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# Koecher-Maaß Series for Jacobi Forms

Dedicated to Holger P. Petersson
on the occasion of his 60th birthday

by

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## 1. Introduction

In the fifties Maaß [M1] and Koecher [Ko1] were the first who introduced Dirichlet series associated with Siegel modular forms, which are nowadays called Koecher-Maaß series (cf. [Bö]). There were problems with the determination of the poles and residues (cf. [K-P]) but now the analytic properties are well understood mainly due to the work of Arakawa (cf. [Ar1], [Ar2], [Kl2], [M2]) and of Ibukiyama [Ib] in the more general context of tube domains. Up to now however the number theoretical significance has only been investigated in certain particular cases (cf. [Bö], [Kr1]).

There also exist attempts by Berndt [Be] and Martin [Ma] in the case dealt with by Eichler-Zagier [E-Z] and more generally by Arakawa [Ar3] to investigate Koecher-Maaß series for Jacobi forms. In this paper we will extend this approach. Our basic new idea is to associate a vector valued Koecher-Maaß series to each Jacobi form in the sense of Ziegler [Z]. Then we obtain the analytic properties and a natural functional equation in the usual way.

Moreover we will investigate their numbrer theoretical significance in the case of the Maaß space for orthogonal groups (cf. [G], [Kr3]). Here we deal with the half-spaces associated with the circular cones, i.e. with the simple formally real Jordan algebras of rank 2 (cf. [Ko2]). In this particular situation we are able to describe a relation between the Koecher-Maaß series of the Jacobi form and of its lift to the orthogonal group. Moreover we derive an Euler product expansion of the Koecher-Maaß series when considering the Cayley half-plane of degree 2 (cf. [E]).

## 2. Jacobi forms

We will define Jacobi forms in the general sense of Ziegler [Z]. Therefore let $\mathrm{Sym}_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}); X = X'\}$ denote the vector space of real symmetric $n \times n$ matrices, prime the transpose and $\mathcal{P}_n$ the cone of positive definite matrices in $\mathrm{Sym}_n(\mathbb{R})$. Then $\mathcal{H}_n =$

$\mathrm{Sym}_n(\mathbb{R}) + i\mathcal{P}_n$ stands for the Siegel half-space of degree $n$ and $\Gamma_n$ for the Siegel modular group of degee $n$ acting on $\mathcal{H}_n$ by $Z \mapsto M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Moreover fix $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \Gamma_n$, where $I$ will always be the identity matrix, inducing the transformation $Z \mapsto -Z^{-1}$. Denote by $\mathrm{Sym}_n^*(\mathbb{Z})$ the lattice of even $n \times n$ matrices.

We fix some positive definite $S \in \mathrm{Sym}_m^*(\mathbb{Z})$ of level $q$, i.e.

$$q = \min\{l \in \mathbb{N};\, lS^{-1} \in \mathrm{Sym}_m^*(\mathbb{Z})\}\,.$$

Given a function $\phi : \mathcal{H}_n \times \mathbb{C}^{(m,n)} \to \mathbb{C}$ we associate the function

$$\phi_S^* : \mathcal{H}_{n+m} \to \mathbb{C}\,, \quad Z = \begin{pmatrix} Z_1 & Z_2' \\ Z_2 & Z_4 \end{pmatrix} \mapsto \phi(Z_1, Z_2)e^{\pi i \mathrm{tr}(SZ_4)}\,.$$

Given $k \in \mathbb{N}_0$ let $J_{k,n}(S)$ stand for the vector space of *Jacobi forms of weight k, degree n and index* $\frac{1}{2}S$ (cf. [Z], [Kl2]), which consists of all holomorphic functions $\phi : \mathcal{H}_n \times \mathbb{C}^{(m,n)} \to \mathbb{C}$ satisfying

(1) $$\phi_S^*|_k M(Z) := \det(CZ+D)^{-k}\phi_S^*(M\langle Z\rangle) = \phi_S^*(Z)$$

$$\text{for all} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & I^{(m)} \end{pmatrix} \in \Gamma_{n+m}$$

and the usual condition of boundedness if $n = 1$. Each such $\phi$ possesses a Fourier expansion of the form

$$\phi_S^*(Z) = \sum_{T = \begin{pmatrix} * & * \\ * & S \end{pmatrix} \in \mathrm{Sym}_{n+m}^*(\mathbb{Z}),\, T \geq 0} \alpha(T)e^{\pi i \mathrm{tr}(TZ)}\,.$$

The subspace $J_{k,n}^{cusp}(S)$ of *cusp forms* is characterized by the property that

$$(\det Im\, Z)^{k/2}|\phi_S^*(Z)|$$

is bounded on $\mathcal{H}_{n+m}$.

We use the abbreviation $A[B] := B'AB$. Given $R \in \mathbb{Z}^{(m,n)}$ we consider the theta series

(2) $$\theta_R(Z_1, Z_2) := \sum_{G \in \mathbb{Z}^{(m,n)}} e^{\pi i \mathrm{tr}(S[G+S^{-1}R]Z_1 + 2(G+S^{-1}R)'SZ_2)}\,.$$

Fix a set of representatives

$$R_1, \cdots, R_d \quad \text{of} \quad \mathbb{Z}^{(m,n)}/S\mathbb{Z}^{(m,n)}\,, \quad d = (\det S)^n\,,$$

and consider the vector

$$\Theta := (\theta_{R_1}, \cdots, \theta_{R_d})'\,.$$

From Shintani [Sh] and Shimura [S] we conclude that there is a mapping $\chi : \Gamma_n \to \mathcal{U}_d$, where $\mathcal{U}_d$ stands for the unitary group of $d \times d$ matrices, such that

$$\Theta_S^*\,|_{m/2}(M \times I^{(2m)}) = \chi(M)\Theta_S^* \quad \text{for all} \quad M \in \Gamma_n\,.$$

Note that to define this operation $\det(CZ+D)^{1/2}$ stands for a $\mathbb{C}$-valued holomorphic function on $\mathcal{H}_n$ satisfying

$$(\det(CZ+D)^{1/2})^2 = \det(CZ+D)\,.$$

Clearly $\chi(M)$ depends on the choice of the branch of $\det(CZ+D)^{1/2}$ if $m$ is odd. Dealing with vectors we apply the operation (1) to each component using the same notation.

Considering

$$f_R(Z_1) := \sum_{T_1 : T = \begin{pmatrix} T_1 & R' \\ R & S \end{pmatrix} \geq 0} \alpha(T) e^{\pi i \operatorname{tr}((T_1 - S^{-1}[R])Z_1)} \tag{3}$$
$$= \sum_{\substack{T^* \in \operatorname{Sym}_n(\mathbb{Q}),\, T^* \geq 0 \\ T^* \equiv -S^{-1}[R] \operatorname{mod} \operatorname{Sym}_n^*(\mathbb{Z})}} a_R(T^*) e^{\pi i \operatorname{tr}(T^* Z_1)}$$

we conclude

$$f_{R+SG} = f_R \quad \text{for} \quad G \in \mathbb{Z}^{(m,n)}\,.$$

Introducing the vector

$$f = (f_{R_1}, \cdots, f_{R_d})'$$

we obtain

$$\phi = f' \cdot \Theta = \sum_{R : \mathbb{Z}^{(m,n)}/S\mathbb{Z}^{(m,n)}} f_R \cdot \theta_R\,. \tag{4}$$

This (cf. [Z]) yields the

PROPOSITION. *Given $\phi \in J_{k,n}(S)$ one has*

$$f\,\big|_{k-m/2} M = \overline{\chi(M)} f \quad \textit{for all} \quad M \in \Gamma_n\,.$$

*In particular all the $f_R$ are modular forms of weight $k-m/2$ on some congruence subgroup of $\Gamma_n$.*

Using (3) the Proposition implies

$$a_R(T^*) = \mathcal{O}((\det T^*)^{k-m/2}) \quad \text{for} \quad T^* > 0\,. \tag{5}$$

Moreover consider the subgroup

$$\mathcal{S} := \mathcal{S}_{n,q} := \{V \in \mathcal{S}L_n(\mathbb{Z});\, V \equiv I \bmod q\}$$

of $GL_n(\mathbb{Z})$. Then (2) and (4) lead to

$$f_R(Z_1[V]) = f_R(Z_1) \quad \text{for all} \quad V \in \mathcal{S}\,.$$

## 3. Koecher-Maaß series

We start with $\phi \in J_{k,n}(S)$ as in section 2 and consider the $\mathcal{S}$-classes

$$\{T^*[V];\, V \in \mathcal{S}\}\,, \quad T^* \in \operatorname{Sym}_n(\mathbb{Q})\,.$$

Let $\{T^*\} > 0$ stand for a set of representatives of these classes in $\mathcal{P}_n$. Moreover let

$$\sharp(T^*) := \sharp\{V \in \mathcal{S};\, T^*[V] = T^*\}\,, \quad T^* > 0\,,$$

denote the order of the $\mathcal{S}$-automorphism group. Then we consider the Dirichlet series

$$D_R(\phi, s) := \sum_{\{T^*\}>0} \frac{a_R(T^*)}{\sharp(T^*)} (\det T^*)^{-s} \tag{6}$$

using (3). In view of (5) and the fact that $\mathcal{S}$ has finite index in $GL_n(\mathbb{Z})$ we conclude that $D_R(\phi, s)$ converges absolutely for $Re(s) > k + \frac{n+1-m}{2}$. Hence

$$D(\phi, s) := (D_{R_1}(\phi, s), \cdots, D_{R_d}(\phi, s))'$$

is called the vector valued *Koecher-Maaß series associated with* $\phi$.

Let $dv = (\det Y)^{-(n+1)/2} dY$ denote the $GL_n(\mathbb{R})$-invariant volume element of $\mathcal{P}_n$. Moreover let

$$D_k := (\det Y)^{-k} \hat{M}_n (\det Y)^k M_n\,, \quad M_n = (\det Y)\left(\det \frac{\partial}{\partial Y}\right)$$

just as in [M2]. Then exactly the same calculations as in [M2], §15, resp. [Kr2], Lemma 2, lead to the

LEMMA. *Given $\phi \in J_{k,n}(S)$, $R \in \mathbb{Z}^{(m,n)}$ and $s \in \mathbb{C}$ with $Re(s) > k + \frac{n+1-m}{2}$ one has*

$$\int_{\mathcal{S}\backslash\mathcal{P}_n} (\det Y)^s D_{k-m/2} f_R(iY) dv$$
$$= \delta_q (-1)^n \pi^{\frac{n(n-1)}{4} - ns} \left( \prod_{j=0}^{n-1} \left( s - k + \frac{m+j}{2} \right) \Gamma\left( s + 1 - \frac{j}{2} \right) \right) D_R(\phi, s)\,,$$

*where $\delta_1 = \delta_2 = 2$ and $\delta_q = 1$ for $q \geq 3$.*

For the value of $\delta_q$ note that $-I \in \mathcal{S}$ holds if and only if $q \leq 2$ which also implies $n$ to be even.

Hence consider

$$\mathbb{D}(\phi, s) := \pi^{-ns} \left( \prod_{j=0}^{n-1} \Gamma\left( s - \frac{j}{2} \right) \right) D(\phi, s)\,.$$

THEOREM 1. *Given $\phi \in J_{k,n}(S)$, then the Koecher-Maaß series $D(\phi, s)$ possesses a meromorphic continuation to the whole $s$-plane, which is holomorphic except for possible simple poles at $s = k - \frac{m+j}{2}$, $j = 0, 1, \cdots, n-1$. The function $\mathbb{D}(\phi, s)$ possesses a meromorphic continuation to the whole $s$-plane with poles at most at $s = k - \frac{m+j}{2}$, $s = \frac{j}{2}$, $j = 0, 1, \cdots, n-1$, and satisfies the functional equation*

$$\mathbb{D}\left(\phi, k - \frac{m}{2} - s\right) = i^{n(k-m/2)} \overline{\chi(J)} \mathbb{D}(\phi, s)\,.$$

*If $k > n-1$ all the poles are simple ones. If $\phi$ is a cusp form then $D(\phi, s)$ and $\mathbb{D}(\phi, s)$ are entire functions of $s$.*

*Proof.* We apply the Lemma and fix a fundamental domain $\mathcal{F}$ of $\mathcal{S} \backslash \mathcal{P}_n$, which consists of finitely many images of the set of Minkowski reduced matrices under unimodular transformations. Then we can decompose $\mathcal{F}$ by the hypersurface $\det Y = 1$. In view of

$$f(iY^{-1}) = i^{n(k-m/2)}(\det Y)^{k-m/2}\overline{\chi(J)}f(iY)$$

we obtain

$$\int_{\mathcal{F}} (\det Y)^s D_{k-m/2} f(iY) dv$$

$$= \int_{\mathcal{F},\det Y \geq 1} ((\det Y)^s I + i^{n(k-m/s)}(\det Y)^{k-(m/2)-s}\overline{\chi(J)}) D_{k-m/2} f(iY) dv\,.$$

Hence the assertion follows along the classical lines in [M2], §15. For the functional equation note that $f \neq 0$ implies $(i^{n(k-m/2}\overline{\chi(J)})^2 = I$. □

If one considers the sum of the components of the vectors on both sides one obtains Arakawa's result [Ar3], Theorem 8.

If $S$ is moreover unimodular, then $f$ is a Siegel modular form of weight $k - m/2$ and $\chi$ is trivial. In this case we obtain Koecher's classical result (cf. [Ko1]).

## 4. Modular forms on the orthogonal group

Now we consider the case $n = 1$. The existence of a Maaß lift from $J_{k,1}(S)$ to modular forms on the orthogonal group was proved by Gritsenko [G] and in [Kr3].

Consider the matrix

$$S_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -S & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

of signature $(1, m+1)$ as well as the attached bilinear form

$$\mu(a, b) := \frac{1}{2}a' S_0 b\,, \quad a, b \in \mathbb{R}^{m+2}\,.$$

$\mathbb{R}^{m+2}$ endowed with the product

$$a \cdot b := \mu(a, e)b + \mu(b, e)a - \mu(a, b)e\,, \quad e = e_1 + e_{m+2} = (1, 0, \cdots, 0, 1)'\,,$$

becomes a formally real Jordan algebra of rank 2 with unit element $e$ and domain of positivity

$$\mathcal{P}_S := \{v \in \mathbb{R}^{m+2};\ \mu(v, e) > 0, \mu(v, v) > 0\}$$

(cf. [B-K] or [Ko2]). The attached half-space is

$$\mathcal{H}_S = \{w = u + iv \in \mathbb{C}^{m+2};\ v \in \mathcal{P}_S\}$$

with the main involution

$$w \mapsto -w^{-1} = \frac{1}{\mu(w, w)} V w\,, \quad V = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathcal{O}(S_0; \mathbb{Z})\,.$$

The vector space $M_k(\Gamma_S)$ of modular forms of weight $k \in \mathbb{Z}$ consists of all holomorphic functions $\varphi : \mathcal{H}_S \to \mathbb{C}$ satisfying

$$\varphi(w + \lambda) = \varphi(w) \quad \text{for all} \quad \lambda \in \mathbb{Z}^{m+2} \quad \text{and} \quad \varphi(-w^{-1}) = \mu(w, w)^k \varphi(w)\,.$$

Each such $\varphi$ possesses a Fourier expansion of the form

$$(7) \qquad \varphi(w) = \sum_{\rho\in\mathbb{Z}^{m+2},\, S_0^{-1}\rho\in\overline{\mathcal{P}_S}} \alpha_\varphi(\rho) e^{2\pi i\rho' w}\,.$$

Let $\mathcal{G}$ denote the subgroup of $\mathcal{O}(S_0;\mathbb{Z})$ generated by the matrices

$$(8) \qquad \begin{pmatrix} 1 & \lambda' S & \frac{1}{2}S[\lambda] \\ 0 & I & \lambda \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ \lambda & I & 0 \\ \frac{1}{2}S[\lambda] & \lambda' S & 1 \end{pmatrix}, \quad \lambda \in \mathbb{Z}^m\,.$$

Then we have

$$\varphi(Uw) = \varphi(w) \quad \text{and} \quad \alpha_\varphi(U'^{-1}\rho) = \alpha_\varphi(\rho)$$

for all $U \in \mathcal{G}$ due to [Kr3], sect. 4.

Let $\{\rho\} > 0$ stand for a set of representatives of the $\mathcal{G}$-classes

$$\{U'^{-1}\rho;\, U \in \mathcal{G}\}\,, \quad \rho \in \mathbb{Z}^{m+2}\,, \quad S_0^{-1}\rho \in \mathcal{P}_S\,,$$

and denote the order of the $\mathcal{G}$-automorphism group of $\rho$ by $\sharp(\rho)$. Then we formally associate the *Koecher-Maaß series* to $\varphi$

$$K_\varphi(s) := \sum_{\{\rho\}>0} \frac{\alpha_\varphi(\rho)}{\sharp(\rho)} \mu(\rho,\rho)^{-s}\,.$$

This type of Koecher-Maaß series was investigated in a more general context by Ibukiyama [Ib].

The *Maaß space* $M_k^*(\Gamma_s)$ consists of all $\varphi \in M_k(\Gamma_s)$ with the Fourier expansion (7) satisfying

$$(9) \qquad \alpha_\varphi(\rho) = \sum_{\delta\mid \gcd(\rho)} \delta^{k-1}\alpha_\varphi\begin{pmatrix} 1 \\ \lambda/\delta \\ ll^*/\delta^2 \end{pmatrix} \quad \text{for all} \quad \rho = \begin{pmatrix} l^* \\ \lambda \\ l \end{pmatrix} \neq 0\,.$$

The Maaß space $M_k^*(\Gamma_S)$ is isomorphic to the space $J_{k,1}(S)$ (cf. [Kr3], Theorem 3). Let $\phi$ be the first Fourier-Jacobi coefficient of $\varphi$, i.e.

$$\phi(\tau, z) = \sum_{l=0}^{\infty} \sum_{\lambda\in\mathbb{Z}^m,\, S^{-1}[\lambda]\le 2l} \alpha_\varphi\begin{pmatrix} 1 \\ \lambda \\ l \end{pmatrix} e^{2\pi i(l\tau+\lambda' z)}\,,$$

$\tau \in \mathcal{H}_1$, $z \in \mathbb{C}^m$. The Koecher-Maaß series associated with $\phi$ are given by

$$D_r(\phi, s) = \sum_{2l>S^{-1}[r]} \alpha_\varphi\begin{pmatrix} 1 \\ r \\ l \end{pmatrix} (2l - S^{-1}[r])^{-s}\,, \quad r : \mathbb{Z}^m/S\mathbb{Z}^m\,,$$

according to (6).

We will derive a relation between both types of Koecher-Maaß series in the form of a Rankin type convolution which were studied in detail by Bump [Bu], sect. 1. Therefore we introduce the "class-number" function

$$H_r(n) = \sum_{\substack{\{\rho\}>0,\mu(\rho,\rho)=n \\ \rho\equiv\begin{pmatrix}0\\r\\0\end{pmatrix} \bmod S_0\mathbb{Z}^{m+2}}} \frac{1}{\sharp(\rho)}, \quad r \in \mathbb{Z}^m .$$

THEOREM 2. *Let $\varphi \in M_k^*(\Gamma_S)$ with its first Fourier-Jacobi coefficient $\phi \in J_{k,1}(S)$. Then one has*

$$K_\varphi(s) = \zeta(2s+1-k) \cdot \sum_{r:\mathbb{Z}^m/S\mathbb{Z}^m} \sum_{n=1}^{\infty} \alpha_\varphi \begin{pmatrix} 1 \\ r \\ n+\frac{1}{2}S[r] \end{pmatrix} H_r(n) n^{-s} .$$

*Proof.* Due to the Maaß condition (9) we have

$$\begin{aligned} K_\varphi(s) &= \sum_{\{\rho\}>0} \frac{\alpha_\varphi(\rho)\cdot}{\sharp(\rho)} \mu(\rho,\rho)^{-s} \\ &= \sum_{\{\rho\}>0} \frac{1}{\sharp(\rho)} \sum_{\delta|\gcd(\rho)} \delta^{k-1} \alpha_\varphi \begin{pmatrix} 1 \\ \lambda/\delta \\ ll^*/\delta^2 \end{pmatrix} \mu(\rho,\rho)^{-s} . \end{aligned}$$

We interchange the summation and note that $\sharp(\delta\rho) = \sharp(\rho)$ as well as $\mu(\delta\rho,\delta\rho) = \delta^2\mu(\rho,\rho)$. Hence we obtain

$$\begin{aligned} K_\varphi(s) &= \sum_{\delta=1}^{\infty} \sum_{\substack{\{\rho\}>0 \\ \delta|\gcd(\rho)}} \frac{1}{\sharp(\rho)} \delta^{k-1} \alpha_\varphi \begin{pmatrix} 1 \\ \lambda/\delta \\ ll^*/\delta \end{pmatrix} \mu(\rho,\rho)^{-s} \\ &= \zeta(2s+1-k) \cdot \sum_{\{\rho\}>0} \frac{1}{\sharp(\rho)} \alpha_\varphi \begin{pmatrix} 1 \\ \lambda \\ ll^* \end{pmatrix} \mu(\rho,\rho)^{-s} \\ &= \zeta(2s+1-k) \cdot \sum_{r:\mathbb{Z}^m/S\mathbb{Z}^m} \sum_{n=1}^{\infty} \alpha_\varphi \begin{pmatrix} 1 \\ r \\ n+\frac{1}{2}S[r] \end{pmatrix} H_r(n) n^{-s} \end{aligned}$$

in view of

$$\alpha_\varphi \begin{pmatrix} 1 \\ \lambda \\ n+\frac{1}{2}S[\lambda] \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ r \\ n+\frac{1}{2}S[r] \end{pmatrix} \quad \text{if } \lambda \equiv r \bmod S\mathbb{Z}^m$$

and $\mu(\rho,\rho) = ll^* - \frac{1}{2}S[\lambda]$ for $\rho = \begin{pmatrix} l^* \\ \lambda \\ l \end{pmatrix}$. □

## 5. Applications

If $m = 1$ and $S = (2)$ section 4 refers to Siegel modular forms of degree 2. In this case Theorem 2 is due to Böcherer [Bö].

If $m = 2$ and $\frac{1}{2}S$ is the matrix of the norm form of an imaginary quadratic number field, the results refer to Hermitian modular forms of degree 2 and to Hermitian Jacobi forms as investigated by Haverkamp [Ha].

Now let us consider $m = 4$ and $S = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$. In this case we deal with modular forms of degree 2 over the Hurwitz quaternions $\mathcal{O}$ (cf. [Kr1], [Kr2]). The group $\mathcal{G}$ in section 4 then is isomorphic to $SL_2(\mathcal{O})$ (cf. [K-W]) which is of index 3 in $GL_2(\mathcal{O})$. Moreover $0, e_2, e_3, e_4$ is a set of representatives of $\mathbb{Z}^4/S\mathbb{Z}^4$. A slight modification of the calculations in [Kr2], Theorem 5, yields

$$H_0(l) = \begin{cases} \dfrac{1}{640}\left[\sigma_2(l) - \dfrac{10}{3}\sigma_2(l/2)\right] & \text{if } l \text{ is even}, \\ 0 & \text{if } l \text{ is odd}, \end{cases}$$

$$H_{e_2}(l) = H_{e_3}(l) = H_{e_4}(l) = \begin{cases} \dfrac{1}{640}\sigma_2(l) & \text{if } l \text{ is odd}, \\ 0 & \text{if } l \text{ is even}. \end{cases}$$

Hence we obtain the results from [Kr2], in particular the existence of an Euler product expansion for $K_\varphi(s)$ provided that $\varphi$ is a simultaneous Hecke eigenform.

Moreover it is clear that one can deal with the Maaß space for the non-trivial multiplier system (cf. [Kr4]) in the same way. One merely has to insert the character in the definition of the Koecher-Maaß series.

Next assume that $S$ is unimodular. Hence $f$ is an elliptic modular form of weight $k - m/2$. If its Fourier coefficients are denoted by $\alpha_f(n)$ Theorem 2 says that

$$K_\varphi(s) = \zeta(2s+1-k) \cdot \sum_{n=1}^{\infty} \alpha_f(n) H_0(n) n^{-s}, \quad H_0(n) = \sum_{\{\rho\}>0, \mu(\rho,\rho)=n} \frac{1}{\sharp(\rho)}.$$

Now let in particular $m = 8$ and let $\frac{1}{2}S$ be the Gram matrix of the norm form with respect to the integral Cayley numbers $\mathcal{O}$ in [E] and [E-K]. Note that

$$\sum_{0 \neq a \in \mathcal{O}} N(a)^{-s} = 240\zeta(s)\zeta(s-3).$$

Now we observe that the matrices (8) induce the transformations

$$Z \mapsto (\bar{U}'Z)U = \bar{U}'(ZU), \quad U = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad \lambda \in \mathcal{O},$$

on the Cayley half-plane of degree 2. Moreover the properties of the Cayley numbers in [Eb], 9.1.3, lead to

$$U(a\bar{a}')\bar{U}' = (Ua)\overline{(Ua)}' \quad \text{for} \quad a \in \mathcal{O}^2, \quad U = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad \lambda \in \mathcal{O}.$$

Thus we consider the unique modular form of weight 8, which was described as a theta series in [E-K], Theorem 2. In view of $\sharp(\mathcal{O}/a\mathcal{O}) = N(a)^4$ for $0 \neq a \in \mathcal{O}$ the same procedure as in [Kr2], sect. 6, yields

$$\sum_{n=1}^{\infty} \sigma_3(n) H_0(n) n^{-s} = \gamma \cdot \frac{\zeta(s)\zeta(s-3)\zeta(s-4)\zeta(s-7)}{\zeta(2s-7)}$$

with a certain constant $\gamma \in \mathbb{Q}$, $\gamma > 0$. Hence we obtain

$$H_0(n) = \gamma \cdot \sigma_4(n), \quad n \in \mathbb{N},$$

from [Bu], (1.1.1).

Now we start with $\varphi \in M_k^*(\Gamma_S)$ and consider the attached elliptic modular form $\Omega(\varphi) = f$ of weight $k-4$ in [E-K], (16). Denote the standard Hecke $L$-function associated to $f$ by $L_f(s)$ (cf. [K-K], IV. 4.4). Moreover let $E_k$ stand for the Eisenstein series of weight $k$ $(= 18l)$ on the Cayley half-plane of degree 2 arising from the one on the 27-dimensional exceptional domain by the Siegel operator (cf. [E-K], sect. 3). Then the standard properties of elliptic modular forms (cf. [K-K], IV. 4.8) just as in [Kr2], Theorem 6, lead to

THEOREM 3. *Let $\varphi \in M_k^*(\Gamma_S)$ such that $\Omega(\varphi) = f$ is a normalized simultaneous Hecke eigenform. Then the Koecher-Maaß series $K_\varphi(s)$ possesses an Euler product expansion and satisfies*

$$K_\varphi(s) = \gamma \cdot L_f(s) \cdot L_f(s-4).$$

*In the case of the Eisenstein series one has*

$$K_{E_k}(s) = \gamma_k \cdot \zeta(s) \cdot \zeta(s-4) \cdot \zeta(s+5-k) \cdot \zeta(s+1-k), \quad \gamma_k \in \mathbb{Q}^*.$$

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