamburger’s Theorem ([16], see the monographs by Akhiezer [1], Shohat and Tamarkin [28], or Berg, Christensen, and Ressel [3], or the beautiful lecture notes Momentproblemet (in Danish) by Christian Berg, Copenhagen University, 1979) but which is not perfect since there exist indeterminate moment sequences, such as the example \( n \mapsto (4n + 3)! \) given by Stieltjes in the year of his death [29]. It was shown by Sz.-Nagy [30] that \( \mathbb{N}_0 \) is even ‘operator semiperfect’.

For \( k \geq 2 \) the semigroup \( (\mathbb{N}_0^k, +) \) has several involutions, and it is not semiperfect for any of them. For the identical involution, this was shown by Berg, Christensen, and Jensen [2] and independently by Schmüdgen [27]. The case of \( \mathbb{N}_0^k \) with the switching involution \( (m, n)^s = (n, m) \) can be found in [3], Section 6.3. The case of \( \mathbb{N}_0^k \) with arbitrary involution
for arbitrary $k \geq 2$ follows by noting that an arbitrary involution on $\mathbb{N}_0^k$ consists in switching certain pairwise disjoint pairs of coordinates.

A homomorphism $h$ of one $*$-semigroup into another is a $*$-homomorphism if $h(s^*) = h(s)^*$ for all $s$ in the domain. Every $*$-homomorphic image of a $*$-semigroup that is perfect or semiperfect has the same property ([11, Proposition 1 and Theorem 1]).

A complex-valued function on $SS$ (for some $*$-semigroup $S$) is singular if it vanishes on $SS$. Simple considerations show that no nonzero moment function is singular. A $*$-semigroup is flat if it admits no nonzero singular positive definite function. By the preceding, every semiperfect semigroup (in particular, every perfect semigroup) is flat.

A $*$-semigroup $S$ is determinate if for one, hence for all, $n \in \mathbb{N}$, if $\mu$, $\nu$ are measures on $A(S^*)$ such that $\int \hat{s} \, d\mu = \int \hat{s} \, d\nu$ ($\mu$ and $\nu$ assumed to integrate the integrand) for all $s \in \hat{S} \cdots \hat{S}$ then $\mu = \nu$. For the equivalence, see the paper by Sakakibara and the author [12]. A $*$-semigroup $S$ is quasi-perfect (in a new sense!) if it is determinate and for one, hence for all, $n \geq 3$, for each $\varphi \in \mathcal{P}(S)$ there is some (hence, by determinacy, a unique) $\mu \in F_+(S^*)$ such that $\varphi(s) = \int \hat{s} \, d\mu$ for all $s \in \hat{S} \cdots \hat{S}$. For the equivalence, see [12].

If $S$ is a quasi-perfect semigroup then each $\varphi \in \mathcal{P}(S)$ admits a unique decomposition $\varphi = \varphi_3 + \varphi_m$ where $\varphi_3$ is a singular positive definite function and $\varphi_m$ is a moment function [9]. It easily follows that a $*$-semigroup is perfect if and only if it is quasi-perfect and flat.

A $*$-subsemigroup of a $*$-semigroup is a subsemigroup stable under the involution.

An ideal of a $*$-semigroup $S$ is a nonempty $*$-stable subset $I$ of $S$ such that $SI \subset I$. If $I$ is an ideal of $S$ then, in particular, $II \subset I$, so $I$ is a $*$-subsemigroup of $S$. Every ideal of a quasi-perfect semigroup is quasi-perfect [12]. The analogue with 'perfect' instead of 'quasi-perfect' is false, as shown by the example of $H = [1, \infty]$ and $S = \{0\} \cup H$ where $S$ (with addition and the identical involution) is perfect ([11] and [21]) but its ideal $H$ is not ([3], Chapter 8).

For every $*$-semigroup $S$ we denote by $\hat{S}$ the $*$-semigroup obtained by adjoining to $S$ an identity exterior to $S$, i.e., we choose some element $e$ outside $S$, define $e^2 = e = e^*$ and $es = s$ for all $s \in S$, and let $\hat{S} = S \cup \{e\}$. Note that $S$ is an ideal of $\hat{S}$. Now $S$ is quasi-perfect if and only if $S$ is perfect [12].

A $*$-semigroup $H$ is $*$-archimedean if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathbb{N}$ such that $(x^n)y^n = yz$. A $*$-archimedean component of a $*$-semigroup $S$ is a $*$-archimedean $*$-subsemigroup of $S$ which is maximal for the inclusion ordering. Every $*$-semigroup is the disjoint union of its $*$-archimedean components, and every $*$-archimedean $*$-subsemigroup of a $*$-semigroup $S$ is contained in a unique $*$-archimedean component of $S$. We drop the ‘$*$-’ of $*$-archimedean’ if the involution is the identity. An abelian semigroup $H$ is archimedean if and only if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathbb{N}$ such that $x^n = yz$.

A $*$-semigroup $S$ is quasi-perfect if and only if each $*$-archimedean component of $S$ is quasi-perfect [12]. Hence, a $*$-semigroup $S$ having an identity $e$ is perfect if and only if $H \cup \{e\}$ is perfect for each $*$-archimedean component $H$ of $S$. This was first shown in [6].
A \( \ast \)-semigroup \( S \) is \( \mathbb{R}_+ \)-separative if \( S^+ \) separates points in \( S \). For every \( \ast \)-semigroup \( S \) we denote by \( \rho \) (or \( \rho_S \), if \( S \) has to be specified) the quotient mapping of \( S \) onto its greatest \( \mathbb{R}_+ \)-separative \( \ast \)-homomorphic image, that is, the quotient \( \ast \)-semigroup \( S/\sim \) where \( \sim \) is the congruence relation in \( S \) defined by the condition that \( s \sim t \) if and only if \( \sigma(s) = \sigma(t) \) for all \( \sigma \in S^+ \). It was shown in [6] that a \( \ast \)-semigroup \( S \) with identity \( e \) is perfect if and only if \( \rho(H \cup \{ e \}) \) is Stieltjes perfect for each \( \ast \)-archimedean component \( H \) of \( S \), and that every perfect semigroup with identity is Stieltjes perfect. Conversely, it was shown in the paper by Sakakibara and the author [13] that every Stieltjes perfect semigroup (with identity or not) is perfect. For semigroups without identity, the two concepts are not equivalent; see [13].

We call a real-valued function \( \varphi \) on a subset \( X \) of a rational vector space midpoint convex if
\[
 f\left( \frac{1}{2}(x + y) \right) \leq \frac{1}{2}(f(x) + f(y))
\]
whenever \( x, y \in X \) and \( \frac{1}{2}(x + y) \in X \).

Let us agree to redefine the terms ‘convex’ and ‘cone’ using rational scalars instead of real ones. Consider every rational vector space \( U \) with the topology of finitely open sets, defined by the condition that a subset \( G \) of \( U \) is open in \( U \) if and only if for each finite-dimensional linear subspace \( E \) of \( U \) the set \( G \cap E \) is open in \( E \) with respect to the trace of the standard topology on the enveloping real vector space of \( E \).

A \( \ast \)-semigroup \( S \) is \( \mathbb{R}_+ \)-separative if and only if \( S \) carries the identical involution and each archimedean component of \( S \) is embeddable in a torsion-free abelian group, or equivalently, in a rational vector space (Lauritzen [20], Theorem 0.1 p. 135). An equivalent condition is that \( S \) carries the identical involution and is torsion-free in the sense that if \( a, b \in S, k \in \mathbb{N}, a^k = b^k \) then \( a = b \), cf. [5], Theorem 1. Call a \( \ast \)-semigroup \( S \) densely cosetlike if \( S \) is \( \mathbb{R}_+ \)-separative and for each archimedean component \( H \) of \( S \) the enveloping rational vector space of \( H \) is the sum of those of its linear subspaces \( U \) such that \( H \) contains a nonempty open subset of a coset (in \( U \)) of a dense subgroup of \( U \).

**Theorem 1.** A \( \ast \)-semigroup \( S \) is perfect if \( S \) is flat and \( \rho(S) \) is densely cosetlike.

**Remarks.** (1) The concept of flatness is of course not defined in terms of algebraic structure; it has to be verified each time one applies the theorem. However, the flatness problem disappears if one restricts attention to semigroups with neutral element since every such semigroup is flat. More generally, a \( \ast \)-semigroup \( S \) is flat if \( S = SS \) since it follows that \( SSS = S \). (2) Nor is the concept of a densely cosetlike semigroup defined purely algebraically. It involves topology. (3) The important thing to note is that the theorem contains all examples of perfect semigroups known to us, which is to say the classical ones (Herglotz’ Theorem, etc.) and those published by the school of Berg, Christensen, and Ressel [3]. Verifying this in detail would involve a large amount of work, and the outcome of it would have merely historical interest as soon as someone discovers a perfect semigroup which is not covered by the theorem. We merely remark that if \( G \) is a dense subgroup of \( \mathbb{Q} \) then every ‘\( G \)-conelike’ subsemigroup of a rational vector space (in the sense of [13])
is trivially densely cosetlike. The perfectness of semi-$*$-divisible semigroups is a corollary (see [13]). The perfectness of $*$-divisible semigroups is a corollary. The perfectness of abelian inverse semigroups is a corollary, and the discrete version of the Bochner–Weil theorem is a corollary.

2. A set-theoretical lemma

The purpose of the present section is to prove the next lemma. It is probably well-known, but we cannot give a reference. For us it grew out of Gert Kjærgaard Pedersen’s functional analysis class at Copenhagen University in the early 1980’s. The point is that the proof does not appeal to the axiom of choice.

LEMMA 1. Suppose $(X, \leq)$ is a partially ordered set in which each nonempty totally ordered subset $Y$ has a least upper bound, say $\bigvee Y$. Suppose $x \mapsto x'$ is a mapping of $X$ into itself such that $x \leq x'$ for all $x \in X$. Suppose $a \in X$. Let $X_a$ be the least subset of $X$ such that $a \in X_a$, if $x \in X_a$ then $x' \in X_a$, and if $Y$ is a nonempty totally ordered subset of $X_a$ then $\bigvee Y \in X_a$. Then $X_a$ is well-ordered and has a greatest element, say $b$. Finally, $b = b'$.

Proof. If $A$ is a subset of $X$, say that a subset $B$ of $A$ is saturated in $A$ if firstly, $a \in B$, secondly, if $x \in B$ and $x' \in A$ then $x' \in B$, and thirdly, if $Y$ is a nonempty totally ordered subset of $B$ such that $\bigvee Y \in A$ then $\bigvee Y \in B$. Let $A$ be the set of all subsets of $X$ such that $a \in A$ is well-ordered in the inherited ordering and such that if $A \neq \emptyset$ then $a \in A$ and every saturated subset of $A$ is equal to $A$.

Let us first show that if $A \neq \emptyset$ then $a$ is the least element of $A$. By hypothesis, $a$ is an element of $A$. We only have to show that it is the least one. So let $B$ be the set of those $x \in A$ such that $a \leq x$; we have to show that $B = A$. It suffices to show that $B$ is saturated in $A$. Clearly $a \in B$. If $x \in B$ and $x' \in A$ then $a \leq x \leq x'$, so $x' \in B$. Finally, if $Y$ is a nonempty subset of $B$ such that $\bigvee Y \in A$ then we can choose $y \in Y$, and then $a \leq y \leq \bigvee Y$, so $\bigvee Y \in B$.

Next, let us show that if $A \neq \emptyset$ and if $b$ is an element of $A$ which is not a greatest element of $A$ then $b'$ is the immediate successor of $b$ in $A$. Suppose not; we shall derive a contradiction. Let $b$ be the least element of $A$ such that $b'$ is not the immediate successor of $b$ in $A$. By hypothesis, $b$ is not a greatest element of $A$. Define $B_0 = \{x \in A \mid x \leq b\}$, $B_1 = \{x \in A \mid x \geq b'\}$, and $B = B_0 \cup B_1$.

First suppose $b' \notin A$; we shall derive a contradiction by showing that $B_0 = A$, implying that $b$ is a greatest element of $A$. It suffices to show that $B_0$ is saturated in $A$. Since $b \in A$ then $A \neq \emptyset$, so $a$ is the least element of $A$. Since, in particular, $a \leq b$ then $a \in B_0$. If $x \in B_0$ and $x' \in A$ then $x \leq b$. We cannot have $x = b$ since $b' \notin A$. Thus $x < b$. By the definition of $b$, $x'$ is the immediate successor of $x$ in $A$, so $x' \leq b$, hence $x' \in B_0$. Finally, if $Y$ is a nonempty subset of $B_0$ such that $\bigvee Y \in A$ then for each $y \in Y$ we have $y \leq b$. It follows that $\bigvee Y \leq b$, so $\bigvee Y \in B_0$. Thus $B_0 = A$.

If $b' \in A$, then $B_0 = A$, and $b$ is the greatest element of $A$. Thus $b' \notin A$.
The contradiction shows \( b' \in A \). We can now show \( B = A \). It suffices to show that \( B \) is saturated in \( A \). We saw that \( a \in B_0 \subset B \). Suppose \( x \in B \) and \( x' \in A \); we have to show \( x' \in B \). For \( x \in B_0 \), this is as in the preceding paragraph except that if \( x = b \) then \( x' = b' \in B_1 \subset B \). If \( x \in B_1 \) then \( b' \leq x \leq x' \), so \( x' \in B_1 \subset B \). Finally, suppose \( Y \) is a nonempty subset of \( B \) such that \( \bigvee Y \in A \); we have to show \( \bigvee Y \in B \). If \( Y \subset B_0 \) then \( y \leq b \) for all \( y \in Y \), hence \( \bigvee Y \leq b \), that is, \( \bigvee Y \in B_0 \subset B \). Thus we may assume \( Y \cap B_1 \neq \emptyset \). Choosing \( y \in Y \cap B_1 \), we have \( b' \leq y \leq \bigvee Y \), that is, \( \bigvee Y \in B_1 \subset B \). Thus \( B = A \), showing that \( b' \) is, after all, the immediate successor of \( b \) in \( A \), the desired contradiction.

We can now show that if \( A \in A \) and if \( Y \) is a nonempty subset of \( A \) which has an upper bound in \( A \) then \( \bigvee Y \in A \). Suppose not; we shall derive a contradiction. Let \( B \) be the set of those \( x \in A \) such that \( x \leq y \) for some \( y \in Y \); we shall show \( B = A \). Since \( \emptyset \neq Y \subset A \) then \( a \) is the least element of \( A \). Choosing \( y \in Y \), we have \( a \leq y \), so \( a \in B \). Suppose \( x \in B \) and \( x' \in A \); we shall show \( x' \in B \). Suppose not; we shall derive a contradiction. Since \( x' \notin B \), that is, \( y < x' \) for all \( y \in Y \), and since \( x' \) is the immediate successor of \( x \) in \( A \), then \( Y \) is bounded above by \( x \). Since \( x \in B \) then \( x \) is a greatest element of \( Y \), so \( \bigvee Y = x \in A \), a contradiction. Thus \( x' \in B \). Finally, suppose \( Z \) is a nonempty subset of \( B \) such that \( \bigvee Z \in A \); we shall show \( \bigvee Z \in B \). Suppose not; we shall derive a contradiction. Since \( \bigvee Z \notin B \) then there is no \( y \in Y \) such that \( \bigvee Z \leq y \). In other words, for each \( y \in Y \) there is some \( z \in Z \) such that \( z \leq y \), which is equivalent to \( z > y \) since the ordering is total. On the other hand, for each \( z \in Z \) there is obviously some \( y \in Y \) such that \( z \leq y \). It follows that \( \bigvee Y = \bigvee Z \in A \), a contradiction.

The contradiction shows that \( B = A \), that is, \( A \) is bounded above by \( \bigvee Y \). On the other hand, \( Y \) is bounded above in \( A \) by hypothesis, so \( \bigvee Y \) must be a greatest element of \( A \), contradicting the hypothesis that \( \bigvee Y \notin A \).

Let us show that if \( A, B \in A \) and \( A \neq B \) then either \( A = \{ x \in B \mid x < b \} \) for some \( b \in B \) or similarly with \( A \) and \( B \) interchanged. Since \( A \neq B \) then \( A \setminus B \neq \emptyset \) or \( B \setminus A \neq \emptyset \). By symmetry, we may assume \( A \setminus B \neq \emptyset \). Let \( b \) be the least element of \( B \setminus A \); we claim that \( A = C \) where \( C = \{ x \in B \mid x < b \} \). We have \( C \subset A \) by the definition of \( b \). If \( A = \emptyset \) then \( b \) is the least element of \( B \), so \( C = \emptyset = A \). Thus we may assume \( A \neq \emptyset \). It now suffices to show that \( C \) is saturated in \( A \). Since \( A \neq \emptyset \) then \( a \) is the least element of \( A \). Since \( a \notin B \setminus A \) then \( b > a \), so \( a \in C \). Suppose \( x \in C \) and \( x' \in A \); we shall show \( x' \in C \). Since \( x \in C \) then \( x < b \). Since \( x \) is thus not a greatest element of \( B \) then \( x' \) is the immediate successor of \( x \) in \( B \), so \( x' \leq b \). It is impossible that equality holds since \( b \notin A \). Thus \( x' < b \), that is, \( x' \notin C \). Finally, suppose \( Y \) is a nonempty subset of \( C \) such that \( \bigvee Y \in A \); we shall show \( \bigvee Y \in C \). Since \( Y \) is bounded above in \( B \) (by \( b \)) then \( \bigvee Y \in B \), so we only have to show \( \bigvee Y < b \). Again, since \( Y \) is bounded above by \( b \) then \( \bigvee Y \leq b \). Equality cannot hold since \( b \notin A \).

The fact shown in the preceding paragraph implies that the set \( \bigcup A \) is well-ordered. Indeed, suppose \( P \) is a nonempty subset of \( \bigcup A \); we have to show that \( P \) has a least element. Since \( P \) is a nonempty subset of \( \bigcup A \) then there is some \( A \in A \) such that \( P \cap A \neq \emptyset \). Let \( p \) be the least element of \( P \cap A \); we claim that \( p \) is a least element of \( P \).
Suppose $q \in P$; we have to show $p \leq q$. Since $q \in P \subset \bigcup A$ then $q \in B$ for some $B \in A$. If $B \subset A$, the desired inequality follows from the definition of $p$. Thus, by the preceding paragraph, we may assume that $A = \{x \in B \mid x < b\}$ for some $b \in B$. In particular, $p < b$. The desired equality follows if $q \geq b$. Thus we may assume $q < b$. But then $q \in A$, so the desired inequality follows from the definition of $p$.

Let us show that $\bigcup A \in A$. Since $\bigcup A$ is well-ordered, this amounts to showing that if $\bigcup A \neq \emptyset$ then $a \in \bigcup A$ and every saturated subset of $\bigcup A$ is equal to $\bigcup A$. Since clearly $\{a\} \in A$ then $a \in \bigcup A$. Suppose $B$ is a saturated subset of $\bigcup A$; we have to show $B = \bigcup A$. It is completely trivial to verify that if $A \in A$ and $A \neq \emptyset$ then the set $B \cap A$ is saturated in $A$. Since $A \in A$ it follows that $B \cap A = A$, that is, $A \subset B$. This being so for all such $A$, it follows that $B = \bigcup A$, as desired.

We can now show that $\bigcup A$ has a greatest element. Indeed, since $\bigcup A$ is, in particular, totally ordered then we may define $b = \bigvee_{A}$. Clearly $b$ is an upper bound on $\bigcup A$. Thus we only have to show $b \in \bigcup A$. Suppose not; we shall derive a contradiction. Define $C = (\bigcup A) \cup \{b\}$. Clearly $C$ is well-ordered. Since $a \in \bigcup A$ then $a \in C$. Suppose $B$ is a saturated subset of $C$; we shall show $B = C$. It is completely trivial to verify that the set $B \cap \bigcup A$ is saturated in $\bigcup A$. Since $\bigcup A \in A$ it follows that $B \cap \bigcup A = \bigcup A$, that is, $\bigcup A \subset B$. Thus we only have to show $b \in B$. But since $B$ is saturated in $C$, this follows from the facts that $\bigcup A$ is a subset of $B$ and $b = \bigvee \bigcup A \in C$. Thus $C \in A$, whence $b \in \bigcup A$, a contradiction. This proves that $\bigcup A$ has a greatest element.

To see that $X_a$ is well-ordered, it now suffices to show $X_a = \bigcup A$. To see that $\bigcup A \subset X_a$ it suffices to show that if $A \in A$ then $A \subset X_a$. This is trivial if $A = \emptyset$; so assume $A \neq \emptyset$. Define $B = A \cap X_a$; we have to show $B = A$. It suffices to show that $B$ is saturated in $A$. But this is trivial.

To see that $X_a \subset \bigcup A$, it suffices to show that, firstly, $a \in \bigcup A$, secondly, if $x \in \bigcup A$ then $x' \in \bigcup A$, and thirdly, if $Y$ is a nonempty subset of $\bigcup A$ then $\bigvee Y \in \bigcup A$. To see that $a \in \bigcup A$ it suffices to note that $\{a\} \in A$. Suppose $x \in \bigcup A$; we have to show $x' \in \bigcup A$. Choose $A \in A$ such that $x \in A$. If $x$ is not a greatest element of $A$ then (as we saw) it follows that $x' \in A \subset \bigcup A$. Thus we may assume that $x$ is a greatest element of $A$. We then define $B = A \cup \{x'\}$. It now suffices to show that $B \in A$. Clearly, $B$ is well-ordered. Suppose $C$ is a saturated subset of $B$; we have to show $C = B$. It is completely trivial to verify that $C \cap A$ is a saturated subset of $A$. Since $A \in A$ it follows that $C \cap A = A$, that is, $A \subset C$. We now only have to show that $x' \in C$. But this follows from the facts that $x \in A \subset C$ and $x' \in B$. This proves $x' \in \bigcup A$. Finally, suppose $Y$ is a nonempty subset of $\bigcup A$; we have to show $\bigvee Y \in \bigcup A$. But this follows (as we saw) from the facts that $\bigcup A \in A$ and that the subset $Y$ is upper bounded in $\bigcup A$ since the latter set has a greatest element.

Since $X_a = \bigcup A$ then $X_a$ is well-ordered and has a greatest element $b$. Since $b' \in X_a$ (by the definition of $X_a$) then $b' \leq b$. Since $b \leq b'$, it follows that $b = b'$. This completes the proof.
3. Proof of Theorem 1

A *-semigroup $S$ is Stieltjes determinate if the mapping $L: F_+(S^*_\ast) \to C^S$ is one-to-one. Clearly, a *-semigroup is Stieltjes perfect if and only if it is Stieltjes semiperfect and Stieltjes determinate.

Suppose $S$ is a flat semigroup such that $\sigma(S)$ is densely cosetlike; we have to show that $S$ is perfect. Since a *-semigroup is perfect if and only if it is flat and its greatest $R_+$-separative *-homomorphic image is quasi-perfect [13] then it suffices to show that $\rho(S)$ is quasi-perfect.

In other words, it suffices to show that every densely cosetlike semigroup is quasi-perfect.

Since a *-semigroup is quasi-perfect if and only if each of its *-archimedean components so is, and since every *-archimedean component of a densely cosetlike semigroup is obviously densely cosetlike then it suffices to show that every archimedean densely cosetlike semigroup is quasi-perfect.

Before proceeding with the proof, we prove a lemma which may have independent interest. It is a generalization of [6], Theorem 4.3, which was concerned with subsemigroups of $Q_\ast$. A simple example with a subsemigroup of $Q_\ast^2$ will show that one cannot omit the hypothesis that $H$ is archimedean, a hypothesis that was automatically satisfied in [6]. The hypothesis that $H$ is archimedean is not a serious one since in order to characterize perfect subsemigroups of rational vector spaces containing the zero of the space it suffices to do so for semigroups of the form $H \cup \{0\}$ where $H$ is archimedean [6].

A measure is concentrated on a set $X$ if it vanishes on every measurable set disjoint with $X$. If $\mu$ is a measure and $f$ is a mapping then we denote by $\mu_f$ the image measure of $\mu$ under $f$ whenever this makes sense. If $T$ is a *-subsemigroup of a *-semigroup $S$ then we define $\pi_{S,T}: S^* \to T^* \cup \{0\}$ by $\pi_{S,T}(\sigma) = \sigma|T$ for $\sigma \in S^*$. Note that although a character on $S$ is nonzero by definition, it may happen to vanish on $T$. If $\mu$ is a measure on $A(S^*)$ then by abuse of notation we denote by $\mu^{\pi_{S,T}}$ the image measure of $\mu|\pi_{S,T}^{-1}(T^*)$ under the mapping $\pi_{S,T}|\pi_{S,T}^{-1}(T^*)$. This slightly complicated definition is necessary since we do not wish to consider measures with mass at the ‘zero character’. Clearly, if $\mu \in F_+(S^*)$ then $\mu^{\pi_{S,T}} \in F_+(T^*)$ and $L(\mu^{\pi_{S,T}}) = (L\mu)|TT$.

**Lemma 2.** If $H$ is an archimedean subsemigroup of a rational vector space and if the semigroup $S = H \cup \{0\}$ has a perfect subsemigroup which spans the same space then $S$ is perfect.

**Proof.** Choose a perfect subsemigroup $T$ of $S$ which spans the same space. Since the semigroup $T \cup \{0\}$ is likewise perfect [26] and spans the same space then we may assume $0 \in T$.

Since $S$ has a zero then it suffices (and is also necessary) to show that $S$ is Stieltjes perfect. So suppose $\varphi \in P_c(S)$; we have to show that $\varphi$ is a Stieltjes determinate Stieltjes moment function. For $s \in S$ the function $\varphi_s: S \to R_+$ defined by $\varphi_s(x) = \varphi(s + x)$
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is clearly completely positive definite. In particular, \( \psi_{|T} \) is completely positive definite. Since \( T \) is perfect and has a zero then \( T \) is Stieltjes perfect. Hence, for each \( s \in S \) there is a unique measure \( \lambda_s \in \mathcal{M}_+(T^*_+) \) such that \( \psi_s(T) = L\lambda_s \), that is,

\[
\psi(s + t) = \int_{T^*_+} \tau(t) \, d\lambda_s(\tau) \quad (1)
\]

for \( t \in T \). If \( a \in T \) then \( \int_{T^*_+} \tau(t) \, d\lambda_{s+a}(\tau) = \psi(s + a + t) = \int_{T^*_+} \tau(a + t) \, d\lambda_s(\tau) = \int_{T^*_+} \tau(a) \tau(t) \, d\lambda_s(\tau) \) for all \( t \in T \). By the uniqueness of \( \lambda_{s+a} \) it follows that

\[
\lambda_{s+a} = \tau(a) \, d\lambda_s(\tau). \quad (2)
\]

As in [6], Theorem 4.3, the family \( \{\lambda_s\}_{s \in S} \) is completely positive definite in the sense that for each \( A \in \mathcal{A}(T^*_+) \) the function \( s \mapsto \lambda_s(A): S \to \mathbb{R}_+ \) is completely positive definite. (Note that the measures \( \lambda_s \) \( s \in S \) are finite since \( T \) has a zero.)

For definiteness, we consider only the most difficult case, which is the case that \( 0 \notin H \).

Let \( S^x \) be the set of all everywhere positive characters on \( S \), and similarly for \( T \) instead of \( S \). Since every character on \( H \) is nowhere zero ([7], Lemma 2) then \( S^x = S^x \cup \{\theta_S\} \) where \( \theta_S \) is the indicator function of the set \( \{0\} \) as a subset of \( S \). (We cannot claim similarly for \( T \) instead of \( S \).)

The hypothesis that \( T \) spans the same space as \( S \) implies that the group \( (S - S)/(T - T) \) is a torsion group. Indeed, since \( S - S \) is a group then the space spanned by \( S \) is the set \( \{n^{-1}g \mid g \in S - S, n \in \mathbb{N}\} \), and similarly for \( T \) instead of \( S \). Since the two spaces are the same then for each \( g \in S - S \) there is some \( n \in \mathbb{N} \) such that \( ng \in T - T \). That is, \( (S - S)/(T - T) \) is a torsion group.

The mapping \( \sigma \mapsto \sigma|T: S^x \to T^x \) is a bijection of \( S^x \) onto \( T^x \). To see this, it suffices to note that the set of all everywhere positive characters \( \sigma \) on a subsemigroup \( S \) of a rational vector space is in a canonical one-to-one correspondence with the set of all additive real-valued functions \( \xi \) on the rational vector space spanned by \( S \), given by \( \sigma = e^\xi|S \). (Use the hypothesis that \( S \) and \( T \) span the same space.) The mapping mentioned is an isomorphism between the measurable spaces \( (S^x, \mathcal{A}(S^x)) \) and \( (T^x, \mathcal{A}(T^x)) \). Indeed, it is clearly measurable. The proof that the inverse mapping is measurable is an easy consequence of the facts that for \( s \in S \) there is some \( n \in \mathbb{N} \) such that \( ns \in T - T \) and that for each \( n \in \mathbb{N} \) the mapping \( s \mapsto \sqrt[n]{x}: \mathbb{R}_+ \to \mathbb{R}_+ \) is Borel measurable.

Let us show that for \( s \in H \) the measure \( \lambda_s \) is concentrated on \( T^x \). Suppose \( A \in \mathcal{A}(T^*_+) \) and \( A \cap T^x = 0 \); we have to show \( \lambda_s(A) = 0 \). Choose \( n \in \mathbb{N} \) such that \( ns \in T - T \). Recalling that the family \( \{\lambda_s\}_{s \in S} \) is completely positive definite, for \( j = 1, \cdots, n - 1 \) we have by the Cauchy–Schwarz inequality, \( \lambda_{js}(A)^2 \leq \lambda_{j(j-1)s}(A)\lambda_{(j+1)s}(A) \). By induction it follows that we need only show \( \lambda_{ns}(A) = 0 \). Choose \( t', t'' \in T \) such that

\[
ns = t' - t''. \quad (3)
\]

By (2),

\[
\tau(t') \, d\lambda_0(\tau) = \lambda_{t'} = \lambda_{ns + t''} = \tau(t'') \, d\lambda_{ns}(\tau). \quad (4)
\]
Since \( H \) is archimedean and \( ns \in H \) then we can choose \( r \in H \) and \( m \in \mathbb{N} \) such that \( mns = r + t'' \). (If \( t'' \in H \), this is so by the definition of the archimedean property; otherwise, trivially.) By (2), \( \lambda_{mns} = \tau(t'') d\lambda_r(\tau) \), showing that \( \lambda_{mns} \) is concentrated on the set of those \( \tau \in T^*_+ \) such that \( \tau(t'') > 0 \), a set which we denote by \( G_{t''} \) (and similarly for any other element of \( T \)). Repeating the argument with the Cauchy–Schwarz inequality, we see that \( \lambda_s \) is concentrated on \( G_{t''} \). Now the equality (3) continues to hold if we add to both \( t' \) and \( t'' \) the same element of \( T \). Thus, repeating the argument, we see that for each \( a \in T \) the measure \( \lambda_s \) is concentrated on the set \( G_{t''}+a = G_{t''} \cap G_a \subset G_a \). The next step is to get from this to the conclusion that \( \lambda_s \) is concentrated on the set \( T^\times = \bigcap_{a \in T} G_a \).

This is not trivial since \( T \) may be uncountable. However, note that by a formula in [11],

\[
A(T^*_+) = \bigcup_{U \in \mathcal{D}(T)} \pi_{T,U}^{-1}(A(U^*_+))
\]

where \( \mathcal{D}(T) \) is the set of all countable *-subsemigroups of \( T \). Thus it suffices to show that if \( U \in \mathcal{D}(T) \) and if \( B \in A(U^*_+) \) is such that \( \pi_{T,U}^{-1}(B) \cap T^\times = \emptyset \) then \( 0 = \lambda_s(\pi_{T,U}^{-1}(B)) = \lambda_{s,T,U}^\times \). Now each everywhere positive character on \( U \) extends to an everywhere positive character on \( T \). (By the above correspondence between everywhere positive characters on a subsemigroup of a rational vector space and additive real-valued functions on the space it spans, this is just another way of stating that every additive real-valued function on a rational vector space extends to a similar mapping on every rational superspace; which is well-known.) If \( B \cap U^\times \neq \emptyset \) then we can choose \( \omega \in B \cap U^\times \). Extending \( \omega \) to an everywhere positive character \( \tau \) on \( T \), we then have \( \tau \in \pi_{T,U}^{-1}(B) \cap T^\times \), a contradiction. Thus \( B \cap U^\times = \emptyset \). It therefore suffices to show that the measure \( \lambda_{s,T,U}^\times \) is concentrated on the set

\[
U^\times = \bigcap_{u \in U} D_u
\]

where \( D_u = \{ \omega \in U^*_+, \omega(u) > 0 \} \) for \( u \in U \). Since \( U \) is countable then it suffices to show that \( \lambda_{s,T,U}^\times \) is concentrated on \( D_u \) for each \( u \in U \). But this follows from the fact that \( \lambda_s \) is concentrated on the set \( G_a = \pi_{T,U}^{-1}(D_a) \).

We have shown that \( \lambda_s \) is concentrated on \( T^\times \). Hence, we can uniquely define a measure \( \lambda_s' \) on \( A(T^\times) \) by the condition that \( \lambda_s'(A \cap T^\times) = \lambda_s(A) \) for each \( A \in A(T^*_+) \). Let \( \tau \mapsto \tilde{\tau}: T^\times \rightarrow S^\times \) be the inverse of the mapping \( \sigma \mapsto \sigma|T: S^\times \rightarrow T^\times \); we saw above that it is well-defined and measurable. Now let \( \mu_s \) be the image measure of \( \lambda_s' \) under the mapping \( \tau \mapsto \tilde{\tau} \). By the definition of \( S^\times \), the measure

\[
\kappa_s = \sigma(s)^{-1} d\mu_s(\sigma)
\]

is well-defined. We shall show that all the measures \( \kappa_s (s \in H) \) are one and the same measure. Let us first show that \( \kappa_s \) is finite for each \( s \in H \). For \( n \in \mathbb{N} \) let \( G_{s,n} \) be the set of
those $\sigma \in S^\times$ such that $\sigma(s) > 1/n$. Then

$$\kappa_s(G_{s,n}) = \int_{G_{s,n}} \sigma(s)^{-1} \, d\mu_s(\sigma) = \int_{F_{s,n}} \tilde{\tau}(s)^{-1} \, d\lambda_s(\tau) \quad (5)$$

where $F_{s,n} = \{ \sigma \mid \sigma \in G_{s,n} \}$. If $j \in \mathbb{N}$ then the matrix of measures

$$\begin{pmatrix} \lambda_{(j-1)s} & \lambda_{js} \\ \lambda_{js} & \lambda_{(j+1)s} \end{pmatrix}$$

is positive semidefinite since the family $(\lambda_r)_{r \in S}$ is completely positive definite. Hence, if $f$ and $g$ are bounded measurable real-valued functions then

$$\left( \int f^2 \, d\lambda_{(j-1)s} + \int g^2 \, d\lambda_{(j+1)s} + 2 \int fg \, d\lambda_s \right) \geq 0 .$$

(Apply [6], Lemma 3.7, noting that $f$ and $g$ are square integrable since the measures are finite.) This being so for all such $f$ and $g$, by the Cauchy–Schwarz inequality we have

$$\left( \int f \, d\lambda_{js} \right)^2 \leq \int f^2 \, d\lambda_{(j-1)s} \int g^2 \, d\lambda_{(j+1)s}$$

for all such $f$ and $g$. Applying this to the functions $f$ and $g$ defined by

$$f(\tau) = \begin{cases} \tilde{\tau}(s)^{-1} & \text{if } \tau \in F_{s,n} \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(\tau) = \begin{cases} \tilde{\tau}(s)^{-1} & \text{if } \tau \in F_{s,n} \\ 0 & \text{otherwise} \end{cases}$$

(which are bounded by the definition of $F_{s,n}$), we see that the function $j \mapsto \int_{F_{s,n}} \tilde{\tau}(s)^{-1} \, d\lambda_{js}(\tau) : \mathbb{N}_0 \to \mathbb{R}_+$ is logarithmically convex. Hence, by (5),

$$\kappa_s(G_{s,n}) = \int_{F_{s,n}} \tilde{\tau}(s)^{-1} \, d\lambda_s(\tau) \leq \lambda_0(F_{s,n})^{(n-1)/n} \left( \int_{F_{s,n}} \tilde{\tau}(s)^{-n} \, d\lambda_{ns}(\tau) \right)^{1/n} \quad (6)$$

for each $n \in \mathbb{N}$. Choosing $n \in \mathbb{N}$ and $t', t'' \in T$ such that (3) holds, by (4) we have

$$\lambda_{ns} = \tau(t') \tau(t'')^{-1} \, d\lambda_0(\tau) = \tilde{\tau}(s)^n \, d\lambda_0(\tau) ,$$

so (6) reduces to $\kappa_s(G_{s,n}) \leq \lambda_0(F_{s,n})$, whence by letting $n \to \infty$,

$$\kappa_s(S^\times) \leq \lambda_0(T^+_n) < \infty .$$

As in [6], Theorem 4.3, the family $(\kappa_s)_{s \in H}$ is again completely positive definite.

Suppose $A \in \mathcal{A}(S^\times)$; we shall show that the function $f : H \to \mathbb{R}_+$ defined by $f(s) = \kappa_s(A)$ for $s \in H$ is constant. Note that it is completely positive definite. Let $\pi$ be the quotient mapping of $S - S$ onto $(S - S)/(T - T)$. Since $\pi(H)$ is a semigroup that generates the torsion group $\pi(S - S)$ then it is equal to it. We now define $g : \pi(H) \to \mathbb{R}_+$ by the condition that $g(\pi(s)) = f(s)$ for $s \in H$. We have to show that it is well-defined. So suppose $r, s \in H$ are such that $\pi(r) = \pi(s)$; we have to show that $f(r) = f(s)$. Since
\( \pi(r) = \pi(s) \) then there exist \( t, u \in T \) such that \( r + t = s + u \). By (2), \( \tau(t) d\lambda_r(\tau) = \lambda_{r+t} = \lambda_{s+u} = \tau(u) d\lambda_s(\tau) \). Hence similarly with primes on the measures. Taking the image under the mapping \( \tau \mapsto \tilde{\tau} \) we obtain \( \sigma(t) d\mu_r(\sigma) = \sigma(u) d\mu_s(\sigma) \). Since \( r + t = s + u \) then \( \sigma(r) \sigma(t) = \sigma(r + t) = \sigma(s + u) = \sigma(s) \sigma(u) \) for all \( \sigma \in S^0 \). By the definition of \( k_q \) for \( q \in H \) it now follows that \( k_r = k_s \). In particular, \( f(r) = k_r(A) = k_s(A) = f(s) \), as desired. Thus \( g \) is well-defined. Since \( \pi \) is a homomorphism then \( g \) is again completely positive definite. Since \( \pi(H) \) is a torsion group then by [6], Lemma 4.6, it follows that \( g \) is a constant. Hence so is the function \( f = g \circ \pi \).

Since this applies to all \( A \in A(S^\times) \) then we have shown that all the measures \( k_r \) \( (s \in H) \) are one and the same measure, say \( k \). For \( s \in H \) we have by (1)

\[
\varphi(s) = \varphi(s + 0) = \int_{T^+_s} \tau(0) d\lambda_s(\tau) = \lambda_s(T^+_s) = \lambda_s' (T^\times) = \mu_s(S^\times) = \int_{S^\times} \sigma(s) d\kappa(\sigma).
\]

Noting that \( S^\times \) is in a one-to-one correspondence with \( H^+_s \), this proves that the function \( \varphi|H \) is a Stieltjes moment function. Let us show that it is Stieltjes determinate. Suppose \( \theta \) is another measure on \( S^\times = H^+_s \) such that \( \varphi|H = \mathcal{L}\theta \). Denoting by \( h \) the mapping \( \sigma \mapsto \sigma[T : S^\times \to T^\times] \), the measures \( k^h \) and \( \theta^h \) represent the same function on the Stieltjes determinate semigroup \( T \) and so are equal. By the isomorphism of measurable spaces mentioned above it follows that \( \theta = \kappa \).

Thus \( \varphi|H \) is a Stieltjes determinate Stieltjes moment function. The remainder of the proof consists of a repetition of the Nakamura–Sakakibara argument [21], also used by Sakakibara in his proof that if \( H \) is a perfect semigroup then the semigroup \( S = H \cup \{0\} \) is again perfect [26]. This completes the proof.

We note a corollary which we shall not need. It is a kind of converse to the result that every ideal of a quasi-perfect semigroup is quasi-perfect [12]. The hypothesis about the \(*\)-archimedean property cannot be omitted since, e.g., the semigroup \( N_0 \cup \{\infty\} \) with the usual addition in \( N_0, n + \infty = \infty + n = \infty + \infty = \infty \) for all \( n \in N_0 \), and the identical involution has the perfect ideal \( \{\infty\} \) but is not perfect since it has the face \( N_0 \) which is not perfect. (If \( S \) is a perfect semigroup then so is every \(*\)-subsemigroup \( X \) of \( S \) which is a face of \( S \) in the sense that if \( x, y \in S \) and \( x + y \in X \) then \( x, y \in X \) [25].) This is not a serious restriction since in order to characterize quasi-perfect semigroups it suffices to do so for \(*\)-archimedean semigroups [12].

**COROLLARY 1.** Every \(*\)-archimedean semigroup which has a quasi-perfect ideal is quasi-perfect.

**PROOF.** We use additive notation. Suppose \( H \) is a \(*\)-archimedean semigroup which has a quasi-perfect ideal, say \( I \); we have to show that \( H \) is quasi-perfect. It is easy to see that every \(*\)-archimedean semigroup which has a zero is a group. A group has no ideal other than itself. Thus the whole thing is trivial if \( H \) has a zero. Assume the contrary. We precisely have to show that the \(*\)-semigroup \( S = H \cup \{0\} \) is perfect. Since this semigroup has a zero then it suffices to show that \( \rho(S) \) is perfect [13]. Now \( \rho(S) \) can be identified with \( \rho(H) \cup \{0\} \) [7], Lemma 3, applied to \( R^+_\text{-separativity} \) instead of \( R^\text{-separativity} \). Since \( H \)
is *-archimedean, so is its *-homomorphic image $\rho(H)$. Since $\rho(H)$ carries the identical involution, this means that $\rho(H)$ is archimedean. Being an archimedean $R_+$-separative semigroup, $\rho(H)$ is embeddable in a rational vector space. Since $I$ is an ideal of $H$ then $\rho(I)$ is an ideal of $\rho(H)$. Being a *-homomorphic image of the quasi-perfect semigroup $I$, the semigroup $\rho(I)$ is quasi-perfect.

In other words, we may assume that $H$ is a subsemigroup of a rational vector space carrying the identical involution. Since $I$ is quasi-perfect then the semigroup $T = I \cup \{0\}$ is perfect. By Lemma 2, it now suffices to show that $T$ spans the same space as $S$. Of course, $I$ is an ideal of $S$. Choosing $a \in I$, we have $S + a \subset S + I \subset I$, hence $S - S = (S + a) - (S + a) \subset I - I \subset T - T$. The converse being trivial, we have $S - S = T - T$. Since the rational vector space spanned by $S$ is the same as that spanned by $S - S$, and similarly for $T$, we are done.

Returning to the proof of the Theorem, suppose $H$ is an archimedean densely cosetlike semigroup; we have to show that $H$ is quasi-perfect. Since $H$ is densely cosetlike then it is $R_+$-separative by definition. Recalling that each archimedean component of an $R_+$-separative semigroup is embeddable in a rational vector space, and noting that $H$, being archimedean, has only one archimedean component (viz., itself), we see that $H$ is embeddable in a rational vector space. Let $V$ be the enveloping rational vector space of $H$.

If $H$ contains the zero of $V$ then we precisely have to show that $H$ is perfect. Otherwise, the semigroup $\hat{H}$ can be identified with $H \cup \{0\}$, and we precisely have to show that $\hat{H}$ is perfect.

Thus, in every case, we precisely have to show that the semigroup $S = H \cup \{0\}$ is perfect.

Let $U$ be the set of all linear subspaces of $V$ such that $H$ contains a nonempty open subset of a coset (in $U$) of a dense subgroup of $U$. By hypothesis, $V = \sum_{U \in U} U$.

Suppose we show that for each $U \in U$ the semigroup $S \cap U$ is perfect. Since every *-semigroup with identity which is generated by the union of its perfect *-subsemigroups containing the identity is perfect [11], it follows that the semigroup $T = \sum_{U \in U} (S \cap U)$ is perfect. Since $H$ is archimedean then it follows from Lemma 2 that in order to infer that $S$ is perfect we need only verify that $T$ spans $V$. Since $V = \sum_{U \in U} U$ then it suffices to show that if $U \in U$ then $H \cap U$ spans $U$. So let $W$ be the linear subspace of $U$ spanned by $H \cap U$; we have to show $W = U$. Since every linear subspace of a rational vector space carrying the topology of finitely open sets is closed then in particular, $W$ is closed, so $\overline{W \cap H} \subset W$. Choose a nonempty open subset $A$ of $U$ and a coset $C$ (in $U$) of a dense subgroup of $U$ such that $A \cap C \subset H$. Of course, $A \cap C \subset H \cap U$. Since $A$ is open in $U$ then $A \cap C \subset H \cap U \subset W$. But since $C$ is a coset in $U$ of a dense subgroup of $U$ then $C = U$. (The proof of this uses only the fact that translations are continuous; note that the topology of finitely open sets is not in general a vector space topology.) Thus $A \subset W$. By the definition of the topology, $A \cap E$ is open in $E$ for each finite-dimensional linear subspace $E$ of $U$. Since $A \neq \emptyset$ then for all sufficiently large $E$ the set $A \cap E$ is nonempty. Being a nonempty open subset of $E$, it spans $E$. Since $W$ is a vector space it follows that $E \subset W$. This being so for all such $E$, we have $U \subset W$. The converse is trivial.
Before proceeding with the proof, we prove a simple lemma.

**Lemma 3.** If $H$ is an archimedean subsemigroup of an abelian group $G$ then for every subgroup $F$ of $G$ the semigroup $H \cap F$ is archimedean.

**Proof.** Suppose $x, y \in H \cap F$; we have to show that there exist $z \in H \cap F$ and $n \in \mathbb{N}$ such that $nx = y + z$. Since $H$ is archimedean, there exist $z \in H$ and $n \in \mathbb{N}$ such that $nx = y + z$. Since $x, y \in F$ then $z = nx - y \in F$, so $z \in H \cap F$.

Returning to the proof of the theorem, recall that it suffices to show that $S \cap U$ is perfect for each $U \in \mathcal{U}$. Now $H \cap U$ is archimedean, by the lemma.

In other words, it suffices to show that if $H$ is an archimedean subsemigroup of a rational vector space $U$ such that $H$ contains a nonempty open subset of a coset of a dense subgroup of $U$ then the semigroup $S = H \cup \{0\}$ is perfect.

Choose a nonempty open subset $A$ of $U$ and a coset $C$ of a dense subgroup of $U$ such that $A \cap C \subset H$. If we show that the subsemigroup $T$ of $U$ generated (as a semigroup with zero) by $A \cap C$ is perfect then by Lemma 2 it follows that so is $S$. (The proof that $T$ spans $U$ is as above.)

In other words, we may assume that $S$ is generated (as a semigroup with zero) by $A \cap C$. Since $S$ has a zero then showing that it is perfect is equivalent to showing that it is Stieltjes perfect. Clearly, this can be done by showing (a) each completely positive definite function on $S$ extends to a unique completely positive definite function on the subsemigroup $T$ of $U$ generated (as a semigroup with zero) by $A \cap C$ is perfect; and (c) the mapping $\tau \mapsto \tau | S : T_{+}^{*} \to S_{+}^{*}$ is an isomorphism between the measurable spaces $(T_{+}^{*}, \mathcal{A}(T_{+}^{*}))$ and $(S_{+}^{*}, \mathcal{A}(S_{+}^{*}))$.

We shall prove (a) and (b). We do not know whether (c) follows from the hypotheses, but at the end we shall give an argument to replace it.

Concerning (a): Let $G$ be the group $C - C$, which is dense in $U$ by hypothesis. Choosing $a \in A \cap C$, we have $C = a + G$. Let $\mathcal{E}$ be the set of all finite-dimensional linear subspaces of $U$. For every subgroup $K$ of $U$, define

$$K' = \bigcup_{E \in \mathcal{E}} K \cap E.$$  

If $x, y \in K'$ then we can choose $D, E \in \mathcal{E}$ such that $x \in K \cap D$ and $y \in K \cap E$. With $F = D + E$ we have $F \in \mathcal{E}$ and $x, y \in K \cap F$. Since the canonical topology on a finite-dimensional space is a vector space topology it follows that $x - y \in K \cap F \subset K'$. Thus $K'$ is a group. Clearly $K \subset K'$. For $E \in \mathcal{E}$ we have $K \cap E \subset K$. Hence $K' \subset K$.

Let $\mathcal{K}$ be the least set of subgroups of $U$ such that, firstly, $G \in \mathcal{K}$, secondly, if $K \in \mathcal{K}$ then $K' \in \mathcal{K}$, and thirdly, if $\mathcal{J}$ is a nonempty totally ordered subset of $\mathcal{K}$ then $\bigcup \mathcal{J} \in \mathcal{K}$. By Lemma 1, $\mathcal{K}$ is well-ordered under the inclusion ordering and has a greatest element, say $M$, satisfying $M = M'$. By the definition of $M'$ it follows that $M$ is closed. Since $G \subset M$ it follows that $G \subset M$. On the other hand, it is clear that every element of $\mathcal{K}$ is contained in $G$. (Let $M$ be the set of all subgroups of $G$ and note that $G \in M$, if $K \in M$)
then $\mathcal{K}' \subset \overline{\mathcal{K}} \subset \overline{\mathcal{G}}$, that is, $\mathcal{K}' \in \mathcal{M}$, and if $\mathcal{J}$ is a nonempty totally ordered subset of $\mathcal{M}$ then $J \subset \overline{\mathcal{G}}$ for all $J \in \mathcal{J}$, hence $\bigcup \mathcal{J} \subset \overline{\mathcal{G}}$, that is, $\bigcup \mathcal{J} \in \mathcal{M}$; by the definition of $\mathcal{K}$ it follows that $\mathcal{K} \subset \mathcal{M}$.) Thus $M = \overline{G}$.

Suppose we show that for every subgroup $\mathcal{K}$ of $U$, every completely positive definite function on the subsemigroup $S_\mathcal{K}$ of $U$ generated (as a semigroup with zero) by $A \cap (a + \mathcal{K})$ extends to a unique completely positive definite function on $S_{\mathcal{K}'}$. Let $\mathcal{L}$ be the set of all subgroups $L$ of $U$, containing $G$, such that every completely positive definite function on $S_G$ extends to a unique completely positive definite function on $S_L$. Clearly, $G \in \mathcal{L}$. If $L \in \mathcal{L}$ then (by what we are supposed to show) $L' \subseteq \mathcal{L}$. If $\mathcal{N}$ is a nonempty totally ordered subset of $\mathcal{L}$ and if $\phi$ is a completely positive definite function on $S_G$ then for each $\mathcal{N} \in \mathcal{N}$ there is a unique completely positive definite function $\phi_{\mathcal{N}}$ on $S_N$ which extends $\phi$. If $N, P \in \mathcal{N}$ are such that $N \subseteq P$ then the function $\phi_P|S_N$ is a completely positive definite function on $S_N$ which extends $\phi$, hence equal to $\phi_N$. Since the ordering is total it follows that the function $\bigcup_{N \in \mathcal{N}} \phi_{\mathcal{N}}$ is a completely positive definite function on $S_{\bigcup \mathcal{N}}$ extending $\phi$. It is unique since $\bigcup \mathcal{N} \subseteq \mathcal{L}$. Thus $\bigcup \mathcal{N} \in \mathcal{L}$. By the definition of $\mathcal{K}$ it follows that $\mathcal{K} \subseteq \mathcal{L}$. In particular, $\overline{G} \in \mathcal{L}$, that is, each completely positive definite function on $S_G$ extends to a unique completely positive definite function on $S_{\overline{G}}$.

But $\overline{G} = U$. Thus the proof of (a) will be complete when we have shown that if $\mathcal{K}$ is a subgroup of $U$ then each completely positive definite function on the semigroup $S$ with zero generated by $A \cap (a + \mathcal{K})$ extends to a unique completely positive definite function on the semigroup $T$ with zero generated by $A \cap (a + \mathcal{K}')$.

Suppose we show that for each $E \in \mathcal{E}$, each completely positive definite function on the semigroup $S_E$ with zero generated by $A \cap (a + (K \cap E))$ extends to a unique completely positive definite function on the semigroup $T_E$ with zero generated by $A \cap (a + K \cap E)$. If $\phi$ is a completely positive definite function on $S$ then for each $E \in \mathcal{E}$ the function $\phi|S_E$ is a completely positive definite function on $S_E$ and so extends to a unique completely positive definite function $\phi_E$ on $T_E$. If $E, F \in \mathcal{E}$ are such that $E \subseteq F$ then the function $\phi_F|S_E$ is a completely positive definite function on $S_E$ extending $\phi$, hence equal to $\phi_E$ by uniqueness. Hence, the function $\bigcup_{E \in \mathcal{E}} \phi_E$ is a completely positive definite function on $T$ extending $\phi$ and clearly unique.

Thus the proof of (a) will be complete when we have shown that if $E$ is a finite-dimensional rational vector space, if $A$ is an open subset of $E$, and if $C$ is a coset of a subgroup of $E$ then each completely positive definite function on the semigroup $S$ with zero generated by $A \cap C$ extends to a unique completely positive definite function on the semigroup $T$ with zero generated by $A \cap \overline{C}$. Since addition is continuous then $T \subset \overline{T}$.

If we show that each completely positive definite function on $S$ extends to a continuous function on $T$ then the proof of (a) will be complete since the extension is completely positive definite by continuity and is unique since the argument applied to $S$ may be applied to $T$, showing that each completely positive definite function on $T$ is continuous.

Let $G$ be the group $C = C$. Choosing $a \in A \cap C$, we have $C = a + G$. (If we cannot choose such an $a$, i.e., if $A \cap C = \emptyset$, then since $A$ is open it follows that $A \cap \overline{C} = \emptyset$, so
$S = T = \{0\}$, and the whole thing is trivial.) By [4], Lemma 2, $\overline{G} = D \oplus F$ where $D$ is a linear subspace of $E$ and $F$ is a free subgroup of $E$. Since $D$ is open (hence closed) in $\overline{G}$ then continuity of a function on $T$ amounts to continuity on each coset of $D$. Note that $G \cap D$ is dense in $D$.

Since $A$ is open then it is a union of open convex sets. If $B'$ and $B''$ are open convex subsets of cosets of $D$ and if $C'$ and $C''$ are cosets of $G \cap D$ that intersect $B'$ and $B''$, respectively, then

$$(B' \cap C') + (B'' \cap C'') = (B' + B'') \cap (C' + C'').$$

Indeed, by translations we may assume that $B'$ and $B''$ are open convex subsets of $D$ and that $C' = C'' = G \cap D$. The inclusion of the left-hand side of $(\ast)$ in the right-hand side is trivial. For the converse, suppose $x \in (B' + B'') \cap G \cap D$; we have to find $y \in B' \cap G \cap D$ and $z \in B'' \cap G \cap D$ such that $x = y + z$. Since $G \cap D$ is a group, this is equivalent to finding $y \in B' \cap G \cap D$ such that $x - y \in B''$. In other words, we only have to show that the set $G \cap D \cap B$ is nonempty where $B = B' \cap (x - B'')$. Since $B$ is open and $G \cap D$ is dense in $D$ then it suffices to show $B \not= \emptyset$. But this is just another way of stating the fact that $x \in B' + B''$.

Since the sum of two open convex sets is an open convex set, and since the sum of two cosets of $G \cap D$ is a coset of $G \cap D$, then we see from $(\ast)$ that the set $B$ of all nonempty intersections of open convex subsets of cosets of $D$ and cosets of $G \cap D$ is stable under the formation of the sum of two sets. Since $S$ is generated by a union of sets in $B$ then it follows that, 0 apart, $S$ is a union of sets in $B$.

Suppose we show that if $B \in B$ and if $\varphi$ is a completely positive definite function on the semigroup with zero generated by $B$ then $\varphi|B$ extends to a continuous function on the convex hull of $B$. If now $\varphi$ is a completely positive definite function on $S$ then this may be applied to each $B \in B$ such that $B \subset S$. The extensions that we get agree on the intersections of their domains of definition since if $B'$ and $B''$ are in $B$ then so is $B' \cap B''$ if nonempty.

In other words, it suffices to show that if $E$ is a finite-dimensional rational vector space, if $D$ is a linear subspace of $E$, if $G$ is a dense subgroup of $D$, if $A$ is an open convex subset of a coset of $D$, if $C$ is a coset of $G$, and if $\varphi$ is a completely positive definite function on the semigroup with zero generated by $A \cap C$ then $\varphi|(A \cap C)$ extends to a continuous function on $A$. It clearly suffices to show that $A$ is covered by open convex subsets $B$ such that

$$|\varphi(x) - \varphi(y)| \leq P \|x - y\|^{1/4}$$

for all $x, y \in B \cap C$ for some $P > 0$. We shall show that this is so. We make two remarks in passing. (1) The exponent $1/4$ can be replaced by 1, but we have no direct proof of this. (2) A condition of the form (7) holds on every set $B$ of the form $B = a + U$ where $U$ is

* As the anonymous referee kindly pointed out, this point is well worth clarifying. Choose $x' \in B' \cap C'$ and $x'' \in B'' \cap C''$. Define $B'_x = B' - x', C'_x = C' - x'$, and similarly with double primes. Then $B'_x$ and $B''_x$ are open convex subsets of $D$ (both containing 0) while $C'_x$ and $C''_x$ are both equal to $G \cap D$. If now $(\ast)$ holds for the starred objects then by adding $x' + x''$ we get $(\ast)$ as desired.
the open unit ball with respect to some norm such that the set \( a + tU \) is contained in \( A \) for some \( t > 1 \). However, as the greatest such \( t \) tends to 1, the best possible constant \( P \) may tend to \( \infty \). (Otherwise, we would get a contradiction with the fact that a completely positive definite function on an open convex cone in \( \mathbb{Q}_+^k \) does not necessarily extend to a continuous function on the closure.)

The essence of the proof is [4], Lemma 4, by which midpoint convex functions on certain subsets of subgroups of rational vector spaces extend to continuous functions on the closure. Suppose \( B \) is a subset of \( A \) of the form \( B = a + U \) where \( U \) is the open unit ball for some norm \( \| \cdot \| \) on \( D \) and where \( a \in C \) (so \( C = a + G \)). Consider the problem of showing that the function \( \varphi|F \) is midpoint convex, where \( F = (B \cap C) + (B \cap C) + (B \cap C) = (B + B + B) \cap (C + C + C), \text{ cf. } (\#). \) Suppose \( x, y \in F \) are such that \( \frac{1}{2}(x + y) \in F \); we wish to show

\[
\varphi\left(\frac{1}{2}(x + y)\right) \leq \frac{1}{2}(\varphi(x) + \varphi(y)).
\]

In fact it is true, but we have no direct proof of this. Suppose we can find \( r, s, t \in B \cap C \) such that

\[
x = r + 2s, \quad \frac{1}{2}(x + y) = r + s + t, \quad \text{ and } \quad y = r + 2t.
\]  \hspace{1cm} (8)

Note that if any two of these equations hold, so does the third. Since \( \varphi \) is completely positive definite then the function \( z \mapsto \varphi(r + z) \) is positive definite, so by the Cauchy–Schwarz inequality, \( \varphi(\frac{1}{2}(x + y)) = \varphi(r + s + t) \leq \sqrt{\varphi(r + 2s)\varphi(r + 2t)} = \sqrt{\varphi(x)\varphi(y)} \leq \frac{1}{2}(\varphi(x) + \varphi(y)) \), as desired.

By translations with multiples of \( a \), the problem is equivalent to the following. Suppose \( x, y \in G \cap 3U \) are such that \( \frac{1}{2}(x + y) \in G \); find \( r, s, t \in G \cap U \) such that \( x = r + 2s \) and \( y = r + 2t \). If there is a solution then \( \|x - y\| = \|(r + 2s) - (r + 2t)\| = 2\|x - t\| < 4 \). So strengthen the assumption to \( x, y \in G \cap 2U \). Then there is a solution. Indeed, the problem is to find \( s \in G \cap U \) such that \( x - 2s, s + \frac{1}{2}(y - x) \in U \). In other words, we only have to show \( G \cap V \neq \emptyset \) where \( V = U \cap \frac{1}{2}(x + U) \cap \left(\frac{1}{2}(x - y) + U\right) \). Since \( V \) is open and \( G \) is dense, this is equivalent to showing \( V \neq \emptyset \). But \( \frac{1}{2}x \in V \).

Translating back, we get the midpoint convexity of \( \varphi \) on any set of the form \( (3a + 2U) \cap (C + C + C) \) where \( a \in A \cap C \) and where \( U \) is the open unit ball for some norm on \( D \) such that \( a + U \subset A \). Restricting to a norm that is the supremum norm for some identification of \( D \) with \( \mathbb{Q}_+^k \) for some \( k \in \mathbb{N} \), by the proof of [4], Lemma 4, we get a bound of the form

\[
|\varphi(x) - \varphi(y)| \leq M\|x - y\|
\]

for \( x, y \in (3a + \lambda U) \cap (C + C + C) \) for some \( M > 0 \) for some \( \lambda > 0 \) (in fact, for every \( \lambda < 1 \)). Note that the balls \( 3a + \lambda U \) (for all such \( a, U, \) and \( \lambda \)) cover \( A \).

The remainder of the proof of (a) is now easy. Indeed, since \( \varphi \) is completely positive definite then for \( r \in A \cap C \) the function \( s \mapsto \varphi(r + s) \) is positive definite, so by the Cauchy–Schwarz inequality, for \( x, y \in A \cap C \) that both belong to some small enough ball,
This proves that the set \((A \cap C) + (A \cap C) = (A + A) \cap (C + C)\) (cf. \((\ast)\)) is covered by balls \(B\) such that
\[
|\varphi(x) - \varphi(y)| \leq N\|x - y\|^{1/2}
\]
for all \(x, y \in B \cap C\) for some \(N > 0\).

To get (7), repeat the argument with \(r = 0\).

This proves (a).

Now to prove (b). Suppose \(U\) is a rational vector space and \(A\) is a nonempty open subset of \(U\); we have to show that the semigroup \(T\) generated (as a semigroup with zero) by \(A\) is Stieltjes perfect, or equivalently, perfect. Let \(E\) be the set of all finite-dimensional linear subspaces of \(U\). For \(E \in E\) the set \(A \cap E\) is open in \(E\). Let \(T_E\) be the semigroup generated (as a semigroup with zero) by \(A \cap E\). Clearly, \(T = \bigcup_{E \in E} T_E\). Since every \(*\)-semigroup which is generated by the union of its perfect \(*\)-subsemigroups is perfect, it suffices to show that each \(T_E\) is perfect.

Thus it suffices to show that if \(A\) is a nonempty open subset of a finite-dimensional rational vector space \(E\) then the semigroup \(T\) generated (as a semigroup with zero) by \(A\) is perfect. Since \(T\) has a zero then by a result of Nishio and Sakakibara [22] it suffices to show that \(T\) is conelike in the sense that for each \(t \in T\) there is some \(a > 0\) such that \(at \in T\) for all \(a \in \mathbb{Q}\) such that \(a > a\). This is trivial if \(t = 0\), so suppose \(t \neq 0\). Since \(T\) is generated (as a semigroup with zero) by an open set then \(T \setminus \{0\}\) is open. Hence, the set \(J = \{\alpha \in \mathbb{Q}^+ \setminus \{0\} \mid \alpha t \in T\}\) is an open subset of \(\mathbb{Q}_+ \setminus \{0\}\). Since \(T\) is a semigroup, so is \(J\). Note that \(1 \in J\). It is easy to see that every subsemigroup of \(\mathbb{Q}_+\) containing a nonempty open set contains all sufficiently large rationals. Thus \(J\) contains all sufficiently large rationals. This proves (b).

We can now complete the proof of Theorem 1 by showing that \(S\) is indeed Stieltjes perfect. If \(\varphi\) is a completely positive definite function on \(S\) then by (a), \(\varphi\) extends to a (unique) completely positive definite function on \(T\), say \(\Phi\). By (b) there is a (unique) measure \(M \in F_+(T^*_+)\) such that \(\Phi = \mathcal{L}M\). Hence \(\varphi = \Phi|S = (\mathcal{L}M)|S = \mathcal{L}(M^{\ast r,s})\). Thus \(\varphi\) is a Stieltjes moment function. This proves that \(S\) is Stieltjes semiperfect.

It remains to be shown that if \(\mu, \nu \in F_+(S^*_+)\) are such that \(\mathcal{L}\mu = \mathcal{L}\nu\) then \(\mu = \nu\). We note that \(S^\times\) is a measurable subset of \(S^*_+\). Indeed, it is equal to \(G_s := \{\sigma \in S^*_+ \mid \sigma(s) > 0\}\) for each \(s \in H\). This is because every character on a \(*\)-archimedean semigroup is nowhere zero. For definiteness, we consider only the most difficult case, which is the case that \(0 \notin H\). Then \(S^*_+ = S^\times \cup \{\theta_S\}\) where \(\theta_S\) is the indicator function of the set \(\{0\}\) as a subset of \(S\). If we like, we may identify \(S^\times\) with \(H^*_+ (= H^\times)\) by identifying \(\sigma \in S^\times\) with \(\sigma|H\). Denoting by \(\varepsilon_s\) the Dirac measure at a point \(x,\) we can write uniquely
\[
\mu = a \varepsilon_{\theta_S} + \mu' \quad \text{and} \quad \nu = b \varepsilon_{\theta_S} + \nu'
\]
where \(a, b \geq 0\) and \(\mu', \nu' \in F_+(S^\times)\). Since, in particular, \(\mathcal{L}\mu(0) = \mathcal{L}\nu(0)\) then
\[
a + \mu'(S^\times) = b + \nu'(S^\times).
\]
If we show \(\mu' = \nu'\) then it follows that \(a = b\), so \(\mu = \nu\).
Thus we only have to show \( \mu' = \nu' \). We saw that \( S \) and \( T \) span the same space. As in the proof of Lemma 2 it follows that the mapping \( \pi_{T,S} : T^\infty \rightarrow S^\infty \) is an isomorphism of the measurable spaces \( (T^\infty, \mathcal{A}(T^\infty)) \) and \( (S^\infty, \mathcal{A}(S^\infty)) \). Let \( j \) be its inverse. It then suffices to show \( (\mu')^j = (\nu')^j \).

We must admit that we have not been able to decide whether from the hypothesis that \( 0 \notin H \) it follows that \( 0 \notin A \). Be that as it may, let \( \langle A \rangle \) be the subsemigroup of \( T \) generated by \( A \). Then \( T = \langle A \rangle \cup \{0\} \). Now \( \langle A \rangle \), being an ideal of the Stieltjes determinate semigroup \( T \), is Stieltjes determinate (cf. the proof in [12] that every ideal of a determinate semigroup is determinate). Hence it suffices to show \( [\mathcal{L}((\mu')^j)](\langle A \rangle) = [\mathcal{L}((\nu')^j)](\langle A \rangle) \). We have

\[
\langle A \rangle = \bigcup_{n=1}^{\infty} A + \cdots + A,
\]

so it suffices to show

\[
[\mathcal{L}((\mu')^j)] A + \cdots + A = [\mathcal{L}((\nu')^j)] A + \cdots + A
\]

for each \( n \in \mathbb{N} \). Now the functions involved are continuous. (By the definition of the topology, this is another way of saying that they are continuous on every finite-dimensional linear subspace; but it is well-known that if \( E \) is a finite-dimensional rational vector space, if \( \text{Hom}(E, \mathbb{R}) \) is the set of all additive real-valued functions on \( E \), if \( G \) is an open subset of \( E \), and if \( \lambda \) is a measure on \( \text{Hom}(E, \mathbb{R}) \) such that \( \int e^{\xi(s)} d\lambda(\xi) < \infty \) for all \( s \in G \), then the function \( s \mapsto \int e^{\xi(s)} d\lambda(\xi) : G \rightarrow \mathbb{R}_+ \) is continuous.) Since the set

\[
H_n = \overline{(A \cap C) + \cdots + (A \cap C)} = \overline{A + \cdots + A \cap C + \cdots + C}
\]

(cf. (9)) is dense in \( A + \cdots + A \) then it suffices to show that the two functions involved coincide on \( H_n \). But \( H_n \) is a subset of \( H \), and the two functions involved coincide on \( H \) since, firstly, \( [\mathcal{L}((\mu')^j)]|H = [\mathcal{L}((\mu')^j)]|S|H[\mathcal{L}((\mu')^j)\pi_{T,S}]|H = (\mathcal{L} \mu')|H \) and similarly for \( \nu' \) instead of \( \mu' \), and secondly, \( (\mathcal{L} \nu')|H = (\mathcal{L} \nu')|H \) since for \( s \in H, \theta_s(s) = 0 \), so \( \mathcal{L} \mu'(s) = \mathcal{L}(\mu - a \varepsilon_s)(s) = \mathcal{L}(\mu)(s) - \theta_s(s) = \mathcal{L}(\mu)(s) \) and similarly for \( \nu' \) instead of \( \mu' \), whence the desired equality by the hypothesis that \( L \mu L \nu \). This completes the proof of Theorem 1.

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