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Lectures on Dispersionless Integrable Hierarchies

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Preface

This note is based on a series of lectures delivered at Rikkyo University, Tokyo, from the 24th till the 28th of June, 2013. The aim was to give a short survey on dispersionless integrable hierarchies and their connection with complex analysis. In particular there was a pedagogical purpose to show students how complex analysis which they had learned was applied to other areas of mathematics.

As most of audiences were graduate/undergraduate students who were not familiar with integrable systems and physics, I added a brief introduction on the integrable systems at the beginning of the lecture. To make this note readable also to those who have never heard about integrable systems, I included this introductory part here. It is too elementary for experts, but some historical facts might be interesting to young researchers.

On the contrary, I wrote about complex analysis (the Riemann mapping theorem, the Löwner type equations, the harmonic moments, the Laplacian growth problem and so on) in the latter half of this note but quite inadequately, since I am merely a novice in this domain.

I am very grateful to Professor Michio Jimbo for giving me the opportunity to deliver the lecture at Rikkyo University. I also thank Saburo Kakei, Yoko Shigyo and Teruhisa Tsuda who attended the lecture and helped me in preparing this note. Michiaki Onodera, Jun'ichi Shiraishi and Anton Zabrodin gave me helpful information and comments to the draft, to which I express special gratitude. I was partly supported by "The National Research University Higher School of Economics' Academic Fund Program in 2013-2014, research grant No.12-01-0075".

Moscow, Russia — Takashi Takebe
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CHAPTER 1

Introduction to integrable hierarchies

In this chapter we give a brief overview of the theory of integrable systems by describing its history, emphasising "why such systems are interesting". Since the whole world of integrable systems is so enormous, it is impossible to mention all of important theories. We have to restrict ourselves to those facts which will be necessary to the main subject of this lecture note, namely, the dispersionless integrable hierarchies.

1.1. Linear wave equation

There are various phenomena called "*waves*": water waves, electromagnetic waves (light, radio waves, X-ray,...), sound waves, seismic waves and so on. Figure 1.1.1 shows typical examples, water wave and longitudinal wave (for example, sound wave), symbolically.

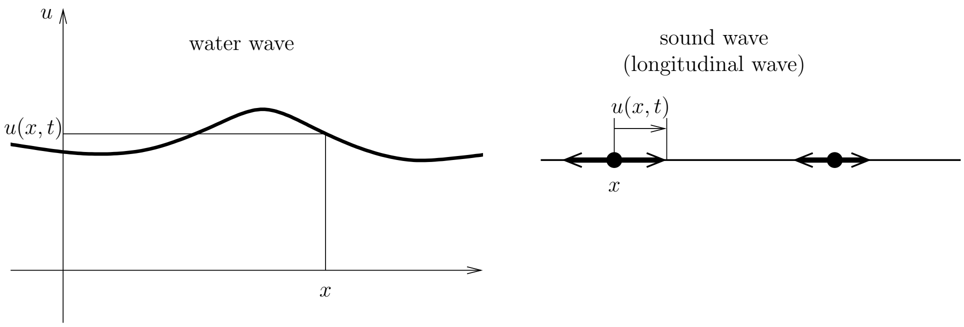

FIGURE 1.1.1. Wave profiles (at time t).

The behaviour of any wave is approximately described by one and the same equation, if its amplitude is sufficiently small. We examine the simplest case, namely the wave travelling in one-dimensional space. Let x be the coordinate in the one-dimensional space and $u(x,t)$ be the amplitude of the wave at time t. Then it satisfies the (one-dimensional) *linear wave equation*:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.1.2}$$

Here the constant c is a parameter determined by properties of the medium in which the wave propagates.

As an example, let us derive this equation for a limit of a one-dimensional system of particles. Assume that there is a chain of particles with mass m connected by springs (Figure 1.1.3).

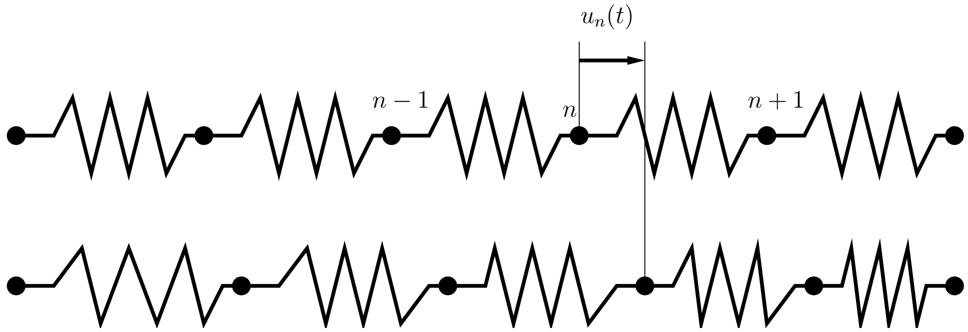

FIGURE 1.1.3. Vibrating chain of springs.

The "amplitude" in this case is the displacement $u_n(t)$ of the n-th particle from its original position. The spring between the n-th and the $n+1$-st particles gets extended by $u_{n+1} - u_n$. (If it is negative, it means that the spring is compressed.) We assume that the force of the spring extended by u is equal to $F(u)$. Then, according to Newton's second law the equation of motion of the n-th particle is

$$m\frac{d^2u_n}{dt^2} = F(u_n - u_{n-1}) - F(u_{n+1} - u_n). \tag{1.1.4}$$

The first term in the right hand side is the force by the spring left to the particle in Figure 1.1.3, while the second term is by the right spring.

It is known that, if the displacement u of the spring is small, the force is proportional to u (*Hooke's law*): $F(u) = -ku$, where k is a constant factor characteristic of the spring. In this case the equation of motion (1.1.4) becomes

$$m\frac{d^2u_n}{dt^2} = -k(u_n - u_{n-1}) + k(u_{n+1} - u_n). \tag{1.1.5}$$

In order to take the continuous limit of this equation, we change the coordinate from the discrete $n \in \mathbb{Z}$ to the continuous $x \in \mathbb{R}$: we denote the distance between neighbouring particles at the equilibrium position, i.e., when the particles are at rest, by Δ and assume that the n-th particle's coordinate is $x = n\Delta$. Then the displacement $u_n(t)$ is interpreted as the amplitude $u(x,t) = u(n\Delta, t)$ of a continuous "chain" or "string".

In the limit $\Delta \to 0$ the extension $u_{n+1} - u_n$ of the spring becomes

$$\begin{aligned} u_{n+1}(t) - u_n(t) &\to u(x+\Delta,t) - u(x,t) \\ &= u(x,t) + \Delta\frac{\partial u}{\partial x}(x,t) + \frac{\Delta^2}{2!}\frac{\partial^2 u}{\partial x^2}(x,t) + O(\Delta^3) - u(x,t) \\ &= \Delta\frac{\partial u}{\partial x}(x,t) + \frac{\Delta^2}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + O(\Delta^3), \end{aligned}$$

because of Taylor's theorem. Similarly we have

$$u_n(t) - u_{n-1}(t) = \Delta\frac{\partial u}{\partial x}(x,t) - \frac{\Delta^2}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + O(\Delta^3)$$

Substituting the above expressions into (1.1.5), we have

$$m\frac{\partial^2 u}{\partial t^2} = k\Delta^2\frac{\partial^2 u}{\partial x^2} + O(\Delta^3).$$

Scaling m and k so that the ratio $k\Delta^2/m$ is constant (which eventually becomes c^2), we take the limit $\Delta \to 0$ and obtain the wave equation (1.1.2).

The most important feature of the equation (1.1.2) is its *linearity*. The equation is linear in u and hence the solution space (= the kernel of the linear operator $\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$) is a linear space. Therefore the sum $u_1 + u_2$ of two solutions u_1 and u_2 is a solution, and a scalar multiple λu of a solution u ($\lambda \in \mathbb{R}$) is also a solution. This property is called the *superposition principle* in physics.

Because of this linearity we can analyse the linear wave equation (1.1.2) thoroughly and understand its solution quite well.

For example, let us assume that the domain of (x,t) is the whole plane $\mathbb{R}^2$. Then any solution is written in the form

$$u(x,t) = f(x-ct) + g(x+ct) \tag{1.1.6}$$

with appropriate functions f and g in one variable[1]. In fact, by changing the coordinate system from (x,t) to $(\xi,\eta) = (x-ct, x+ct)$, we can rewrite the equation (1.1.2) in the form

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} = 0.$$

This means that the function

$$\frac{\partial u}{\partial \eta} = G(\eta) \tag{1.1.7}$$

[1]Of course, f and g should be sufficiently differentiable, but we do not linger on this kind of conditions here.

is a function solely in η and does not depend on ξ. Integrating (1.1.7) with respect to η, we have

$$u = \int_0^\eta G(\eta')\, d\eta' = g(\eta) - g(0),$$

where g is an indefinite integral of G. The integration constant $g(0)$ cannot depend on η but may depend on ξ. Denoting $-g(0)$ by $f(\xi)$, we have $u(\xi, \eta) = f(\xi) + g(\eta)$, which is nothing but (1.1.6). This solution is called *d'Alembert's* solution.

The first part $f(x-ct)$ in (1.1.6) is a wave propagating to the right $x \to \infty$ (cf. Figure 1.1.8), while $g(x+ct)$ is a wave propagating to the left.

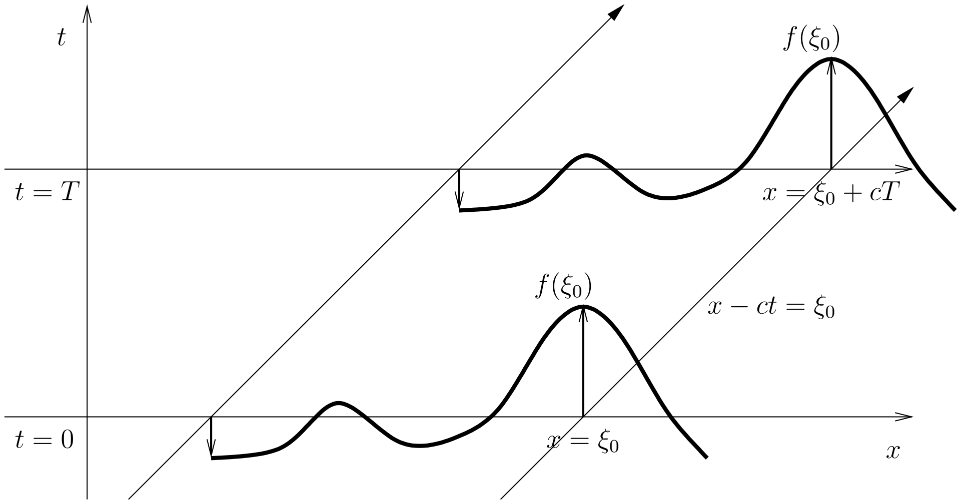

FIGURE 1.1.8. Right moving wave.

Actually, the right mover $f(x-ct)$ is a solution of the equation

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u(x,t) = 0, \tag{1.1.9}$$

which is a "factor" of the wave equation (1.1.2) in the sense

$$\begin{aligned}\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \\ &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right).\end{aligned}$$

In the same way $g(x+ct)$ satisfies another factor of (1.1.2),

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x,t) = 0. \tag{1.1.10}$$

1.2. Discovery of integrable systems

However, there is no way that the real world is so naive and simple! In fact, if the amplitude of a wave gets large, the system obeys *non*-linear equation, solutions of which no longer form a linear space. There is no good theory applicable to all general nonlinear partial differential equations and has to attack the problem one by one, or use the numerical method.

When the very first electronic computers became available to physics in the middle of the twentieth century, three physicists, Fermi, Pasta and Ulam, made use of this opportunity to compute the behaviour of the nonlinear lattice. They added perturbation terms (quadratic or cubic in u_n's) to (1.1.5) and solved the equation by a computer.

From their intuition as physicists, they expected that the system would become chaotic because of interaction of modes (states which were independent when the system were linear) by the nonlinear terms. But, contrary to this intuitive expectation, their result [**FPU**][2] quite unexpectedly showed a quasi-periodic behaviour.

It was so stimulating to physicists and as a result two important questions were posed. One is "what about continuous system?" and the other is "is there a nonlinear lattice which has periodic solutions rigorously (i.e., not only confirmed numerically but *proved* mathematically)?". The former was by Zabusky and Kruskal [**ZK**] and the latter was by Toda [**To1**].

Zabusky and Kruskal numerically solved the KdV equation[3],

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta^2 \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.2.1}$$

which is one of continuous limits of Fermi-Pasta-Ulam's lattices. The coefficient δ is a parameter, which Zabusky and Kruskal set to 0.022 in their experiment, but by scaling the variables x, t and u we can change the coefficients in this equation arbitrarily. They took $u(x, t = 0) = \cos \pi x$ as the initial value and found that this initial sine curve decomposed into several waves. These waves travel like "particles". They have their own speed (the taller, the faster) and after collisions each wave returns to its original shape.

Because of this particle-like property, Zabusky and Kruskal named each wave a "soliton" = <u>solit</u>ary wave + <u>on</u>.

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[2] See also [**Dau**], according to which there is another person, Tsingou, who contributed this result essentially.

[3] The KdV equation was first introduced by Boussinesq [**Bo**] (p.360) and rediscovered by Korteweg and de Vries [**KdV**], who found a solitary wave solution and a periodic solution.

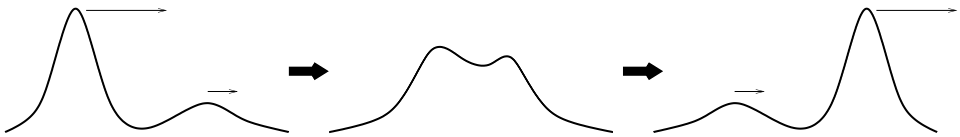

FIGURE 1.2.2. Collision of two solitons.

When one soliton is isolated, it has a form

$$u(x,t) = \frac{c}{\cosh^2 \alpha(x - ct)}. \tag{1.2.3}$$

Hence the speed is proportional to the height c. Actually, the first two terms $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x}$ in (1.2.1) corresponds to the equation of the right moving wave (1.1.9) with the speed c of the wave replaced by u, which means that the wave travels faster when the amplitude is larger. The last term $\delta^2\dfrac{\partial^3 u}{\partial x^3}$ in (1.2.1) is called the *dispersion* term. Because of the dispersion term, the speed of the wave differs by its wavelength. The balance of dispersion and nonlinear effect makes soliton solutions stable.

In 1967 Gardener, Greene, Kruskal and Miura [**GGKM**] found analytic method of solving the KdV equation, so-called the *inverse scattering method*. There are also solutions expressed in terms of elliptic functions or theta functions.

One can find details of the inverse scattering method and elliptic/theta function solutions of these systems in references on integrable systems like [**TD, To2, To3, DJ, To4, Kasm**].

Toda took a different way from the result of Fermi-Pasta-Ulam. He searched for a nonlinear lattice which possesses a periodic solution *rigorously*. The linear lattice (1.1.5) has a periodic solution expressed in terms of trigonometric functions. His idea was, “How about elliptic functions? They have addition formulae like trigonometric functions.” In the summer vacation in 1966 he was at a summer resort in Japan by the sea. Among a few books on mathematics, which he had there, was an encyclopedia of mathematics with a list of formulae of elliptic functions at the end. After several trials with the help of those formulae he found that a lattice (1.1.4) with $F(u) = a(e^{-bu} - 1)$ (a and b are constants) has a periodic solution expressed in terms of elliptic functions. (Details of this discovery are in [**To5**].) When we normalise the constants a, b and m to 1, the equation of motion (1.1.4) becomes

$$\frac{d^2u_n}{dt^2} = e^{u_{n-1}-u_n} - e^{u_n-u_{n+1}}, \tag{1.2.4}$$

which is now called the *Toda lattice*. Soliton solutions of this equation were also found later.

1.3. Why are integrable systems solvable?

It is natural to ask why those equations (*integrable systems*) like the KdV equation and the Toda lattice have good analytic solution in spite of their nonlinearity.

One answer to this question is the existence of conserved quantities. In many typical dynamical systems energy and momentum (and sometimes angular momentum also) are conserved. It was discovered that integrable systems have infinitely many conserved quantities in addition to energy and momentum. (See, for example, Ch.14 of [**To2**], Ch.13 of [**To4**].) These infinitely many conserved quantities constrain the system as parameters of a solution so tightly that the system becomes "solvable".

Another answer is that the system possesses infinite-dimensional symmetries. Actually these two answers are two sides of the same coin, because Noether's theorem guarantees that symmetry is always associated with a conserved quantity. For example, conservation of energy (respectively, momentum, angular momentum) is a consequence of the symmetry of the system with respect to translation in time direction (respectively, translation in space direction, rotation).

One good framework[4] to explain the solvability of an integrable system was proposed in 1981 by Mikio Sato [**Sat1, SS**]. He introduced the *KP hierarchy* (= the Kadomtsev-Petviashvili hierarchy), which is a sort of universal family of integrable nonlinear partial differential equations and includes the KdV equation as a special case. The *Sato theory* claims that the solution space of the KP hierarchy is an infinite-dimensional Grassmann manifold (the *Sato Grassmann manifold*). The "solvability" is explained by linearising the flow of the nonlinear equations on the Grassmann manifold. It also explains

- existence of infinitely many conserved quantities; in fact, since each point of the Sato Grassmann manifold corresponds to a

[4]Because of my insufficient knowledge I cannot mention other frameworks like the Gelfand-Dickey hierarchy [**GDi**], the Drinfeld-Sokolov hierarchy [**DS**], Hamiltonian formalism (for example, [**FT**]) and so on and so forth, which have their own importance.

solution of the KP hierarchy, the coordinates serve as conserved quantities of solutions.
- infinite-dimensional symmetry; as the finite-dimensional Grassmann manifold is a homogeneous space $GL(N)/P$, where P is the subgroup of $GL(N)$ consisting of block upper triangular matrices, the Sato Grassmann manifold is also a sort of homogeneous space. From this view point, Date, Kashiwara, Jimbo and Miwa [**DJKM**] constructed the theory of transformation groups for soliton equations. (See also [**DJM**].)

Later in 1984 Ueno and Takasaki [**UT**] introduced the *Toda lattice hierarchy*, which is the Toda lattice version of the KP hierarchy.

Chapter 2 of this lecture note is a brief review of the Sato Theory of the KP hierarchy, and Chapter 3 is on the Toda lattice hierarchy.

1.4. Dispersionless integrable systems

Now, relax. Let us watch the KdV equation[5]

$$\frac{\partial u}{\partial t} - 3u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3} = 0, \tag{1.4.1}$$

"relaxed from afar". By this we mean introducing so-called "slow variables": $X = \varepsilon x$, $T = \varepsilon t$, where ε is a small parameter. In slow variables equation (1.4.1) becomes

$$\frac{\partial u}{\partial T} - 3u\frac{\partial u}{\partial X} - \varepsilon^2\frac{1}{4}\frac{\partial^3 u}{\partial X^3} = 0 \tag{1.4.2}$$

The limit $\varepsilon \to 0$ of this equation is called the dispersionless limit because the dispersion term $\dfrac{\partial^3 u}{\partial X^3}$ vanishes:

$$\frac{\partial u}{\partial T} - 3u\frac{\partial u}{\partial X} = 0. \tag{1.4.3}$$

This is the *dispersionless KdV equation* or the dKdV equation for short. Such an equation was studied first by Lebedev and Manin [**LM**] and by Zakharov [**Zakh**] independently around 1980. Krichever [**Kr1**] got to the same equation when he studied Whitham's averaging method.

As we mentioned in the previous section, the soliton solution of the KdV equation exists thanks to the balance of the nonlinear term and the dispersion term. So, the dKdV equation (1.4.3) has no longer soliton solutions. Nevertheless this equation is *solvable*, i.e.,

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[5] We choose the coefficients so that it is embedded in the KP hierarchy without scaling. See a remark after (1.2.1).

- has explicit solutions [**KG**] (by Tsarev's generalised hodograph method), [**Kr2**] (by a Riemann-Hilbert type method);
- has conserved quantities [**LM**];
- the inverse scattering method can be applied [**Zakh**].

The Toda lattice version of such kind of equations was found in [**BF, GDa**] in the study of general relativity. It also appeared in [**GKR, SV, KSSV**] from completely different context (Lie algebras with continual gradings).

In 1990's it turned out that such dispersionless type integrable systems have deep connection with the string theory and the topological conformal field theory in elementary particle physics. There are too many references on this subject, so we refer to the introduction of [**TT1**] or to the introduction of [**TT3**] for references till 1994. The main idea in this connection is that the dispersionless hierarchies describe the leading term in the asymptotic expansion of solutions of the original hierarchies. In particular, an important quantity called the free energy (logarithm of the tau function) of a dispersionless integrable hierarchy describes the tree level part of the partition functions or the matrix integrals.

I beg readers' permission to tell my personal history on this subject. In 1991 during my stay in Leningrad as an exchange graduate student between Japan and the USSR, I visited Moscow for a few weeks, where Krichever showed me the draft of [**Kr2**], which analysed the dKdV equation appearing in the topological conformal field theory[6]. I was quite puzzled by his method of constructing solutions of the dKdV equation. Just after that, Takasaki and I attended a conference in Turku, Finland, where I showed Takasaki my hand-written note of Krichever's draft. (It was still difficult to make photocopies in the USSR then.) This was the start of our collaboration. We could generalise Krichever's construction of solutions to the dispersionless KP hierarchy (Theorem 5.3.1), when we participated the RIMS Research Project on Infinite Analysis in the summer of 1991. In subsequent years we were able to make some contributions to this area, which we summarised in [**TT3**]. Most of the contents in Chapter 4, Chapter 5 and Chapter 8 of this note are taken from this paper.

Around the turn of the century came a real surprise (at least for me). Gibbons and Tsarev [**GTs2**] found a differential equation which

[6]The topological field theory and the theory of Frobenius manifolds are also very closely related to the dispersionless integrable hierarchies. See, for example, [**Dub**] for details.

describes a reduction of the dispersionless KP hierarchy (or the Benney equations) This equation turned out to be a differential equation satisfied by conformal mappings of slit domains and is now called the *chordal Löwner equation*. On the other hand, Mineev-Weinstein, Wiegmann and Zabrodin [**MiWZ**] found that certain interface dynamics of two-dimensional fluid (the *Hele-Shaw flow*) is described by the dispersionless Toda hierarchy. Further study revealed that the Riemann mapping theorem and related complex analysis are deeply involved with the dispersionless integrable hierarchies. But we still do not know why such systems *should* appear in complex analysis.

In Chapter 7 and Chapter 9 we shall give examples of solutions coming from the Riemann mapping theorem. This is the main goal of this note.

CHAPTER 2

KP hierarchy and Sato theory

In this chapter we briefly review Sato's theory of the KP hierarchy. We omit many details, which can be found in many references, for example, [**DJKM, DJM, SN, SS, NT**].

The name "KP hierarchy" comes from the *KP equation* (the *Kadomtsev-Petviashvili equation*),

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - 3u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right), \qquad u = u(x,y,t), \tag{2.0.1}$$

which is a generalisation of the KdV equation. Indeed the KdV equation is contained in the parentheses in the right hand side of (2.0.1) and any solution $u(x,t)$ of the KdV equation automatically satisfies the KP equation when we regard $u(x,t)$ as a function of (x,y,t) not depending on y. This equation is also a soliton equation and integrable.

Mikio Sato's theory of the KP hierarchy tells us that we can reveal the true nature of this integrable equation when we consider not only this equation alone but also infinitely many equations (which corresponds to the infinitely many conserved quantities) together.

2.1. KP hierarchy

The KP hierarchy depends on infinitely many independent variables $t = (t_1, t_2, t_3, \dots)$ and x. We usually identify t_1 with x for simplicity, but sometimes we need to distinguish them. The unknown functions are also infinitely many: $u_i(t)$ $(= u_i(t,x))$ $(i = 2, 3, \dots)$. They are encapsulated into a generating series called the *Lax operator*,

$$L = \partial + u_2(t)\partial^{-1} + u_3(t)\partial^{-2} + \cdots = \sum_{i=0}^{\infty} u_i(t)\partial^{1-i}, \tag{2.1.1}$$

where we set $u_0 = 1$, $u_1 = 0$.

Here $\partial = \dfrac{\partial}{\partial x}$, but what are $\partial^{-1}, \partial^{-2}, \dots$? They are called *microdifferential operators*[1]. Though they are named "operators", they are nothing more than "symbols", or exactly speaking, elements of an algebra. We consider objects like

$$A = a_N(t,x)\partial^N + a_{N-1}(t,x)\partial^{N-1} + \cdots = \sum_{n \le N} a_n(t,x)\partial^n.$$

The integer N in this expression is called the *order* of A, when $a_N \neq 0$.

The sum of microdifferential operators is defined as usual and the product is defined by the following rules:

$$\partial^m \circ \partial^n = \partial^{m+n},$$
$$\partial^n \circ f(x) = \sum_{r=0}^{\infty} \binom{n}{r} f^{(r)} \partial^{n-r}. \tag{2.1.2}$$

Here $f^{(r)}$ is the r-th derivative of f and the binomial coefficient is defined by

$$\binom{n}{r} := \begin{cases} 1 & (r=0), \\ \dfrac{n(n-1)\cdots(n-r+1)}{r!} & (r \neq 0). \end{cases} \tag{2.1.3}$$

Note that this definition is valid for any $n \in \mathbb{Z}$ (even for $n < 0$). Of course the above definition coincides with the ordinary composition of differential operators when $n \geqq 0$. For example, let $n = 2$ and consider the action of the operator $\partial^2 \circ f(x)$ on a function $g(x)$. Because of the Leibniz rule, we have

$$\begin{aligned}(\partial^2 \circ f)g &= \partial^2(fg) \\ &= (\partial^2 f)g + 2(\partial f)\,(\partial g) + f\,(\partial^2 g) \\ &= (f\partial^2 + 2f'\partial + f'')g,\end{aligned}$$

namely $\partial^2 \circ f = f\partial^2 + 2f'\partial + f''$, which coincides with (2.1.2) for $n = 2$. Hereafter we omit $\circ$ for the product.

Since this multiplication is noncommutative, the commutator,

$$[A, B] := AB - BA, \tag{2.1.4}$$

[1] The name "pseudo-differential operator" seems to be more popular. In this book, however, following Sato's original works [**Sat1**] and [**SS**], we use the name "microdifferential operator" as in [**Scha**] and [**Kash**].

is not trivial. For example, the commutator of ∂ and L of the form (2.1.1) is

$$[\partial, L] = \frac{\partial L}{\partial x} := \sum_{i=0}^{\infty} \frac{\partial u_i}{\partial x} \partial^{1-i}. \tag{2.1.5}$$

Another operation to a microdifferential operator is the *truncation* to the differential operator part:

$$A = \sum_{n \in \mathbb{Z}} a_n \partial^n \mapsto A_+ = \sum_{0 \le n} a_n \partial^n.$$

We sometimes use the truncation to the other side: $A_- := A - A_+$.

Now let us proceed to the definition of the KP hierarchy.

Definition 2.1.1. We call the system of differential equations

$$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad B_n = (L^n)_+, \quad n = 1, 2, \dots, \tag{2.1.6}$$

the *Kadomtsev-Petviashvili hierarchy* or the *KP hierarchy* for short.

The left hand side of (2.1.6) is the derivation of L with respect to t_n, which means the differentiation of coefficients of L:

$$\frac{\partial L}{\partial t_n} = \sum_{i=0}^{\infty} \frac{\partial u}{\partial t_n} \partial^{1-i}.$$

Remark 2.1.2. Equation (2.1.6) for $n = 1$ is the reason why we identify t_1 with x. In fact, $B_1 = (L)_+ = \partial$ and therefore (2.1.6) for $n = 1$ is $\dfrac{\partial L}{\partial t_1} = \dfrac{\partial L}{\partial x}$, which means L depends on t_1 and x only through combination $t_1 + x$ and does not depend on $t_1 - x$.

The equations of the form (2.1.6) are called the *Lax equation* and therefore the above definition of the KP hierarchy is called the *Lax representation*. In fact there are other formulation of the KP hierarchy.

Proposition 2.1.3. *(i) When L satisfies* (2.1.6)*, then the following equations hold for all m and n:*

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0. \tag{2.1.7}$$

(ii) Conversely, if B_n ($n = 1, 2, \dots$) satisfy (2.1.7)*, then L satisfies* (2.1.6)*.*

Definition 2.1.4. We call the system (2.1.7) the *Zakharov-Shabat representation* of the KP hierarchy.

PROOF OF PROPOSITION 2.1.3. (i) Because of the Leibniz rule and the Lax equation (2.1.6), we have

$$\begin{aligned}\frac{\partial L^m}{\partial t_n} &= \frac{\partial L}{\partial t_n}L^{m-1} + L\frac{\partial L}{\partial t_n}L^{m-2} + \cdots + L^{m-1}\frac{\partial L}{\partial t_n} \\ &= (B_nL - LB_n)L^{m-1} + L(B_nL - LB_n)L^{m-2} + \cdots \\ &\qquad \cdots + L^{m-1}(B_nL - LB_n) \\ &= [B_n, L^m].\end{aligned} \tag{2.1.8}$$

for all m and n. Decomposing L^m as $L^m = B_m - B_m^c$, where $B_m^c = -(L^m)_-$, we can rewrite (2.1.8) as

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_m^c}{\partial t_n} = [B_n, B_m] - [B_n, B_m^c]. \tag{2.1.9}$$

Interchanging m and n in (2.1.8) and substituting $B_m = L^m + B_m^c$ and $L^n = B_n - B_n^c$ into it, we obtain

$$\begin{aligned}\frac{\partial B_n}{\partial t_m} - \frac{\partial B_n^c}{\partial t_m} &= [L^m + B_m^c, L^n] = [B_m^c, L^n] \\ &= [B_m^c, B_n] - [B_m^c, B_n^c].\end{aligned} \tag{2.1.10}$$

From (2.1.9) and (2.1.10) follows

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = \frac{\partial B_m^c}{\partial t_n} - \frac{\partial B_n^c}{\partial t_m} + [B_m^c, B_n^c]. \tag{2.1.11}$$

Note that the left hand side is the differential operator (a linear combination of ∂^k $(k \geqq 0)$) while the right hand side is the microdifferential operator of negative order (a linear combination of ∂^k $(k < 0)$). Hence both hand sides are zero, which proves (2.1.7).

(ii) As we do not use this statement later, we omit the proof. □

In the above proof we obtained following equations for $m, n = 1, 2, \dots,$ together with (2.1.7):

$$\frac{\partial B_m^c}{\partial t_n} - \frac{\partial B_n^c}{\partial t_m} + [B_m^c, B_n^c] = 0. \tag{2.1.12}$$

EXERCISE 2.1.5. Show that, if we put $u = u_2$, $t_1 = x$, $t_2 = y$, $t_3 = t$, the equation (2.1.7) for $(m, n) = (2, 3)$ implies the KP equation (2.0.1). (Hint: When one expands (2.1.7), $(m, n) = (2, 3)$, it has the form $f(x)\partial + g(x) = 0$, which is equivalent to $f = g = 0$. Eliminate u_3 in these equations.)

EXERCISE 2.1.6. The KP hierarchy with the condition "$L^2 = \partial^2 + u(t)$" (i.e., L^2 does not have the negative order part) is called the *KdV hierarchy*. Show that

(i) L does not depend on t_{2n} ($n = 1, 2, 3, \dots$).

(ii) $u = u_2$ satisfies the KdV equation (1.4.1) with respect to $t_1 = x$ and $t_3 = t$. (Hint: expand (2.1.6) for $n = 3$.)

2.2. Wave function

Equations (2.1.7) and (2.1.12) might be more impressive if we write them as follows:

$$\left[\frac{\partial}{\partial t_m} - B_m, \frac{\partial}{\partial t_n} - B_n\right] = 0, \tag{2.2.1}$$

$$\left[\frac{\partial}{\partial t_m} - B^c_m, \frac{\partial}{\partial t_n} - B^c_n\right] = 0, \tag{2.2.2}$$

which are called the *zero-curvature conditions.* Actually the Lax equation (2.1.6) can be also rewritten in the form of commutator:

$$\left[L, \frac{\partial}{\partial t_n} - B_n\right] = 0. \tag{2.2.3}$$

The equations (2.2.3) and (2.2.1) are the compatibility conditions of the linear system,

$$L\Psi(t; z) = z\Psi(t; z), \tag{2.2.4}$$

$$\frac{\partial \Psi}{\partial t_n}(t; z) = B_n \Psi(t; z), \qquad n = 1, 2, \dots . \tag{2.2.5}$$

Hence if L is a solution of the KP hierarchy, the above system possesses a solution $\Psi(t; z)$.

Remark 2.2.1. Let V be a linear space and $A(x, y)$ and $B(x, y)$ be linear operators on V depending on (x, y). Then, under certain "good" conditions like unique solvability of ordinary differential equations, the system of linear partial differential equations for a V-valued function $f(x, y)$,

$$\left(\frac{\partial}{\partial x} - A(x, y)\right) f(x) = 0, \qquad \left(\frac{\partial}{\partial y} - B(x, y)\right) f(x) = 0,$$

has a unique solution for any initial value $f(0, 0) = f_0 \in V$, if the *compatibility condition*,

$$\left[\frac{\partial}{\partial x} - A(x, y), \frac{\partial}{\partial y} - B(x, y)\right] = 0,$$

holds. This is also true for the system with more than two variables.

Actually, since L and B_n have specific forms, we can prove more precise statement.

PROPOSITION 2.2.2. *Let L be a solution of the KP hierarchy* (2.1.6). *Then there exists a solution of the system* (2.2.4), (2.2.5) *of the form*

$$\begin{aligned} \Psi(t;z) &= \hat{w}(t;z)e^{\zeta(t;z)}, \\ \hat{w}(t;z) &= 1 + w_1(t)z^{-1} + w_2(t)z^{-2} + \cdots, \\ \zeta(t;z) &= \sum_{n=1}^{\infty} t_n z^n. \end{aligned} \tag{2.2.6}$$

This solution is called the *wave function*[2] of the KP hierarchy.

To such a wave function one can assign a microdifferential operator

$$W = 1 + w_1(t)\partial^{-1} + w_2(t)\partial^{-2} + \cdots, \tag{2.2.7}$$

which is called the *wave operator* or the *dressing operator*. Defining the action of ∂^n on $e^{\zeta(t;z)}$ naturally by $\partial^n e^{\zeta(t;z)} = z^n e^{\zeta(t;z)}$, we have

$$\Psi(t;z) = We^{\zeta(t;z)}. \tag{2.2.8}$$

Moreover, it is easy to see from (2.2.4) and (2.2.5) that W satisfies

$$L = W\partial W^{-1}, \tag{2.2.9}$$

$$\frac{\partial W}{\partial t_n} = B_n W - W\partial^n, \qquad (n = 1, 2, \dots). \tag{2.2.10}$$

EXERCISE 2.2.3. Show that if W satisfies (2.2.9) then Ψ defined by (2.2.8) is a solution of (2.2.4). Likewise, show that equation (2.2.10) implies (2.2.5).

In fact, we can prove the existence of W satisfying (2.2.9) and (2.2.10) from the compatibility condition (2.1.12) and then Proposition 2.2.2 is proved as in Exercise 2.2.3.

2.3. Tau function

Now we are in a position to introduce the tau function, one of the most important notion in the Sato theory.

THEOREM 2.3.1. *(i) If $\Psi(t;z)$ is a wave function of the KP hierarchy, it is expressed in terms of a function $\tau(t)$ as*

$$\Psi(t;z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} e^{\zeta(t;z)}, \tag{2.3.1}$$

[2] Recently it is also called the *Baker-Akhiezer function*, but originally this name was for a function on a Riemann surface satisfying certain properties, which makes it a wave function in the above sense.

where

$$t - [z^{-1}] := \left(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, t_3 - \frac{1}{3z^3}, \dots \right), \tag{2.3.2}$$

and the function $\tau(t)$ satisfies the following equation for any t and t'.

$$\oint_{z=\infty} \tau(t - [z^{-1}])\, \tau(t' + [z^{-1}])\, e^{\zeta(t;z) - \zeta(t';z)} \frac{dz}{2\pi i} = 0. \tag{2.3.3}$$

(ii) Conversely, if a function $\tau(t)$ satisfies (2.3.3)*, we obtain a solution of the KP hierarchy by tracing back* $\tau \xrightarrow[(2.3.1)]{} \Psi \xrightarrow[(2.2.6)]{(2.2.7)} W \xrightarrow[(2.2.9)]{} L$.

In (2.3.3) the symbol $\oint \frac{dz}{2\pi i}$ means taking the residue of the power series in z:

$$\oint_{z=\infty} \left(\sum_{n \in \mathbb{Z}} a_n z^n \right) \frac{dz}{2\pi i} = a_{-1}.$$

Definition 2.3.2. The function $\tau(t)$ in Theorem 2.3.1 is called the *tau function* of the KP hierarchy. The equation (2.3.3) is called the bilinear residue identity.

If we compute the residue in (2.3.3), expanding $\tau(t - [z^{-1}])$ and $\tau(t' + [z^{-1}])$, we obtain bilinear differential equations for the tau function. The first two non-trivial equations are

$$\begin{gathered} (D_{t_1}^4 + 3D_{t_2}^2 - 4D_{t_1}D_{t_3})\tau(t) \cdot \tau(t) = 0, \\ (D_{t_1}^3 D_{t_2} + 2D_{t_2}D_{t_3} - 3D_{t_1}D_{t_4})\tau(t) \cdot \tau(t) = 0, \end{gathered} \tag{2.3.4}$$

where $P(D_t)\tau(t) \cdot \tau(t)$ is Hirota's bilinear operator defined by

$$P(D_t) f(t) \cdot g(t) = P(\partial_a)\big(f(t+a)\, g(t-a)\big)|_{a=0}.$$

The generating function of all the bilinear equations obtained from the bilinear residue identity (2.3.3) is

$$\sum_{j=0}^{\infty} p_j(-2a)\, p_{j+1}(\tilde{D}_t) e^{\sum_{l=1}^{\infty} a_l D_t} \tau(t) \cdot \tau(t) = 0. \tag{2.3.5}$$

Here $p_j(t)$ is a polynomial defined by

$$\sum_{j=0}^{\infty} p_j(t) z^j = e^{\zeta(t;z)}, \text{ or } p_j(t) = \sum_{i_1 + 2i_2 + 3i_3 + \cdots = j} \frac{t_1^{i_1}}{i_1!} \frac{t_2^{i_2}}{i_2!} \frac{t_3^{i_3}}{i_3!} \cdots .$$

The equations (2.3.4) or (2.3.5) are called *Hirota's bilinear equations* of the KP hierarchy.

In 1970's Hirota [**H1**] found that soliton equations can be transformed into bilinear differential equations of this kind by suitable change of dependent variables. (See, for example, [**H2**].) Sato, Miwa and Jimbo extended Hirota's bilinearisation to higher order equations in the hierarchy. As a result Sato revealed what mathematical structure was hidden behind them by the following theorem[3] in [**Sat1**].

THEOREM 2.3.3. *The derivatives* $\{s_\lambda(\tilde{\partial})\tau(t)\}_{\lambda\in Y}$ *of the tau function* $\tau(t)$ *form a set of the Plücker coordinates of a point in an infinite-dimensional Grassmann manifold, which is called the* Sato Grassmann manifold. *Hirota's bilinear equations are the Plücker relations among them.*

Conversely, any point of the Sato Grassmann manifold gives a solution of the KP hierarchy.

Here Y is the set of all Young diagrams and $s_\lambda(t)$ is the Schur polynomial corresponding to the Young diagram $\lambda = (\lambda_k \geqq \lambda_{k-1} \geqq \cdots \geqq \lambda_1)$:

$$s_\lambda(t) = \det(p_{\lambda_i - i + j})_{i,j=1,\dots,k}. \tag{2.3.6}$$

Details of this theory are in [**SS, SN, Sat2, Sat3**] or [**NT**].

As is well-known, the finite-dimensional Grassmann manifold is a homogeneous space $GL(N)/P$, where P is a maximal parabolic subgroup of $GL(N)$ and consequently has the $GL(N)$-symmetry. Likewise the Sato Grassmann manifold can be regarded as an infinite-dimensional homogeneous space "$GL(\infty)/P$", but we need to pay attention to possible divergence caused by infinite-dimensionality.

Date, Jimbo, Kashiwara and Miwa considered the infinitesimal $gl(\infty)$-symmetry instead of the group symmetry and constructed the theory of transformation groups of the KP hierarchy in [**DJKM**]. (See [**DJM**], too.) They found that the action of generators of $gl(\infty)$ can be expressed in terms of the so-called vertex operators and that soliton solutions are generated by means of them.

[3] Those who are not familiar with words like "Grassmann manifold" or the "Plücker coordinates/relations" are referred to, for example, [**Fu**] or [**DJM**].

CHAPTER 3

Toda lattice hierarchy

In this chapter we briefly review the Toda lattice hierarchy. Stimulated by Sato's work, Ueno and Takasaki introduced the Toda lattice version of the KP hierarchy in [**UT**]. Exactly speaking, what they introduced contains the two-dimensional analogue of the Toda lattice (1.2.4). The one-dimensional original Toda lattice is obtained by the reduction procedure, which imposes a condition to solutions. (See Exercise 3.1.2.) We omit detailed proofs, which can be found in the original paper [**UT**] or in Takasaki's book [**Taka**].

3.1. Toda lattice hierarchy

The independent variables of the Toda lattice hierarchy are two sets of continuous variables $t = (t_1, t_2, \dots)$, $\bar{t} = (\bar{t}_1, \bar{t}_2, \dots)$ and one discrete variable $s \in \mathbb{Z}$. The unknown functions, $u_i = u_i(t, \bar{t}; s)$, $\bar{u}_i = \bar{u}_i(t, \bar{t}; s)$, are encapsulated in two difference operators, the *Lax operators* L and $\bar{L}$, instead of one microdifferential operator L of the KP hierarchy:

$$
\begin{aligned}
L &= e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + u_3 e^{-2\partial_s} + \cdots , \\
\bar{L} &= \bar{u}_0 e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \bar{u}_3 e^{2\partial_s} + \cdots .
\end{aligned}
\tag{3.1.1}
$$

Here, $e^{n\partial_s}$ is a difference operator acting on a function $f(s)$ of s as

$$
(e^{n\partial_s} f)(s) = f(s+n).
$$

This symbol of the exponentiated differential operator comes from the Taylor expansion formula:

$$
f(s+\hbar) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \partial_s^n f(s) = (e^{\hbar\partial_s} f)(s).
\tag{3.1.2}
$$

Later we shall use this "shift by $\hbar$" but in this chapter we use shifts by integers, which makes it possible to write down everything in terms of matrices. Let us denote a function $f(s)$ of $s \in \mathbb{Z}$ as a column vector

$\begin{pmatrix} \vdots \\ f(-1) \\ f(0) \\ f(1) \\ \vdots \end{pmatrix}$. Then the operator e^{∂_s} corresponds to the shift matrix Λ:

$$\Lambda = \left(\begin{array}{cccc|cccc} \ddots & \ddots & & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & 1 & & & & \\ & & & 0 & 1 & & & \\ \hline & & & & 0 & 1 & & \\ & & & & & 0 & 1 & \\ & & & & & & 0 & \ddots \\ & & & & & & & \ddots \end{array}\right) = (\delta_{i+1,j})_{i,j\in\mathbb{Z}}. \tag{3.1.3}$$

(The horizontal and the vertical lines divide indices into negative and non-negative parts.) In this notation, the Lax operators are matrices of the following forms:

$$L = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ * & 1 & & & & & & \\ * & * & 1 & & & & & \\ * & * & * & 1 & & & & \\ \hline * & * & * & * & 1 & & & \\ * & * & * & * & * & 1 & & \\ * & * & * & * & * & * & \ddots & \end{array}\right), \qquad \bar{L} = \left(\begin{array}{cccc|ccc} \ddots & * & * & * & * & * & * \\ & * & * & * & * & * & * \\ & & * & * & * & * & * \\ \hline & & & * & * & * & * \\ & & & & * & * & * \\ & & & & & * & * \\ & & & & & & \ddots \end{array}\right), \tag{3.1.4}$$

We need two truncation operations, $()_{\geq 0}$ and $()_{<0}$ for difference operators:

$$\left(\sum_{n\in\mathbb{Z}} a_n e^{n\partial_s}\right)_{\geq 0} = \sum_{n\geq 0} a_n e^{n\partial_s}, \qquad \left(\sum_{n\in\mathbb{Z}} a_n e^{n\partial_s}\right)_{<0} = \sum_{n<0} a_n e^{n\partial_s}. \tag{3.1.5}$$

In the matrix notation, the former truncates a matrix to its upper triangular part, while the latter to its strictly lower triangular part.

The definition of the Toda lattice hierarchy is almost the same as that of the KP hierarchy.

DEFINITION 3.1.1. We call the system of difference-differential equations

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad \frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L],$$
$$\frac{\partial \bar{L}}{\partial t_n} = [B_n, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{L}], \tag{3.1.6}$$

the *Toda lattice hierarchy*. Here the difference operators B_n and $\bar{B}_n$ $(n = 1, 2, \dots)$ are defined by

$$B_n = (L^n)_{\geq 0}, \qquad \bar{B}_n = (\bar{L}^n)_{<0}. \tag{3.1.7}$$

As in the KP case, the above definition by the Lax type equations (3.1.6) is called the *Lax representation*, which is equivalent to the *Zakharov-Shabat representation*,

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0, \tag{3.1.8}$$

$$\frac{\partial \bar{B}_m}{\partial t_n} - \frac{\partial B_n}{\partial \bar{t}_m} + [\bar{B}_m, B_n] = 0, \tag{3.1.9}$$

$$\frac{\partial \bar{B}_m}{\partial \bar{t}_n} - \frac{\partial \bar{B}_n}{\partial \bar{t}_m} + [\bar{B}_m, \bar{B}_n] = 0. \tag{3.1.10}$$

EXERCISE 3.1.2. (i) Show that, if we put $\bar{u}_0(s) = \exp(\varphi(s) - \varphi(s-1))$ and $u_1(s) = \partial_{\bar{t}_1}\varphi(s)$, then the Zakharov-Shabat equation (3.1.9) for $m = n = 1$ gives the *two-dimensional Toda equation* (or the *Toda field equation*),

$$\frac{\partial^2}{\partial t_1\, \partial \bar{t}_1}\varphi(s) = e^{\varphi(s)-\varphi(s-1)} - e^{\varphi(s+1)-\varphi(s)}. \tag{3.1.11}$$

(Exactly speaking, "the above substitution of $\bar{u}_0(s)$ and $u_1(s)$ is consistent with (3.1.9) and in addition gives the equation (3.1.11).")

(ii) Show that, if $\varphi(s)$ satisfies

$$\frac{\partial \varphi(s)}{\partial t_1} + \frac{\partial \varphi(s)}{\partial \bar{t}_1} = 0,$$

then $\varphi(s)$ satisfies

$$-\frac{1}{4}\frac{\partial^2 \varphi(s)}{\partial^2 t_0} = e^{\varphi(s)-\varphi(s-1)} - e^{\varphi(s+1)-\varphi(s)}.$$

with respect to the variable $t_0 := (t_1 - \bar{t}_1)/2$. This is the Toda lattice equation (1.2.4) with scaled variables. ($t \mapsto 2t_0$, $n \mapsto s$, $u_n \mapsto -\varphi(s)$.)

Despite the symmetry of the roles of L and $\bar{L}$ in the equations (3.1.6), their forms (3.1.4) are not quite "symmetric". In fact we can change their forms by gauge transformation and make them "symmetric" or even exchange their forms (up to transposition) [**Take1**]. Let g be a diagonal matrix $\operatorname{diag}_s(e^{\alpha\varphi(s)})$, where $\varphi(s)$ is the Toda field in Exercise 3.1.2. The gauge transformation of the Toda lattice hierarchy by g is defined by

(3.1.12)
$$\begin{aligned}
&L \mapsto L^g := g^{-1}Lg, && \bar{L} \mapsto \bar{L}^g := g^{-1}\bar{L}g,\\
&B_n \mapsto B_n^g := g^{-1}B_n g - g^{-1}\frac{\partial g}{\partial t_n}, && \bar{B}_n \mapsto \bar{B}_n^g := g^{-1}\bar{B}_n g - g^{-1}\frac{\partial g}{\partial \bar{t}_n},\\
\text{or } &g^{-1}\left(\frac{\partial}{\partial t_n} - B_n\right)g = \frac{\partial}{\partial t_n} - B_n^g, && g^{-1}\left(\frac{\partial}{\partial \bar{t}_n} - \bar{B}_n\right)g = \frac{\partial}{\partial \bar{t}_n} - \bar{B}_n^g,
\end{aligned}$$

We can replace L, $\bar{L}$, B_n and $\bar{B}_n$ in (3.1.6) with L^g, $\bar{L}^g$, B_n^g and $\bar{B}_n^g$ respectively. When we use the $\alpha = 1/2$-gauge, then (3.1.1) and (3.1.7) change to "symmetric form",

$$\begin{aligned}
L &= u_0 e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + u_3 e^{-2\partial_s} + \cdots,\\
\bar{L} &= \bar{u}_0 e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \bar{u}_3 e^{2\partial_s} + \cdots,\\
u_0(s) &= \bar{u}_0(s+1),\\
B_n &:= (L^n)_{>0} + \frac{1}{2}(L^n)_0, \qquad \bar{B}_n := (\bar{L}^n)_{<0} + \frac{1}{2}(\bar{L}^n)_0.
\end{aligned} \tag{3.1.13}$$

(The truncations $()_{>0}$, $()_0$ are defined similarly as in (3.1.5).) The $\alpha = 1$-gauge exchanges the roles of L and $\bar{L}$ completely:

$$\begin{aligned}
L &= u_0 e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + u_3 e^{-2\partial_s} + \cdots,\\
\bar{L} &= e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \bar{u}_3 e^{2\partial_s} + \cdots,\\
B_n &:= (L^n)_{>0}, \qquad \bar{B}_n := (\bar{L}^n)_{\le 0}.
\end{aligned} \tag{3.1.14}$$

Later we shall encounter the symmetric gauge ($\alpha = 1/2$) in the application of the dispersionless Toda hierarchy in Chapter 9.

3.2. Wave matrices and tau function

In the Toda lattice hierarchy there are two *wave operators* W and $\bar{W}$ instead of one wave operator W for the KP hierarchy. They are of the form

$$W = 1 + w_1(t,\bar{t};s)e^{-\partial_s} + w_2(t,\bar{t};s)e^{-2\partial_s} + \cdots, \tag{3.2.1}$$

$$\bar{W} = \bar{w}_0(t,\bar{t};s) + \bar{w}_1(t,\bar{t};s)e^{\partial_s} + \bar{w}_2(t,\bar{t};s)e^{2\partial_s} + \cdots, \tag{3.2.2}$$

and characterised by the equations

$$L = We^{\partial_s}W^{-1}, \qquad \bar{L} = \bar{W}e^{-\partial_s}\bar{W}^{-1}, \tag{3.2.3}$$

$$\frac{\partial W}{\partial t_n} = B_n W - We^{n\partial_s}, \qquad \frac{\partial \bar{W}}{\partial t_n} = B_n \bar{W}, \tag{3.2.4}$$

$$\frac{\partial W}{\partial \bar{t}_n} = \bar{B}_n W, \qquad \frac{\partial \bar{W}}{\partial \bar{t}_n} = \bar{B}_n \bar{W} - \bar{W}e^{-n\partial_s}, \tag{3.2.5}$$

which corresponds to (2.2.9) and (2.2.10). Multiplying them to $\exp(\zeta(t; e^{\partial_s}))$ and $\exp(\zeta(\bar{t}; e^{-\partial_s}))$, where $\zeta(t; e^{\partial_s}) = \sum_{n=1}^{\infty} t_n e^{n\partial_s}$ and $\zeta(\bar{t}; e^{-\partial_s}) = \sum_{n=1}^{\infty} \bar{t}_n e^{-n\partial_s}$, we obtain the *wave matrices* of the Toda lattice hierarchy,

$$\Psi(t, \bar{t}; s) = W \exp(\zeta(t; e^{\partial_s})), \qquad \bar{\Psi}(t, \bar{t}; s) = \bar{W} \exp(\zeta(\bar{t}; e^{-\partial_s})), \tag{3.2.6}$$

satisfying

$$L\Psi = \Psi e^{\partial_s}, \qquad \bar{L}\bar{\Psi} = \bar{\Psi}e^{-\partial_s}, \tag{3.2.7}$$

$$\frac{\partial \Psi}{\partial t_n} = B_n \Psi, \qquad \frac{\partial \bar{\Psi}}{\partial t_n} = B_n \bar{\Psi}, \tag{3.2.8}$$

$$\frac{\partial \Psi}{\partial \bar{t}_n} = \bar{B}_n \Psi, \qquad \frac{\partial \bar{\Psi}}{\partial \bar{t}_n} = \bar{B}_n \bar{\Psi}. \tag{3.2.9}$$

Actually the Toda field $\varphi(s)$, which appeared in the gauge transformation (3.1.12) (recall that $g = e^{\alpha\varphi(s)}$), is hidden in $\bar{W}$ as $\bar{w}_0(s) = e^{\varphi(s)}$, or equivalently $\varphi(s) = \log \bar{w}_0(s)$.

The rest of the story is almost parallel to (but longer than) that of the KP hierarchy. See [**UT**] or [**Taka**]. We can define the tau function $\tau(t, \bar{t}; s)$, which satisfies the Hirota type bilinear equations. The Sato Grassmann manifold is replaced with a sort of "$GL(\infty)$" ([**ITEP**], [**GM**], [**Take2**]).

CHAPTER 4

Formal asymptotic analysis of KP hierarchy

In the introduction Chapter 1, we have mentioned the dispersionless KdV equation as the equation obtained by limiting procedure "dispersion $\to 0$" from the KdV equation. We can take such a limit of the whole KP hierarchy, but it is more convenient to regard this limit as a quasi-classical limit of a quantum mechanical system.

4.1. $\hbar$-dependent KP hierarchy

First note the resemblance of the equations in the KP hierarchy to the equations in quantum mechanics. (If the reader is not familiar with quantum mechanics, just believe me, there is a great science which describes the microscopic world and is governed by equations shown below.) The Lax equation corresponds to the Heisenberg equation for an operator $\hat{A}$:

$$\frac{\partial L}{\partial t_n} = [B_n, L] \quad \longleftrightarrow \quad -i\hbar\frac{\partial \hat{A}}{\partial t} = [\hat{H}, \hat{A}], \tag{4.1.1}$$

and the linear equations for the wave function (2.2.5) corresponds to the Schrödinger equations for the wave function Ψ:

$$\frac{\partial \Psi}{\partial t_n} = B_n\Psi \quad \longleftrightarrow \quad i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi. \tag{4.1.2}$$

Here the Hamiltonian operator is denoted by $\hat{H}$.

The lacking ingredients in the KP side are $\hbar$ = the Planck constant, a very "small" physical constant, and $i = \sqrt{-1}$. The classical mechanics is recovered when we take the "classical limit" $\hbar \to 0$ and replace the commutator of operators $[\hat{A}, \hat{B}]$ by the Poisson bracket of observables: $\{A, B\}$.

Now let us introduce a "small" parameter $\hbar$ into the KP hierarchy and take the "quasi-classical limit" $\hbar \to 0$. (We do not care about whether the variables take real values or complex values, so we do not introduce $i = \sqrt{-1}$, which complicates the equations.) Taking the resemblance with the Heisenberg equation into account, we consider the following $\hbar$-dependent KP hierarchy.

$$\hbar \frac{\partial L}{\partial t_n} = [B_n, L], \qquad B_n = (L^n)_+, \quad n = 1, 2, \dots . \tag{4.1.3}$$

Just writing this equation makes no sense, since we have not yet specified how L should depend on $\hbar$. In order to obtain a meaningful system for "small $\hbar$", i.e., around $\hbar = 0$, it is sufficient to assume that L has a form

$$L = \hbar\partial + \sum_{n=1}^{\infty} u_{n+1}(\hbar, t, x)(\hbar\partial)^{-n}, \tag{4.1.4}$$

where the coefficients $u_n(\hbar, t, x)$ are formally regular with respect to $\hbar$. This means that they have an expansion of the form $u_n(\hbar, t, x) = u_n^{(0)}(t, x) + \hbar u_n^{(1)}(t, x) + \hbar^2 u_n^{(2)}(t, x) + \cdots$ as $\hbar \to 0$.

Let us import several notions from the theory of linear partial differential equations with suitable modifications, which are convenient to handle "leading terms" in the limit $\hbar \to 0$. The first is the order of the microdifferential operators, but since we have to assign non-zero order to $\hbar$, we call it the *$\hbar$-order*:

$$\operatorname{ord}^\hbar \left(\sum a_{n,m}(t, x)\hbar^n \partial^m \right) := \max\{m - n \mid a_{n,m}(t, x) \neq 0\}. \tag{4.1.5}$$

In particular, $\operatorname{ord}^\hbar \hbar = -1$, $\operatorname{ord}^\hbar \partial = 1$, $\operatorname{ord}^\hbar \hbar\partial = 0$. We define $\operatorname{ord}^\hbar(0) = -\infty$ for convenience. For example, the condition which we imposed on the coefficients $u_n(\hbar, t, x)$ can be restated as $\operatorname{ord}^\hbar(L) = 0$.

Once an order = a filtration is introduced, one of the must-do things in mathematics is to examine its leading term: in algebra we take "graded linear spaces" of filtered linear spaces; in analysis "principal symbols" of linear partial differential operators determine their main property. What we are going to examine is leading terms of the $\hbar$-dependent KP hierarchy in the $\hbar$-order.

As the leading term of a microdifferential operator with respect to the ordinary order is called the symbol, we call the leading term with respect to the $\hbar$-order the *$\hbar$-symbol*. In fact we need two kinds of symbols:

$$\sigma^\hbar \left(\sum a_{n,m}(t, x)\hbar^n \partial^m \right) := \sum_{m-n=\operatorname{ord}(A)} a_{n,m}(t, x)\xi^m \tag{4.1.6}$$

$$\sigma_l^\hbar \left(\sum a_{n,m}(t, x)\hbar^n \partial^m \right) := \sum_{m-n=l} a_{n,m}(t, x)\xi^m. \tag{4.1.7}$$

The first one, $\sigma^\hbar(A)$, is the *principal symbol*, while the second one, $\sigma_l^\hbar(A)$, is the *symbol of order l*. For example, $\sigma^\hbar((\hbar\partial)^2 + \hbar^2\partial) = \xi^2$.

If $\mathrm{ord}^\hbar(A) = l$, then $\sigma^\hbar(A) = \sigma^\hbar_l(A)$, but when $\mathrm{ord}^\hbar(A) < l$, then $\sigma^\hbar_l(A) = 0$ while $\sigma^\hbar(A) \neq 0$. (We do not use $\sigma^\hbar_l(A)$ when $\mathrm{ord}^\hbar(A) > l$.) When it is clear from the context, we sometimes write $\sigma^\hbar$ instead of $\sigma^\hbar_l$.

Basic properties of the $\hbar$-order and the $\hbar$-symbols are as follows:

LEMMA 4.1.1. *Let P_i $(i = 1, 2)$ be two microdifferential operators of finite $\hbar$-order:* $\mathrm{ord}^\hbar(P_i) = l_i$.

(1) $\mathrm{ord}^\hbar(P_1 + P_2) \leqq \max(l_1, l_2) =: l$ *and*

$$\sigma^\hbar_l(P_1 + P_2) = \sigma^\hbar_l(P_1) + \sigma^\hbar_l(P_2). \tag{4.1.8}$$

In particular, $\mathrm{ord}^\hbar(P_1 + P_2) < \max(l_1, l_2)$ *only when* $l_1 = l_2$ *and* $\sigma^\hbar(P_1) + \sigma^\hbar(P_2) = 0$.

(2) $\mathrm{ord}^\hbar(P_1 P_2) = l_1 + l_2$ *and*

$$\sigma^\hbar(P_1 P_2) = \sigma^\hbar(P_1)\sigma^\hbar(P_2). \tag{4.1.9}$$

(3) $\mathrm{ord}([P_1, P_2]) \leqq l_1 + l_2 - 1$ *and*

$$\sigma^\hbar_{l_1+l_2-1}([P_1, P_2]) = \{\sigma^\hbar(P_1), \sigma^\hbar(P_2)\}, \tag{4.1.10}$$

where $\{\ ,\ \}$ *is the* Poisson bracket *defined by*

$$\{f(\xi, x), g(\xi, x)\} = \frac{\partial f(\xi, x)}{\partial \xi}\frac{\partial g(\xi, x)}{\partial x} - \frac{\partial f(\xi, x)}{\partial x}\frac{\partial g(\xi, x)}{\partial \xi}. \tag{4.1.11}$$

For example, the principal symbol of the commutation relation $[\hbar\partial, x] = \hbar$ is the relation $\{\xi, x\} = 1$.

EXERCISE 4.1.2. Prove Lemma 4.1.1.

Now let us take the principal symbols of the $\hbar$-dependent KP hierarchy (4.1.3). First, The principal symbol of the L-operator is the series

$$\mathcal{L} = \sigma^\hbar(L) = \xi + \sum_{n=1}^{\infty} u^{(0)}_{n+1}(t, x)\xi^{-n}, \tag{4.1.12}$$

where $u^{(0)}_n(t, x) := \sigma^\hbar_0(u_n(\hbar, t, x))$. The principal symbols of the Lax equations (4.1.3) are

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \qquad \mathcal{B}_n = (\mathcal{L}^n)_+, \quad n = 1, 2, \dots, \tag{4.1.13}$$

because of Lemma 4.1.1. Here the truncation $(\ )_+$ of a Laurent series in ξ is defined by

$$\left(\sum_{n\in\mathbb{Z}} a_n \xi^n\right)_+ := \sum_{0\leq n} a_n \xi^n. \tag{4.1.14}$$

This system (4.1.13) of partial differential equations for $\mathcal{L}$ is called the *dispersionless KP hierarchy* and this shall be one of our main topics, but for the moment, we keep assuming that we have L satisfying (4.1.3).

The Zakharov-Shabat representation of the $\hbar$-dependent KP hierarchy is

$$\hbar\frac{\partial B_m}{\partial t_n} - \hbar\frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0, \quad m, n = 1, 2, \dots, \tag{4.1.15}$$

as is easily proved. (The proof is the same as that of Proposition 2.1.3.) The principal symbols of them are

$$\frac{\partial \mathcal{B}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial t_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0, \quad m, n = 1, 2, \dots, \tag{4.1.16}$$

which is called the Zakharov-Shabat representation of the dispersionless KP hierarchy.

4.2. Formal WKB analysis of the linear equations

The next important ingredient of the KP hierarchy was the wave operator (2.2.7) and the wave function (2.2.6) which satisfies the linear equations (2.2.4) and (2.2.5). What form should they have in the presence of $\hbar$?

As in the previous section, we introduce $\hbar$ into the linear equations as follows.

$$L\Psi(t; z) = z\Psi(t; z), \tag{4.2.1}$$

$$\hbar\frac{\partial \Psi}{\partial t_n}(t; z) = B_n\Psi(t; z), \qquad n = 1, 2, \dots . \tag{4.2.2}$$

As the L-operator has the form $L = \hbar\partial + u_1(\hbar\partial)^{-1} + \cdots$, we can infer that the equation (2.2.9) should be replaced by

$$L = W(\hbar\partial)W^{-1}. \tag{4.2.3}$$

For those who are familiar with Lie groups and Lie algebras it is evident that W in an operation like $\hbar\partial \mapsto \mathrm{Ad}(W)\hbar\partial := W(\hbar\partial)W^{-1}$ should have the form $W = \exp(\tilde{X})$. If W has such a form, namely, if

$$W = e^{\tilde{X}} = \sum_{n=0}^{\infty} \frac{1}{n!}\tilde{X}^n, \text{ or } \tilde{X} = \log W = -\sum_{k=1}^{\infty} \frac{(1-W)^k}{k}, \tag{4.2.4}$$

then we can rewrite (4.2.3) as follows.

$$L = e^{\mathrm{ad}(\tilde{X})}\hbar\partial := \sum_{n=0}^{\infty} \frac{(\mathrm{ad}(\tilde{X}))^n}{n!}\hbar\partial, \tag{4.2.5}$$

where $\mathrm{ad}(A)B := [A, B]$. Let us explain this for those who have never seen this kind of expression before. We denote the left (respectively, right) multiplication operation $B \mapsto AB$ (respectively, $B \mapsto BA$) by L_A (respectively, R_A). With these notations (4.2.3) is expanded as

$$\begin{aligned} L = W(\hbar\partial)W^{-1} = e^{\tilde{X}}(\hbar\partial)e^{-\tilde{X}} &= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \tilde{X}^m \hbar\partial(-\tilde{X})^n \\ &= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} L_{\tilde{X}}^m R_{-\tilde{X}}^n \hbar\partial \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m+n=k} \frac{k!}{m!n!} L_{\tilde{X}}^m R_{-\tilde{X}}^n \right) \hbar\partial. \end{aligned}$$

Note that L_A and $R_{A'}$ always commute for any A and A'. Hence we have

$$L = \sum_{k=0}^{\infty} \frac{1}{k!} (L_{\tilde{X}} + R_{-\tilde{X}})^k \hbar\partial$$

by the binomial theorem. The operation $L_A + R_{-A}$ is nothing but the commutator $B \mapsto AB - BA = [A, B] = \mathrm{ad}(A)B$. Thus we obtain (4.2.5).

Recall that the commutation operation decrease the $\hbar$-order by 1 because of Lemma 4.1.1 (3): $\mathrm{ord}^\hbar([\tilde{X}, P]) = \mathrm{ord}^\hbar(P) + \mathrm{ord}^\hbar(\tilde{X}) - 1$. Therefore in order to obtain an operator L of the $\hbar$-order 0 by the operation (4.2.5), it is sufficient that the operator $\tilde{X}$ should have the $\hbar$-order 1. Hence we consider W of the form

$$\begin{gathered} W = e^{\frac{1}{\hbar}X}, \qquad \mathrm{ord}^\hbar(X) = 0, \\ X = \chi_1(t)(\hbar\partial)^{-1} + \chi_2(t)(\hbar\partial)^{-2} + \cdots . \end{gathered} \tag{4.2.6}$$

Proposition 4.2.1. *(i) A solution L of the $\hbar$-dependent KP hierarchy* (4.1.3) *is expressed as $L = W(\hbar\partial)W^{-1}$ by an operator of the form* (4.2.6) *satisfying the equations*

$$\hbar \frac{\partial W}{\partial t_n} = B_n W - W(\hbar\partial)^n, \qquad n = 1, 2, \dots . \tag{4.2.7}$$

Conversely, if W of the form (4.2.6) *satisfies* (4.2.7)*, then $L = W(\hbar\partial)W^{-1}$ is a solution of* (4.1.3).

(ii) If an operator W has the form (4.2.6),

$$W e^{\frac{1}{\hbar}\zeta(t;z)} = \exp\left(\frac{1}{\hbar} \big(S(t; z) + O(\hbar^1) \big) \right). \tag{4.2.8}$$

Here the function $S(t; z)$ *has an expansion*

$$S(t; z) = \sum_{n=1}^{\infty} S_n(t) z^{-n} + \zeta(t; z), \qquad \zeta(t; z) = \sum_{n=1}^{\infty} t_n z^n. \tag{4.2.9}$$

(Note that $S(t; z)$ *does not depend on* $\hbar$*.)*

(iii) For the operator W *in (i) the function*

$$\Psi(\hbar; t; z) = W e^{\frac{1}{\hbar}\zeta(t;z)} \tag{4.2.10}$$

satisfies the system (4.2.1), (4.2.2).

Idea of the proof of Proposition 4.2.1. (i) Although the equation is the same as (2.2.10), the proof of the existence of such an operator is more complicated than that of the ordinary KP hierarchy, since we specify the form (4.2.6). The detailed proof can be found in [**TT3**], Proposition 1.7.5.

(ii) By definition $(\hbar\partial)^k e^{\zeta(t;z)/\hbar} = z^k e^{\zeta(t;z)/\hbar}$, which means that applying an operator to $e^{\zeta(t;z)/\hbar}$ replaces $\hbar\partial$ in it by z. In view of this observation, the statement (ii) might seem obvious, but in fact it is by no means trivial. This is because multiplication in the definition of the exponential in the left hand side of (4.2.8),

$$W = e^{X/\hbar} = \sum_{n=0}^{\infty} \frac{X(\hbar, x, \hbar\partial)^n}{\hbar^n n!}, \tag{4.2.11}$$

is non-commutative, while the exponential in the right hand side,

$$e^{(S+\cdots)/\hbar} = \sum_{n=0}^{\infty} \frac{(S+\cdots)^n}{\hbar^n n!}, \tag{4.2.12}$$

is defined by commutative multiplication. The proof is rather lengthy. See Proposition 3.1 of [**TT6**]. (The key idea comes from the theory of symbols of microdifferential operators of infinite order by Aoki, [**Ao**].)

(iii) This is a direct computation as in Exercise 2.2.3. □

Remark 4.2.2. In quantum mechanics solutions of the Schrödinger equation of the form $\Psi = \exp\left(\frac{1}{i\hbar}S\right)$ are called the WKB solutions. The wave function Ψ in Proposition 4.2.1 corresponds to the WKB solutions.

As we discussed above, the L-operator in Proposition 4.2.1 is $L = \hbar\partial + \sum_{n=1}^{\infty} u_{n+1}(\hbar\partial)^{-n} = \exp\left(\frac{1}{\hbar}\operatorname{ad} X\right)\hbar\partial$. Taking the principal symbol of this formula, we have

$$
\begin{aligned}
\mathcal{L} := \sigma^\hbar(L) &= \xi + \sum_{n=1}^{\infty} u_{n+1}^{(0)} \xi^{-n} \\
&= \exp\bigl(\operatorname{ad}_{\{,\}} X_0\bigr)\xi = \sum_{k=0}^{\infty} \frac{(\operatorname{ad}_{\{,\}} X_0)^k}{k!}\xi,
\end{aligned} \tag{4.2.13}
$$

where $u_{n+1}^{(0)} := \sigma^\hbar(u_{n+1})$, $X_0 := \sigma^\hbar(X)$ and $\operatorname{ad}_{\{,\}}(f)(g) := \{f, g\}$. Therefore X_0 plays a role of the wave operator for the dispersionless KP hierarchy.

The next question is: What is the "leading term" of the linear equations (4.2.1) and (4.2.2)? Note that Ψ has the form (4.2.8), which makes it impossible to pick up the leading term naively from the expansion as a power series in $\hbar$. We need first apply operators L, $\hbar\partial/\partial t_n$ and B_n to Ψ and then divide the results by the exponential factor, only after which we can extract the leading term.

To begin with, let us apply $\hbar\partial$ to Ψ.

$$
\begin{aligned}
(\hbar\partial)\Psi &= \hbar\frac{\partial}{\partial x} e^{\frac{1}{\hbar}(S + O(\hbar^1))} \\
&= (\partial S + O(\hbar^1))\Psi.
\end{aligned}
$$

Applying $\hbar\partial$ once more, we obtain

$$
\begin{aligned}
(\hbar\partial)^2\Psi &= \hbar\frac{\partial}{\partial x}(\partial S + O(\hbar^1))\Psi \\
&= (\hbar\partial(\partial S + O(\hbar^1)) + (\partial S + O(\hbar^1))^2)\Psi \\
&= ((\partial S)^2 + O(\hbar^1))\Psi.
\end{aligned}
$$

Inductively we have

$$
(\hbar\partial)^n\Psi = ((\partial S)^n + O(\hbar^1))\Psi, \tag{4.2.14}
$$

for $n \geqq 0$. In fact, this formula is valid also for $n < 0$. The proof is rather technical[1].

Now using the formula (4.2.14), we apply $L = \hbar\partial + \sum_{n=1}^{\infty} u_{n+1}(\hbar\partial)^{-n}$ to Ψ. The result is

$$
L\Psi = (\partial S + u_2^{(0)}(t)(\partial S)^{-1} + u_3^{(0)}(t)(\partial S)^{-2} + \cdots + O(\hbar^1))\Psi. \tag{4.2.15}
$$

[1]Let $n > 0$ and consider $(\hbar\partial)^{-n}\Psi$. By the symbol calculus (Lemma 3.3 of [**TT6**]; $a = (\hbar\partial)^{-n}$, $p = 0$, $b = 1$, $e^{q/\hbar} = W$), we have $(\hbar\partial)^{-n}\Psi = f(\hbar; t; z)\Psi$ for some function f of *finite* $\hbar$-order. Applying $(\hbar\partial)^n$ to the both sides and using the Leibniz rule to the right hand side, one can show that $1 = f(\partial S)^n + O(\hbar)$.

On the other hand, the linear equation (4.2.1) says that this is equal to $z\Psi$. Hence, dividing by Ψ and taking the limit $\hbar \to 0$, we have

$$z = \partial S + u_2^{(0)}(t)(\partial S)^{-1} + u_3^{(0)}(t)(\partial S)^{-2} + \cdots = \mathcal{L}|_{\xi \mapsto \partial S}, \tag{4.2.16}$$

where $\mathcal{L} = \sigma^\hbar(L)$ as in (4.2.13). In a similar manner we obtain

$$\frac{\partial S}{\partial t_n} = \mathcal{B}_n|_{\xi \mapsto \partial S} \tag{4.2.17}$$

from (4.2.2). We regard (4.2.16) and (4.2.17) as the leading terms of the linear equations (4.2.1) and (4.2.2).

Remark 4.2.3. The above argument somehow follows derivation of the Hamilton-Jacobi equations from the correspondence of quantum and classical mechanics.

4.3. Tau function and differential Fay identity

The tau function of the $\hbar$-dependent KP hierarchy is introduced by the formula

$$\Psi(\hbar; t; z) = \frac{\tau(t - \hbar[z^{-1}])}{\tau(t)} e^{\frac{1}{\hbar}\zeta(t;z)}, \tag{4.3.1}$$

which replaces (2.3.1). Comparing the asymptotic form (4.2.8) with (4.3.1), we have

$$\frac{1}{\hbar}(S(t;z) + O(\hbar^1)) = \log\tau(t - \hbar[z^{-1}]) - \log\tau(t) + \frac{1}{\hbar}\zeta(t;z). \tag{4.3.2}$$

As the shift of the argument $f(t_n) \mapsto f(t_n - \hbar z^{-n})$ is expressed by the operator $e^{-\hbar z^{-n}\partial_{t_n}} f(t_n)$ (cf. (3.1.2)), the shift $t \mapsto t - \hbar[z^{-1}]$ is realised by the operator $e^{-\hbar D(z)}$, where

$$D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{t_n}. \tag{4.3.3}$$

Hence the right hand side of the formula (4.3.2) is

$$\begin{aligned} \log\tau(t - \hbar[z^{-1}]) - \log\tau(t) &= e^{-\hbar D(z)} \log\tau(t) - \log\tau(t) \\ = &- \hbar D(z)\log\tau(t) + \frac{(-\hbar D(z))^2}{2!}\log\tau(t) + \cdots \\ = &- \hbar D(z)\log\tau(t) + O(\hbar^2). \end{aligned}$$

Therefore, substituting this into (4.3.2), we have

$$S(t;z) + O(\hbar^1) = -\hbar^2 D(z)\log\tau(t) + (\text{lower order terms}) + \zeta(t;z). \tag{4.3.4}$$

Hence, in order for this to make sense, $\log\tau$ should have an expansion like

$$\text{(4.3.5)}\quad \log\tau(t) = \frac{1}{\hbar^2}F_0(t) + \frac{1}{\hbar}F_1(t) + F_2(t) + \hbar F_3(t) + \cdots = \sum_{n=0}^{\infty} \hbar^{n-2} F_n,$$

and by the expansion (4.2.9), the leading term in (4.3.4) becomes

$$\text{(4.3.6)}\qquad \sum_{n=0}^{\infty} S_n(t) z^{-n} = -D(z)F_0, \text{ i.e., } S_n = -\frac{1}{n}\frac{\partial F_0}{\partial t_n}.$$

This F_0 shall play the role of the tau function (exactly speaking, its logarithm) in the theory of the dispersionless KP hierarchy.

We know that the tau function of the KP hierarchy satisfies the bilinear identity (2.3.3). In the presence of $\hbar$ it is modified as

$$\text{(4.3.7)}\qquad \oint_{z=\infty} \tau(t-\hbar[z^{-1}])\,\tau(t'+\hbar[z^{-1}])\,e^{\frac{1}{\hbar}(\zeta(t;z)-\zeta(t';z))}\frac{dz}{2\pi i} = 0.$$

This characterise the tau function of the $\hbar$-dependent KP hierarchy, but there are infinite terms of both positive and negative powers of $\hbar$ in the expansion of this equation. Therefore in order to find a reasonable "leading term" we have to extract a part which possesses only finitely many negative powers of $\hbar$.

Let us substitute

$$t' = t - \hbar[u^{-1}] - \hbar[v^{-1}] - \hbar[w^{-1}],$$

in (4.3.7), where u, v and w are parameters. Using the Taylor expansion formula of the logarithm,

$$\text{(4.3.8)}\qquad -\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n},$$

we have

$$e^{\frac{1}{\hbar}(\zeta(t;z)-\zeta(t';z))} = e^{\sum \frac{z^n}{nu^n} + \sum \frac{z^n}{nv^n} + \sum \frac{z^n}{nw^n}} = \frac{1}{1-\frac{z}{u}}\frac{1}{1-\frac{z}{v}}\frac{1}{1-\frac{z}{w}}.$$

Thus, substituting this formula into (4.3.7) and taking the residue, we have the *Hirota-Miwa equation* (with $\hbar$),

$$\text{(4.3.9)}\qquad \begin{aligned} &(w-v)\tau(t-\hbar[u^{-1}])\,\tau(t-\hbar[v^{-1}]-\hbar[w^{-1}]) \\ +&(u-w)\tau(t-\hbar[v^{-1}])\,\tau(t-\hbar[w^{-1}]-\hbar[u^{-1}]) \\ +&(v-u)\tau(t-\hbar[w^{-1}])\,\tau(t-\hbar[u^{-1}]-\hbar[v^{-1}]) = 0. \end{aligned}$$

One can use this equation to find the leading terms, but for later use we further reduce the Hirota-Miwa equation to the *differential Fay identity*. This is obtained by multiplying w^{-1} to (4.3.9), differentiating the resulting equation by $a = w^{-1}$ and setting a to 0. For example, we need formulae like

$$\left.\frac{\partial}{\partial a}\right|_{a=0} \tau(t - \hbar[a]) = \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial a}\frac{-\hbar a^n}{n}\right) \left.\frac{\partial \tau}{\partial t_n}(t - \hbar[a])\right|_{a=0} = -\hbar \partial_{t_1} \tau(t).$$

PROPOSITION 4.3.1 (Differential Fay identity [**AvM**]). *If $\tau(t)$ is a tau function of the $\hbar$-dependent KP hierarchy, then it satisfies the following equation.*

(4.3.10)
$$\begin{aligned} &\hbar\partial_{t_1}\tau(t-\hbar[u^{-1}])\,\tau(t-\hbar[v^{-1}]) - \tau(t-\hbar[u^{-1}])\,\hbar\partial_{t_1}\tau(t-\hbar[v^{-1}]) \\ &= (v-u)\big(\tau(t-\hbar[u^{-1}])\,\tau(t-\hbar[v^{-1}]) - \tau(t)\,\tau(t-\hbar[u^{-1}]-\hbar[v^{-1}])\big). \end{aligned}$$

Dividing this equation by $(v-u)\tau(t-\hbar[u^{-1}])\,\tau(t-\hbar[v^{-1}])$, we can rewrite it into the form

$$\begin{aligned} &\frac{\tau(t-\hbar[u^{-1}]-\hbar[v^{-1}])\,\tau(t)}{\tau(t-\hbar[u^{-1}])\,\tau(t-\hbar[v^{-1}])} \\ &\qquad = 1 + \frac{1}{u-v}\left(\frac{\partial}{\partial t_1}\log\tau(t-\hbar[u^{-1}]) - \frac{\partial}{\partial t_1}\log\tau(t-\hbar[v^{-1}])\right). \end{aligned}$$

The logarithm of this equation has an expansion in $\hbar$ with the highest order $\hbar^{-2}$ in view of the asymptotic form of the tau function (4.3.5). Picking up the leading terms, we obtain the following:

PROPOSITION 4.3.2 ([**TT3**]). *If the tau function of the form* (4.3.5) *satisfies the $\hbar$-dependent KP hierarchy, the leading term $F = F_0$ satisfies the* dispersionless Hirota equation*,*

(4.3.11)
$$\sum_{m,n=1}^{\infty} u^{-m}v^{-n}\frac{1}{mn}\frac{\partial^2}{\partial t_m \partial t_n}F = \log\left(1 - \sum_{n=1}^{\infty}\frac{u^{-n}-v^{-n}}{u-v}\frac{1}{n}\frac{\partial^2}{\partial t_1 \partial t_n}F\right),$$

namely,

$$D(u)\,D(v)F = \log\left(1 - \frac{D(u)-D(v)}{u-v}\frac{\partial F}{\partial t_1}\right). \tag{4.3.12}$$

REMARK 4.3.3. Although we obtained the Hirota-Miwa equation (4.3.9) and the differential Fay identity (4.3.10) by specialising variables in the bilinear residue identity (4.3.7), they turn out to be equivalent to the original whole KP hierarchy. It was first proved in [**TT3**] and a shorter proof for the Hirota-Miwa equation was obtained in [**Sh**]. Actually an equivalent statement (without using the tau function) was already mentioned in [**SS**] by Sato and Sato without a proof.

CHAPTER 5

Dispersionless KP hierarchy

In Chapter 4 we considered the leading terms of the KP hierarchy depending on a parameter $\hbar$ and obtained equations like (4.1.13), (4.1.16), (4.2.16), (4.2.17), (4.3.11), which are expressed solely in terms of the leading terms. In fact we can reconstruct the theory without considering the lower order terms. This is the theory of the *dispersionless KP hierarchy*.

5.1. Dispersionless KP hierarchy

The set of independent variables of the dispersionless KP hierarchy is the same as that of the KP hierarchy, namely, x and $t = (t_1, t_2, \dots)$. We can identify x and t_1 almost always (cf. Remark 5.1.2 below), but sometimes we need to distinguish them (in (5.1.6), for example). The unknown functions are $u_i(t)$ $(i = 2, 3, \dots)$ as in the KP case, but actually they correspond to $u_i^{(0)}(t)$ in (4.1.12) of the $\hbar$-dependent KP hierarchy. The Lax operator L of the KP hierarchy is replaced by the *Lax series* $\mathcal{L}$,

$$\mathcal{L} = \mathcal{L}(t;\xi) == \xi + u_2(t)\xi^{-1} + u_3(t)\xi^{-2} + \cdots = \sum_{i=0}^{\infty} u_i(t)\xi^{1-i}, \tag{5.1.1}$$

where $u_0 = 1$, $u_1 = 0$.

Definition 5.1.1. We call the system of differential equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \qquad \mathcal{B}_n = (\mathcal{L}^n)_+, \quad n = 1, 2, \dots, \tag{5.1.2}$$

the *dispersionless KP hierarchy* (or the *dKP hierarchy* for short)[1].

Here the Poisson bracket $\{\ ,\ \}$ is defined by (4.1.11) and the truncation operation $(\ )_+$ is defined by (4.1.14).

Remark 5.1.2. Since $\mathcal{B}_1 = \xi$ and $\{\xi, (\cdot)\} = \partial(\cdot)$, equation (5.1.2) for $n = 1$ means that $\mathcal{L}$ depends on t_1 and x only through combination $t_1 + x$ as in the KP hierarchy. See Remark 2.1.2.

[1] In some literatures the abbreviation "dKP" means the discrete KP hierarchy, but in this lecture note we use "dKP" only for the dispersionless KP hierarchy.

This is the Lax representation of the dispersionless KP hierarchy and it is equivalent to the *Zakharov-Shabat representation*,

$$\frac{\partial \mathcal{B}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial t_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0, \quad m, n = 1, 2, \dots . \tag{5.1.3}$$

The equivalence of (5.1.2) and (5.1.3) is proved in the same way as Proposition 2.1.3.

The substitute of the wave (dressing) operator W in the dispersionless KP hierarchy should be the symbol $\sigma^\hbar(X)$ of the operator X in (4.2.6). It is characterised by the leading order equation of (4.2.5) and (4.2.7).

PROPOSITION 5.1.3. *(i) Let $\mathcal{L}$ be a solution of the dispersionless KP hierarchy* (5.1.2)*. Then there exists a Laurent series χ (dressing function) of the form $\chi(t) = \sum_{n=1}^\infty \chi_n(t)\xi^{-n}$, which satisfies the equations*

$$\mathcal{L} = e^{\mathrm{ad}_{\{,\}}\chi}(\xi), \tag{5.1.4}$$

$$\nabla_{t_n}\chi = -(e^{\mathrm{ad}_{\{,\}}\chi}(\xi^n))_{\le -1}, \ \textit{i.e.,}\ \nabla_{t_n}\chi = \mathcal{B}_n - \mathcal{L}^n, n = 1, 2, \dots, \tag{5.1.5}$$

where

$$\mathrm{ad}_{\{,\}}\chi(\psi) = \{\chi, \psi\}, \qquad \nabla_{t_n}\chi = \sum_{m=0}^\infty \frac{(\mathrm{ad}_{\{,\}}\chi)^m}{(m+1)!}\frac{\partial \chi}{\partial t_n}.$$

Such Laurent series χ is unique up to change $\chi \mapsto H(\chi, \eta)$, where η is a constant Laurent series $\eta = \sum_{n=1}^\infty \eta_n \xi^{-n}$ ($\eta_n \in \mathbb{C}$) and $H(X, Y)$ is the Hausdorff series defined by

$$\exp(\mathrm{ad}_{\{,\}} H(X, Y)) = \exp(\mathrm{ad}_{\{,\}} X)\exp(\mathrm{ad}_{\{,\}} Y).$$

(ii) Conversely, if $\chi(t) = \sum_{n=1}^\infty \chi_n(t)\xi^{-n}$ satisfies (5.1.5)*, then $\mathcal{L}$ defined in* (5.1.4) *is a solution of the dispersionless KP hierarchy* (5.1.2)*.*

This is Proposition 1.2.1 of [**TT3**]. We omit the proof.

In principle, every ingredient of the dispersionless KP hierarchy can be expressed in terms of the dressing function χ, but it is convenient to use the following *Orlov-Schulman series* instead.

$$\mathcal{M} := e^{\mathrm{ad}_{\{,\}}\chi}\left(x + \sum_{n=1}^\infty n t_n \xi^{n-1}\right) = e^{\mathrm{ad}_{\{,\}}\chi} e^{\mathrm{ad}_{\{,\}}\zeta(t;\xi)}(x). \tag{5.1.6}$$

(Here we have to distinguish t_1 and x: $\{\xi, x\} = 1$ but $\{\xi, t_1\} = 0$.) Since (5.1.4) can be rewritten as

$$\mathcal{L} = e^{\mathrm{ad}_{\{,\}}\chi} e^{\mathrm{ad}_{\{,\}}\zeta(t;\xi)}(\xi),$$

it is obvious that $\mathcal{M}$ is the canonical conjugate of $\mathcal{L}$:

$$\{\mathcal{L}, \mathcal{M}\} = 1, \tag{5.1.7}$$

by virtue of the formula $\{e^{\operatorname{ad}_{\{,\}} A} X, e^{\operatorname{ad}_{\{,\}} A} Y\} = e^{\operatorname{ad}_{\{,\}} A}\{X, Y\}$. Moreover, because of (5.1.5), the Orlov-Schulman series satisfies the same Lax equation as $\mathcal{L}$.

$$\frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}, \quad n = 1, 2, \dots . \tag{5.1.8}$$

(From this equation for $n = 1$ follows that, once $\mathcal{M}$ is defined, we do not have to distinguish t_1 and x. See Remark 5.1.2.)

By the definition (5.1.6) the Orlov-Schulman series is a Laurent series in $\mathcal{L}$ as follows.

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + x + \sum_{i=1}^{\infty} v_i \mathcal{L}^{-i-1}, \tag{5.1.9}$$

where $v_i = v_i(t)$ is a function of t.

Remark 5.1.4. Orlov and Schulman introduced an operator corresponding to $\mathcal{M}$ in [**OrS**],

$$M := W \left(x + \sum_{n=1}^{\infty} n t_n \xi^{n-1} \right) W^{-1} = W e^{\zeta(t;\partial)} x \big(W e^{\zeta(t;\partial)} \big)^{-1}.$$

in order to study the symmetry of the KP hierarchy.

Let us define a two-form ω by

$$\omega := d\xi \wedge dx + \sum_{n=1}^{\infty} d\mathcal{B}_n \wedge dt_n. \tag{5.1.10}$$

(When t_1 and x are identified, the first term $d\xi \wedge dx$ is omitted.) Here the exterior derivative d is defined with respect to all the variables x, t_n ($n = 1, 2, \dots$) and ξ:

$$df := \frac{\partial f}{\partial x} dx + \sum_{n=1}^{\infty} \frac{\partial f}{\partial t_n} dt_n + \frac{\partial f}{\partial \xi} d\xi.$$

Proposition 5.1.5. *(i) If $\mathcal{L}$ is a solution of the dispersionless KP hierarchy* (5.1.2) *and $\mathcal{M}$ is the Orlov-Schulman series* (5.1.6)*, then*

$$\omega = d\mathcal{L} \wedge d\mathcal{M}. \tag{5.1.11}$$

(ii) Conversely, if $\mathcal{L}$ of the form (5.1.1) *and $\mathcal{M}$ of the form* (5.1.9) *satisfy* (5.1.11)*, then $\mathcal{L}$ is a solution of the dispersionless KP hierarchy* (5.1.2) *and $\mathcal{M}$ is a corresponding Orlov-Schulman series.*

PROOF. Two-forms $d\xi \wedge dx$, $d\xi \wedge dt_n$, $dx \wedge dt_n$ and $dt_m \wedge dt_n$ ($m, n = 1, 2, \ldots, m < n$) form a basis of the space of two-forms. The coefficients of $d\xi \wedge dx$ in (5.1.11) are

$$1 = \frac{\partial \mathcal{L}}{\partial \xi}\frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial \xi} = \{\mathcal{L}, \mathcal{M}\}. \tag{5.1.12}$$

Comparing the coefficients of $d\xi \wedge dt_n$ and $dx \wedge dt_n$, we have

$$\begin{aligned}
\begin{pmatrix} \dfrac{\partial \mathcal{B}_n}{\partial \xi} \\ \dfrac{\partial \mathcal{B}_n}{\partial x} \end{pmatrix} &= \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial \xi}\dfrac{\partial \mathcal{M}}{\partial t_n} - \dfrac{\partial \mathcal{L}}{\partial t_n}\dfrac{\partial \mathcal{M}}{\partial \xi} \\ \dfrac{\partial \mathcal{L}}{\partial x}\dfrac{\partial \mathcal{M}}{\partial t_n} - \dfrac{\partial \mathcal{L}}{\partial t_n}\dfrac{\partial \mathcal{M}}{\partial x} \end{pmatrix} \\
&= \begin{pmatrix} -\dfrac{\partial \mathcal{M}}{\partial \xi} & \dfrac{\partial \mathcal{L}}{\partial \xi} \\ -\dfrac{\partial \mathcal{M}}{\partial x} & \dfrac{\partial \mathcal{L}}{\partial x} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial t_n} \\ \dfrac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix}
\end{aligned}$$

Let us multiply the inverse of the matrix in the last expression. Note that its determinant is $\left(-\dfrac{\partial \mathcal{M}}{\partial \xi}\right)\left(\dfrac{\partial \mathcal{L}}{\partial x}\right) - \left(\dfrac{\partial \mathcal{L}}{\partial \xi}\right)\left(-\dfrac{\partial \mathcal{M}}{\partial x}\right) = \{\mathcal{L}, \mathcal{M}\} = 1$. So the result is

$$\begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial t_n} \\ \dfrac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial x} & -\dfrac{\partial \mathcal{L}}{\partial \xi} \\ \dfrac{\partial \mathcal{M}}{\partial x} & -\dfrac{\partial \mathcal{M}}{\partial \xi} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathcal{B}_n}{\partial \xi} \\ \dfrac{\partial \mathcal{B}_n}{\partial x} \end{pmatrix} = \begin{pmatrix} \{\mathcal{B}_n, \mathcal{L}\} \\ \{\mathcal{B}_n, \mathcal{M}\} \end{pmatrix}.$$

Namely we obtain the Lax equations (5.1.2) and (5.1.8). Similarly we obtain the Zakharov-Shabat equations from the coefficients of $dx \wedge dt_n$ and $dt_m \wedge dt_n$ in (5.1.11). □

REMARK 5.1.6. In our early work [**TT1**] the equation (5.1.11) was the starting point. The dressing operation in Proposition 5.1.3 and the definition (5.1.6) of $\mathcal{M}$ were first introduced in [**TT3**].

5.2. S-function and tau function

As we have seen in the previous section, the equation of two-forms (5.1.11) is a compact form of the dispersionless KP hierarchy. Moreover it is an important step to define the S-function and the tau function. Let us rewrite (5.1.11) in the following form:

$$d(\mathcal{M}\, d\mathcal{L} + \xi\, dx + \sum_{n=1}^{\infty} \mathcal{B}_n\, dt_n) = 0.$$

According to Poincaré's lemma this means that there exists a function S such that

$$\mathcal{M}\, d\mathcal{L} + \xi\, dx + \sum_{n=1}^{\infty} \mathcal{B}_n\, dt_n = dS. \tag{5.2.1}$$

In other words,

$$\mathcal{M} = \left.\frac{\partial S}{\partial \mathcal{L}}\right|_{x,t \text{ fixed}}, \quad \xi = \left.\frac{\partial S}{\partial x}\right|_{\mathcal{L},t \text{ fixed}}, \quad \mathcal{B}_n = \left.\frac{\partial S}{\partial t_n}\right|_{\mathcal{L},x,t_m (m\neq n) \text{ fixed}}. \tag{5.2.2}$$

Fixing the variable $\mathcal{L}$ is equivalent to regarding $z = \mathcal{L}$ as an independent variable. Solving the second equation in (5.2.2) with respect to z gives

$$\mathcal{L}|_{\xi=\partial_x S} = z, \tag{5.2.3}$$

which is (4.2.16) in Section 4.2. In this context the third equation in (5.2.2) is

$$\left.\frac{\partial S}{\partial t_n}\right|_{\xi=\partial_x S} = \mathcal{B}_n, \tag{5.2.4}$$

which is nothing but (4.2.17).

Remark 5.2.1. We can interpret S as a generating function of the canonical transformation $(\xi, x) \mapsto (\mathcal{L}, \mathcal{M})$.

Actually we can write down S directly in terms of $\mathcal{L}$ and $\mathcal{M}$.

Proposition 5.2.2 ([**TT1**] Proposition 3). *S is given by*

$$S = \sum_{i=1}^{\infty} t_i \mathcal{L}^i + x\mathcal{L} + \sum_{i=1}^{\infty} S_i \mathcal{L}^{-i}, \tag{5.2.5}$$

$$S_i = -\frac{1}{i} v_i. \tag{5.2.6}$$

Here v_i's are coefficients in the expansion (5.1.9) *of $\mathcal{M}$.*

Proof. If S is defined by (5.2.5) and (5.2.6), then the first equation in (5.2.2) is obvious by construction. The essential part of the proof is to show the third and the second equations, which can be rewritten as

$$\mathcal{B}_n = \mathcal{L}^n - \sum_{i=1}^{\infty} \frac{1}{i} \frac{\partial v_i}{\partial t_n} \mathcal{L}^{-i}, \tag{5.2.7}$$

$$\xi = \mathcal{L} - \sum_{i=1}^{\infty} \frac{1}{i} \frac{\partial v_i}{\partial x} \mathcal{L}^{-i}. \tag{5.2.8}$$

In order to prove (5.2.7) let us differentiate (5.1.9) with respect to t_n.

$$\frac{\partial \mathcal{M}}{\partial t_n} = n\mathcal{L}^{n-1} + \sum_{i=1}^{\infty} \frac{\partial v_i}{\partial t_n}\mathcal{L}^{-i-1} + \left.\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\right|_{x,t\text{ fixed}} \frac{\partial \mathcal{L}}{\partial t_n}. \tag{5.2.9}$$

We can extract $\dfrac{\partial v_i}{\partial t_n}$ from this equation as follows.

$$\frac{\partial v_i}{\partial t_n} = \operatorname{Res} \mathcal{L}^i \left(\frac{\partial \mathcal{M}}{\partial t_n} - \left.\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\right|_{x,t\text{ fixed}} \frac{\partial \mathcal{L}}{\partial t_n} \right) d_\xi \mathcal{L}. \tag{5.2.10}$$

Here d_ξ is the exterior differential with respect to ξ and the residue in this expression means picking up the coefficient of ξ^{-1}. Thanks to the invariance of the residue with respect to the coordinate change,

$$\operatorname{Res} f(\xi) d\xi = \operatorname{Res} f(g(\xi)) d_\xi g(\xi),$$

the residue in (5.2.10) is the coefficient of $\mathcal{L}^{-i-1}$ in the power series of $\mathcal{L}$ in the parentheses, and thus (5.2.10) holds.

A part of the right hand side of (5.2.10) is rewritten as

$$\begin{aligned}
&\left(\frac{\partial \mathcal{M}}{\partial t_n} - \left.\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\right|_{x,t\text{ fixed}} \frac{\partial \mathcal{L}}{\partial t_n} \right) d_\xi \mathcal{L} \\
&= \{\mathcal{B}_n, \mathcal{M}\} d_\xi \mathcal{L} - \{\mathcal{B}_n, \mathcal{L}\} d_\xi \mathcal{M} \\
&= \left(\frac{\partial \mathcal{B}_n}{\partial \xi}\frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{B}_n}{\partial x}\frac{\partial \mathcal{M}}{\partial \xi} \right) \frac{\partial \mathcal{L}}{\partial \xi} d\xi - \left(\frac{\partial \mathcal{B}_n}{\partial \xi}\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \mathcal{B}_n}{\partial x}\frac{\partial \mathcal{L}}{\partial \xi} \right) \frac{\partial \mathcal{M}}{\partial \xi} d\xi \\
&= \frac{\partial \mathcal{B}_n}{\partial \xi}\{\mathcal{L}, \mathcal{M}\}\, d\xi = \frac{\partial \mathcal{B}_n}{\partial \xi} d\xi = d_\xi \mathcal{B}_n,
\end{aligned}$$

because of the Lax equations (5.1.2), (5.1.8) and the canonical conjugation relation (5.1.7). Substituting this into (5.2.10), we have

$$\frac{\partial v_i}{\partial t_n} = \operatorname{Res} \mathcal{L}^i d_\xi \mathcal{B}_n = -\operatorname{Res} \mathcal{B}_n d_\xi(\mathcal{L}^i) = -i \operatorname{Res} \mathcal{B}_n \mathcal{L}^{i-1} d_\xi \mathcal{L}. \tag{5.2.11}$$

Here we used the formula

$$\operatorname{Res} f(\xi)\,(d_\xi g(\xi)) + \operatorname{Res}(d_\xi f(\xi))\, g(\xi) = 0.$$

(This is a consequence of $\operatorname{Res} d_\xi a(\xi) = 0$ and the Leibniz rule $d_\xi(fg) = f\,(d_\xi g) + (d_\xi f)\, g$.)

The equation (5.2.11) means that the coefficient of $\mathcal{L}^{-i}$ in the expansion of $\mathcal{B}_n$ as a power series in $\mathcal{L}$ is $-\dfrac{1}{i}\dfrac{\partial v_i}{\partial t_n}$. Namely,

$$\mathcal{B}_n = \sum_{m=0}^{\infty} b_{mn} \mathcal{L}^m - \sum_{i=1}^{\infty} \frac{1}{i}\frac{\partial v_i}{\partial t_n}\mathcal{L}^{-i}. \tag{5.2.12}$$

By projecting this to the positive power part in ξ, we obtain

$$\mathcal{B}_n = \sum_{m=0}^{\infty} b_{mn} \mathcal{B}_m.$$

Since $\mathcal{B}_n = \xi^n +$ (lower order terms), $\mathcal{B}_m$'s are linearly independent. Thus we have $b_{mn} = \delta_{mn}$ and substituting this into (5.2.12), we have (5.2.7).

If one differentiate (5.1.9) with respect to x, then we obtain

$$\frac{\partial v_i}{\partial x} = \operatorname{Res} \mathcal{L}^i d\xi$$

instead of (5.2.11), from which follows (5.2.8). □

The tau function $\tau_{\mathrm{dkp}} = \tau_{\mathrm{dkp}}(t)$ (or its logarithm) for the dispersionless KP hierarchy is defined by the following equation.

$$d \log \tau_{\mathrm{dkp}} = \sum_{n=1}^{\infty} v_n(t)\, dt_n. \tag{5.2.13}$$

(Here x is set to 0, or identified with t_1.)

Lemma 5.2.3 (Proposition 6 of [**TT1**]). *There exists* $\log \tau_{\mathrm{dkp}}(t)$ *satisfying* (5.2.13).

Proof. Due to the equation (5.2.11) we have

$$\begin{aligned} &\frac{\partial v_m}{\partial t_n} - \frac{\partial v_n}{\partial t_m} \\ &= \operatorname{Res}\left(\mathcal{L}^m d_\xi (\mathcal{L}^n)_+ + (\mathcal{L}^m)_+ \, d_\xi \mathcal{L}^n\right) \\ &= \operatorname{Res}\left(2(\mathcal{L}^m)_+ d_\xi (\mathcal{L}^n)_+ + (\mathcal{L}^m)_- d_\xi (\mathcal{L}^n)_+ + (\mathcal{L}^m)_+ \, d_\xi (\mathcal{L}^n)_-\right), \end{aligned} \tag{5.2.14}$$

where the truncation $(\ )_-$ is defined by

$$\left(\sum_{n\in\mathbb{Z}} a_n \xi^n\right)_- := \sum_{n\in\mathbb{Z}} a_n \xi^n - \left(\sum_{n\in\mathbb{Z}} a_n \xi^n\right)_+ = \sum_{n<0} a_n \xi^n. \tag{5.2.15}$$

It is easy to see that $-(\mathcal{L}^m)_+ d_\xi (\mathcal{L}^n)_+$ and $(\mathcal{L}^m)_- \, d_\xi (\mathcal{L}^n)_-$ do not have residues. Adding them to (5.2.14), we obtain

$$\begin{aligned} \frac{\partial v_m}{\partial t_n} - \frac{\partial v_n}{\partial t_m} &= \operatorname{Res}(\mathcal{L}^m d_\xi \mathcal{L}^n) \\ &= n \operatorname{Res}(\mathcal{L}^{m+n-1} d_\xi \mathcal{L}) = \frac{n}{m+n} \operatorname{Res} d_\xi (\mathcal{L}^{m+n}) = 0. \end{aligned}$$

This is the integrability of the system

$$\frac{\partial}{\partial t_n} \log \tau_{\mathrm{dkp}} = v_n, \tag{5.2.16}$$

which guarantees the existence of the function $\log \tau_{\rm dkp}$ satisfying (5.2.13). □

Substituting (5.2.16) into (5.2.6), we obtain

$$S_n = -\frac{1}{n}\frac{\partial}{\partial t_n}\log \tau_{\rm dkp}, \tag{5.2.17}$$

which corresponds to (4.3.6) under the identification $\log \tau_{\rm dkp} = F_0$.

5.3. Riemann-Hilbert type construction of solutions

Instead of Sato's correspondence between the solution space of the KP hierarchy and the Sato-Grassmann manifold, we have a correspondence between solutions of the dispersionless KP hierarchy and canonical transformations.

THEOREM 5.3.1. *(i) Let $\mathcal{L}$ and $\mathcal{M}$ be series of the form* (5.1.1) *and* (5.1.9) *respectively. Assume that there exists a pair $(f(x,\xi), g(x,\xi))$ of functions (formal power series in ξ with coefficients depending on x) satisfying*

(1) $\{f, g\} = 1$,
(2) $f(\mathcal{M},\mathcal{L})$ *and* $g(\mathcal{M},\mathcal{L})$ *do not contain negative powers of* ξ, *i.e.,*

$$f(\mathcal{M},\mathcal{L})_- = g(\mathcal{M},\mathcal{L})_- = 0. \tag{5.3.1}$$

Then, $\mathcal{L}$ is a solution of the dispersionless KP hierarchy and $\mathcal{M}$ is a corresponding Orlov-Schulman series.

(ii) Conversely, there exists a pair $(f(x,\xi), g(x,\xi))$ satisfying the conditions in (i) for a pair $(\mathcal{L},\mathcal{M})$ of a solution of the dispersionless KP hierarchy and its Orlov-Schulman series.

The statement (i) is Proposition 7 of [**TT1**] (cf. Proposition 1.5.1 of [**TT3**]). We omit the proof because we shall give detailed proof for the similar theorem, Theorem 8.2.1, for the dispersionless Toda hierarchy later. The technique of the proof is almost the same as that in the proof of Proposition 5.1.5. The statement (ii) is Proposition 1.5.2 of [**TT3**], the proof of which uses the dressing operation discussed in Section 5.1.

This construction is a sort of Riemann-Hilbert decomposition. In fact we "decompose" a map $(f,g) : (\mathcal{L},\mathcal{M}) \mapsto (f(\mathcal{L},\mathcal{M}), g(\mathcal{L},\mathcal{M}))$ into two maps, $(\mathcal{L},\mathcal{M}) \mapsto (\xi, x)$ and $(\xi, x) \mapsto (f(\mathcal{L},\mathcal{M}), g(\mathcal{L},\mathcal{M}))$. The former of these maps is meromorphically extended to $\xi = \infty$ by condition on the forms, (5.1.1), (5.1.9). The latter map is extended to $\xi = 0$ because of the condition (5.3.1).

Very roughly speaking, the Riemann-Hilbert decomposition is factorisation of a function into two parts with prescribed poles, singularity or regularity. This method is very useful in the theory of integrable systems. See, for example, §§16–18 of [**Fad**] or Chapter II of [**FT**].

Remark 5.3.2. The above construction of solutions is a generalisation of Krichever's construction of solutions of the dispersionless KdV hierarchy in [**Kr2**].

A similar theorem for the $\hbar$-dependent KP hierarchy was proved in [**TT3**] (Proposition 1.7.11 and 1.7.12). This can be regarded as a canonical quantisation of Theorem 5.3.1.

CHAPTER 6

Dispersionless Hirota equation

In Section 4.3 we derived the dispersionless Hirota equation (4.3.11), or (4.3.12), by taking the quasi-classical limit of the differential Fay identity (4.3.10). In this chapter we derive the same equation, staying solely in the realm of the dispersionless hierarchy. Since the differential Fay identity is equivalent to the whole KP hierarchy as mentioned in Remark 4.3.3, we can expect that the dispersionless Hirota equation is equivalent to the dKP hierarchy. It is indeed so, which was first proved by Boyarsky-Marshakov-Ruchyayskiy-Wiegmann-Zabrodin [**BMRWZ**], but here we show Teo's very clear proof in [**Te**], using terminology in the theory of univalent functions. See also [**CK**].

6.1. Faber polynomials and Grunsky coefficients

In this section we review several notions in the theory of univalent functions. For details we refer to [**Dur**] or [**P**].

Let $g(z)$ be a power series in z of the form

$$g(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots . \tag{6.1.1}$$

In the theory of univalent functions this should be a holomorphic univalent function outside of the unit disk $|z| > 1$, but here we regard it just as a formal series.

We call the coefficients b_{mn} in the following expansion the *Grunsky coefficients* of $g(z)$:

$$\log \frac{g(z) - g(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} b_{mn} z^{-m} \zeta^{-n}. \tag{6.1.2}$$

They are polynomials in b_k's and can be computed using the formula, $\log(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$. Since this generating function is symmetric in z and ζ, the Grunsky coefficients are symmetric:

$$b_{mn} = b_{nm}. \tag{6.1.3}$$

If we replace $g(\zeta)$ in the left hand side of (6.1.2) with w and expand as a power series in z, we obtain,

$$
\begin{aligned}
\log \frac{g(z)-w}{z-\zeta} &= \log \frac{g(z)-w}{z} - \log \frac{z-\zeta}{z} \\
&= \log \left(1 + \sum_{n=1}^{\infty} b_n z^{-n-1} - \frac{w}{z}\right) - \log\left(1 - \frac{\zeta}{z}\right) \\
&= -\sum_{n=1}^{\infty} \Phi_n(w) \frac{z^{-n}}{n} + \sum_{n=1}^{\infty} \zeta^n \frac{z^{-n}}{n}.
\end{aligned}
\tag{6.1.4}
$$

It is easy to see that $\Phi_n(w)$ is a monic polynomial in w of degree n. This polynomial is called the *Faber polynomial.*

EXERCISE 6.1.1. Compute the Grunsky coefficients b_{mn} and the Faber polynomials $\Phi_n(w)$ for $m, n = 1, 2, 3$ explicitly. (The reader will find the answer for b_{m1} and $\Phi_1(w)$ below.)

REMARK 6.1.2. The Grunsky coefficients were introduced by Grunsky [**G**] in 1939 in the context of geometric function theory and used, for example, to prove special cases of the Bieberbach conjecture, which we shall mention later. The Faber polynomials were introduced by Faber [**Fab**] much earlier in 1903 in the study of approximation of analytic functions by polynomials.

Combining the two expressions (6.1.2) and (6.1.4) of $\log(g(z) - g(\zeta))/(z-\zeta)$, we have

$$
\sum_{m,n=1}^{\infty} b_{mn} z^{-m} \zeta^{-n} = \sum_{n=1}^{\infty} \frac{1}{n} (\Phi_n(g(\zeta)) - \zeta^n) z^{-n}.
$$

from which follows

$$
\Phi_n(g(\zeta)) = \zeta^n + \sum_{m=1}^{\infty} n b_{mn} \zeta^{-m}.
\tag{6.1.5}
$$

This is an important relation between the Faber polynomials and the Grunsky coefficients. When $n = 1$, this equation should have the form

$$
g(\zeta) + c = \zeta + \sum_{m=1}^{\infty} b_{m1} \zeta^{-m}.
$$

with a constant c, because $\Phi_n(w)$ is a monic polynomial of degree one. Actually, as there is no constant term in the right hand side, $c = 0$, i.e., $\Phi_1(w) = w$, and we have

$$g(\zeta) = \zeta + \sum_{m=1}^{\infty} b_{m1}\zeta^{-m}. \tag{6.1.6}$$

Comparing this with (6.1.1), we can deduce $b_{m1} = b_m$.

6.2. Dispersionless Hirota equation

Identity (6.1.5) means that a certain n-th degree polynomial of $g(\zeta)$ is equal to ζ^n+ (negative powers of ζ). The reader might recall that we have encountered a similar expression before. When we defined the S-function in Section 5.2, the positive part $\mathcal{B}_n$ of $\mathcal{L}^n$ was expanded as (5.2.7), or

$$\mathcal{B}_n = \mathcal{L}^n - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \log\tau}{\partial t_m \partial t_n}\mathcal{L}^{-m}, \tag{6.2.1}$$

due to (5.2.16). (In this section we denote τ_{dkp} by τ for simplicity.) The left hand side in (6.2.1) is a monic polynomial in ξ of degree n and the right hand side is $\mathcal{L}^n$ minus a series of negative powers of $\mathcal{L}$. This exactly corresponds to (6.1.5)! The second derivatives of the logarithm of the tau function corresponds to the Grunsky coefficients, or, in other words, the logarithm of the tau function (the free energy) is the potential of the Grunsky coefficients.

Solving the expansion of $\mathcal{L}$ with respect to ξ, we obtain (5.2.8), or

$$\xi = \mathcal{L} - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \log\tau}{\partial t_m \partial t_1}\mathcal{L}^{-m}. \tag{6.2.2}$$

This is the function g in our case: $\xi = g(\mathcal{L})$. Namely we have the following correspondence.

function $g(z)$	$z - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \log\tau}{\partial t_m \partial t_1} z^{-m}$
variables z, w	$\mathcal{L}$, ξ
Grunsky coefficient b_{mn}	$-\frac{1}{mn}\frac{\partial^2 \log\tau}{\partial t_m \partial t_n}$
Faber polynomial $\Phi_n(w)$	$\mathcal{B}_n$

Let us substitute these data into the generating function (6.1.2) of the Grunsky coefficients.

$$\log \frac{1}{z-\zeta}\left(\left(z-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2 \log\tau}{\partial t_m \partial t_1} z^{-m}\right) - \left(\zeta-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2 \log\tau}{\partial t_m \partial t_1} \zeta^{-m}\right)\right)$$
$$= \sum_{m,n=1}^{\infty}\frac{1}{mn}\frac{\partial^2 \log\tau}{\partial t_m \partial t_n} z^{-m}\zeta^{-n}.$$

This gives

$$\log\left(1-\sum_{m=1}^{\infty}\frac{z^{-m}-\zeta^{-m}}{z-\zeta}\frac{1}{m}\frac{\partial^2 \log\tau}{\partial t_m \partial t_1}\right) = \sum_{m,n=1}^{\infty}\frac{1}{mn}\frac{\partial^2 \log\tau}{\partial t_m \partial t_n} z^{-m}\zeta^{-n}. \tag{6.2.3}$$

Using the operator $D(z)$ defined by (4.3.3), we can rewrite it in the following form.

$$\log\left(1-\frac{D(z)-D(\zeta)}{z-\zeta}\frac{\partial \log\tau}{\partial t_1}\right) = D(z)\,D(\zeta)\log\tau. \tag{6.2.4}$$

These are nothing but the *dispersionless Hirota equations* (4.3.11) and (4.3.12), which we derived in Section 4.3 as the limit of the differential Fay identity (4.3.10). Thus we have proved the dispersionless Hirota equation, starting from the dispersionless KP hierarchy itself and not using the $\hbar$-dependent KP hierarchy.

Conversely let us assume that the function $\log\tau(t)$ satisfies the dispersionless Hirota equation (6.2.3). We define a function $g(z) = g(t;z)$ by

$$g(t;z) := z-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2 \log\tau}{\partial t_m \partial t_1} z^{-m}. \tag{6.2.5}$$

Note that the left hand side of the dispersionless Hirota equation (6.2.3) is equal to $\log\dfrac{g(z)-g(\zeta)}{z-\zeta}$. Comparing it with the definition (6.1.2) of the Grunsky coefficients, we have

(6.2.6) The (m,n)-th Grunsky coefficient of $g(z) = -\dfrac{1}{mn}\dfrac{\partial^2 \log\tau}{\partial t_m \partial t_n}$.

Therefore it follows from the relation (6.1.5) that the n-th Faber polynomial of $g(z)$ is

$$\Phi_n(g(z)) = z^n - \sum_{m=1}^{\infty} \frac{1}{mn} \frac{\partial^2 \log \tau}{\partial t_m \partial t_n} z^{-m}. \tag{6.2.7}$$

Let $\mathcal{L}(\xi) = \mathcal{L}(t;\xi)$ be the inverse function of $g(t;z)$ with respect to z: $\mathcal{L}(t; g(t;z)) = z$, $g(t;\mathcal{L}(t;\xi)) = \xi$. Such a function should have the expansion of the following form.

$$\mathcal{L} = g^{-1}(\xi) = \xi + u_2(t)\xi^{-1} + u_3(t)\xi^{-2} + \cdots. \tag{6.2.8}$$

Substituting $z = \mathcal{L}(t;\xi)$ into (6.2.7), we have

$$\Phi_n(\xi) = \mathcal{L}^n - (\text{sum of negative powers of } \mathcal{L}).$$

Since the left hand side is a polynomial in ξ, $\Phi_n(\xi)$ should be equal to the polynomial part of $\mathcal{L}^n$. Hence we have

$$\Phi_n(\xi) = (\mathcal{L}^n)_+ = \mathcal{B}_n, \tag{6.2.9}$$

and together with (6.2.7) we obtain (6.2.1).

Now, let us differentiate $\xi = g(t;\mathcal{L})$ with respect to t_n. By the chain rule and the derivation rule of the inverse function it is easy to see that

$$\begin{aligned} 0 = \frac{\partial}{\partial t_n}\big(g(t;\mathcal{L})\big) &= \frac{\partial g}{\partial \mathcal{L}}(t;\mathcal{L}) \frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial g}{\partial t_n}(t;\mathcal{L}) \\ &= \left(\frac{\partial \mathcal{L}}{\partial \xi}\right)^{-1} \frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial g}{\partial t_n}(t;\mathcal{L}), \end{aligned}$$

where $\dfrac{\partial g}{\partial t_n}(t;\mathcal{L})$ in the last expression means $\left.\dfrac{\partial g}{\partial t_n}(t;z)\right|_{z\mapsto\mathcal{L}}$. Substituting the definition (6.2.5) of $g(z)$, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_n} &= -\frac{\partial \mathcal{L}}{\partial \xi}\frac{\partial g}{\partial t_n} \\ &= \frac{\partial \mathcal{L}}{\partial \xi} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^3 \log \tau}{\partial t_n \partial t_m \partial t_1} \mathcal{L}^{-m}. \end{aligned} \tag{6.2.10}$$

On the other hand, the derivative of (6.2.1) with respect to $x = t_1$ becomes

$$\frac{\partial \mathcal{B}_n}{\partial x} = \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial x} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^3 \log \tau}{\partial t_1 \partial t_m \partial t_n} \mathcal{L}^{-m}. \tag{6.2.11}$$

By this formula we can replace the infinite sum in (6.2.10) by derivations of $\mathcal{B}_n$ as follows.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial t_n} &= \frac{\partial \mathcal{L}}{\partial \xi}\left(-\frac{\partial \mathcal{B}_n}{\partial x} + \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial x}\right) \\ &= \frac{\partial \mathcal{B}_n}{\partial \xi}\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \mathcal{B}_n}{\partial x}\frac{\partial \mathcal{L}}{\partial \xi} = \{\mathcal{B}_n, \mathcal{L}\},\end{aligned}$$

which is nothing but the Lax equation of the dispersionless KP hierarchy (5.1.2).

Thus we have proved the following theorem.

THEOREM 6.2.1 ([**Te**], Proposition 3.5). *The tau function $\tau(t)$ of the dispersionless KP hierarchy satisfies the dispersionless Hirota equation* (6.2.3).

Conversely, if the function $\tau(t)$ satisfies the dispersionless Hirota equation (6.2.3)*, it is the tau function of the dispersionless KP hierarchy.*

In other words, $\tau(t)$ is the tau function of the dispersionless KP hierarchy if and only if $\log\tau(t)$ *is the potential of the Grunsky coefficients of the function $g(z)$ defined by* (6.2.5) *in the sense of* (6.2.6).

REMARK 6.2.2. In fact the Grunsky coefficients already appeared in [**SS**] in the context of the (original) KP hierarchy: J_{mn} in p.270, which is directly related to the statement mentioned in Remark 4.3.3 of this lecture note. It would be much interesting if the KP hierarchy and complex analysis would be connected in this way.

CHAPTER 7

Dispersionless KP hierarchy and Löwner equation

Around the turn of the millennium it turned out that the dKP hierarchy is related with the theory of univalent functions (i.e., with the Riemann mapping theorem) unexpectedly deeply. One example is appearance of the Grunsky coefficients and the Faber polynomials in the proof of the equivalence of the dispersionless Hirota equation with the dKP hierarchy, as is shown in Chapter 6. The other examples are

- The dispersionless hierarchies and the Löwner type equations (found by Gibbons and Tsarev [**GTs2**]).
- The Laplacian growth problem and the dispersionless Toda hierarchy (found by Mineev-Weinstein, Wiegmann and Zabrodin [**MiWZ**]).

We shall discuss the former example in this chapter and the latter in Chapter 9.

7.1. One-variable reduction of the dispersionless KP hierarchy

The Löwner equations are differential equations characterising one-parameter families of conformal mappings between families of domains with growing slits and fixed reference domains. It was discovered by the seminal work by Gibbons and Tsarev [**GTs2**] that one of such equations (the chordal Loewner equation) describes the reduction of the dispersionless KP hierarchy. Soon after that similar examples were found by [**MMAM, M, TTZ, TT5, Take3**] and others.

We postpone details of the Löwner equations in the context of complex analysis to the next section and explain here how the chordal Löwner equation arises from the dKP hierarchy. We follow the arguments in [**TTZ**].

The solution $\mathcal{L}$ of the dispersionless KP hierarchy depends on infinite number of variables $t = (t_1, t_2, \dots)$ by definition, but what if it depends essentially only on one variable? This is the *one-variable reduction* of the dispersionless KP hierarchy.

THEOREM 7.1.1. *(i) Suppose that $\mathcal{L}(t;\xi)$ is a solution of the dispersionless KP hierarchy whose dependence on $t = (t_1, t_2, \dots)$ is through a single variable λ. Namely, there exists a function $f(\lambda;\xi)$ of ξ and λ of the form*

$$f(\lambda;\xi) = \xi + u_2(\lambda)\xi^{-1} + u_3(\lambda)\xi^{-2} + \cdots, \tag{7.1.1}$$

and a function $\lambda(t)$ of t such that

$$\mathcal{L}(t;\xi) = f(\lambda(t);\xi). \tag{7.1.2}$$

We assume $\dfrac{du_2}{d\lambda} \neq 0$ and $\dfrac{\partial\lambda}{\partial t_1} \neq 0$. Let the function $g(\lambda;z)$ of the form

$$g(\lambda;z) = z + v_2(\lambda)z^{-1} + v_3(\lambda)z^{-2} + \cdots \tag{7.1.3}$$

be the inverse function of $f(\lambda;\xi)$ with respect to ξ: $g(\lambda;f(\lambda;\xi)) = \xi$, $f(\lambda;g(\lambda;z)) = z$. Then $g(\lambda;z)$ satisfies the following equation (the chordal Löwner equation*) with respect to λ.*

$$\frac{\partial g}{\partial\lambda}(\lambda;z) = \frac{1}{U(\lambda) - g(\lambda;z)}\frac{du}{d\lambda}. \tag{7.1.4}$$

Here $U(\lambda)$ is a function of λ, not depending on z, and $u(\lambda) = u_2(\lambda)$.

The function $\lambda(t)$ is characterised by the following system: for any $n \in \mathbb{N}$,

$$\frac{\partial\lambda}{\partial t_n} = \chi_n(\lambda)\frac{\partial\lambda}{\partial t_1}. \tag{7.1.5}$$

The function $\chi_n(\lambda)$ is defined by

$$\chi_n(\lambda) := \frac{\partial\Phi_n}{\partial w}(\lambda;U(\lambda)), \tag{7.1.6}$$

where $\Phi_n(\lambda;w)$ is the n-th Faber polynomial of $g(\lambda;z)$ (cf. Section 6.1).

(ii) Conversely, suppose that $f(\lambda;\xi)$ is a function of the form (7.1.1) *and its inverse function $g(\lambda;z)$ satisfies the differential equation* (7.1.4)*. If $\lambda(t)$ is a solution of* (7.1.5)*, then $\mathcal{L}(t;\xi) := f(\lambda(t);\xi)$ is a solution of the dispersionless KP hierarchy.*

REMARK 7.1.2. As we shall see in Section 7.2, the chordal Löwner equation (7.1.4) is an equation satisfied by a one-parameter family of holomorphic mappings on slit domains.

REMARK 7.1.3. The system (7.1.5) can be solved in the following way (*Tsarëv's generalised hodograph method*). Let $R(\lambda)$ be an arbitrary (sufficiently differentiable) function. Define $\lambda = \lambda(t)$ by the relation

$$t_1 + \sum_{n=1}^{\infty}\chi_n(\lambda)t_n = R(\lambda) \tag{7.1.7}$$

as an implicit function. Then $\lambda(t)$ satisfies (7.1.5).

EXERCISE 7.1.4. Prove the statement of Remark 7.1.3. (Hint: differentiate (7.1.7) by t_1 and t_n and compare the results.)

PROOF OF THEOREM 7.1.1. (i) Let $\tau(t)$ be the tau function of this solution. Recall that the inverse function of $\mathcal{L}(t;\xi)$ with respect to ξ is expressed in terms of $\log\tau(t)$ as in (6.2.2). Hence

$$\xi = g(z,\lambda(t)) = z - D(z)\frac{\partial}{\partial t_1}\log\tau(t), \tag{7.1.8}$$

where $D(z)$ is the differential operator $D(z) = \sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial}{\partial t_n}$ defined by (4.3.3). This formula implies

$$D(z_1)\,g(\lambda(t);z_2) = D(z_2)\,g(\lambda(t);z_1) = -D(z_1)\,D(z_2)\,\frac{\partial}{\partial t_1}\log\tau(t). \tag{7.1.9}$$

Because $g(\lambda;z)$ is the inverse function of f of the form (7.1.1), the coefficient $v_2(\lambda)$ of z^{-1} in (7.1.3) is equal to $-u_2=-u$. Therefore the coefficients of z_2^{-1} in (7.1.9) are

$$-D(z_1)u(\lambda(t)) = \frac{\partial}{\partial t_1}g(\lambda(t);z_1).$$

By the chain rule we can rewrite this as follows.

$$-\frac{du}{d\lambda}D(z_1)\,\lambda(t) = \frac{\partial g}{\partial\lambda}\,\frac{\partial\lambda}{\partial t_1}.$$

Thus we obtain

$$D(z)\,\lambda(t) = -\frac{\partial g}{\partial\lambda}\,\frac{\partial_{t_1}\lambda}{\partial_\lambda u}. \tag{7.1.10}$$

Using this formula together with the chain rule again, we can rewrite the left hand side of (7.1.9) as

$$\begin{aligned} D(z_1)\,g(\lambda(t);z_2) &= \frac{\partial g}{\partial\lambda}(\lambda(t);z_2)D(z_1)\,\lambda(t) \\ &= -\frac{\partial g}{\partial\lambda}(\lambda(t);z_1)\frac{\partial g}{\partial\lambda}(\lambda(t);z_2)\frac{\partial_{t_1}\lambda}{\partial_\lambda u}. \end{aligned} \tag{7.1.11}$$

Having prepared these formulae, let us differentiate the dispersionless Hirota equation (6.2.4) with respect to t_1. The result is

$$\frac{\partial_\lambda g(\lambda(t);z_1) - \partial_\lambda g(\lambda(t);z_2)}{g(\lambda(t);z_1) - g(\lambda(t);z_2)}\,\frac{\partial\lambda}{\partial t_1} = D(z_1)\,D(z_2)\,\frac{\partial}{\partial t_1}\log\tau(t), \tag{7.1.12}$$

and its right hand side is equal to
(7.1.13)
$$D(z_1)\,D(z_2)\,\frac{\partial}{\partial t_1}\log\tau(t) = -D(z_1)\,g(\lambda(t);z_2) = \frac{\partial g}{\partial\lambda}(\lambda(t);z_1)\frac{\partial g}{\partial\lambda}(\lambda(t);z_2)\frac{\partial_{t_1}\lambda}{\partial_\lambda u},$$
because of (7.1.8) and (7.1.11). As we assume $\dfrac{\partial\lambda}{\partial t_1} \neq 0$, we can divide (7.1.12) and (7.1.13) by $\dfrac{\partial\lambda}{\partial t_1} \neq 0$ and obtain
$$\left(\frac{\partial g}{\partial\lambda}(\lambda(t);z_1) - \frac{\partial g}{\partial\lambda}(\lambda(t);z_2)\right)\frac{\partial u}{\partial\lambda}$$
$$= (g(\lambda(t);z_1) - g(\lambda(t);z_2))\frac{\partial g}{\partial\lambda}(\lambda(t);z_1)\frac{\partial g}{\partial\lambda}(\lambda(t);z_2).$$
Putting all terms with z_1 in the left side and all terms with z_2 in the right side, we have
$$\frac{\partial_\lambda u}{\partial_\lambda g(\lambda(t);z_1)} + g(\lambda(t);z_1) = \frac{\partial_\lambda u}{\partial_\lambda g(\lambda(t);z_2)} + g(\lambda(t);z_2).$$
That is to say, the function
$$U(\lambda) := \frac{\partial_\lambda u}{\partial_\lambda g(\lambda;z)} + g(\lambda;z) \tag{7.1.14}$$
does not depend on z.

The definition (7.1.14) is equivalent to the equation
$$\frac{\partial g}{\partial\lambda} = \frac{1}{U(\lambda)-g}\frac{du}{d\lambda},$$
which is the chordal Löwner equation (7.1.4). Substituting this equation to (7.1.10), we have
$$\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial\lambda}{\partial t_n} = \frac{1}{g(\lambda;z(t)) - U(\lambda(t))}\frac{\partial\lambda}{\partial t_1}. \tag{7.1.15}$$
Note that the definition (6.1.4) of the Faber polynomials yields
$$\frac{-1}{g(z)-w} = -\sum_{n=1}^{\infty}\Phi_n'(w)\frac{z^{-n}}{n},$$
by differentiation with respect to w. Therefore the fraction in the right hand side of (7.1.15) can be rewritten as
$$\frac{1}{g(\lambda;z) - U(\lambda)} = \sum_{n=1}^{\infty}\Phi_n'(U(\lambda))\frac{z^{-n}}{n}, \tag{7.1.16}$$

and thus the equation (7.1.15) gives

$$\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial \lambda}{\partial t_n} = \sum_{n=1}^{\infty} \Phi'_n(U(\lambda)) \frac{z^{-n}}{n} \frac{\partial \lambda}{\partial t_1}, \tag{7.1.17}$$

which is the generating function of the relations (7.1.5).

(ii) We denote the Grunsky coefficients of $g(\lambda(t); z)$ by $b_{mn}(t)$. Differentiating the definition (6.1.2) (with $z = z_1$, $\zeta = z_2$) of the Grunsky coefficients by t_k, we obtain

$$\frac{\partial_\lambda g(\lambda(t); z_1) - \partial_\lambda g(\lambda(t); z_2)}{g(\lambda(t); z_1) - g(\lambda(t); z_2)} \frac{\partial \lambda}{\partial t_k} = - \sum_{m,n=1}^{\infty} \frac{\partial b_{mn}}{\partial t_k}(t) z_1^{-m} z_2^{-n}. \tag{7.1.18}$$

Because of the chordal Löwner equation (7.1.4) and the equation (7.1.5), the left hand side of (7.1.18) is equal to

$$\begin{aligned} &\frac{1}{g(\lambda(t); z_1) - g(\lambda(t); z_2)} \times \\ &\left(\frac{1}{U(\lambda(t)) - g(\lambda(t); z_1)} - \frac{1}{U(\lambda(t)) - g(\lambda(t); z_2)} \right) \frac{du}{d\lambda}(\lambda(t))\, \chi_k(\lambda(t)) \frac{\partial \lambda}{\partial t_1}(t) \\ &= \frac{1}{(U(\lambda(t)) - g(\lambda(t); z_1))(U(\lambda(t)) - g(\lambda(t); z_2))} \frac{du}{d\lambda}(\lambda(t))\, \chi_k(\lambda(t)) \frac{\partial \lambda}{\partial t_1}(t). \end{aligned} \tag{7.1.19}$$

Using this equation and (7.1.16) which follows directly from the definition (6.1.4) of the Faber polynomials, we can rewrite (7.1.18) as

$$\begin{aligned} \sum_{m,n=1}^{\infty} \frac{z_1^{-m} z_2^{-n}}{mn} \chi_m(\lambda(t))\, \chi_n(\lambda(t))\, \chi_k(\lambda(t)) \frac{du}{d\lambda}(\lambda(t)) \frac{\partial \lambda}{\partial t_1}(t) \\ = - \sum_{m,n=1}^{\infty} \frac{\partial b_{mn}}{\partial t_k}(t) z_1^{-m} z_2^{-n}. \end{aligned}$$

Thus we obtain

$$-mn \frac{\partial b_{mn}}{\partial t_k} = \chi_m(\lambda(t))\, \chi_n(\lambda(t))\, \chi_k(\lambda(t)) \frac{du}{d\lambda}(\lambda(t)) \frac{\partial \lambda}{\partial t_1}(t).$$

Note that the right hand side of this equation is symmetric in indices (m, n, k). Hence we have a system

$$\frac{\partial}{\partial t_k}(-mn\, b_{mn}(t)) = \frac{\partial}{\partial t_m}(-kn\, b_{kn}(t))$$

for all k, m, n, which is the compatibility condition of the system

$$\frac{\partial G_n}{\partial t_m} = -mn\, b_{mn}(t). \tag{7.1.20}$$

Since the right hand side of (7.1.20) is symmetric in (m, n) because of (6.1.3),

$$\frac{\partial G_n}{\partial t_m} = \frac{\partial G_m}{\partial t_n},$$

which is the compatibility condition of

$$\frac{\partial F}{\partial t_n} = G_n. \tag{7.1.21}$$

Combining (7.1.20) and (7.1.21), we have shown the existence of a function $F(t)$ satisfying

$$\frac{\partial^2 F}{\partial t_m \, \partial t_n} = -mn \, b_{mn}(t).$$

Since b_{mn}'s are the Grunsky coefficients, $F(t)$ is the logarithm of a tau function $\tau(t)$ of the dispersionless KP hierarchy due to Theorem 6.2.1. (Note that (6.2.5) is automatically satisfied because of (6.1.6).)

This completes the proof of the theorem. □

Thanks to this theorem, once we have a solution of the chordal Löwner equation, we have a solution of the dispersionless KP hierarchy. Then, how can we find a solution of the chordal Löwner equation? In fact, it is known that certain families of Riemann's conformal mapping functions satisfy it, so the problem of solving the differential equation is turned into a problem of conformal mappings in complex analysis! We discuss it in the next section.

Remark 7.1.5. Theorem 7.1.1 (ii) can be generalised to the N-*variable reduction*. Namely, we can construct a solution of the dispersionless KP hierarchy depending on $t = (t_1, t_2, \dots)$ through N variables $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$. In this case, in addition to the chordal Löwner equations for each variable λ_i, we need compatibility condition for them, which is called the *Gibbons-Tsarev system* first proposed in [**GTs1**].

We refer details of the N-variable reductions to [**GTs1, GTs2, MMAM, M, TT4**]. There are many references on the Gibbons-Tsarev systems. See, for example, [**OdS**] besides the first papers by Gibbons and Tsarev [**GTs1, GTs2**].

7.2. Löwner equations in complex analysis

The Riemann mapping theorem, one of the fundamental theorem in complex analysis, states that any simply connected domain $D \subsetneq \mathbb{C}$ is bijectively mapped to the unit disk Δ (or to the upper half plane H, or to any other simply connected domain $D' \subsetneq \mathbb{C}$) by a holomorphic function (Figure 7.2.1). Such a holomorphic function $f : D \to \Delta$ is unique if one imposes the appropriate condition. (For example, if the target domain is Δ, one can require $f(z_0) = 0$, $f'(z_0) > 0$ for a fixed point z_0 in D.) See, for example, Chapter 6 of [**Ah**] for details.

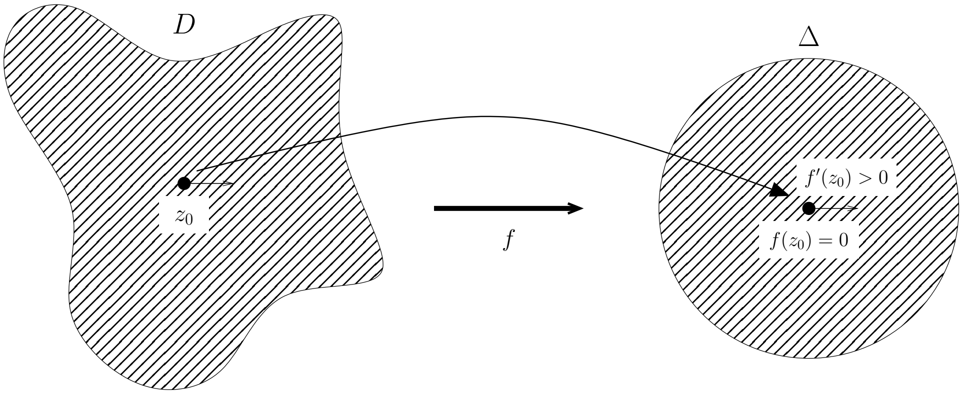

FIGURE 7.2.1. The Riemann mapping theorem.

We consider a family of conformal mappings between slit domains and the upper half plane H. *Slit domains* are, roughly speaking, domains obtained by pulling out slits from a domain. Exactly speaking, let D be a simply connected domain with the infinity on its boundary. Let Γ be a Jordan arc $\Gamma : [a, b] \to D$ lying in D except for $\Gamma(a) \in \partial D$. For $\lambda \in [a, b]$ we denote $\Gamma([a, \lambda])$ by Γ_λ and consider the conformal mappings between $D \smallsetminus \Gamma_\lambda$ to the upper half plane H.

$$
\begin{aligned}
g_\lambda : D \smallsetminus \Gamma_\lambda \ni z &\mapsto g_\lambda(z) = g(\lambda; z) \in H, \\
f_\lambda : \qquad H \ni \xi &\mapsto f_\lambda(\xi) = f(\lambda; \xi) \in D \smallsetminus \Gamma_\lambda.
\end{aligned}
$$

They are mutually inverse,

$$
g(\lambda; f(\lambda; \xi)) = \xi, \qquad f(\lambda; g(\lambda; z)) = z,
$$

and normalised by the following condition (the *hydrodynamic normalisation*) on the expansion at infinity.

$$
\begin{aligned}
g(\lambda; z) &= z - u(\lambda) z^{-1} + O(z^{-2}), \\
f(\lambda; \xi) &= \xi + u(\lambda) \xi^{-1} + O(\xi^{-2}).
\end{aligned}
\tag{7.2.2}
$$

This normalisation fixes the ambiguity of the maps coming from the automorphisms of H like a shift $z \mapsto z + a$, a dilation $z \mapsto cz$ $(c > 0)$ and the inversion $z \mapsto -z^{-1}$.

THEOREM 7.2.1. *In the above situation there exists a continuous function $U : [a, b] \to \mathbb{R}$ (the* driving function*) such that g satisfies the* *chordal Löwner equation*

$$\frac{\partial g}{\partial \lambda} = \frac{1}{U(\lambda) - g(\lambda; z)} \frac{du}{d\lambda}. \tag{7.2.3}$$

The point $U(\lambda)$ on the real axis is the image of the tip of the curve $\Gamma(\lambda)$, if the map g can be properly extended to Γ_λ (Figure 7.2.4).

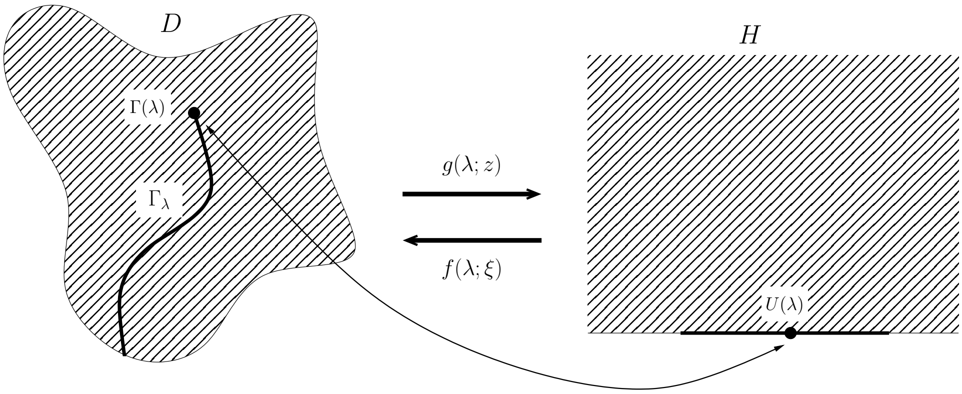

FIGURE 7.2.4. The slit mappings between $D \setminus \Gamma_\lambda$ and H.

This is how the chordal Löwner equation appears in complex analysis. It was first proved by Kufarev, Sobolev and Sporševa in [**KSS**]. It was rediscovered in the context of integrable systems by Gibbons and Tsarev [**GTs2**] and became well-known when Schramm [**Schr**] discovered it independently again and studied random curves in the plane. (In the celebrated *stochastic/Schramm Löwner evolution* (SLE) the driving function $U(\lambda)$ is a Brownian motion.) For details we refer to [**ABCD**]. In the next section we briefly explain how such a differential equation arises in this context, following [**dMG**].

Then, how can it be named after "Löwner"? In fact there is another version of the Löwner equation with different normalisation. The original Löwner's equation is for mappings between slit domains and the unit disk (Figure 7.2.5).

Here the normalisation is

$$g(\lambda; z) = e^{-\phi(\lambda)} z + O(z^2) \tag{7.2.6}$$

at $z = 0$. Note that a fixed inner point $z = 0$ is mapped to a fixed inner point $\xi = 0$ in this case, while in the chordal Löwner case a fixed boundary point $z = \infty$ corresponds to a fixed boundary point $\xi = \infty$.

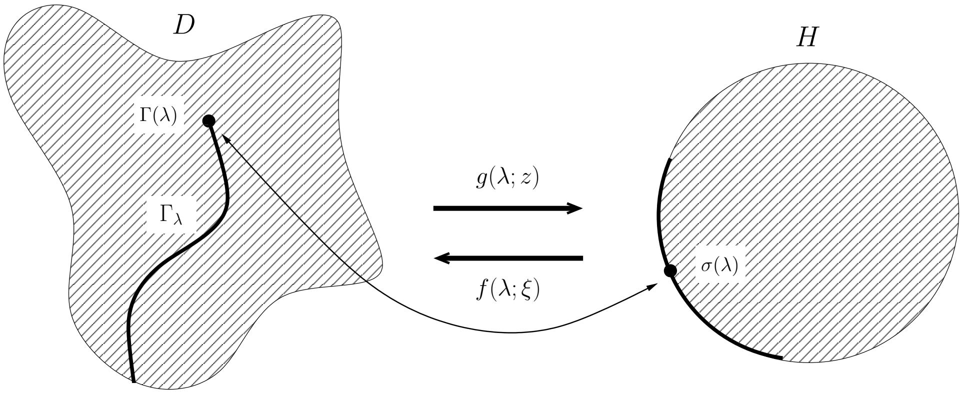

FIGURE 7.2.5. The slit mappings between $D \setminus \Gamma_\lambda$ and Δ.

This difference of normalisation is the essential difference between the chordal Löwner equation and the original Löwner equation[1].

In this case the role of the driving function $U(\lambda)$ is played by a function $\sigma : [a, b] \to \partial\Delta = \{\xi \mid |\xi| = 1\}$ and $g = g(\lambda; z)$ satisfies

$$\frac{\partial g}{\partial \lambda} = g \frac{\sigma(\lambda) + g}{\sigma(\lambda) - g} \frac{d\phi}{d\lambda}, \tag{7.2.7}$$

which is now called the *radial Löwner equation* nowadays to distinguish it from the chordal case. This equation was found by Löwner[2] in 1923. It is a powerful tool in geometric function theory, for example, to evaluate coefficients of univalent functions. One of the examples, to which the Löwner equation was successfully applied, is the Bieberbach conjecture posed by Bieberbach in 1916.

As the Bieberbach conjecture is related also to the Grunsky coefficients frequently used in this lecture, it is probably worth mentioning what that conjecture is, digressing a little bit. Let us consider a univalent holomorphic function $f(z)$ on the unit disk $\Delta = \{|z| < 1\}$. (Univalence means that $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$.) It has a Taylor expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots,$$

but if we are interested only in the "shape" of the image, the image can be shifted ($f(z) \mapsto f(z) + c$, $c \in \mathbb{C}$), rotated ($f(z) \mapsto e^{i\theta} f(z)$, $\theta \in \mathbb{R}$) or resized ($f(z) \mapsto r\, f(z)$, $r > 0$). So, we may assume that $f(z)$ has

[1]Whether the reference domain is the upper half plane or the unit disk, is not essential, since we can map one to the other by a fractional linear transformation.

[2]He changed the spelling of his family name to "Loewner" after he immigrated to the USA, but the author of [**Lö**] on this subject is "Karl Löwner".

the following expansion.

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{7.2.8}$$

Namely, we normalised $f(z)$ by $f(z) \mapsto a_1^{-1}(f(z) - a_0)$, so that $a_0 = 0$, $a_1 = 1$. Note that $a_1 = f'(0) \neq 0$ because of the univalence.

Bieberbach [**Bi**] proved that an inequality $|a_2| \leqq 2$ follows from the univalence and that $|a_2| = 2$ if and only if $f(z)$ is the Koebe function,

$$k(z) := \frac{z}{(1-\varepsilon z)^2} = z + 2\varepsilon z^2 + 3\varepsilon^2 z^3 + \cdots , \qquad (|\varepsilon| = 1). \tag{7.2.9}$$

He also conjectured in the same paper that $|a_n| \leqq n$ for any $n \in \mathbb{N}$. This is the *Bieberbach conjecture*. Löwner proved this conjecture for $n = 3$ by using the Löwner equation (7.2.7). The Grunsky coefficients were used for the proof of the cases $n = 4$ and $n = 6$. The final proof of the conjecture (for any n) was by de Branges [**dB**], who also used the Löwner equation. (See also [**FP**].)

EXERCISE 7.2.2. Show that $k(z)$ defined by (7.2.9) with $\varepsilon = 1$ maps the unit disk Δ to the domain $D := \mathbb{C} \setminus (-\infty, -\frac{1}{4})$. (Hint: it is easier to construct a conformal map from D to Δ: first shift $w \in D$ by $\frac{1}{4}$, then take the square root and open the slit to get the upper half plane, and then map it to Δ by a fractional linear transform.)

As the one-variable reduction of the dispersionless KP hierarchy was characterised by the chordal Löwner equation, the radial Löwner equation is related to the one-variable reduction of the dispersionless Toda hierarchy ([**TTZ**] and references therein), but we do not go into details.

Now, let us return to the chordal Löwner equation and construct solutions of the dispersionless KP hierarchy, using conformal mappings. (These examples are from [**TTZ**].)

EXAMPLE 7.2.3. Let us consider the maps between the upper half plane H and the domain $D_\lambda := H \setminus \{U + i\sqrt{2t} \mid t \in [0, \lambda]\}$, $g_\lambda : D_\lambda \to H$ and $f_\lambda : H \to D_\lambda$ (Figure 7.2.10).

It is easy to see that those holomorphic functions satisfying the normalisation condition (7.2.2) are given by

$$g_\lambda(z) = g(\lambda; z) = U + \sqrt{(z-U)^2 + 2\lambda} = z + \frac{\lambda}{z} + \frac{\lambda U}{z^2} + \cdots ,$$

$$f_\lambda(\xi) = f(\lambda; \xi) = U + \sqrt{(\xi-U)^2 - 2\lambda} = \xi - \frac{\lambda}{\xi} - \frac{\lambda U}{\xi} + \cdots .$$

Here the driving function $U(\lambda)$ is the constant function $U(\lambda) \equiv U$ and $u(\lambda) = -\lambda$.

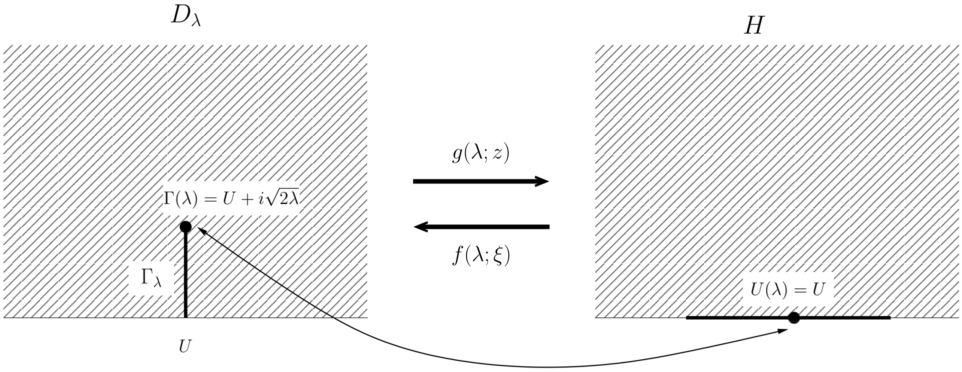

FIGURE 7.2.10. The slit mapping between $D_\lambda := H \setminus (U + i[0, \sqrt{2\lambda}])$ and H.

The first three Faber polynomials ($\Phi_n(\lambda;\xi) = (f(\lambda;\xi)^n)_+$) are

$$\Phi_1(\lambda;\xi) = \xi, \quad \Phi_2(\lambda;\xi) = \xi^2 - 2\lambda, \quad \Phi_3(\lambda;\xi) = \xi^3 - 3\lambda\xi - 3\lambda U,$$

and the corresponding $\chi_n(\lambda)$'s are

$$\chi_1(\lambda) = 1, \qquad \chi_2(\lambda) = 2U, \qquad \chi_3(\lambda) = 3U^2 - 3\lambda.$$

(Recall that $\chi_n(\lambda) = \Phi_n'(\lambda; U(\lambda))$.)

Let us solve the system (7.1.5) by the generalised hodograph method (Remark 7.1.3; we take $R(\lambda) = 0$): we fix $t_n = 0$ ($n > 3$) in (7.1.7) and obtain

$$t_1 + 2Ut_2 + (3U^2 - 3\lambda)t_3 = 0,$$

from which follows

$$\lambda(t)|_{t_n=0\ (n>3)} = \frac{t_1 + 2Ut_2 + 3U^2 t_3}{3t_3}. \tag{7.2.11}$$

Therefore in this case the solution (with $t_n = 0$ ($n > 3$) fixed) is

$$\begin{aligned} \mathcal{L}(t_1, t_2, t_3; \xi) &= f(\lambda(t_1, t_2, t_3); \xi) \\ &= U + \sqrt{(\xi - U)^2 - \frac{2(t_1 + 2Ut_2 + 3U^2 t_3)}{3t_3}} \\ &= U + \sqrt{\xi^2 - 2U\xi - U^2 - \frac{2(t_1 + 2Ut_2)}{3t_3}}. \end{aligned} \tag{7.2.12}$$

EXERCISE 7.2.4. Check that the above g_λ and f_λ give the conformal mappings between H and D_λ and that g_λ satisfies the chordal Löwner equation.

EXAMPLE 7.2.5. Usually the chordal Löwner equation is considered for the conformal mappings between the upper half plane and the domain obtained from the upper half plane by subtracting a growing curve. Here we consider a different kind of domain, $D_\lambda := \mathbb{C} \setminus ((-\infty, 0] \cup [4\lambda, +\infty))$ (Figure 7.2.13). In this case the driving function is $U(\lambda) = 3\lambda$.

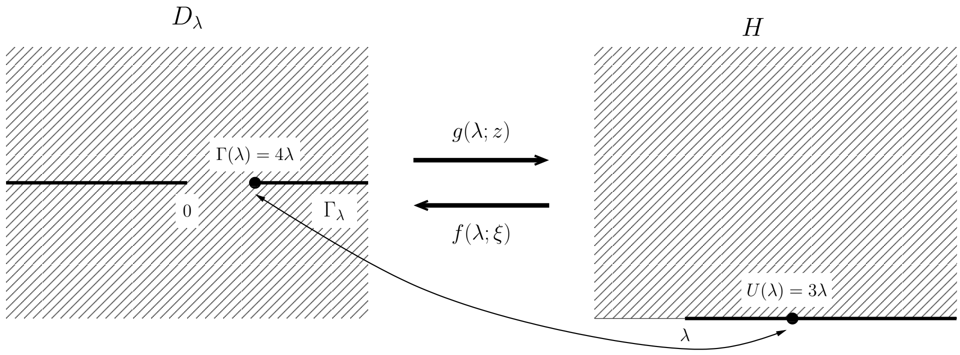

FIGURE 7.2.13. The slit mapping between $D_\lambda := \mathbb{C} \setminus ((-\infty, 0] \cup [4\lambda, +\infty))$ and H.

The conformal mappings between D_λ and H are

$$g_\lambda(z) = g(\lambda; z) = \lambda + \frac{z + \sqrt{(z-2\lambda)^2 - 4\lambda^2}}{2},$$
$$f_\lambda(\xi) = f(\lambda; \xi) = \xi + \frac{\lambda^2}{\xi - 2\lambda} = \xi + \lambda^2 \xi^{-1} + 2\lambda^3 \xi^{-2} + \cdots .$$

EXERCISE 7.2.6. (i) Check that the above g_λ and f_λ are indeed conformal mappings between D_λ and H.

(ii) Check that g_λ satisfies the chordal Löwner equation.

(iii) The slit $\Gamma_\lambda = [4\lambda, +\infty)$ is "shrinking" when λ grows. Apparently this violates the conditions we considered, but nevertheless (as we have checked in (ii)) the chordal Löwner equation holds. Why? (Hint: change the coordinates: $z' = 1/z$, $\lambda' = 1/\lambda$. The chordal Löwner equation is covariant under the change of the coordinates.)

As is readily seen from the explicit expressions, $u(\lambda) = \lambda^2$ and the Faber polynomials are

$$\Phi_1(\lambda; \xi) = \xi, \quad \Phi_2(\lambda; \xi) = \xi^2 + 2\lambda^2, \quad \Phi_3(\lambda; \xi) = \xi^3 + 3\lambda^3\xi + 6\lambda^3.$$

Thus

$$\chi_1(\lambda) = 1, \qquad \chi_2(\lambda) = 6\lambda, \qquad \chi_3(\lambda) = 30\lambda^2.$$

We solve the system (7.1.5) with $t_n = 0$ $(n > 2)$ by the generalised hodograph method with $R(\lambda) = 0$ in (7.1.7) again. The generalised hodograph relation is

$$t_1 + 6t_2\lambda = 0, \text{ i.e., } \lambda(t_1, t_2) = -\frac{t_1}{6t_2},$$

and the solution of the dispersionless KP hierarchy is

$$\mathcal{L}(t_1, t_2; \xi) = \xi + \frac{t_1^2}{12(3t_2\xi + t_1)}.$$

7.3. Idea of proof of the chordal Löwner equation

There are several proofs of Theorem 7.2.1. Those found in the context of the SLE require techniques like the Brownian motion ([**La**]) or the extremal length ([**LSW**]). There is a simple explanation by physicists (for example, [**C**]), which is not rigorous. Recently del Monaco and Gumenyuk published a rigorous elementary direct proof [**dMG**]. In this section, following this paper, we explain how a differential equation arises from a family of slit conformal mappings.

When the slit is determined by the map $\Gamma : [a, b] \ni \lambda \mapsto \Gamma(\lambda) \in D$, we define a family of conformal maps by

$$\varphi_{s,t}(\xi) := g(s; f(t; \xi)) \tag{7.3.1}$$

for $a < s < t < b$. This maps the upper half plane H to $g(s; D \smallsetminus \Gamma([a, t])) = H \smallsetminus g(s; \Gamma((s, t]))$. We denote $g(t; \Gamma[s, t])$ by $C_{s,t}$.

To make the idea of the proof clear, in Figure 7.3.2 we assume that the maps can be continuously extended to the boundaries in appropriate sense. In particular, the segment C_t is the double-valued "image" of $\Gamma([a, t])$ by $g(t; z)$.

Note that $\varphi_{s,t}(\xi)$ has an expansion of the form

$$\varphi_{s,t}(\xi) = \xi - u_\varphi(s, t)\xi^{-1} + O(\xi^{-2}), \qquad u_\varphi(s, t) = u(s) - u(t), \tag{7.3.3}$$

because of the normalisation (7.2.2) of g and f.

If we can show that this map $\varphi_{s,t}$ satisfies the chordal Löwner equation

$$\frac{\partial \varphi_{s,t}}{\partial s}(\xi) = \frac{1}{U(s) - \varphi_{s,t}(\xi)} \frac{\partial u_\varphi(s, t)}{\partial s}, \tag{7.3.4}$$

substituting $f(t; \xi)$ by z and s by λ, we obtain the chordal Löwner equation for g, (7.1.4). The advantage to use $\varphi_{s,t}$ is that both the domain of definition and the image are almost the upper half plane.

The basic tool of the proof is the following *Schwarz integral formula*.

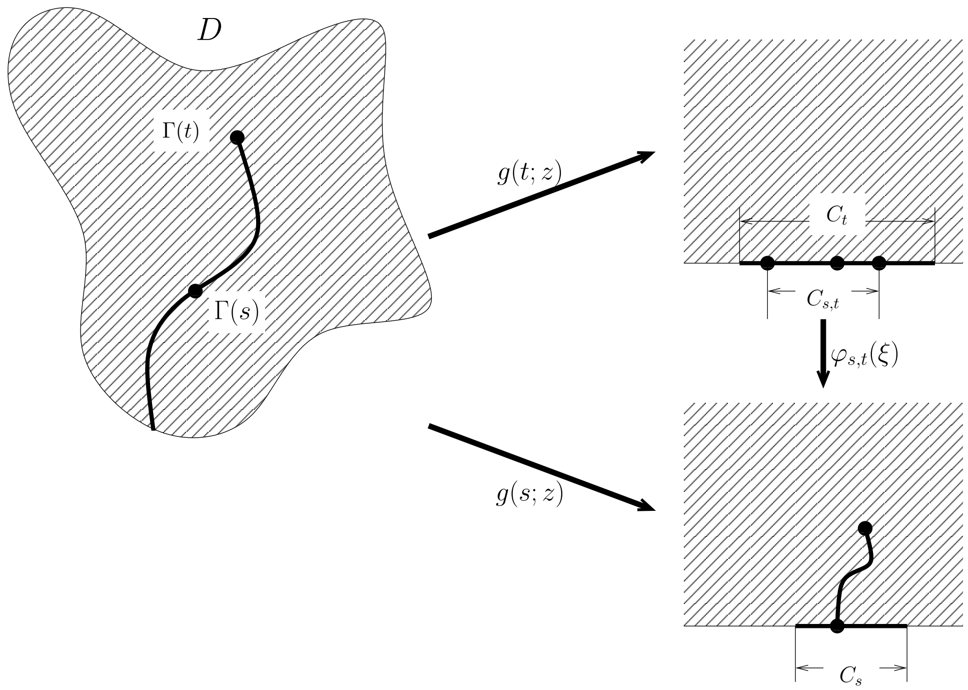

FIGURE 7.3.2. Definition of $\varphi_{s,t}$ and $C_{s,t}$.

LEMMA 7.3.1. *Let $f : H \to \mathbb{C}$ be a holomorphic function and assume that*

(1) *f has a continuous extension from $\bar{H} = H \cup \mathbb{R}$ to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$;*
(2) *$f(\infty) = 0$;*
(3) *for any $\xi \in H$*

$$\int_{\mathbb{R}} \left| \frac{\operatorname{Im} f(x)}{x - i} \right| dx < +\infty.$$

Then,

$$f(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im} f(\xi)}{\xi - z} d\xi \tag{7.3.5}$$

for all $z \in H$.

This can be regarded as a limit of the usual Cauchy integral formula, as is explained, for example, in [**Nak**] (p.9). The rigorous proof consists of the Cayley transform $z \mapsto i\dfrac{1+z}{1-z}$ from the unit disk $\Delta = \{|z| < 1\}$ to the upper half plane H and another version of the Schwarz integral formula for a holomorphic function on Δ,

$$f(z) = i\,\mathrm{Im}\bigl(f(0)\bigr) + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\bigl(f(e^{i\theta})\bigr)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta, \tag{7.3.6}$$

which is a direct consequence of the Gauss' mean value theorem for harmonic functions. (Another proof of (7.3.6) is found in [**Y**], §31).

Note that $\varphi_{s,t}(\xi)$ takes real value on $\mathbb{R} \smallsetminus C_{s,t}$ if continuously extended to the boundary. Moreover $\varphi_{s,t}(\xi) - \xi$ satisfies the conditions in Lemma 7.3.1. Therefore for $\xi \in H$ we have

$$\begin{aligned} \varphi_{s,t}(\xi) &= \xi + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{Im}(\varphi_{s,t}(x) - x)}{x-\xi}\,dx \\ &= \xi + \frac{1}{2\pi}\int_{C_{s,t}} \frac{\mathrm{Im}(\varphi_{s,t}(x) - x)}{x-\xi}\,dx. \end{aligned} \tag{7.3.7}$$

Substituting $\xi = iy$ in (7.3.7) and using the expansion (7.3.3), we obtain

$$u(t) - u(s) + O((iy)^{-1}) = \frac{1}{2\pi}\int_{C_{s,t}} \frac{iy}{x-iy}\,\mathrm{Im}\,\varphi_{s,t}(x)\,dx.$$

The limit $y \to \infty$ gives us

$$-u_\varphi(s,t) = u(t) - u(s) = \frac{-1}{2\pi}\int_{C_{s,t}} \mathrm{Im}\,\varphi_{s,t}(x)\,dx. \tag{7.3.8}$$

Let us assume $s < s' < t$. It is easy to see that $\varphi_{s,t} = \varphi_{s,s'} \circ \varphi_{s',t}$. (See Figure 7.3.9.)

Hence, applying (7.3.7) to $\varphi_{s,s'}$, we have

$$\begin{aligned} \varphi_{s,t}(\xi) - \varphi_{s',t}(\xi) &= \varphi_{s,s'}(\varphi_{s',t}(\xi)) - \varphi_{s',t}(\xi) \\ &= \frac{1}{2\pi}\int_{C_{s,s'}} \frac{\mathrm{Im}\,\varphi_{s,s'}(x)}{x-\varphi_{s',t}(\xi)}dx. \end{aligned} \tag{7.3.10}$$

Here is the punchline of the proof: since $C_{s,s'}$ shrinks to one point $U(s)$ when $s' \to s$, the ratio of (7.3.10) and (7.3.8),

$$\frac{\varphi_{s,t}(\xi) - \varphi_{s',t}(\xi)}{u(s)-u(s')} = \frac{\frac{1}{2\pi}\int_{C_{s,s'}} \frac{\mathrm{Im}\,\varphi_{s,s'}(x)}{x-\varphi_{s',t}(\xi)}dx}{\frac{1}{2\pi}\int_{C_{s,s'}} \mathrm{Im}\,\varphi_{s,s'}(x)\,dx},$$

converges to

$$\frac{\frac{\partial\varphi_{s,t}}{\partial s}}{\frac{\partial u_\varphi}{\partial s}(s,t)} \sim \frac{\frac{\mathrm{Im}\,\varphi_{s,s'}(x)}{x-\varphi_{s',t}(\xi)}\,|C_{s,s'}|}{\mathrm{Im}\,\varphi_{s,s'}(x)\,|C_{s,s'}|} \to \frac{1}{U(s)-\varphi_{s,t}(\xi)}, \tag{7.3.11}$$

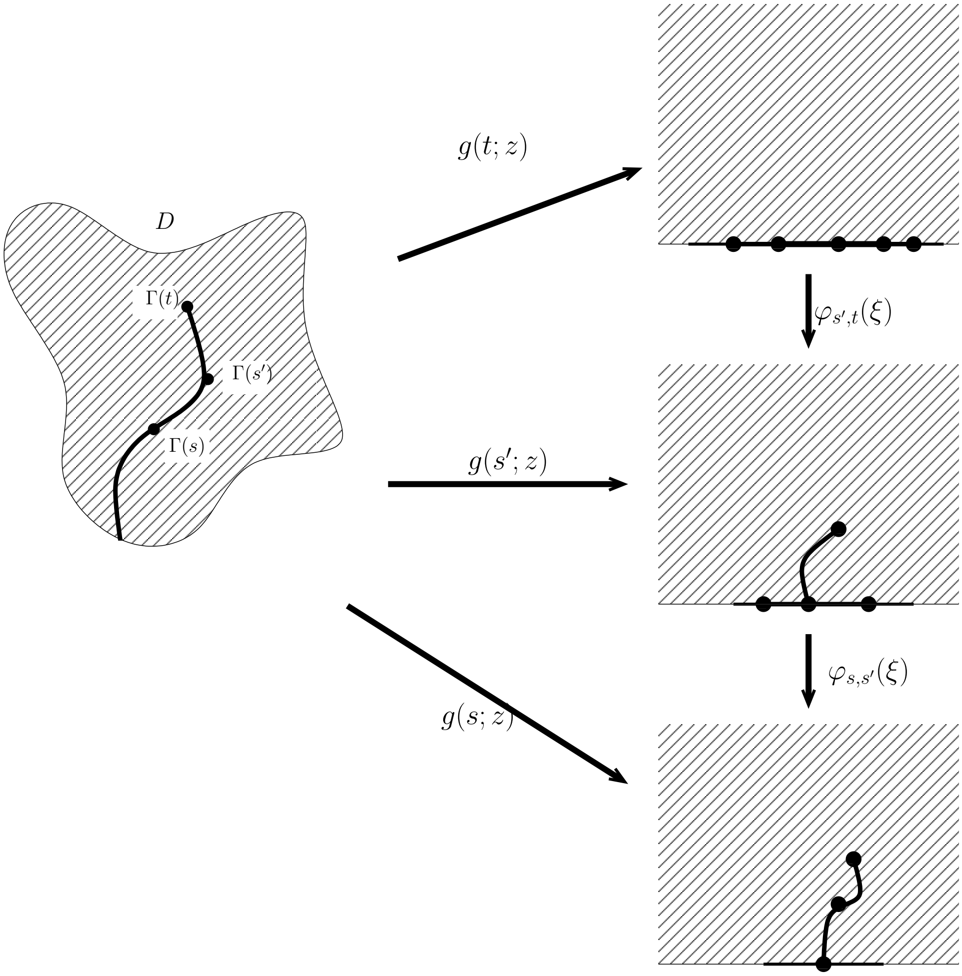

FIGURE 7.3.9. $\varphi_{s,t} = \varphi_{s,s'} \circ \varphi_{s',t}$.

from which follows the chordal Löwner equation (7.3.4) for $\varphi_{s,t}$. Here $|C_{s,s'}|$ is the length of $C_{s,s'}$. (We assumed $s < s' < t$ and $s' \searrow s$ but if $s' < s < t$ and $s' \nearrow s$, the limit (7.3.11) is the same.)

This is the outline of the proof of Theorem 7.2.1. We omitted important details like extendability of conformal maps to boundaries or the convergence $C_{s,s'} \to \{U(s)\}$, for which we refer to the original paper [**dMG**].

CHAPTER 8

Dispersionless Toda hierarchy

Recall that the Toda lattice hierarchy is a system of differential-difference equations. For example, one of the equations is the two-dimensional Toda lattice equation (3.1.11),

$$\frac{\partial^2}{\partial t_1\,\partial \bar{t}_1}\varphi(s) = e^{\varphi(s)-\varphi(s-1)} - e^{\varphi(s+1)-\varphi(s)}.$$

for the Toda field $\varphi(t,\bar{t};s)$. In this differential-difference equation the shift of the s variable is fixed to 1. Let us replace this with a small parameter $\hbar$ and replace t_1, $\bar{t}_1$ to $t_1/\hbar$, $\bar{t}_1/\hbar$ as we did in Chapter 4.

$$\hbar^2\frac{\partial^2}{\partial t_1\,\partial \bar{t}_1}\varphi(s) = e^{\varphi(s)-\varphi(s-\hbar)} - e^{\varphi(s+\hbar)-\varphi(s)}.$$

We can take the limit $\hbar \to 0$, scaling φ appropriately, and obtain the equation

$$\frac{\partial^2}{\partial t_1\,\partial \bar{t}_1}\varphi(s) + \frac{\partial}{\partial s}\exp\left(\frac{\partial\varphi}{\partial s}\right) = 0.$$

This is called the two-dimensional dispersionless Toda lattice equation or the continual Toda lattice, which was studied in late 80's in the context of the "continual Cartan subalgebra" ([**GKR, SV, KSSV**]).

In principle the above procedure is how we derive the dispersionless Toda hierarchy from the Toda lattice hierarchy. Probably the reader has noticed resemblance with the arguments in Chapter 4. In fact, we can proceed just like in Chapter 4 and Chapter 5 to develop the theory of the $\hbar$-dependent/dispersionless Toda lattice hierarchy parallel to the KP case therein. So we omit most details and refer to [**TT2**] and [**TT3**].

8.1. Dispersionless Toda hierarchy

For the Toda lattice hierarchy the procedure in Chapter 4 and Chapter 5 are changed as follows. We first replace the difference operator $e^{\partial_s} : f(s) \mapsto f(s+1)$ by $e^{\hbar\partial_s} : f(s) \mapsto f(s+\hbar)$. We introduce the symbol map $\sigma^\hbar$ defined by

$$\sigma^\hbar(f(s)e^{n\hbar\partial_s}) = f(s)p^n. \tag{8.1.1}$$

By taking the symbols the canonical commutation relation of the difference operator is mapped to the canonical Poisson relation:

$$\begin{array}{ccc} [e^{\hbar\partial_s}, s] = \hbar e^{\hbar\partial_s} & \longrightarrow & \{p, s\} = p, \\ [P, Q] & \longrightarrow & \{\sigma^\hbar(P), \sigma^\hbar(Q)\}, \end{array}$$

where the Poisson bracket is defined by

$$\{f(s,p), g(s,p)\} := p\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial s} - \frac{\partial f}{\partial s}\frac{\partial g}{\partial p}\right). \tag{8.1.2}$$

Note that for the dispersionless KP hierarchy the canonical Poisson relation is $\{\xi, x\} = 1$, but in the present case $\{p, s\} = p$.

DEFINITION 8.1.1. The *dispersionless Toda hierarchy* (or the *dToda hierarchy* for short) is the system of differential equations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{L}\}, & \frac{\partial \mathcal{L}}{\partial \bar{t}_n} &= \{\bar{\mathcal{B}}_n, \mathcal{L}\}, \\ \frac{\partial \bar{\mathcal{L}}}{\partial t_n} &= \{\mathcal{B}_n, \bar{\mathcal{L}}\}, & \frac{\partial \bar{\mathcal{L}}}{\partial \bar{t}_n} &= \{\bar{\mathcal{B}}_n, \bar{\mathcal{L}}\}, \end{aligned} \tag{8.1.3}$$

where $\mathcal{L}$ and $\bar{\mathcal{L}}$ are generating functions of unknown functions $u_i = u_i(t, \bar{t}; s)$, $\bar{u}_i = \bar{u}_i(t, \bar{t}; s)$,

$$\begin{aligned} \mathcal{L} &= p + u_1 + u_2 p^{-1} + u_3 p^{-2} + \cdots, \\ \bar{\mathcal{L}}^{-1} &= \bar{u}_0 p^{-1} + \bar{u}_1 + \bar{u}_2 p + \bar{u}_3 p^2 + \cdots, \end{aligned} \tag{8.1.4}$$

and $\mathcal{B}_n$, $\bar{\mathcal{B}}_n$ are defined by

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \qquad \bar{\mathcal{B}}_n = (\bar{\mathcal{L}}^{-n})_{<0}. \tag{8.1.5}$$

Here the truncation operations $()_{\geq 0}$ and $()_{<0}$ are defined by taking the polynomial part and the negative degree part in p.

REMARK 8.1.2. In contrast to the $\bar{L}$-operator (3.1.1) of the Toda lattice hierarchy, $\bar{\mathcal{L}}$ is defined by its inverse in (8.1.4). This is just for later convenience (for the Riemann-Hilbert type construction of solutions). If one would define $\bar{\mathcal{L}}$ as the second series in (8.1.4), then $\bar{\mathcal{B}}_n$ should be $\bar{\mathcal{B}}_n = (\bar{\mathcal{L}}^n)_{<0}$ but the equations in (8.1.3) are the same.

REMARK 8.1.3. As in the case of the Toda lattice hierarchy, in which we have freedom of gauge transformation (3.1.12), the dispersionless Toda hierarchy has different definition equivalent to the above definition (8.1.3) up to gauge transformation. We omit details, which can be found in §2.1 of [**TT3**]. What we shall use later is the following "symmetric gauge", which is the dispersionless counterpart of (3.1.13).

$$
\begin{aligned}
\mathcal{L} &= u_0 p + u_1 + u_2 p^{-1} + u_3 p^{-2} + \cdots, \\
\bar{\mathcal{L}}^{-1} &= \bar{u}_0 p^{-1} + \bar{u}_1 + \bar{u}_2 p + \bar{u}_3 p^2 + \cdots, \\
u_0 &= \bar{u}_0, \\
\mathcal{B}_n &:= (\mathcal{L}^n)_{>0} + \frac{1}{2}(\mathcal{L}^n)_0, \qquad \bar{\mathcal{B}}_n := (\bar{\mathcal{L}}^{-n})_{<0} + \frac{1}{2}(\bar{\mathcal{L}}^{-n})_0.
\end{aligned} \tag{8.1.6}
$$

Here the truncation operations $()_{>0}$, $()_0$ and $()_{<0}$ are defined as taking the positive power/constant/negative power part of the Laurent series in p.

As is shown for the Toda lattice hierarchy Section 3.1, the Lax representation (8.1.3) is equivalent to the *Zakharov-Shabat representation*:

$$\frac{\partial \mathcal{B}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial t_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0, \tag{8.1.7}$$

$$\frac{\partial \bar{\mathcal{B}}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial \bar{t}_m} + \{\bar{\mathcal{B}}_m, \mathcal{B}_n\} = 0, \tag{8.1.8}$$

$$\frac{\partial \bar{\mathcal{B}}_m}{\partial \bar{t}_n} - \frac{\partial \bar{\mathcal{B}}_n}{\partial \bar{t}_m} + \{\bar{\mathcal{B}}_m, \bar{\mathcal{B}}_n\} = 0. \tag{8.1.9}$$

There are dressing operations, by means of which the Lax functions are expressed in the form,

$$\mathcal{L} = e^{\operatorname{ad}_{\{,\}} \chi} p, \qquad \bar{\mathcal{L}} = e^{\operatorname{ad}_{\{,\}} \varphi} e^{\operatorname{ad}_{\{,\}} \bar{\chi}} p.$$

See §2.2 of [**TT3**][1] for the proof and details. Using these operations, we define the Orlov-Schulman series as follows.

$$
\begin{aligned}
\mathcal{M} &:= e^{\operatorname{ad}_{\{,\}} \chi} \left(s + \sum_{n=1}^{\infty} n t_n p^n \right) = e^{\operatorname{ad}_{\{,\}} \chi} e^{\operatorname{ad}_{\{,\}} \zeta(t;p)} (s), \\
\bar{\mathcal{M}} &:= e^{\operatorname{ad}_{\{,\}} \varphi} e^{\operatorname{ad}_{\{,\}} \bar{\chi}} \left(s + \sum_{n=1}^{\infty} n \bar{t}_n p^{-n} \right) \\
&= e^{\operatorname{ad}_{\{,\}} \varphi} e^{\operatorname{ad}_{\{,\}} \bar{\chi}} e^{\operatorname{ad}_{\{,\}} \zeta(\bar{t};p^{-1})} (s).
\end{aligned} \tag{8.1.10}
$$

The series $\zeta(t;p)$ is as is defined by (2.2.6): $\zeta(t;p) = \sum_{n=1}^{\infty} t_n p^n$. We have two Orlov-Schulman series $\mathcal{M}$ and $\bar{\mathcal{M}}$ corresponding to two Lax series

[1]In [**TT3**] we used symbols φ, ϕ and $\bar{\varphi}$ instead of χ, φ and $\bar{\chi}$.

$\mathcal{L}$ and $\bar{\mathcal{L}}$. They have the following expansion in $\mathcal{L}$ and $\bar{\mathcal{L}}$.

$$\begin{aligned}\mathcal{M} &= \sum_{n=1}^{\infty} nt_n \mathcal{L}^n + s + \sum_{i=1}^{\infty} v_i(t,\bar{t};s)\mathcal{L}^{-i},\\ \bar{\mathcal{M}} &= -\sum_{n=1}^{\infty} n\bar{t}_n \bar{\mathcal{L}}^{-n} + s + \sum_{i=1}^{\infty} \bar{v}_i(t,\bar{t};s)\bar{\mathcal{L}}^{i}.\end{aligned} \tag{8.1.11}$$

(In fact, we can start from the quadruple $(\mathcal{L}, \bar{\mathcal{L}}, \mathcal{M}, \bar{\mathcal{M}})$ without introducing the dressing operators, as we did in the earlier work [**TT2**].)

The Orlov-Schulman series satisfy the canonical Poisson relations with $\mathcal{L}$ and $\bar{\mathcal{L}}$,

$$\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}, \qquad \{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}}. \tag{8.1.12}$$

(These are direct consequences of $\{p, s\} = p$.) They also satisfy the same Lax equations as $\mathcal{L}$ and $\bar{\mathcal{L}}$,

$$\begin{aligned}\frac{\partial \mathcal{M}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{M}\}, & \frac{\partial \mathcal{M}}{\partial \bar{t}_n} &= \{\bar{\mathcal{B}}_n, \mathcal{M}\},\\ \frac{\partial \bar{\mathcal{M}}}{\partial t_n} &= \{\mathcal{B}_n, \bar{\mathcal{M}}\}, & \frac{\partial \bar{\mathcal{M}}}{\partial \bar{t}_n} &= \{\bar{\mathcal{B}}_n, \bar{\mathcal{M}}\}.\end{aligned} \tag{8.1.13}$$

There are two S-functions S and $\bar{S}$ instead of one S-function of the dispersionless KP hierarchy (5.2.1), Proposition 5.2.2. Existence of the tau function $\tau_{\mathrm{dtoda}}(t, \bar{t}; s)$ is proved similarly as Lemma 5.2.3. Since we do not use them in this lecture note, we omit details, which can be found in §2.4 of [**TT3**]. The dispersionless Hirota equation for the dispersionless Toda hierarchy was proved in [**Te**].

Remark 8.1.4. As is shortly remarked in Section 7.2, the one-variable reduction of the dispersionless Toda hierarchy is reduced to the radial Löwner equation (7.2.7). In fact we should use the symmetric gauge (8.1.6) of the dispersionless Toda hierarchy. The series $\mathcal{L}$ corresponds to the conformal map outside of the unit disk Δ, while the series $\bar{\mathcal{L}}$ corresponds to the conformal map inside Δ. For details we refer to §5.2 of [**TTZ**].

8.2. Riemann-Hilbert type construction of solutions

The Riemann-Hilbert type construction of solutions of the dispersionless KP hierarchy is a correspondence of solutions and canonical transformations. There is a same kind of correspondence for the dispersionless Toda hierarchy.

Theorem 8.2.1. *(i) Let* $(\mathcal{L}, \bar{\mathcal{L}}, \mathcal{M}, \bar{\mathcal{M}})$ *be a quadruplet of series of the form* (8.1.4) *and* (8.1.11). *Assume that there exists a quadruplet*

$(f(s,p), g(s,p), \bar{f}(s,p), \bar{g}(s,p))$ *of functions (formal power series in* p *with coefficients depending on* s*) satisfying*

(1) $\{f, g\} = f$, $\{\bar{f}, \bar{g}\} = \bar{f}$;

(2) *the following equations hold:*

$$f(\mathcal{M}, \mathcal{L}) = \bar{f}(\bar{\mathcal{M}}, \bar{\mathcal{L}}), \qquad g(\mathcal{M}, \mathcal{L}) = \bar{g}(\bar{\mathcal{M}}, \bar{\mathcal{L}}). \tag{8.2.1}$$

Then, the pair $(\mathcal{L}, \bar{\mathcal{L}})$ *is a solution of the dispersionless Toda hierarchy and the pair* $(\mathcal{M}, \bar{\mathcal{M}})$ *is the corresponding pair of Orlov-Schulman series.*

(ii) Conversely, if $(\mathcal{L}, \bar{\mathcal{L}})$ *is a solution of the dispersionless Toda hierarchy and the pair* $(\mathcal{M}, \bar{\mathcal{M}})$ *is the corresponding pair of Orlov-Schulman series, there exists a quadruplet* $(f(s,p), g(s,p), \bar{f}(s,p), \bar{g}(s,p))$ *satisfying the conditions in (i).*

In the present lecture note we omitted the proof of the corresponding theorem for the dispersionless KP hierarchy. Actually the reader can easily translate the following proof for the dispersionless Toda hierarchy to the KP case.

PROOF. We prove only the statement (i), following the proof of Proposition 2.5.1 of [**TT3**]. The proof of the second statement requires the dressing operation. See Proposition 2.5.2 of [**TT3**].

Differentiation of the condition

$$\begin{pmatrix} f(\mathcal{M}, \mathcal{L}) \\ g(\mathcal{M}, \mathcal{L}) \end{pmatrix} = \begin{pmatrix} \bar{f}(\bar{\mathcal{M}}, \bar{\mathcal{L}}) \\ \bar{g}(\bar{\mathcal{M}}, \bar{\mathcal{L}}) \end{pmatrix} \tag{8.2.2}$$

with respect to p and s gives the equation

$$\begin{aligned} &\begin{pmatrix} \dfrac{\partial f}{\partial \mathcal{L}}(\mathcal{L}, \mathcal{M}) & \dfrac{\partial f}{\partial \mathcal{M}}(\mathcal{L}, \mathcal{M}) \\ \dfrac{\partial g}{\partial \mathcal{L}}(\mathcal{L}, \mathcal{M}) & \dfrac{\partial g}{\partial \mathcal{M}}(\mathcal{L}, \mathcal{M}) \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial p} & \dfrac{\partial \mathcal{L}}{\partial s} \\ \dfrac{\partial \mathcal{M}}{\partial p} & \dfrac{\partial \mathcal{M}}{\partial s} \end{pmatrix} \\ &\qquad = \begin{pmatrix} \dfrac{\partial \bar{f}}{\partial \bar{\mathcal{L}}}(\bar{\mathcal{L}}, \bar{\mathcal{M}}) & \dfrac{\partial \bar{f}}{\partial \bar{\mathcal{M}}}(\bar{\mathcal{L}}, \bar{\mathcal{M}}) \\ \dfrac{\partial \bar{g}}{\partial \bar{\mathcal{L}}}(\bar{\mathcal{L}}, \bar{\mathcal{M}}) & \dfrac{\partial \bar{g}}{\partial \bar{\mathcal{M}}}(\bar{\mathcal{L}}, \bar{\mathcal{M}}) \end{pmatrix} \begin{pmatrix} \dfrac{\partial \bar{\mathcal{L}}}{\partial p} & \dfrac{\partial \bar{\mathcal{L}}}{\partial s} \\ \dfrac{\partial \bar{\mathcal{M}}}{\partial p} & \dfrac{\partial \bar{\mathcal{M}}}{\partial s} \end{pmatrix} \end{aligned} \tag{8.2.3}$$

by the chain rule. The determinant of the left hand side of this equation is equal to

$$\left(p^{-1}\{f, g\}\right)\Big|_{s \mapsto \mathcal{M},\ p \mapsto \mathcal{L}} \times p^{-1}\{\mathcal{L}, \mathcal{M}\},$$

and the determinant of the right hand side of (8.2.3) is the same with replacement $(f, g, \mathcal{L}, \mathcal{M}) \mapsto (\bar{f}, \bar{g}, \bar{\mathcal{L}}, \bar{\mathcal{M}})$. Therefore, using the relations

$\{f, g\} = f$, $\{\bar{f}, \bar{g}\} = \bar{f}$ and $f(\mathcal{M}, \mathcal{L}) = \bar{f}(\bar{\mathcal{M}}, \bar{\mathcal{L}})$, we can rewrite the determinants of the equation (8.2.3) as

$$\mathcal{L}^{-1}\{\mathcal{L}, \mathcal{M}\} = \bar{\mathcal{L}}^{-1}\{\bar{\mathcal{L}}, \bar{\mathcal{M}}\}. \tag{8.2.4}$$

By the assumption of the form of $(\mathcal{L}, \bar{\mathcal{L}})$, the right hand side of this equation is

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\mathcal{L}, \sum_{n=1}^{\infty} nt_n \mathcal{L}^n + s + \sum_{i=1}^{\infty} v_i \mathcal{L}^{-i}\right\} \\
&= \mathcal{L}^{-1}\left(\{\mathcal{L}, s\} + \sum_{i=1}^{\infty}\{\mathcal{L}, v_i\}\mathcal{L}^{-i}\right) \\
&= (p^{-1} + (\text{lower degree})) \\
&\quad \times \Bigg((p + (\text{lower degree})) \\
&\qquad + \sum_{i=1}^{\infty}\left(\frac{\partial v_i}{\partial s} p + (\text{lower degree})\right)(p^{-i} + (\text{lower degree}))\Bigg) \\
&= 1 + O(p^{-1}).
\end{aligned}$$

Similarly the left hand side of (8.2.4) contains only the non-negative powers of p. Therefore the both hand sides of (8.2.4) are equal to 1, which proves the canonical Poisson relations $\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}$ and $\{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}}$.

The Lax equations are proved as follows. First, differentiating (8.2.2) by t_n gives

$$\begin{aligned}
&\begin{pmatrix} \dfrac{\partial f}{\partial \mathcal{L}}(\mathcal{L}, \mathcal{M}) & \dfrac{\partial f}{\partial \mathcal{M}}(\mathcal{L}, \mathcal{M}) \\ \dfrac{\partial g}{\partial \mathcal{L}}(\mathcal{L}, \mathcal{M}) & \dfrac{\partial g}{\partial \mathcal{M}}(\mathcal{L}, \mathcal{M}) \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial t_n} \\ \dfrac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} \\
&\qquad = \begin{pmatrix} \dfrac{\partial \bar{f}}{\partial \bar{\mathcal{L}}}(\bar{\mathcal{L}}, \bar{\mathcal{M}}) & \dfrac{\partial \bar{f}}{\partial \bar{\mathcal{M}}}(\bar{\mathcal{L}}, \bar{\mathcal{M}}) \\ \dfrac{\partial \bar{g}}{\partial \bar{\mathcal{L}}}(\bar{\mathcal{L}}, \bar{\mathcal{M}}) & \dfrac{\partial \bar{g}}{\partial \bar{\mathcal{M}}}(\bar{\mathcal{L}}, \bar{\mathcal{M}}) \end{pmatrix} \begin{pmatrix} \dfrac{\partial \bar{\mathcal{L}}}{\partial t_n} \\ \dfrac{\partial \bar{\mathcal{M}}}{\partial t_n} \end{pmatrix}.
\end{aligned} \tag{8.2.5}$$

Using (8.2.3), we can rewrite this equation as

$$
\begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial p} & \dfrac{\partial \mathcal{L}}{\partial s} \\ \dfrac{\partial \mathcal{M}}{\partial p} & \dfrac{\partial \mathcal{M}}{\partial s} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial t_n} \\ \dfrac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \bar{\mathcal{L}}}{\partial p} & \dfrac{\partial \bar{\mathcal{L}}}{\partial s} \\ \dfrac{\partial \bar{\mathcal{M}}}{\partial p} & \dfrac{\partial \bar{\mathcal{M}}}{\partial s} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial \bar{\mathcal{L}}}{\partial t_n} \\ \dfrac{\partial \bar{\mathcal{M}}}{\partial t_n} \end{pmatrix}. \tag{8.2.6}
$$

The inverse matrices in this formula are easily calculated because of the canonical Poisson relations. Indeed, for example, the determinant of the matrix in the left hand side is

$$
\det \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial p} & \dfrac{\partial \mathcal{L}}{\partial s} \\ \dfrac{\partial \mathcal{M}}{\partial p} & \dfrac{\partial \mathcal{M}}{\partial s} \end{pmatrix} = p^{-1}\{\mathcal{L}, \mathcal{M}\} = p^{-1}\mathcal{L}. \tag{8.2.7}
$$

Thus we obtain

$$
\mathcal{L}^{-1} \begin{pmatrix} \dfrac{\partial \mathcal{M}}{\partial s} & -\dfrac{\partial \mathcal{L}}{\partial s} \\ -\dfrac{\partial \mathcal{M}}{\partial p} & \dfrac{\partial \mathcal{L}}{\partial p} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial t_n} \\ \dfrac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} = \bar{\mathcal{L}}^{-1} \begin{pmatrix} \dfrac{\partial \bar{\mathcal{M}}}{\partial s} & -\dfrac{\partial \bar{\mathcal{L}}}{\partial s} \\ -\dfrac{\partial \bar{\mathcal{M}}}{\partial p} & \dfrac{\partial \bar{\mathcal{L}}}{\partial p} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \bar{\mathcal{L}}}{\partial t_n} \\ \dfrac{\partial \bar{\mathcal{M}}}{\partial t_n} \end{pmatrix}. \tag{8.2.8}
$$

The first component of the left hand side of (8.2.8) is given by

$$
\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{\partial \mathcal{M}}{\partial s}\frac{\partial \mathcal{L}}{\partial t_n} - \frac{\partial \mathcal{M}}{\partial t_n}\frac{\partial \mathcal{L}}{\partial s}\right) \\
&= \mathcal{L}^{-1}\left(\left.\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\right|_{t,v_i \text{ fixed}} \frac{\partial \mathcal{L}}{\partial s} + 1 + \sum_{i=1}^{\infty} \frac{\partial v_i}{\partial s}\mathcal{L}^{-i} \right) \frac{\partial \mathcal{L}}{\partial t_n} \\
&\quad - \mathcal{L}^{-1}\left(\left.\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\right|_{t,v_i \text{ fixed}} \frac{\partial \mathcal{L}}{\partial t_n} + n\mathcal{L}^n + \sum_{i=1}^{\infty} \frac{\partial v_i}{\partial t_n}\mathcal{L}^{-i} \right) \frac{\partial \mathcal{L}}{\partial s} \\
&= -\frac{\partial(\mathcal{L}^n)}{\partial s} + \mathcal{L}^{-1}\left(1 + \sum_{i=1}^{\infty} \frac{\partial v_i}{\partial s}\mathcal{L}^{-i}\right)\frac{\partial \mathcal{L}}{\partial t_n} - \mathcal{L}^{-1}\left(\sum_{i=1}^{\infty} \frac{\partial v_i}{\partial t_n}\mathcal{L}^{-i}\right)\frac{\partial \mathcal{L}}{\partial s}, \\
&= -\frac{\partial(\mathcal{L}^n)}{\partial s} + (\text{negative powers of } p).
\end{aligned} \tag{8.2.9}
$$

Similarly the first component of the right hand side of (8.2.8) is given by

$$
\begin{aligned}
&\bar{\mathcal{L}}^{-1}\left(\frac{\partial\bar{\mathcal{M}}}{\partial s}\frac{\partial\bar{\mathcal{L}}}{\partial t_n}-\frac{\partial\bar{\mathcal{M}}}{\partial t_n}\frac{\partial\bar{\mathcal{L}}}{\partial s}\right)\\
&=\bar{\mathcal{L}}^{-1}\left(\left.\frac{\partial\bar{\mathcal{M}}}{\partial\bar{\mathcal{L}}}\right|_{t,v_i\ \text{fixed}}\frac{\partial\bar{\mathcal{L}}}{\partial s}+1-\sum_{i=1}^{\infty}\frac{\partial\bar{v}_i}{\partial s}\bar{\mathcal{L}}^i\right)\frac{\partial\bar{\mathcal{L}}}{\partial t_n}\\
&\quad-\bar{\mathcal{L}}^{-1}\left(\left.\frac{\partial\bar{\mathcal{M}}}{\partial\bar{\mathcal{L}}}\right|_{t,v_i\ \text{fixed}}\frac{\partial\bar{\mathcal{L}}}{\partial t_n}+\sum_{i=1}^{\infty}\frac{\partial\bar{v}_i}{\partial t_n}\bar{\mathcal{L}}^i\right)\frac{\partial\bar{\mathcal{L}}}{\partial s}\\
&=\bar{\mathcal{L}}^{-1}\left(1-\sum_{i=1}^{\infty}\frac{\partial\bar{v}_i}{\partial s}\bar{\mathcal{L}}^i\right)\frac{\partial\bar{\mathcal{L}}}{\partial t_n}-\bar{\mathcal{L}}^{-1}\left(\sum_{i=1}^{\infty}\frac{\partial\bar{v}_i}{\partial t_n}\bar{\mathcal{L}}^i\right)\frac{\partial\bar{\mathcal{L}}}{\partial s},
\end{aligned}
$$

which does not contain negative powers of p. Hence the first component of (8.2.8) should be equal to the polynomial part of (8.2.9), namely,

$$
\mathcal{L}^{-1}\left(\frac{\partial\mathcal{M}}{\partial s}\frac{\partial\mathcal{L}}{\partial t_n}-\frac{\partial\mathcal{M}}{\partial t_n}\frac{\partial\mathcal{L}}{\partial s}\right)=\left(-\frac{\partial(\mathcal{L}^n)}{\partial s}\right)_{\geq 0}=-\frac{\partial\mathcal{B}_n}{\partial s}. \tag{8.2.10}
$$

In the same manner we can prove that the second component of (8.2.8) is polynomial part of (8.2.9) with $\partial/\partial s$ instead of $\partial/\partial p$, namely,

$$
\mathcal{L}^{-1}\left(\frac{\partial\mathcal{M}}{\partial p}\frac{\partial\mathcal{L}}{\partial t_n}-\frac{\partial\mathcal{M}}{\partial t_n}\frac{\partial\mathcal{L}}{\partial p}\right)=\left(-\frac{\partial(\mathcal{L}^n)}{\partial p}\right)_{\geq 0}=-\frac{\partial\mathcal{B}_n}{\partial p}. \tag{8.2.11}
$$

Putting (8.2.9), (8.2.10) and (8.2.11) together, we have

$$
\begin{aligned}
&\mathcal{L}^{-1}\begin{pmatrix}\dfrac{\partial\mathcal{M}}{\partial s} & -\dfrac{\partial\mathcal{L}}{\partial s}\\ -\dfrac{\partial\mathcal{M}}{\partial p} & \dfrac{\partial\mathcal{L}}{\partial p}\end{pmatrix}\begin{pmatrix}\dfrac{\partial\mathcal{L}}{\partial t_n}\\ \dfrac{\partial\mathcal{M}}{\partial t_n}\end{pmatrix}\\
&=\bar{\mathcal{L}}^{-1}\begin{pmatrix}\dfrac{\partial\bar{\mathcal{M}}}{\partial s} & -\dfrac{\partial\bar{\mathcal{L}}}{\partial s}\\ -\dfrac{\partial\bar{\mathcal{M}}}{\partial p} & \dfrac{\partial\bar{\mathcal{L}}}{\partial p}\end{pmatrix}\begin{pmatrix}\dfrac{\partial\bar{\mathcal{L}}}{\partial t_n}\\ \dfrac{\partial\bar{\mathcal{M}}}{\partial t_n}\end{pmatrix}=\begin{pmatrix}-\dfrac{\partial\mathcal{B}_n}{\partial s}\\ \dfrac{\partial\mathcal{B}_n}{\partial p}\end{pmatrix}.
\end{aligned} \tag{8.2.12}
$$

Due to the canonical Poisson relations again, the inverse of the matrices in (8.2.12) can be explicitly calculated, and we obtain

$$
\begin{pmatrix}\dfrac{\partial\mathcal{L}}{\partial t_n}\\ \dfrac{\partial\mathcal{M}}{\partial t_n}\end{pmatrix}=\begin{pmatrix}\{\mathcal{B}_n,\mathcal{L}\}\\ \{\mathcal{B}_n,\mathcal{M}\}\end{pmatrix},\quad\begin{pmatrix}\dfrac{\partial\bar{\mathcal{L}}}{\partial t_n}\\ \dfrac{\partial\bar{\mathcal{M}}}{\partial t_n}\end{pmatrix}=\begin{pmatrix}\{\mathcal{B}_n,\bar{\mathcal{L}}\}\\ \{\mathcal{B}_n,\bar{\mathcal{M}}\}\end{pmatrix}, \tag{8.2.13}
$$

which is nothing but t-flow part of the Lax equations (8.1.3). The $\bar{t}$-flow part of the Lax equations can be proved in the same way. □

REMARK 8.2.2. In Theorem 8.2.1 we used four functions f, g, $\bar{f}$ and $\bar{g}$, but we may assume that $\bar{f} = p$ and $\bar{g} = s$. In fact, by composing two canonical transformations $(p, s) \mapsto (f, g)$ and $(p, s) \mapsto (\bar{f}, \bar{g})$ as $(\tilde{f}, \tilde{g}) := (\bar{f}, \bar{g})^{-1} \circ (f, g)$, we can rewrite the condition (8.2.1) as

$$\tilde{f}(\mathcal{M}, \mathcal{L}) = \bar{\mathcal{L}}, \qquad \tilde{g}(\mathcal{M}, \mathcal{L}) = \bar{\mathcal{M}}.$$

CHAPTER 9

Hele-Shaw flow and dispersionless Toda hierarchy

The relation of the dispersionless Toda hierarchy and the Riemann mapping theorem, which were found and developed by Mineev-Weinstein, Wiegmann, Zabrodin, Krichever, Kostov, Marshakov and others, has its origin in physics of two-dimensional fluid, the Hele-Shaw flow. In this chapter we briefly introduce this problem and a tool (the Schwarz function) in complex analysis related to it. Then we show that the conformal mapping from outside of the domain to the outside of the unit disk satisfies the equations of the dispersionless Toda hierarchy when we regard the harmonic moments as independent variables.

9.1. Hele-Shaw flow

The Hele-Shaw flow is a fluid flow between two parallel plates separated by a very small gap. This is named after H. S. Hele-Shaw (1851–1941) who invented the Hele-Shaw cell for experiments of such flow. We shall discuss a very special case of this flow, which turns out to be integrable. Interesting reviews on this subject are [**V**] and [**VE**].

Let us assume that two incompressible fluids, one (say, oil) of which is much more viscous than the other (say, water), exist in-between two parallel plates with a narrow gap (the Hele-Shaw cell; Figure 9.1.1). The water is being injected into the cell from a hole at the centre of a plate and the oil is pushed out by water toward infinity.

The problem is the analysis of the time dependence of the interface curve $\Gamma(t)$ between water and oil. We impose several physical and mathematical assumptions.

- Both fluids are incompressible;
- The viscosity of water (injected fluid) is zero;
- The surface tension between two fluids is zero;
- Oil (viscous fluid) is pumped out "uniformly", while water is injected with constant speed;
- The interface curve $\Gamma(t)$ is a real analytic Jordan curve.

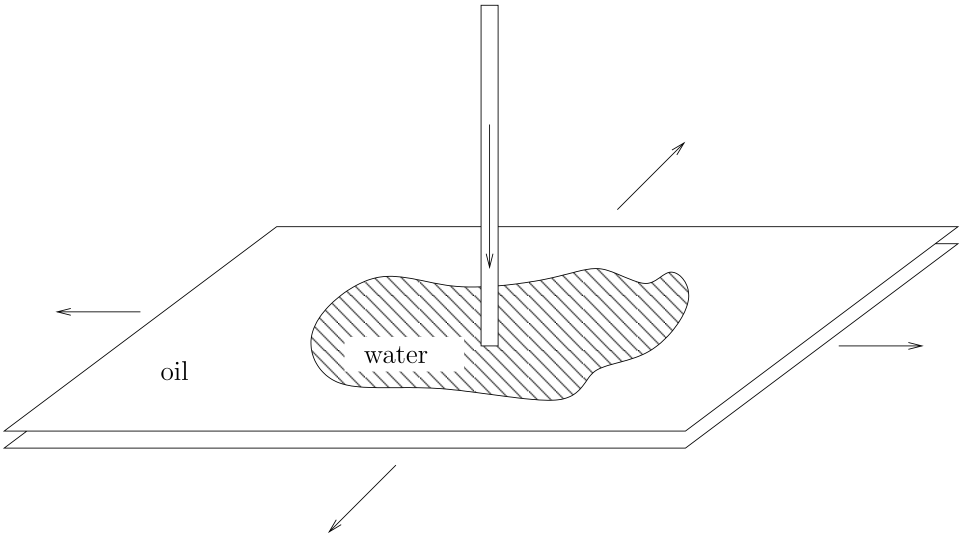

FIGURE 9.1.1. Hele-Shaw cell.

The incompressibility of the fluids imply that the pressure field p is harmonic: $\Delta p = 0$. Hence this kind of problem is called the *Laplacian growth problem*. We refer details to [**Zab2**].

The important property of this flow is existence of infinitely many conserved quantities. For the statement of this theorem we define the harmonic moments of a domain in the plane.

In the probability theory the k-th *moment* of a probability measure $d\mu$ on $\mathbb{R}$ is defined by

$$m_k := \int_{\mathbb{R}} x^k \, d\mu. \tag{9.1.2}$$

For example, m_1 is the mean and $m_2 - m_1^2$ is the variance. The harmonic moment is a complex version of this notion. First, we introduce it by an easy-to-understand definition but not-rigorous (even ill-defined!) way.

DEFINITION 9.1.1. Let Γ be a Jordan curve in the complex plane and D be the outside of Γ. The *k-th harmonic moment* of D is defined by the following area integral.

$$C_k(D) := \begin{cases} -\displaystyle\int_D z^{-k} \, dx \, dy, & (k \geqq 1) \\ \displaystyle\int_{\mathbb{C} \setminus D} z^{-k} \, dx \, dy & (k \leqq 0). \end{cases} \tag{9.1.3}$$

Obviously $C_0(D)$ is the area of the interior of Γ. The parallelism of this definition with (9.1.2) is evident, but the first and second harmonic moments thus defined *diverge*. So, let us give another less intuitive but rigorous definition (9.1.10), which coincides to the above one for $k \neq 1, 2$.

Definition 9.1.1 uses the area integrals, but, as the harmonic moments are determined solely by the Jordan curve Γ, we can expect that they should be expressed in terms of Γ itself. In fact, we can rewrite the harmonic moments in the line integral form by Green's formula.

EXAMPLE 9.1.2. Let us consider the case $k = 0$. The 0-th harmonic moment of D is by definition

$$C_0 = \int_{\mathbb{C} \smallsetminus D} 1 \, dx \wedge dy.$$

We rewrite it as follows, using Green's formula.

$$\begin{aligned} C_0 &= \frac{1}{2i} \int_{\mathbb{C} \smallsetminus D} \left(-\frac{\partial}{\partial y}(x - iy) + \frac{\partial}{\partial x}(i(x - iy)) \right) dx \wedge dy \\ &= \frac{1}{2i} \int_\Gamma \big((x - iy) \, dx + (x - iy) \, i \, dy \big) = \frac{1}{2i} \int_\Gamma \bar{z} \, dz. \end{aligned} \tag{9.1.4}$$

Thus we obtain the expression of C_0 in terms of the contour integral.

Similarly we obtain

$$C_k = \int_{\mathbb{C} \smallsetminus D} z^{-k} \, dx \wedge dy = \frac{1}{2i} \int_\Gamma z^{-k} \bar{z} \, dz, \tag{9.1.5}$$

for $k < 0$.

EXERCISE 9.1.3. Prove (9.1.5).

When $k > 0$, we consider the domain D_R between Γ and an auxiliary circle Γ_R with sufficiently large radius R (Figure 9.1.6).

Applying Green's formula to D_R, we have

$$C_{k,R} := -\int_{D_R} z^{-k} dx \wedge dy = \frac{1}{2i} \left(\int_\Gamma z^{-k} \bar{z} \, dz - \int_{\Gamma_R} z^{-k} \bar{z} \, dz \right). \tag{9.1.7}$$

When $k \geqq 3$, then the limit $R \to \infty$ of (9.1.7) gives

$$C_k = -\int_D z^{-k} dx \wedge dy = \frac{1}{2i} \int_\Gamma z^{-k} \bar{z} \, dz. \tag{9.1.8}$$

When $k = 1$ or $k = 2$, the equation (9.1.7) means that

(9.1.9)

$$C_{k,R} + (\text{constant which depends on } R \text{ but not on } \Gamma) = \frac{1}{2i} \int_\Gamma z^{-k} \bar{z} \, dz.$$

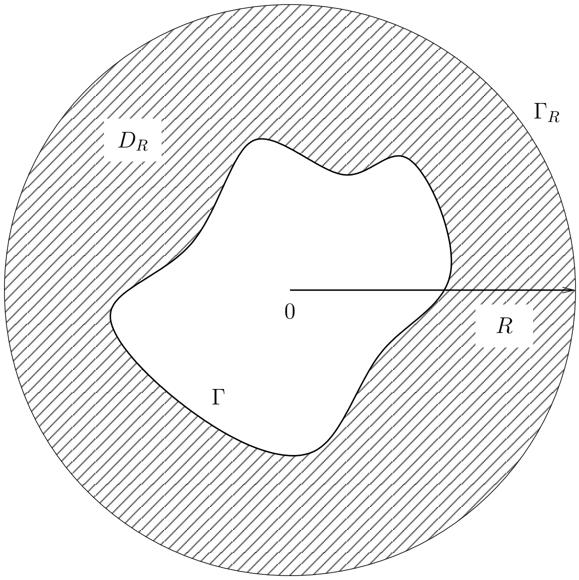

FIGURE 9.1.6. Domain D_R.

In view of the above results, especially (9.1.9), we redefine the harmonic moments by

$$C_k := \frac{1}{2i} \int_\Gamma z^{-k} \bar{z}\, dz. \tag{9.1.10}$$

for all k. As we have shown, this definition coincides with the previous one (9.1.3) for $k \neq 1, 2$. For the case $k = 1$ or $k = 2$, this definition is "renormalisation" of (9.1.3).

The following fact on the harmonic moments is known. (For example, [**Takh**], Theorem 1.5 (i), or [**No**].)

PROPOSITION 9.1.4. *The harmonic moments C_k ($k \geqq 0$) determine the curve Γ locally uniquely. Namely, if there exists a continuous family of curves $\Gamma(x)$ ($x \in (-\varepsilon, \varepsilon)$) with identical harmonic moments C_k ($k \geqq 0$), this family is trivial ($\Gamma(x) = \Gamma(0)$).*

So these quantities can be regarded as "coordinates" in the space of curves in the complex plane.

REMARK 9.1.5. It is also known that C_k's ($k \geqq 0$) determine the curve uniquely, if the inside of the curve is star-shaped, in particular, convex ([**No**]). However, there exists a counterexample in general (Sakai's example [**Sak**]).

Moreover the harmonic moments are conserved by the Hele-Shaw flow.

THEOREM 9.1.6. *Let us denote the outside of* $\Gamma(t)$ *by* $D(t)$ *and the* k*-th harmonic moment by* $C_k(t)$*:* $C_k(t) := C_k(D(t))$*. Then*

$$\frac{dC_k}{dt} = \begin{cases} (\text{non-zero constant}), & k = 0, \\ 0, & k \neq 0. \end{cases} \tag{9.1.11}$$

This is called *Richardson's theorem*. We refer its proof to Richardson's original paper [**R**].

The fact that C_0 grows linearly is an obvious consequence of the assumption of incompressibility and the constant injection speed, because C_0 is the area of the injected water. Richardson's theorem says that the other harmonic moments are conserved quantities. The existence of infinitely many conserved quantities suggests that this system is integrable. We shall show in Section 9.3 that it is indeed described by the dispersionless Toda hierarchy.

9.2. Schwarz function

In the definition (9.1.10) of the harmonic moments, the integrand is not holomorphic, since it contains $\bar{z}$. This is rather inconvenient, as we cannot use powerful machinery of complex analysis, for example, change of the integration contour.

Can we replace the integrand with a holomorphic function? Note that we have only to replace it on the contour Γ in order to have the same integral. If Γ is real analytic, the answer is positive.

DEFINITION 9.2.1. A holomorphic function $S(z)$ in a neighbourhood of a curve Γ is called the *Schwarz function* of Γ, if the curve Γ is equal to $\{z \mid \bar{z} = S(z)\}$ as a set.

EXAMPLE 9.2.2. The trivial example is $S(z) = z$. The curve corresponding to this Schwarz function is the real line: $\mathbb{R} = \{z \mid \bar{z} = z\}$.

EXAMPLE 9.2.3. Let Γ be the circle with radius R: $\Gamma = \{z \mid |z| = R\}$. This means $z\bar{z} = R^2$ on Γ. Hence the Schwarz function for the circle is $S(z) = \dfrac{R^2}{z}$.

The reader can find various examples and applications of the Schwarz function in [**Dav**].

What we need is the Schwarz functions for real analytic curves.

LEMMA 9.2.4. *If* Γ *is a real analytic closed Jordan curve, then there exists a Schwarz function for* Γ*.*

PROOF. Since a real analytic curve is defined by an equation $F(x, y) = 0$, where $F(x, y)$ is a real analytic function, we obtain the Schwarz function $\bar{z} = S(z)$ by solving the equation

$$F\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = 0$$

with respect to $\bar{z}$.

We shall use more explicit expression of the Schwarz function in terms of the conformal mapping, so we give another proof of this lemma here. According to the Riemann mapping theorem (applied to domains including the infinity as an interior point) there exists a holomorphic bijection $\varphi(z)$ from the outside of Γ to the outside of the unit disk $\bar{\Delta} = \{w \mid |w| \leqq 1\}$. We normalise φ so that it has an expansion

$$\varphi(z) = r^{-1}z + \sum_{n=0}^{\infty} c_n z^{-n}, \qquad r > 0, \tag{9.2.1}$$

around the infinity $z = \infty$ for later use. Since we assume that Γ is real analytic, φ has an analytic continuation defined in the neighbourhood of Γ. Since φ maps the boundary to the boundary (Carathéodory's theorem), a point z belongs to Γ if and only if $\varphi(z)$ belongs to the boundary of Δ, i.e., $|\varphi(z)| = 1$. Hence

$$\varphi(z)\,\bar{\varphi}(\bar{z}) = 1, \tag{9.2.2}$$

where $\bar{\varphi}$ is defined by

$$\bar{\varphi}(z) = \overline{\varphi(\bar{z})}, \tag{9.2.3}$$

or, in other words, by conjugating the coefficients in (9.2.1):

$$\bar{\varphi}(z) = r^{-1}z + \sum_{n=0}^{\infty} \bar{c}_n z^{-n}. \tag{9.2.4}$$

Solving (9.2.2) as an equation for $\bar{z}$, we obtain

$$\bar{z} = \bar{\varphi}^{-1}\left(\frac{1}{\varphi(z)}\right).$$

Therefore the Schwarz function in this case is

$$S(z) = \bar{\varphi}^{-1}\left(\frac{1}{\varphi(z)}\right). \tag{9.2.5}$$

□

Replacing $\bar{z}$ in (9.1.10) with the Schwarz function, we obtain

$$C_k = \frac{1}{2i}\int_{\Gamma} z^{-k} S(z)\, dz. \tag{9.2.6}$$

We assume that 0 is inside the curve Γ and that the Schwarz function has the Laurent expansion

$$S(z) = \sum_{n\in\mathbb{Z}} a_n z^n.$$

The coefficient a_n of this expansion is determined by

$$a_n = \frac{1}{2\pi i}\oint_\Gamma S(z)\, z^{-n-1}\, dz.$$

Comparing this with (9.2.6), we have $C_k = \pi a_{k-1}$, i.e.,

$$S(z) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} C_k z^{k-1}. \tag{9.2.7}$$

9.3. Relation to dispersionless Toda hierarchy

Now we return to the problem of the Hele-Shaw flow. Recall that C_k ($k \geqq 0$) serve as coordinates of the space of curves (Proposition 9.1.4). In view of this fact we rename the harmonic moments $C_k(t)$ of $\Gamma(t)$ as follows: for $k \geqq 1$

$$t_k := \frac{C_k}{\pi k}, \qquad v_k := \frac{C_{-k}}{\pi}, \qquad s := \frac{C_0}{\pi}. \tag{9.3.1}$$

The time variable t is essentially the same as $C_0(t) =$ area in $\Gamma(t)$. (See Theorem 9.1.6 and the remark to it.) Hence we identify t with the variable s defined above. Hereafter we use the symbol t not as the time variable of the Hele-Shaw flow, but as the set $t = (t_1, t_2, \dots)$. In this notation, the expansion (9.2.7) is

$$S(s,t;z) = \sum_{k=1}^{\infty} k\, t_k z^{k-1} + \frac{s}{z} + \sum_{n=1}^{\infty} v_n(s,t) z^{-n-1}. \tag{9.3.2}$$

Note that we regard S and its coefficients v_k as functions of $(s,t) = (s, t_1, t_2, \dots)$.

Let us recall that φ in the proof of Lemma 9.2.4 maps the outside of Γ to the outside of $\bar{\Delta}$ and has the expansion (9.2.1). In the present situation Γ depends on s and t_k ($k \geqq 1$) and consequently φ depends on $(s, t_k; k \geqq 1)$:

$$\varphi(s,t;z) = r(s,t)^{-1} z + \sum_{n=0}^{\infty} c_n(s,t) z^{-n}. \tag{9.3.3}$$

Therefore the inverse function of $\varphi(s,t;z)$ with respect to z has an expansion of the form

$$z = \varphi^{-1}(s,t;w) = r(s,t)w + \sum_{j=1}^{\infty} u_j(s,t) w^{1-j}. \tag{9.3.4}$$

The complex conjugate of this on the unit circle $|w|=1$ is

$$\bar{z} = \overline{\varphi^{-1}(s,t;w)} = \overline{\varphi^{-1}}(s,t;\bar{w}) = r(s,t)w^{-1} + \sum_{j=0}^{\infty} \bar{u}_j(s,t) w^{j-1}, \tag{9.3.5}$$

since $\bar{w} = w^{-1}$ on the unit circle and $r(s,t) \in \mathbb{R}$.

Let us put these expansions together with those of $z\,S(z)$ (cf. (9.3.2)) and its complex conjugate:

$$z = r(s,t)w + \sum_{j=1}^{\infty} u_j(s,t) w^{1-j}, \tag{9.3.6}$$

$$\bar{z} = r(s,t)w^{-1} + \sum_{j=1}^{\infty} \bar{u}_j(s,t) w^{j-1}, \tag{9.3.7}$$

$$z\,S(s,t;z) = \sum_{k=1}^{\infty} k\,t_k z^k + s + \sum_{n=1}^{\infty} v_n(s,t) z^{-n}, \tag{9.3.8}$$

$$\bar{z}\,\bar{S}(s,t;z) = \sum_{k=1}^{\infty} k\,\bar{t}_k \bar{z}^k + s + \sum_{n=1}^{\infty} \bar{v}_n(s,t) \bar{z}^{-n}, \tag{9.3.9}$$

The crucial observation in [**MiWZ**] is the following: the above expansions are exactly of the same form as (8.1.6) and (8.1.11)! (Well, up to signature, of course, but essentially the same.)

Now we consider the pair $(t_k, \bar{t}_k)$ as two-dimensional independent variables over $\mathbb{R}$. Hence the derivations $\dfrac{\partial}{\partial t_k}$ and $\dfrac{\partial}{\partial \bar{t}_k}$ should be regarded as the Wirtinger derivatives:

$$\frac{\partial}{\partial t_k} = \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} t_k} - i\frac{\partial}{\partial \operatorname{Im} t_k}\right), \qquad \frac{\partial}{\partial \bar{t}_k} = \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} t_k} + i\frac{\partial}{\partial \operatorname{Im} t_k}\right).$$

Correspondingly we consider the dispersionless Toda hierarchy with variables s, t_k and $\tilde{t}_k$ $(k \geqq 1)$[1]. The above observation can be compiled into Table 1.

---

[1]Note that the variable $\bar{t}_k$ in this section is *not* the variable $\bar{t}_k$ in Chapter 8 but the complex conjugate to t_k and that the variable $\bar{t}_k$ in Chapter 8 corresponds to $\tilde{t}_k$ in this section. We are sorry for this confusing discrepancy of notations.

Hele-Shaw flow	dispersionless Toda
$w,\ t_k,\ \bar{t}_k$	$p,\ t_k,\ -\tilde{t}_k$
$z = \varphi^{-1}(w)$	$\mathcal{L}$
$\bar{z} = \overline{\varphi^{-1}}(w^{-1})$	$\bar{\mathcal{L}}^{-1}$
$z\, S(z)$	$\mathcal{M}$
$\bar{z}\, \overline{S(z)}$	$\bar{\mathcal{M}}$

TABLE 1. Correspondence of the Hele-Shaw problem and the dispersionless Toda hierarchy.

The important point is that the equation of the curve $\bar{z} = S(z)$ and its complex conjugate correspond to

$$\bar{\mathcal{L}}^{-1} = \mathcal{L}\bar{\mathcal{M}}, \qquad \mathcal{L} = \bar{\mathcal{L}}\bar{\mathcal{M}},$$

which reduce to $\bar{\mathcal{L}} = \mathcal{L}\mathcal{M}^{-1}$ and $\bar{\mathcal{M}} = \mathcal{M}$. These are exactly the equations (8.2.1) in Theorem 8.2.1 for the data

$$(f(p,s), g(p,s), \bar{f}(p,s), \bar{g}(p,s)) = (ps^{-1}, s, p, s),$$

which satisfy $\{f, g\} = f$ and $\{\bar{f}, \bar{g}\} = \bar{f}$. Hence the pair $(\mathcal{L} = \varphi^{-1}, \bar{\mathcal{L}} = \bar{\varphi}^{-1})$ is a solution of the dToda hierarchy! Another proof of this fact by Hadamard's variation formula is in [**MaWZ**].

Subsequent developments of this subject are found in [**MiWZ**], [**KKMWZ**], [**Zab1**], [**MaWZ**], [**KMWZ**], [**KMZ**], [**Nat**], [**Zab2**], [**Zab3**], [**NZ**].

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Corrigendum to "Lectures on Dispersionless Integrable Hierarchies"

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The following typo was found after the publication. The author apologises for this error: The equation (5.1.1) in page 37 should be

$$\mathcal{L} = \mathcal{L}(t;\xi) = \xi + u_2(t)\xi^{-1} + u_3(t)\xi^{-2} + \cdots = \sum_{i=0}^{\infty} u_i(t)\xi^{1-i}, \tag{5.1.1}$$

where $u_0 = 1$, $u_1 = 0$.