Explicit Formula of Orbital \( p \)-adic Zeta Functions Associated to Symmetric and Hermitian Matrices

by

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For a prime \( p \), fixed throughout in this paper, let \( F \) be a finite extension of \( \mathbb{Q}_p \) and \( O \) the ring of integers, \( \varpi \) a prime element of \( F \). For \( x \in F \), let \( |x| \) be the absolute value of \( x \) normalized as \( |\varpi|=q^{-1} \) for \( q=|O/\varpi O| \). Let \((G, \rho, X)\) be an irreducible regular reduced prehomogeneous vector space defined over \( F \) and let \( P(x) \) be its basic relative invariant (cf. [S-K]). Here \( G \) is a reductive algebraic groups acting on a finite dimensional vector space \( X \) through \( \rho \), all defined over \( F \), with one Zariski dense orbit over the algebraic closure of \( F \). Then we can consider the zeta function

\[
Z(s)=\int_{\text{X}^{\text{et}}(F)} f(x)|P(x)|^{s-1}dx
\]

associated to \((G, \rho, X)\). Here \( \kappa=\dim X/\deg P \), \( f(x) \) is the characteristic function of \( X(O) \), and \( X^{\text{et}}(F)=X(F)\setminus \{P(x)=0\} \). Except for a few cases, Igusa determined explicitly \( Z(s) \) as a rational function in \( u=q^{-s} \) (cf. [I1]–[I5]). But as was shown in [I-S], to apply the local results to the global cases, we need to know the integral on each orbit of \( X^{\text{et}}(F) \) under the action of \( G(F) \).

The purpose of this paper is to give an explicit expression as rational functions in \( u \) for such orbital zeta functions in the following three types of prehomogeneous vector spaces:

(I) \( G=GL_n \times GL_1, X=S_n, \rho((g, a))x=agx'g \) for \( (g, a)\in G, x\in X \);

(II) \( G=GO(h)^0 \times GL_n, X=M_{n,r}, \rho((g_1, g_2))x=g_1x'g_2, n\geq 2r \), for \( (g_1, g_2)\in G, x\in X \);

(III) \( G=R_{K/F}(GL_n) \times GL_1, X=H_n, \rho((g, a))x=agx'\tilde{g} \), for \( (g, a)\in R_{K/F}(GL_n) \times GL_1, x\in X \),

under the assumption stated below. In (I), \( S_n \) denotes the space of symmetric matrices of size \( n \), and in (II), \( GO(h)^0 \) is the connected component of the unit of the similitude group of a symmetric matrix \( h \) of size \( n \). In (III), \( H_n \) denote the space of hermitian matrices with respect to a quadratic extension \( K \) of \( F \), and \( \tilde{g} \) denotes the componentwise action on \( g \) of the nontrivial element of \( \text{Gal}(K/F) \). These are the prehomogeneous vector spaces of type (2), (15), and (1) in [S-K] respectively, and \( P(x)=\det x, \det(xh'x) \), \( d\) ex accordingly. In all these cases, \( Z(s) \) was determined by Igusa [I1, I3] (under the condition that \( p \neq 2 \) in (I), that \( p \neq 2, h \) is unimodular in (II), and that \( K \) is unramified
in (III) respectively. But orbital zeta functions have not been given except in the cases of (III) with \( n \) odd and an unramified \( K/F \) (cf. [11]) and (I) for \( F=Q_p \) (cf. [I-S]). In the first case, \( X^{s}(F) \) consists of one \( G(F) \) orbit, and \( Z(s) \) coincides with the orbital zeta function.

To explain our result, we take the case (I) for \( n \) even as an example. Then under the action of \( G(F) \), \( X^{s}(F) \) decomposes into the following \( |F^\times/F^\times 2|+1 \) orbits

\[
\bigcup_{a \in F^\times/F^\times 2} X^{s}(F, a) \cup \{ X^{s}(F, (-1)^{n/2}, 1) \cup X^{s}(F, (-1)^{n/2}, -1) \}
\]

where

\[
X^{s}(F, a) = \{ x \in X^{s}(F) \mid \det x \in aF^\times 2 \},
\]

\[
X^{s}(F, a, \pm 1) = \{ x \in X^{s}(F) \mid \det x \in aF^\times 2, \varepsilon(x) = \pm 1 \},
\]

for \( \varepsilon \) the Hasse invariant. Hence we have to calculate the integral on each above orbit. We see easily that it is enough to calculate the integrals

\[
\int_{X^{s}(O, a)} \omega(x)|P(x)|^{s}d\chi,
\]

for \( a \in O^\times/O^\times 2 \), where

\[
X^{s}(O, a) = \{ x \in X(O) \mid \det x \in a\sigma^{m}O^\times 2, m \in Z, m \geq 0 \},
\]

and \( \omega \) is the Hasse invariant or the constant function with the value 1. We will give an explicit expression for such functions.

In the case of (I) and (III), we make no assumption on \( p \) and \( K \), and include the case where \( p=2 \) or \( K \) is ramified. In fact, these cases necessitates the cumbersome calculation, but these are necessary to give a complete result in the global case. In the case of (II), we have to assume \( h \) is even unimodular, and \( p \neq 2 \) when \( n \) is odd. Unless these condition is satisfied, the zeta functions seem to take a more complicated form. The result in this paper allows us to give a complete form of the global zeta function for an even unimodular \( h \) in the case of \( n \) even. This generalizes the result by Ibukiyama and Katsurada on Koecher Maass Dirichlet series associated to Siegel Eisenstein series. We will discuss these global results in a subsequent paper.

§1. Preliminaries

In this section, we recall some known results which are needed in the later sections. First for two variables \( U, a \) and a positive integer \( n \), we set

\[
(U, a)_{n} = \prod_{i=1}^{n} (1 - a^{i-1} U), \quad (a)_{n} = (a, a)_{n}
\]

and \( (U, a)_{0} = (a)_{0} = 1 \). For a non-negative integer \( r \leq n \), we have

\[
(U, a)_{n} = (U, a)_{r}(a^{r} U, a)_{n-r}, \quad (U, a)_{n} = (a^{r-1} U, a^{-1})_{n}.
\]
For \( r \) as above, we set
\[
\binom{n}{r}_a = \frac{(a)_n}{(a)_r (a)_{n-r}}.
\]
Then we have
\[
\binom{n}{r}_a = \binom{n}{r}_{a^{-1}} a^{(a^{-1} - r) r}.
\]
On these polynomials, we have the following equalities.

**Lemma 1.1.** The notation being as above, one has
\[
(1) \quad \sum_{r=0}^{n} \binom{n}{r}_a U^{n-r}(U, a)_r = 1,
\]
and for positive integers \( m > n \)
\[
(2) \quad \sum_{r=0}^{n} \binom{m}{r}_a a^{r^2} U^r \frac{(a^{r+1} U, a)_{n-r}}{(a)_{n-r}} = \frac{(a^{m+1} U, a)_{n}}{(a)_n},
\]
\[
(3) \quad \sum_{r=0}^{n} \binom{m}{r}_a a^{r^2-r} U^r \frac{(a^{r+1} U, a)_{n-r}}{(a)_{n-r}} = \frac{(a^m U, a)_{n}}{(a)_n} + \frac{U(a^{m+1} U, a)_{n-1}}{(a)_{n-1}}.
\]
These are Lemma 5.5 of [I-S] and Lemma 1, 2 of [I3]. We note \( F_{m,n}(x, y), F_{m,n}(x) \) in [I3] can be written as
\[
F_{m,n}(a, U) = \frac{(a^{m+1} U, a)_{n}}{(a)_n}, \quad F_{m,n}(a) = \frac{m}{n}_a
\]
in our notation.

The following simple lemma is useful in the later calculation.

**Lemma 1.2.** Let \( G \) be a finite group acting on a finite set \( S \), and let \( S_0 \) be a subset of \( S \) which satisfies \( G S_0 = S \). Let \( \omega \) be a function on \( S \) which satisfies \( \omega(gs) = \omega(s) \). For \( s_0 \in S_0 \), let
\[
H_{s_0} = \{ g \in G \mid gs_0 \in S_0 \}.
\]
Assume \( |H_{s_0}| \) is independent of \( s_0 \in S_0 \). The one has
\[
\frac{1}{|S|} \sum_{x \in S} \omega(x) = \frac{1}{|S_0|} \sum_{x \in S_0} \omega(s).
\]
Let \( F, O, \mathfrak{m}, |a| \) for \( a \in F \) be as in the introduction. Set \( \mathcal{F} = O/\mathfrak{m} O \approx F_1 \) the finite field with \( q \) elements. For \( a, b \in F \), let \( (a, b) \) denote the Hilbert symbol of \( a \) and \( b \). For an object \( X \) on \( F \), we denote by \( \bar{X} \) the object obtained by reduction modulo \( \mathfrak{m} \). On \( F^n \), we consider the measure such that the volume of \( O^n \) is 1.

The following theorem is a slight generalization of Th.3 of [I2], which can be
proved in the same way.

**Proposition 1.3.** Let \( f(x) \in F[x_1, x_2, \ldots, x_n] \), and let \( G \) be a connected subgroup of \( GL_n \) smooth over \( O \), which preserves \( f(x) \) up to similarities. For \( \xi \in O^n \), let \( N \) be a \( F \)-subspace of \( \text{Aff}^n \) such that \( \dim_F N = \dim_F \bar{N} \) and \( T_\xi(\bar{G}) \oplus \bar{N} = \text{Aff}^n \), where \( T_\xi \) stands for the tangent space of \( \bar{G} \) at \( \xi \). Then one has

\[
\int_{\xi + \sigma O^n} |f(x)|^n dx = q^{-n} \int_{N(O)} |f(\xi + \sigma x^*)|^n dx^*,
\]

where \( dx^* \) is defined by \( N(O) \cong O^r \) with \( r = \dim_F N \).

For \( p = 2 \), we describe \( F^*/F^{*2} \) and quadratic extensions of \( F \). We follow [H-P-S]. Let \( e \) be the ramification index of \( F \), that is, \( 2 \in \sigma O^* \). For a positive integer \( n \), set

\[
U_n = \{ a \in O^* \mid a \equiv 1 \mod \sigma^n \},
\]

and set \( U = O^* \). For \( j, 1 \leq j \leq e \), let \( A_j \) be a system of representatives of \( U_{2j-1}/U_{2j} \). Then \( |A_j| = q \). For \( j = e + 1 \), let \( A_0 \subset \subset U_{2e} \setminus U_{2e+1} \), and \( A_{e+1} = \{ 1, A_0 \} \). Then \( F(\sqrt{\sigma}) \) is the unramified quadratic extension of \( F \), and \( A_{e+1} \) gives a system of representatives of \( U_{2e}/U_{2e+1} \). Lastly set \( A_0 = \{ 1, \sigma \} \). Then

\[
A_0 \times A_1 \times \cdots \times A_{e+1} = \{ a_0 a_1, \ldots, a_{e+1} \mid a_i \in A_i \}
\]
gives a system of representatives of \( F^*/F^{*2} \). For \( \beta \) in the above set, let \( \chi_\beta \) be the quadratic or trivial character given by \( \chi_\beta(x) = (x, \beta) \), which is the character corresponding to the quadratic extension \( F(\sqrt{\beta}) \) unless \( \beta \in F^{*2} \). Then we have

\[
A_j \times \cdots \times A_{e+1} = \{ \beta \in F^*/F^{*2} \mid f(\sigma) \leq 2e + 2 - 2j \},
\]

where \( f(\xi) \) is the exponent of the conductor of \( \chi_\beta \). We note \( U^2 U_{2m} = U^2 U_{2m+1} \) if \( m < e \), and the Hilbert symbol gives a non-degenerate bilinear form on \( F^*/F^{*2} \). Prop. 1.5 in [H-P-S] implies

**Lemma 1.4.** Let \( U/U^2 U \) be the group of characters of \( U/U^2 U \). Then

\[
U/U^2 U = \begin{cases} \{ \chi_\beta \mid \beta \in A_{e-(2l+1) \times \cdots \times A_e} \} & \text{if } l \leq 2e, \\
\{ \chi_\beta \mid \beta \in A_0 \times \cdots \times A_e \} & \text{if } l \geq 2e + 1. 
\end{cases}
\]

**§ 2. The case (I) (the cases of symmetric matrices)**

We use the notation introduced in §1. For a commutative ring \( R \) and a positive integer \( n \), we denote by \( S_n(R) \) the set of symmetric matrices of degree \( n \) with coefficients in \( R \). For a subset \( S \) of \( R \), we set

\[
S_n(R, S) = \{ x \in S_n(R) \mid \det x \in S \}.
\]

For \( x \in X^n(F) = S_n(F, F^*) \), we define the Hasse invariant \( \varepsilon(x) \) by

\[
\varepsilon(x) = \prod_{i \leq j} (x_i, x_j),
\]
when $x$ is equivalent with respect to $GL_n(F)$ to the diagonal matrix with the diagonal elements $x_1, x_2, \cdots, x_n$. We define $\bar{\varepsilon}(x) = \varepsilon((\det x)^{-1}x)$. Then we see

$$\bar{\varepsilon}(x) = (\det x, -1)^{(n+2)/2}) \varepsilon(x).$$

For $x \in S_n(F, F^\times)$, let $d(x)$ be the class of $\det x$ in $F^\times/F^\times^2$. In general, for $x \in S_n(F)$, let $g'xg = \begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix}$ for $x' \in S_n(F, F^\times)$, $g \in GL_n(F)$ with $r \geq 0$. If $r > 0$, set

$$d(x) = d(x'), \quad \varepsilon(x) = \varepsilon(x'), \quad \bar{\varepsilon}(x) = \bar{\varepsilon}(x').$$

If $r = 0$, set $d(x) = \varepsilon(x) = \bar{\varepsilon}(x) = 1$. These are independent of the choice of $g$. We denote by $i$ the constant function on $S_n(F, F^\times)$ with the value 1. When $p = 2$, for $k$, $0 \leq k \leq e$, we set

$$S_n(O, S)_k = \{x = (x_{ij}) \in S_n(O, S) \mid x_{ii} \in \wp^k O\}.$$

Hence $S_n(O, S)_0 = S_n(O, S)$.

The following is an easy consequence of the theory of quadratic forms over local fields.

**Proposition 2.1.** Let $(G, \rho, X)$ be of type (I). Then the $G(F)$ orbits of $X^m(F)$ are given by

$$\{x \in S_n(F, F^\times) \mid \bar{\varepsilon}(x) = 1\} \cup \{x \in S_n(F, F^\times) \mid \bar{\varepsilon}(x) = -1\},$$

if $n$ is odd, and

$$\bigcup_{\varepsilon \in F^\times/F^\times^2} S_n(F, dF^\times^2) \cup \left( \bigcup_{\varepsilon = \pm 1} \{x \in S_n(F, (-1)^{n/2}F^\times^2) \mid \varepsilon(x) = \varepsilon\} \right)$$

if $n$ is even.

In this section, we determine explicitly the following zeta function

$$\int_D \omega(x) \mid \det x \mid^{s-\kappa} dx$$

for

$$D = \bigcup_{n=0}^\infty S_n(O, d\wp^nO^\times^2), \quad \bigcup_{n=0}^\infty S_n(O, d\wp^nO^\times^2)_k.$$

Here $d \in O^\times$, $\kappa = (n+1)/2$ and $\omega$ is a function on $D$, taken as $= 1$, $\varepsilon$, or $\bar{\varepsilon}$ in this section. The second set is taken into consideration to give a formula for the dual lattice $\frac{1}{2} S_n(O)_k$ of $S_n(O)$. When $n$ is odd, it is convenient to take $\bar{\varepsilon}$ as $\omega$ for the application to the global case. These are rational functions in $q^{-s}$, and will be denoted by $Z_n(u, \omega, d)$ or $Z_n(u, \omega, d)$ for $u = q^{-s}$. We denote the odd part and the even part of $Z_n$ by $Z_{n,o}$ and $Z_{n,e}$ respectively, that is,
\[ Z_{n,0}(u, t, d)_k = \frac{1}{2} \left( Z_n(u, t, d)_k + Z_n(-u, t, d)_k \right), \]
\[ Z_{n,e}(u, t, d)_k = \frac{1}{2} \left( Z_n(u, t, d)_k - Z_n(-u, t, d)_k \right), \]
and so on. When \( n \) is even, we give \( Z_{n,0}, Z_{n,e} \) for \( \omega = 1 \).

**Theorem 2.2.** Let \( d \in O^* \).

1. Let \( p \) be odd. If \( n \) is odd, then
\[
Z_n(u, t, d) = 2^{-1}(q^{-1})_n(q^{-2})_{n/2}^{-1}(1 - q^{(n-1)/2}u)^{-1}(qu^2, q^2)_{n/2}^{-1},
\]
\[
Z_n(u, \bar{e}, d) = 2^{-1}(q^{-1})_n(q^{-2})_{n/2}^{-1}(1 + u)(u^2, q^2)_{(n+1)/2}^{-1},
\]
and if \( n \) is even, then
\[
Z_{n,0}(u, t, d) = 2^{-1}(q^{-1})_n(q^{-2})_{n/2}^{-1} - 1(1 - q^{n-1}u)^{-1}(u^2, q^2)_{n/2}^{-1}q^{(n-1)/2}u, \]
\[
Z_{n,e}(u, t, d) = 2^{-1}(q^{-1})_n(q^{-2})_{n/2}^{-1} - 1(1 - q^{n-1}u)^{-1}(u^2, q^2)_{n/2}^{-1}
\times (1 - ((-1)^{n/2}d, \omega)q^{-n/2}(1 - ((-1)^{n/2}d, \omega)q^{-n/2})(qu^2, q^2)_{n/2}^{-1}. \]

2. Let \( p = 2 \). If \( n \) is odd, then
\[
Z_n(u, t, d)_k = \frac{1}{|O^*: O^{*2}|} (q^{-1})_n(q^{-2})_{n/2}^{-1}(1 - q^{(n-1)/2}u)^{-1}(qu^2, q^2)_{n/2}^{-1}q^{-(n-1)/2}u^k, \]
\[
Z_n(u, \bar{e}, d)_k = \frac{q^{-e(n-1)/2}}{|O^*: O^{*2}|} (q^{-1})_n(q^{-2})_{n/2}^{-1}(1 + u)(u^2, q^2)_{(n+1)/2}^{-1}
\times (1 - 1)^{n-1/2}u^k. \]

If \( n \) is even, then
\[
Z_{n,0}(u, t, d)_k = \frac{q^{(n-1)/2}}{|O^*: O^{*2}|} (q^{-1})_n(q^{-2})_{n/2}^{-1} - 1(1 - q^{n-1}u^2)^{-1}(u^2, q^2)_{n/2}^{-1}u^{2k+1}, \]
\[
Z_{n,e}(u, t, d)_k = \frac{1}{|O^*: O^{*2}|} (q^{-1})_n(q^{-2})_{n/2}^{-1}(u^2, q^2)_{n/2}^{-1}
\times \left( (1 - q^{-n})(1 - q^{-n-1}u^2)^{-1}(q^{n-1}u^2)_{u^{2k}} \right)
+ \sum_{\theta \in A_0 \times \cdots \times A_\omega} q^{-\theta(\omega n/2)}\chi_\theta((-1)^{n/2}d)
+ (1 - q^{-n}) \sum_{i=1}^{k-1} \sum_{\theta \in A_0 \times \cdots \times A_\omega, \theta(n) \leq 2i} q^{i}\chi_\theta((-1)^{n/2}d)u^{2k-2i}, \]
\[ Z_n(u, \varepsilon, d) = \frac{(1 + \delta((-1)^{n/2}d))}{2|O^* : O^{*2}|} q^{-e(n-2)/2}(-1, -1)^{(n+2)/8}(q^{1/2})_d(q^{2})_{n/2}^{-1} \]
\[ \times (qu^2, q^{2})_{n/2}^{-1}((-1)^{n/2}d, \varpi)q^{-n/2}((1+((-1)^{n/2}d, \varpi)q^{-n/2}), \]

where

\[ f_k(\chi_d) = \begin{cases} 2k & \text{if } f(\chi_d) \leq 2k, \\ f(\chi_d) & \text{if } f(\chi_d) > 2k, \end{cases} \]
\[ \delta(a) = \begin{cases} 1 & \text{if } a \in U^2 U_{2e}, \\ -1 & \text{otherwise}. \end{cases} \]

When \( n \) is even, \( Z_n(u, \varepsilon, d) \) has a rather complicated form. But for \( k = e \), we have

**Corollary 2.3.** Let \( p = 2, d \in O^* \), and let \( n \) be even. Then

\[ q^{ne}Z_n(u, \varepsilon, d) = \frac{q^e}{|O^* : O^{*2}|} (q^{1/2})_{n/2}^{-1}(1-q^{n-1}u^2)^{-1}(u^2, q^{2})_{n/2}^{-1} \]
\[ \times \left\{ \begin{array}{ll} (1-((-1)^{n/2}d, \varpi)q^{-n/2})^{-1} & \text{if } (-1)^{n/2}d \in O^{*2} U_{2e}, \\ (q^{-(n-1)/2})_{2f(\chi_{-1^{n/2}d})} & \text{otherwise}. \end{array} \right. \]

\[ q^{ne}Z_n(u, \varepsilon, d) = \frac{q^e}{|O^* : O^{*2}|} (q^{1/2})_{n/2}^{-1}(1-q^{n-1}u^2)^{-1}(u^2, q^{2})_{n/2}^{-1}(q^{-(n-1)/2}u)^{2e+1}. \]

When \( p = 2 \), we define

\[ \tilde{Z}_n(u, \omega, d) = q^{n(e-1)}(1-q^{-n})^{-1}u(Z_n(q^{-(n+1)/2}u, \omega, d)_{e-1} - Z_n(q^{-(n+1)/2}u, \omega, d)_{e}). \]

The following will be needed in §3.

**Corollary 2.4.** Let \( p = 2 \). If \( n \) is odd, then

\[ \tilde{Z}_n(u, \varepsilon, d) = \frac{u^e}{|O^* : O^{*2}|} (q^{1/2})_{n-1}(q^{2})_{(n-1)/2}^{-1} \]
\[ \times (1-q^{-1}u)^{-1}(1-q^{-n}u)(q^{-3}u^2, q^{-2})_{(n-1)/2}, \]
\[ \tilde{Z}_n(u, \varepsilon, d) = \frac{q^{-(n-1)/2}u^e}{|O^* : O^{*2}|} (-1, -1)^{(n-2)/8}(q^{1/2})_{n-1}(q^{2})_{(n-1)/2}(q^{-2}u^2, q^{-2})_{(n-2)/2}. \]

If \( n \) is even, then

\[ Z_{n,*}(u, \varepsilon, d) = \frac{q^e}{|O^* : O^{*2}|} (q^{1/2})_{n-1}(q^{2})_{(n-1)/2}^{-1}(1-q^{-1}u^2)^{-1}(q^{-3}u^2, q^{-2})_{(n-1)/2}^{-1} \]
\[ \times \left\{ \begin{array}{ll} (q^{-1}u)^{e+1} & \text{if } * = e, \\ (q^{-1}u)^{f(\chi_{-1^{n/2}d})} & \text{if } * = e, \end{array} \right. \]
\[ \tilde{Z}_n(u, \varepsilon, d) = \frac{(1+\delta((-1)^{n/2}d))q^{e-n/2}}{2|O^* : O^{*2}|} (q^{1/2})_{n-1} \]
\[ \times (q^{-2})_{(n-1)/2}^{-1}(-1, -1)^{p(n+2)/8}((-1)^{n/2}d, \varpi)^{-1}(q^{-2}u^2, q^{-2})_{(n/2)}. \]
To calculate the zeta functions, we decomposes $D$ into subsets according to the type of the Jordan decomposition. Let $n=n_1+n_2+\cdots+n_m$ be a decomposition of $n$ into $m$ positive integers $n_i$. We call this a partition of $n$ of length $m$. We denote this by $\{n_i\}$, and call $\{n_i\}$ is even if all $n_i$ are even, and odd otherwise. A sequence of integers $t_1, t_2, \cdots, t_l$ is called associated to $\{n_i\}$ if $l=m$ and $0\leq t_1 < t_2 < \cdots < t_m$. For $x \in S_n(O, O^\times \setminus \{0\})$, there exists $g \in GL_n(O)$ such that
\[
\begin{pmatrix}
\varpi^{t_1}x_1 & 0 & \cdots & 0 \\
0 & \varpi^{t_2}x_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \varpi^{t_m}x_m
\end{pmatrix},
\]
where $x_i \in S_n(O, O^\times), \{n_i\}$ is a partition of $n$ and $\{t_i\}$ is a sequence associated to $\{n_i\}$. It is known that $\{n_i\}$ and $\{t_i\}$ depend only on $x$ (cf. [K], [O]). We say $x$ has the Jordan decomposition of type $\{n_i\}, \{t_i\}$. We denote by $S_n(O, S, \{n_i\}, \{t_i\})$ the subset of $S_n(O, S)$ consisting of elements of type $\{n_i\}, \{t_i\}$, and by $S_n(O, S, \{n_i\}, \{t_i\})^0$ the subset of $S_n(O, S)$ consisting of all elements of the form (2.1). We define
\[
S_n(O, S, \{n_i\}, \{t_i\})_k, \quad S_n(O, S, \{n_i\}, \{t_i\})^0_k
\]
similarly.

For $d \in O^\times$, let
\[
\lambda(d, \omega, \{n_i\}, \{t_i\}) = \int_{S_n(O, d\varpi^tO^\times, \{n_i\}, \{t_i\})} \omega(x) \det x \varpi^{-t} dx
\]
\[
= q^m t! \int_{S_n(O, d\varpi^tO^\times, \{n_i\}, \{t_i\})} \omega(x) dx.
\]
Here $t = \sum_{i=1}^m n_i t_i$. We define $\lambda(d, \omega, \{n_i\}, \{t_i\})_k$ in the same way. Then we have
\[
Z_n(u, \omega, d) = \sum_{(n_i), (t_i)} \lambda(d, \omega, \{n_i\}, \{t_i\})
\]
and a similar expression for $Z_n(u, \omega, d)_k$ by $\lambda(d, \omega, \{n_i\}, \{t_i\})_k$. Here $\{n_i\}$ and $\{t_i\}$ run through all partitions of $n$ and all sequences associated to them.

For a positive integer $v$, let $O_v = O/\varpi^vO$. For given $\{n_i\}, \{t_i\}$, if $v$ is sufficiently large, we can define
\[
S_n(O_v, d\varpi^tO_v^\times, \{n_i\}, \{t_i\}), \quad S_n(O_v, d\varpi^tO_v^\times, \{n_i\}, \{t_i\})_k
\]
similarly, and for $x \in S_n(O_v, d\varpi^tO_v^\times, \{n_i\}, \{t_i\})$, we can define $\omega(x)$ by taking $\tilde{x} \in S_n(O)$ such that $\tilde{x} \mod \varpi^v = x$.

First we treat the case $m=1, t_1=0$. For $d \in O^\times$, set
\[
\lambda_n(d, \omega) = \lim_{v \to \infty} \frac{2d^{n(n-1)/2}}{|GL_n(O_v)|} \sum_{x \in S_n(O, d\varpi^tO_v^\times)} \omega(x)
\]
and define $\lambda_{n}(d, \omega_{k})$ similarly by

$$
\lambda_{n}(d, \omega_{k}) = (q^{-2})_{n/2}^{-1} \lim_{v \to \infty} \frac{2}{|S_{d}(O_{v}, O_{v}^{*})|} \sum_{x \in S_{d}(O_{v}, dO_{v}^{*})} \omega(x),
$$

Here we used

$$
|S_{d}(O_{v}, O_{v}^{*})| = |GL_{d}(O_{v})|(q^{-2})_{n/2}^{-1}g^{-(n(n-1)/2)v}.
$$

**Lemma 2.5.** Let $d \in O^{*}$.

1. Let $p$ be odd. Then for $\omega = 1, \varepsilon$, one has

$$
\lambda_{n}(d, \omega) = \begin{cases} 
1 & \text{if } n \text{ is odd,} \\
1 + ((-1)^{n/2}d, \sigma)q^{-n/2} & \text{if } n \text{ is even.}
\end{cases}
$$

2. Let $p = 2$. If $n$ is odd, then $\lambda_{n}(d, \omega) = 0$ for $k \geq 1$, and for $k = 0$, one has

$$
\lambda_{n}(d, \omega_{0}) = \frac{2}{|O^{*} : O^{*}x^{2}|} (q^{-2})_{n/2}^{-1}
\times \begin{cases} 
1 & \text{if } \omega = 1, \\
q^{-e(n-1)/2}(-1, -1)^{(n-1)/2}(-1)^{(n+1)/2}d & \text{if } \omega = \varepsilon.
\end{cases}
$$

If $n$ is even, then one has

$$
\lambda_{n}(d, i) = \frac{2}{|O^{*} : O^{*}x^{2}|} (q^{-2})_{n/2}^{-1} \sum_{\theta \in A_{0} \times \cdots \times A_{n}} q^{-f_{k}(\chi_{0})/2} \chi_{0}((-1)^{n/2}d),
$$

$$
\lambda_{n}(d, \varepsilon) = \frac{(1 + \delta((-1)^{n/2}d))}{|O^{*} : O^{*}x^{2}|} (q^{-2})_{n/2}^{-1} g^{-e(n/2-1)-kn/2}
\times (-1, -1)^{(n+2)/2}((-1)^{n/2}d, \sigma)^{4}(1 + ((-1)^{n/2}d, \sigma)q^{-n/2}),
$$

where

$$
f_{k}(\chi_{0}) = \begin{cases} 
2k & \text{if } f(\chi_{0}) \leq 2k, \\
f(\chi_{0}) & \text{if } f(\chi_{0}) > 2k.
\end{cases}
$$

We note if $(1 + \delta((-1)^{n/2}d)) \neq 0$, then $((-1)^{n/2}d, \sigma)$ is well-defined.

**Remark 2.6.** From the above lemma, when $p = 2$ and $n$ is even we have

$$
\lambda_{n}(d, i) = \frac{(1 + \delta((-1)^{n/2}d))q^{-(n-1)e}}{|O^{*} : O^{*}x^{2}|} (1 + ((-1)^{n/2}d, \sigma)q^{-n/2}),
$$

$$
\lambda_{n}(d, \varepsilon) = (-1, -1)^{(n+2)/2}((-1)^{n/2}d, \sigma)^{e} \lambda_{n}(d, i).
$$

Hence for $x \in S_{d}(O, dO^{*})$, we have
\( \epsilon(x) = \frac{(-1, -1)^{n+2}}{(-1)^{n/2}d} \) \( \varphi(x) \).

**Proof of Lemma 2.5.** The case of \( p \) odd is easy. In this case \( \omega(x) = 1 \) for \( x \in S_n(O, O^\times) \), | \( O^\times : O^\times 2 \) | = 2, and for \( d \in O^\times \), we see

\[
| S_n(O^\times, \{d\}) | = | S_n(O^\times, \{d\}) | (q^{-1/2}/2)! q^{-(n-1)/2} \left\{ \begin{array}{ll}
1 & \text{if } n \text{ is odd}, \\
(1 + (-1)^{n/2} d, \varphi) q^{-n/2} & \text{if } n \text{ is even}.
\end{array} \right.
\]

Our assertion follows from this.

Let \( p = 2 \). Let \( S_n(O^\times, S) \) be the subset of \( S_n(O^\times, S) \) consisting of odd elements, that is, the elements equivalent to diagonal matrices with respect to \( GL_n(O^\times) \). We denote by \( S_n(O^\times, S)_0 \) the subset of the diagonal elements in \( S_n(O^\times, S) \). When \( n \) is odd, \( S_n(O^\times, O^\times) = S_n(O^\times, O^\times)_0 \), and when \( n \) is even, we have

\[
S_n(O^\times, O^\times) = S_n(O^\times, O^\times)_0 \cup S_n(O^\times, O^\times)_1,
\]

and for \( k \geq 1 \)

\[
| S_n(O^\times, O^\times)_0 | / | S_n(O^\times, O^\times)_1 | = 1 - q^{-kn}, \quad | S_n(O^\times, O^\times)_k | / | S_n(O^\times, O^\times) | = q^{-kn}.
\]

For \( k \geq 1 \), \( x \in S_n(O^\times, dO^\times 2)_k \) is equivalent to diagonal matrix of size \( n/2 \) with the diagonal elements in \( S_n(O^\times, O^\times)_k \). We denote by \( S_n(O^\times, dO^\times 2)_k \) its subset of all diagonal elements with diagonal elements in \( S_n(O^\times, O^\times)_k \).

First we calculate the contributions of odd elements. Let

\[
\omega(x) = \begin{cases} 
\omega(x) & \text{if } x \in S_n(O^\times, dO^\times 2), \\
0 & \text{otherwise}.
\end{cases}
\]

Applying Lemma 1.2 by taking \( G = GL_n(O^\times) \), \( S = S_n(O^\times, O^\times)_0 \), \( S_0 = S_n(O^\times, O^\times)_0 \), we see

\[
a_n(d, \omega) = \lim_{v \to \infty} \frac{2}{| S_n(O^\times, O^\times)_0 |} \sum_{x \in S_n(O^\times, dO^\times 2)_0} \omega(x)
\]

\[
= \lim_{v \to \infty} \frac{2}{| S_n(O^\times, O^\times)_0 |} \sum_{x \in S_n(O^\times, O^\times)_0} \omega(x)
\]

\[
= \lim_{v \to \infty} \frac{2}{| S_n(O^\times, O^\times)_0 |} \sum_{x \in S_n(O^\times, O^\times)_0} \omega(x)
\]

\[
= \lim_{v \to \infty} \frac{2}{| S_n(O^\times, O^\times)_0 |} \sum_{x \in S_n(O^\times, dO^\times 2)_0} \omega(x).
\]

The condition in Lemma 1.2 can be checked easily by using \( O^\times 1 = O^\times 2 \). Let \( \omega = 1 \). Then we see easily

\[
a_n(d, 1) = \frac{2}{| O^\times : O^\times 2 |}.
\]

Let \( \omega = \epsilon \). Then we see for \( n = 1, 2 \)
\[ a_1(d, \varepsilon) = \frac{2(d, d)}{|O^\times : O^\times 2|}, \quad a_2(d, \varepsilon) = \frac{(1 + \delta(-d))}{|O^\times : O^\times 2|} (-1, -1). \]

For \(n \geq 3\), we have
\[ a_n(d, \varepsilon) = (d, d) q^{-\varepsilon} a_{n-2}(-d, \varepsilon). \]

From this we obtain
\[
\begin{align*}
\frac{1}{|O^\times : O^\times 2|} \left\{ q^{-\varepsilon(n-1)/2} 2(-1, -1)^{(n-1)/2}((-1)^{(n+1)/2}, d) \quad &\text{if} \quad n \text{ is odd}, \\
q^{-\varepsilon(n-2)/2}(1 + \delta((-1)^{n/2}d))(-1, -1)^{(n+2)/8} \quad &\text{if} \quad n \text{ is even}.
\end{align*}
\]

This proves our assertion in the case where \(n\) is odd. We note \(a_n(d, \varepsilon)\) depends only on the class of \(d\) in \(O^\times /O^\times 2U_{2\varepsilon}\), and satisfies

\[ (2.3) \quad \sum_{d_1, d_2 = d} (d_1, d_2) a_m(d_1, \varepsilon) a_n(d_2, \varepsilon) = a_{m+n}(d, \varepsilon), \]

where \(d_1, d_2\) run through \(O^\times /O^\times 2U_{2\varepsilon}\) such that \(d_1d_2 = d\) in \(O^\times /O^\times 2U_{2\varepsilon}\).

To treat even elements, for \(k \geq 1\), we set
\[ b_n(d, \omega)_k = \lim_{v \to \infty} \frac{2}{|S_n(O_v, O^\times)_h|} \sum_{x \in S_n(O_v, dO^\times 2\omega^x)_k} \omega(x). \]

In the same way as in the case of odd elements, we see this is equal to
\[ b_n(d, \omega)_k = \lim_{v \to \infty} \frac{2}{|S_n(O_v, O^\times)_h^0|} \sum_{x \in S_n(O_v, dO^\times 2\omega^x)_k^0} \omega(x). \]

**Sublemma 2.7.** Let \(n\) be even. Then one has
\[
\begin{align*}
b_n(d, 1)_k &= \frac{2g_{hn}}{|O^\times : O^\times 2|} \sum_{\theta \in A_0 \times A_1 \times \cdots \times A_\varepsilon} q^{-f(\varepsilon)g_{hn}/2} \chi_{\theta}((-1)^{n/2}d), \\
b_n(d, \varepsilon)_k &= \frac{1 + \delta((-1)^{n/2}d)}{|O^\times : O^\times 2|} (-1, -1)^{n(n+2)/8} q^{kn/2} q^{-e(n/2 - 1)} \\
&\quad \times ((-1)^{n/2}d, \omega)^k (1 + ((-1)^{n/2}d, \omega) q^{-n/2}).
\end{align*}
\]

**Proof.** First let \(n = 2\), and consider
\[ \Lambda(d, \omega)_k, v = \sum_{x \in S_2(O_v, (d)_h)} \omega(x). \]

Let \(\psi\) be a non-trivial character of the additive group \(F\) which is trivial on \(O\) and not trivial on \(\omega^{-1}O\). Let \(\omega = i\). Then we have
\[ \Lambda(d, i)_k, v = q^{-v} \sum_{\beta \in O_v, \beta \in O_v, \beta \in O_v} \psi((\beta \gamma - \alpha^2 - d)\mu \omega^{-v}). \]

By considering the summation over \(\beta\), we find
\[ A(d, t)_{k,v} = q^{-k} \sum_{\sigma \in O_v \setminus O_v - k, \nu \in O_v \setminus O_v - 2k} \psi((\alpha^2 + d) \mu \sigma^{-\gamma}) \]
\[ = q^{-k} \left( \sum_{i=0}^{y-2k-1} (q-1)q^{y-k-1-i} \sum_{\sigma \in O_v \setminus O_v - 2k} \psi((\alpha^2 + d) \mu \sigma^{-\gamma}) \right) + q^k \sum_{\sigma \in O_v \setminus O_v} \psi((\alpha^2 + d) \mu \sigma^{-\gamma}) \].

We see
\[ \sum_{\sigma \in O_v \setminus O_v - \mu \sigma^{-1}} \psi((\alpha^2 + d) \mu \sigma^{-\gamma}) = q^i \left| \{ \alpha \in O_v \setminus O_v - \mu \sigma^{-1} \} \right| = q^i \left| \{ \alpha \in O_v \setminus O_v - \mu \sigma^{-1} \} \right| = q^i \sum_{\chi} \chi(-d) \],

where \( \chi \) runs through all character of \( O^*/O^{*2} U_l \). By Lemma 1.3 we see
\[ \sum_{\sigma \in O_v \setminus O_v - \mu \sigma^{-1}} \psi((\alpha^2 + d) \mu \sigma^{-\gamma}) = q^i \left( \sum_{\theta \in A_v - \{1/2\} \times \cdots \times A_v} \chi_{\theta}(-d) \right) \]
for \( i \leq 2e \), and
\[ \sum_{\sigma \in O_v \setminus O_v - \mu \sigma^{-1}} \psi((\alpha^2 + d) \mu \sigma^{-\gamma}) = q^i \left( \sum_{\theta \in A_v \times \cdots \times A_v} \chi_{\theta}(-d) \right) \]
for \( i \geq 2e + 1 \).

Summing up these results, we obtain
\[ A(d, t)_{k,v} = \sum_{i=0}^{y-2} (q^{y-2i} - q^{2y-2i-2}) \sum_{\theta \in A_v - i \times \cdots \times A_v} \chi_{\theta}(-d) \]
\[ + (q^{2y-2e} - q^{2y-2e-1}) \sum_{\theta \in A_v \times \cdots \times A_v} \chi_{\theta}(-d) \]
\[ + q^{2y-2e-1} \sum_{\theta \in A_v \times \cdots \times A_v} \chi_{\theta}(-d) \]
\[ = q^{2y} \sum_{\theta \in A_v \times \cdots \times A_v} q^{-f_{\mu}(x_{\theta})} \chi_{\theta}(-d) \],

and
\[ b_2(d, t) = \lim_{v \to \infty} \frac{2}{|S_2(O_v, O_v^0)|} |O_v^*| A(d, t)_{k,v} \]
\[ = \lim_{v \to \infty} \frac{2q^{2k}}{|O_v^* : O_v^{*2}|} \sum_{\theta \in A_v \times \cdots \times A_v} q^{-f_{\mu}(x_{\theta})} \chi_{\theta}(-d) \]
\[ = \frac{2q^{2k}}{|O_v^* : O_v^{*2}|} \sum_{\theta \in A_v \times \cdots \times A_v} q^{-f_{\mu}(x_{\theta})} \chi_{\theta}(-d) \].

For a general \( n \), our assertion for \( t \) follows from
\[ b_n(d, t)_{k,v} = 2^{-n/2 + 1} \sum_{d_1d_2 \cdots d_{n/2} = d} b_2(d_i, t)_{k,v} \],
where $d_i$'s run through $d_i \in O^*/O^{*2}$ such that $d_1d_2 \cdots d_{n/2} = d$ in $O^*/O^{*2}$.

Now let $\omega = \varepsilon$. We begin with the case of $n = 2$. We show $\lambda_2(d, \varepsilon) = 0$ unless $-d \not\in U^2 U_{2e}$. If $-d \notin U^2 U_{2e}$, then there exists $a \in U$, such that $(a, -d) = -1$. Choose $g \in GL_2(O)$ such that $\det g = a$. Then we see easily $g^{-1}axg^{-1} \in S_2(O, dO^{*2})$ for $x \in S_2(O, dO^{*2})$, but $\delta(g^{-1}axg^{-1}) = (a, -d)x = -\delta(x)$. Our assertion follows from this. Henceforth assume $-d \in U^2 U_{2e}$. For $x \in S_2(O, dO^{*2})$, let

$$x = \begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix}.$$  

Then under the assumption $\gamma \neq 0$, $\delta(x) = (-1, -1)(\gamma, -\det x)$. In the same way as in the case of $i$, we see

$$A(d, \varepsilon)_{k,v} = (-1, -1)q^{-v} \sum_{u \in O_v, \delta u \in O_v^{*}, \gamma \neq 0} (\gamma, -d)\psi((\beta\gamma - \alpha^2 - d)u\sigma^{-v})$$

$$= (-1, -1)q^{-k} \sum_{u \in O_v^{*}, \gamma \neq 0, \delta u \in O_v^{*}, (\gamma, -d)\psi((\alpha^2 + d)u\sigma^{-v}).$$

Since $-d \in U^2 U_{2e}$, we have

$$A(d, \varepsilon)_{k,v} = (-1, -1)q^{-k} \left( \sum_{i=0}^{v-k-1} (q-1)q^{v-k-1-i}(-d, \sigma^{i+k}) \psi((\alpha^2 + d)u\sigma^{-v}) \right),$$

where $l = \min(v, 2k + i)$. Substituting

$$\sum_{u \in O_v^{*}, \delta u \in O_v^{*}, (\gamma, -d)\psi((\alpha^2 + d)u\sigma^{-v}) = q^{v} \left\{ \begin{array}{ll} q^{[l/2]} & \text{if } l \leq 2e, \\ (1 + (-d, \sigma)q^{e})q^{e} & \text{if } l \geq 2e + 1, \end{array} \right.$$  

we obtain

$$A(d, \varepsilon)_{k,v} = (-1, -1)q^{2v-k}(-d, \sigma)^{k}(1 + (-d, \sigma)q^{e}) - (1 + (-d, \sigma)q^{-1})q^{e}$$

and

$$b_2(d, \varepsilon)_{k} = (-1, -1)q^{-k} \left( \frac{1 + \delta(-d)}{|O^*/O^{*2}|} \right) (-d, \sigma)^{k}(1 + (-d, \sigma)q^{-1}).$$

As in the case of $i$, we have

$$b_n(d, \varepsilon)_{k} = 2^{1-n/2} \sum_{d_1d_2 \cdots d_{n/2} = d} \prod_{1 < j} (d_i, d_j)b_2(d_i, \varepsilon)_{k}.$$  

Using $\prod_{i<j}(d_i, d_j) = (-1, -1)^{(n-2)/2}$ if $1 + \delta(-d_i) \neq 0$, and $|O^*/O^{*2}| = 2q^{e}$, we obtain our assertion.

We return to the proof of Lemma 2.5. By (2.2), the assertions for $k \geq 1$ follow
from the Sublemma 2.7. For $k=0$, we have

$$
\lambda_\alpha(d,\omega) = (q^{-2})_{[n/2]}^{-1} [1 - q^{-n}] a_n(d,\omega) + q^{-n} b_n(d,\omega),
$$

and the assertion follows also from Sublemma 2.7.

For a partition $\{n_i\}$ of length $m$ and a sequence $\{t_i\}$ associated with it, set

$$
\hat{Q}(\{n_i\}, \{t_i\}) = \frac{n+1}{2} \sum_{i=1}^{m} n_i t_i - \sum_{i=1}^{m} \frac{n_i(n_i+1)}{2} t_i - \sum_{j<i} n_i n_j t_j,
$$

$$
P(a, \{n_i\}) = \prod_{i=1}^{m} (a)_{[n_i/2]}^{-1}, \quad t = \sum_{i=1}^{m} n_i t_i.
$$

Then we can prove

**Proposition 2.8.** Let $d \in O^\times$.

1. Let $p$ be odd. Then

$$
\lambda_\alpha(d, t, \{n_i\}, \{t_i\}) = 2^{-1} u^t q^{\hat{Q}(\{n_i\}, \{t_i\})} (q^{-1})_n P(q^{-2}, \{n_i\}) \left\{ \begin{array}{ll} 1 & \text{if } \{n_i\} \text{ is odd,} \\
(1 + ((-1)^{n/2} d, \omega) q^{-n/2}) & \text{if } \{n_i\} \text{ is even.} \end{array} \right.
$$

If $\{n_i\}$ is odd, $\lambda_\alpha(d, t, \{n_i\}, \{t_i\}) = 0$, unless there exists $t_0$ such that $t_j \equiv t_0 \mod 2$ for $n_j$ odd. If this condition is satisfied, then

$$
\lambda_\alpha(d, t, \{n_i\}, \{t_i\}) = 2^{-1} u^t q^{\hat{Q}(\{n_i\}, \{t_i\})} (q^{-1})_n P(q^{-2}, \{n_i\}) ((-1)^{n(n+1)/2} a_n + 1, \omega) \prod_{n_i \text{ odd, } t_i \equiv t_0 \mod 2} q^{-n_i/2}.
$$

If $\{n_i\}$ is even, then

$$
\lambda_\alpha(d, t, \{n_i\}, \{t_i\}) = 2^{-1} u^t q^{\hat{Q}(\{n_i\}, \{t_i\})} (q^{-1})_n P(q^{-2}, \{n_i\}) \left( \prod_{t_i \text{ odd}} q^{-n_i/2} + ((-1)^{n/2} d, \omega) \prod_{t_i \text{ even}} q^{-n_i/2} \right).
$$

2. Let $p=2$. Then $\lambda_\alpha(d, t, \{n_i\}, \{t_i\}) = 0$ unless $k \leq t_i$ for $n_i$ odd. Assume this condition. Then one has

$$
\lambda_\alpha(d, t, \{n_i\}, \{t_i\}) = \frac{1}{|O^\times : O^\times^2|} u^t q^{\hat{Q}(\{n_i\}, \{t_i\})} (q^{-1})_n P(q^{-2}, \{n_i\})
$$

$$
\times \left\{ \begin{array}{ll} \prod_{t_i < k} q^{-2(k-t_i)n/2} & \text{if } \{n_i\} \text{ is odd,} \\
\sum_{\sigma \in A_0 \times \cdots \times A_p} \prod_{i=1}^{m} q^{-f_{k-t_i}((\sigma_0) n/2)} \chi_\alpha((-1)^{n/2} d) & \text{if } \{n_i\} \text{ is even.} \end{array} \right.
$$

Here $f_{k-t_i}((\sigma_0)) = f((\sigma_0))$ if $k-t_i < 0$.

Let $\omega = \epsilon$. If $\{n_i\}$ is odd, then $\lambda_\alpha(d, t, \{n_i\}, \{t_i\}) = 0$ unless there exists $t_0$ such that $t_j \equiv t_0 \mod 2$ for $n_j$ odd and $k \leq t_i$ for $n_i$ odd. When this condition is satisfied, one has
\[ \lambda_n(d, \varepsilon, \{n_i\}, \{t_i\})_k \]

\[ = u' q^{Q(n_1, t_1)}(q^{-1})_n P(q^{-2}, \{n_i\})((-1)^n(n_1+1)/2, d_{n+1}, \varepsilon)^{\frac{q^{-e(n-2)/2}}{|O^*: O^*|}} \]

\[ \times \left\{ \begin{array}{ll}
(1, -1)^{n+1}/2((-1)^n(n+1)/2, d) & \text{if } n \text{ is odd} \\
(1 + \delta((-1)^{n/2}d))2^{-1}(-1, -1)^{n(n+2)/8} & \text{if } n \text{ is even}
\end{array} \right. \]

\[ \times \prod_{n_i \text{ even}, k > t_i, k \equiv t_0 \mod 2} q^{-(k-t_i)n_i/2} \prod_{n_i \text{ even}, k > t_i, k \neq t_0 \mod 2} q^{-(k-t_i+1)n_i/2} \]

\[ \times \prod_{n_i \text{ even}, t_i \geq k, t_i \equiv t_0 \mod 2} q^{-n_i/2}. \]

If \( \{n_i\} \) is even, then

\[ \lambda_n(d, \varepsilon, \{n_i\}, \{t_i\})_k \]

\[ = u' q^{Q(n_1, t_1)}(q^{-1})_n P(q^{-2}, \{n_i\}) \left( \frac{1 + \delta((-1)^{n/2}d)}{2|O^*: O^*|} \right) q^{-e(n-2)/2}(-1, -1)^{n(n+2)/8} \]

\[ \times \left( \begin{array}{c}
((-1)^{n/2}d, \varepsilon)^k \prod_{t_i < k} q^{-(k-t_i)n_i/2} \prod_{t_i \geq k, t_i \equiv k \mod 2} q^{-n_i/2} \\
+((-1)^{n/2}d, \varepsilon)^{k+1} \prod_{t_i < k} q^{-(k-t_i+1)n_i/2} \prod_{t_i \geq k, t_i \equiv k \mod 2} q^{-n_i/2}
\end{array} \right). \]

**Proof.** As in the proof of Lemma 3.2 of [1-S], we have

\[ \lambda(d, \omega, \{n_i\}, \{t_i\})_k = q^{xu'} \lim_{v \to \infty} \frac{1}{|S_n(O_v)|} \sum_{x \in S_n(O_v, d\sigma O_v^{\pm 2}, \{n_i\}, \{t_i\})_k} \omega(x) \]

\[ = q^{xu'} \lim_{v \to \infty} \frac{|GL_n(O_v)|}{|S_n(O_v)|} q^{-v \sum_{j < i} n_j n_{-j}} \sum_{x \in S_n(O_v, d\sigma O_v^{\pm 2}, \{n_i\}, \{t_i\})_k} \omega(x) \]

\[ \times \prod_{i=1}^m |GL_n(O_v)| \sum_{x \in S_n(O_v, d\sigma O_v^{\pm 2}, \{n_i\}, \{t_i\})_k} \omega(x) \]

\[ = q^{xu'}(q^{-1})_n \lim_{v \to \infty} q^{v(n-1)/2} v^{-\sum_{j < i} n_j n_{-j}} \sum_{x \in S_n(O_v, d\sigma O_v^{\pm 2}, \{n_i\}, \{t_i\})_k} \omega(x) \]

\[ \times \prod_{i=1}^m |GL_n(O_v)| \sum_{x \in S_n(O_v, d\sigma O_v^{\pm 2}, \{n_i\}, \{t_i\})_k} \omega(x). \]

Let \( x \) be of the form (2.1) with \( x_i \in S_n(O, dO^{x_2}) \), and set

\[ \mu(\omega, \{d_i\}, \{n_i\}, \{t_i\}) = \omega(x) / \prod_{i=1}^m \omega(x_i). \]

Then for \( \omega = 1 \), this is equal to 1, and for \( \omega = \varepsilon \), this is equal to
\[ \prod_{i<j} (d_i, \sigma^{n_i t_i}, d_j, \sigma^{n_j t_j}) \prod_{i=1}^{m} (\sigma^{t_i}, (-1)^{n_i(n_i+1)/2} d_i^{n_i+1}) \]
\[ = (-1, \omega)^{\sum_{i<j} n_i n_j t_{i,j} + \sum_{i} (n_i(n_i+1)/2) t_i} \]
\[ \times (d, \omega) \prod_{i=1}^{m} (d_i, \sigma^{n_i}) \prod_{i<j} (d_i, d_j). \]

Hence \( \mu(\omega, \{d_i\}, \{n_i\}, \{t_i\}) \) depends only on \( \{d_i\}, \{n_i\}, \) and \( \{t_i\} \). We note \( x \) of the form (2.1) belongs to \( S_n(O)_k \) if and only if \( x_i \in S_n(O)_{k-t_i} \), for \( k-t_i \geq 0 \). Hence for \( v' = v - \sum_{i=1}^{n} v_i t_i \) we have

\[ \lambda(d, \epsilon, \{n_i\}, \{t_i\}) \]
\[ = \lim_{v \to \infty} q^{2t_1(n_1 t_1)} u'(q^{-1} n_2 - m) \sum_{d_1 d_2 \cdots d_m \in dO^2} \mu(\omega, \{d_i\}, \{n_i\}, \{t_i\}) \]
\[ \times \prod_{i=1}^{m} 2q^{(n_i(n_i-1)/2)(v-t_i)} G_{n_i}(O_{v-t_i}) \sum_{x_i \in S_{n_i}(O_{v-t_i}, d_i O^{s_i t_i})} \lambda_{n_i}(d_i, \omega)_{k-t_i}. \]

Here \( \lambda_{n_i}(d_i, \omega) = \lambda_{n}(d_i, \omega) \), \( S_n(O_{v-t_i}, d_i O^{s_i t_i}) = S_{n_i}(O_{v-t_i}, d_i O^{s_i t_i}) \) if \( l \leq 0 \), and \( d_i \) runs through the classes \( O^x/O^{x^2} \) satisfying \( d_1 d_2 \cdots d_m \in dO^{x^2} \). The rest of the calculation is similar as in [1-S], and we give a proof only for the case \( \{n_i\} \) is odd and \( \omega = \epsilon \) as an example and the rests are left to the readers. By Lemma 2.5, we see

\[ \lambda_{n}(d, \epsilon) = a_n(d, \epsilon) \]
\[ \times \begin{cases} 
1 & \text{if } n \text{ is odd, and } k = 0, \\
0 & \text{if } n \text{ is odd and } k \geq 1, \\
((-1)^{n/2} d, \omega)^{k((1+(-1)^{n/2} d, \omega)q^{-n/2})} & \text{if } n \text{ is even}. 
\end{cases} \]

Let \( d_1 d_2 \cdots d_m = d \) and consider the sum

\[ \sum_{d_1 d_2 \cdots d_m = d} \prod_{i=1}^{m} (d_i, \sigma^{t_i}) \prod_{n_i \text{ even}, k > t_i} ((-1)^{n_i/2} d_i d_i, \sigma) q^{-(k-t_i)n_i/2} \]
\[ \times \prod_{n_i \text{ even}} (1 + ((-1)^{n_i/2} d_i d_i, \sigma) q^{-n_i/2}) \]

where \( d_1', d_2', \cdots, d_m' \) run through \( U_2/\epsilon O^{x^2} \) such that \( d_1' d_2' \cdots d_m' = 1 \). Then we see this sum vanishes unless there exists \( t_0 \) such that

\[ t_i \equiv t_0 \mod 2 \]

for \( n_i \) odd. Assume this condition. Then the above sum is equal to
\[ 2^{m-1}(d, m)^{t_0} \prod_{n \text{ even}, k \equiv t_0 \mod 2} q^{-(k-t) n/2} \times \prod_{n \text{ even}, k > t_1, k \equiv t_0 \mod 2} q^{-(k-t_1+1)/2} \prod_{n \text{ even}, t_1 \geq k, t_1 \equiv t_0 \mod 2} q^{-n/2}. \]

In notice of (2, 3) and \[ t_0 = t \mod 2, \] taking the summation over \( O^n/O^{*2} \mathcal{U}_{2e} \), we obtain our result.

**Proof of Theorem 2.2.** We give a proof only for the case where \( p = 2 \) and \( k \geq 1 \). First let \( \omega = t \). As in [1-S], we set

\[
X_n(u, t) = \sum_{\{n\}, \{t\}, t_1 \geq 1} P(q^{2}, \{n\}) q^{G(t,n,\{t\})} u^t,
\]

\[
Y_n(u, t) = \sum_{\{n\}, \{t\}, t_1 \geq 1} P(q^{2}, \{n\}) q^{G(t,n,\{t\})} u^t,
\]

for \( t = \sum_{i=1}^{m} n_i t_i \), and set \( X_n^0(u, t) = u^{-n} X_n(u, t) \), \( Y_n^0(u, t) = u^{-n} Y_n(u, t) \). Then by Prop. 5.6 of [1-S], we have

\[
X_n^0(u, t) = \begin{cases} \left( q^{u^2}, q^2 \right)_{n/2}^{-1} \left( q^{-2}, q^{2} \right)_{n/2}^{-1} & \text{if } n \text{ is odd}, \\ (1 - q^{-(n+1)/2} u)(u^2, q^2)_{n/2}^{-1} & \text{if } n \text{ is even}, \end{cases}
\]

\[
Y_n^0(u, t) = \left( q^{-2}, q^2 \right)_{n/2}^{-1} (u^2, q^2)_{n/2}^{-1}. \]

For \( k, 0 \leq k \leq e \), set

\[
X_n^0(u, t)_k = \sum_{\{n\}, \{t\}, k \leq t_i \text{ for } n_i \text{ odd}} P(q^{2}, \{n\}) q^{G(t,n,\{t\})} \left( \prod_{i=k}^{m} q^{-(k-t_i)n_i} \right) u^t, \tag{2.4}
\]

\[
U_n^0(u, \theta)_k = \sum_{\{n\}, \{t\}} P(q^{2}, \{n\}) q^{G(t,n,\{t\})} \left( \prod_{i=1}^{m} q^{-(k-1)(x_0)n_i/2} \right) u^t, \]

and \( X_n(u, t)_k = u^n X_n^0(u, t)_k \), \( U_n(u, \theta)_k = u^n U_n^0(u, \theta)_k \). Then we have

\[
X_n^0(u, t)_0 = X_n(u, t), \quad U_n^0(u, \theta)_0 = \left( q^{-f(x_0)n/2} \right) U_n^0(u, \theta),
\]

and for \( k \geq 1 \)

\[
Z_n(u, t, d)_k = \frac{(q^{-1})_n}{|O^n : O^{*2}|} \left( X_n^0(u, t)_k + \sum_{\theta \in A_0 \times \cdots \times A_e} X_n(\theta) \left( 1 - (-1)^{n/2} d \right) U_n^0(u, \theta)_k \right). \tag{2.5}
\]

We divide the sums in (2.4) into two parts according to whether \( t_1 > 0 \), or \( t_1 = 0 \). Then the first summands are \( X_n(u, t)_{k-1} \) and \( U_n(u, \theta)_{k-1} \). For the second summands, set

\[
\begin{align*}
& n_i = n_2, \ldots, n_{m-1} = n_m, \quad t_1 = t_2 - 1, \ldots, t_{m-1} = t_m - 1. 
\end{align*}
\]

Setting \( r = 2n_1 \), for \( k \geq 1 \) we find
\[ X_n^0(u, \theta)_k = \sum_{r=0}^{\left[\frac{n}{2}\right]} (q^{-2})_r^{-1} q^{2r} X_{n-2r}(q^ru, \theta)_{k-1}, \]

(2.6)

\[ U_n^0(u, \theta)_k = \sum_{r=0}^{\left[\frac{n}{2}\right]} (q^{-2})_r^{-1} q^{-f_{\nu}(x_0)r} U_{n-2r}(q^ru, \theta)_{k-1}. \]

The terms for \( r = 0 \) come from those with \( t_1 > 0 \). Here we used

\[
\sum_{i=1}^{m} n_i t_i = n - n_1 + \sum_{i=1}^{m-1} n_i' t_i', \quad \bar{Q}(\{n_i\}, \{t_i\}) = \bar{Q}(\{n_i'\}, \{t_i'\}) + \frac{n_1}{2} \sum_{i=1}^{m-1} n_i' t_i'
\]

for \( t_1 = 0 \).

**Lemma 2.9.** Let \( 1 \leq k \leq e \). Then

\[
X_n^0(u, \theta)_k = (q^{-2})_{\left[\frac{n}{2}\right]}^{-1} \left\{ \begin{array}{ll}
(1 - q^{-(n-1)/2}u)^{-1}(qu^2, q^2)_{[n/2]}(q^{-1}(n-1)/2)u^k & \text{if } n \text{ is odd}, \\
(u^2, q^2)_{[n/2]}(1-q^{-n})(1-q^{-(n-1)/2}u)^{-1}u^{2k} & \text{if } n \text{ is even}, \\
+ (1-q^{-n}) \sum_{i=1}^{k-1} q^{-nk-i}u^{2i} + q^{-kn} & \text{if } n \text{ is even}.
\end{array} \right.
\]

\[
U_n^0(u, \theta)_k = (q^{-2})_{\left[\frac{n}{2}\right]}^{-1}(u^2, q^2)_{\left[\frac{n}{2}\right]}^{-1} \left( q^{-f_{\nu}(x_0)n/2} + (1-q^{-n})q^n \sum_{i=1}^{k-1} q^{-inu^{2k-2i}} \right).
\]

**Proof.** We give a proof only for \( X_n^0(u, \theta)_k \), since the proof for \( U_n^0(u, \theta)_k \) is similar. Let \( n \) be odd. Assume the formula for \( k = 1 \). Then the right hand side of (2.6) is equal to

\[
(q^{-(n-1)/2}u)^{-1}(1-q^{-(n-1)/2}u)^{-1}
\]

\[
\times \sum_{r=0}^{\left[\frac{n}{2}\right]} (q^{-2})_r^{-1}(q^{-2})_{\left[\frac{n}{2}\right]}^{-1} - q^{-2r}(q^ru)^{n-2r}(q^2)^{\left[\frac{n}{2}\right]-r}
\]

\[
= (q^{-2})_{\left[\frac{n}{2}\right]}^{-1}(1-q^{-1/2}u)^{-(n-1)/2}(q^{-1}u)^{(n-1)/2}q^{-n-1/2}u
\]

\[
\times \sum_{r=0}^{\left[\frac{n}{2}\right]} \left( \begin{array}{c}
\left[\frac{n}{2}\right] \\
r
\end{array} \right)
q^{-r}(nu, q^2)_r
\]

\[
= (q^{-2})_{\left[\frac{n}{2}\right]}^{-1}(1-q^{-1/2}u)^{-(n-1)/2}(qu^2, q^2)_{\left[\frac{n}{2}\right]}^{-1}(q^{-1}u)^{-1/2}u^{k},
\]

by (1) of Lemma 1.1. The case of \( n \) even can be proved in the same way.

Substituting the above result into (2.5), we obtain our assertion for \( i \).

To treat the case of \( \epsilon \), we recall the series defined in [1-S, §5]

\[
X_\epsilon(u, \theta) = \sum_{\{n_i\} \{t_i\}} \sum_{\text{1 \leq t_1, t_i even for } n_i \text{ odd}} P(q^{-2}, \{n_i\}) q^{\bar{Q}(\{n_i\}, \{t_i\})} u^t \prod_{n_i \text{ even}, t_i \text{ odd}} q^{-n_i/2},
\]

\[
Y_\epsilon(u, \theta) = \sum_{\{n_i\} \{t_i\}} \sum_{\text{1 \leq t_1, t_i even for } n_i \text{ odd}} P(q^{-2}, \{n_i\}) q^{\bar{Q}(\{n_i\}, \{t_i\})} u^t \prod_{n_i \text{ even}, t_i \text{ even}} q^{-n_i/2}.
\]

We set
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$$X_n^0(u, \varepsilon) = u^{-n}Y_n(u, \varepsilon), \quad Y_n^0(u, \varepsilon) = u^{-n}X_n(u, \varepsilon).$$

These are given explicitly by

$$X_n^0(u, \varepsilon) = (q^{-2})_{[n/2]}^{-1} \begin{cases} u(u^2, q^2)_{(n+1)/2}^{-1} & \text{if } n \text{ is odd}, \\ q^{-n/2}(qu^2, q^2)_{n/2}^{-1} & \text{if } n \text{ is even}, \end{cases}$$

$$Y_n^0(u, \varepsilon) = (q^{-2})_{[n/2]}^{-1} \begin{cases} (u^2, q^2)_{(n+1)/2}^{-1} & \text{if } n \text{ is odd}, \\ (qu^2, q^2)_{n/2}^{-1} & \text{if } n \text{ is even}. \end{cases}$$

For $k$, $0 \leq k \leq e$, similarly as above we set

$$X_n^0(u, \varepsilon)_k = \sum_{\{n_i\} \{t_i\}} \sum_{t_i \geq k \text{ for } n_i \text{ odd}} P(q^{-2}, \{n_i\}) q^{Q(n_i, \{t_i\})} u^t$$

$$\times \prod_{k \leq t_i, t_i \neq 0 \text{ mod } 2, n_i \text{ even}} q^{-n_i/2} \prod_{t_i < k, k \equiv 0 \text{ mod } 2} q^{-(k-t_i)n_i/2}$$

$$\times \prod_{t_i < k, k \equiv 0 \text{ mod } 2} q^{-(k+1-t_i)n_i/2},$$

$$Y_n^0(u, \varepsilon)_k = \sum_{\{n_i\} \{t_i\}} \sum_{t_i \geq k \text{ for } n_i \text{ odd}} P(q^{-2}, \{n_i\}) q^{Q(n_i, \{t_i\})} u^t$$

$$\times \prod_{k \leq t_i, t_i \equiv 0 \text{ mod } 2, n_i \text{ even}} q^{-n_i/2} \prod_{t_i < k, k \neq 0 \text{ mod } 2} q^{-(k-t_i)n_i/2}$$

$$\times \prod_{t_i < k, k \equiv 0 \text{ mod } 2} q^{-(k+1-t_i)n_i/2},$$

and $X_n(u, \varepsilon)_k = u^n Y_n^0(u, \varepsilon)_k, \ Y_n(u, \varepsilon)_k = u^n X_n^0(u, \varepsilon)_k$. Then

$$X_n^0(u, \varepsilon)_0 = X_n^0(u, \varepsilon), \quad Y_n^0(u, \varepsilon)_0 = Y_n^0(u, \varepsilon)$$

and for $k \geq 1$, $Z_n(u, \varepsilon, d)_k$ is equal to

$$q^{-e(n-1)/2} \left( -1, -1 \right)^{a_2 - 1} (X_n^0(u, \varepsilon)_k + Y_n^0(u, \varepsilon)_k)$$

if $n$ is odd, and $Z_n(u, \varepsilon, d)_k$ is equal to

$$\frac{(1 + \delta((-1)^{n/2}d))}{2|O^\times : O^{\times 2}|} q^{-e(n-2)/2} \left( -1, -1 \right)^{n(n+2)/8}$$

$$\times ((-1)^{n/2}d, w)^2((-k+1)/2) X_n^0(u, \varepsilon)_k + ((-1)^{n/2}d, w)^2((-k+1)/2 + 1) Y_n^0(u, \varepsilon)_k,$$

if $n$ is even. In the same way as above, we find

$$X_n^0(u, \varepsilon)_k = \sum_{r=0}^{n/2} (q^{-2})^{-1} X_{n-2r}(u, \varepsilon)_k \begin{cases} q^{-kr} & \text{if } k \equiv 0 \text{ mod } 2, \\ q^{-(k+1)r} & \text{if } k \equiv 1 \text{ mod } 2, \end{cases}$$

$$Y_n^0(u, \varepsilon)_k = \sum_{r=0}^{n/2} (q^{-2})^{-1} Y_{n-2r}(u, \varepsilon)_k \begin{cases} q^{-(k+1)r} & \text{if } k \equiv 0 \text{ mod } 2, \\ q^{-kr} & \text{if } k \equiv 1 \text{ mod } 2. \end{cases}$$
By induction on \(k\), we can prove

**Lemma 2.10.** Let \(1 \leq k \leq e\). If \(n\) is odd, then

\[
X_n(u, \varepsilon)_k = \begin{cases} 
(q^{-2})_{(n-2)/2} u^2, & \text{if } k \equiv 0 \mod 2, \\
u^n_k, & \text{if } k \equiv 1 \mod 2,
\end{cases}
\]

\[
Y_n(u, \varepsilon)_k = \begin{cases} 
(q^{-2})_{(n-2)/2} u^2, & \text{if } k \equiv 0 \mod 2, \\
u^n_{k+1}, & \text{if } k \equiv 1 \mod 2,
\end{cases}
\]

and if \(n\) is even, then

\[
X_n(u, \varepsilon)_k = \begin{cases} 
(q^{-2})_{n/2} (qu^2, q^2)^{1-n/2} u^n, & \text{if } k \equiv 0 \mod 2, \\
q^{-(k+1)/2} u^n, & \text{if } k \equiv 1 \mod 2,
\end{cases}
\]

\[
Y_n(u, \varepsilon)_k = \begin{cases} 
(q^{-2})_{n/2} (qu^2, q^2)^{1-n/2} u^n, & \text{if } k \equiv 0 \mod 2, \\
q^{-(k+1)/2} u^n, & \text{if } k \equiv 1 \mod 2.
\end{cases}
\]

Substituting these results, we obtain our assertion for \(e\). This completes the proof of Th. 2.2.

**§3. The case (II) (the case of representation of symmetric matrices)**

Let \(F, O\) be as in the previous sections, and let \(e\) be as before if \(p = 2\), and set \(e = 0\) otherwise. Let \(h\) be an even element of \(S_\delta(O, O')\). Assume \(h\) is of the form

\[
\begin{pmatrix}
0 & 1_r & 0 \\
1_r & 0 & 0 \\
0 & 0 & h'
\end{pmatrix},
\]

and assume \(h'\) is of the form

\[
h' = \begin{pmatrix}
0 & 1_s & 0 \\
1_s & 0 & 0 \\
0 & 0 & h''
\end{pmatrix}, \quad s = [(n-2r-1)/2],
\]

where \(h'' = 2\) if \(n\) is odd, and \(h'' = H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) or an even element of \(S_2(O, O')\) such that \(\det h'' \in U_{2e} - U_{2e-1}\), that is, such that the quadratic form associated to \(h''\) is the norm form on the maximal order of the unramified quadratic extension of \(F\) if \(n\) is even. Furthermore we assume \(h'' = H\), when \(n = 2r\). Then the orthogonal group of the quadratic form \(Q(v) = \frac{1}{2} v^t H v\) in \(n\) variables is smooth over \(O\). When \(n\) is odd, we set \(\varepsilon_Q = (2, \sigma)\), and when \(n\) is even, we set \(\varepsilon_Q = ((-1)^{n/2} \det h, \sigma)\), that is, \(\varepsilon_Q = 1\) if \(h'' = H\) and \(\varepsilon_Q = -1\) otherwise.

The following can be easily proved.

**Proposition 3.1.** Let \((G, \rho, X)\) be of type (II). Then the \(G(F)\) orbits of \(X^{ss}(F)\) are given by
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\[ \bigcup_{d \in F^*/\{F^{x}\}} \bigcup_{\varepsilon = \pm 1} \{ x \in X^{q}(F) \mid d'(xh) \in dF^{x}, \varepsilon'(xh) = \varepsilon \} \]

if $n$ is odd, by

\[ \bigcup_{d \in F^*/\{v(GO^{0}(F))\}} \bigcup_{\varepsilon = \pm 1} \{ x \in X^{q}(F) \mid d'(xh) \in dv(GO^{0}(F)), \varepsilon'(xh) = \varepsilon \} , \]

if $n$ is even, $r$ is odd, and by

\[ \bigcup_{d \in F^*/\{F^{x} \cap ((-1)^{r/2}, (-1)^{(n-r)/2} d(h))\}} \{ x \in X^{q}(F) \mid d'(xh) \in dF^{x}, \varepsilon'(xh) = \varepsilon \} , \]

if $n$ and $r$ is even, where $v(GO^{0}(F))$ is the group of similitudes of $GO^{0}(F)$.

For $d \in O^{*}$, we consider a subset $D$ of $X(O)$

\[ D = \{ x \in M_{n}(O) \mid \det'(xh) \in d\sigma^{*}O^{x}, n \in \mathbb{Z}, n \geq 0 \} . \]

In this section, we determine explicitly the integral

\[ \int_{D} \omega'(xh)|\det'(xh)|^{\kappa - 1}dx \]

for $\omega = i, \varepsilon, \text{ or } \varepsilon$ and $\kappa = n/2$. The measure $dx$ is normalized so that the volume of $X(O)$ is 1. This integral converges for a sufficiently large $\text{Re}(s)$, and is known to be a rational function in $u = q^{-s}$. We denote it by $Z_{n}(u, \omega, d)$, and as in §2, we denote by $Z_{n,\omega}, Z_{n,\varepsilon}$ the odd, even part of $Z_{n}$ respectively.

Our result for odd $n$ is

**Theorem 3.2.** Let $n$ be odd, and let $d \in O^{*}$. Assume $p \not\equiv 2$.

1. Let $r$ be odd, and let $\varepsilon = ((-1)^{r-1/2}d, \varepsilon)$. Then

\[ Z_{n,\omega}(u, t, d) = \frac{1}{|O^{*} : O^{x}|} (q^{-1})u(q^{-2})_{(n/2)}(q^{-2})_{(n-r/2)}^{-1}(q^{-2})_{(r/2)}^{-1} \times (1 - q^{r-1}u^{2})^{-1}(1 - q^{n-2}u^{2})^{-1}(q^{n-2}u^{2}, q^{-2})_{(r/2)}(q^{r-2}u^{2}, q^{-2})_{(r/2)}^{-1} \times \begin{cases} q^{n-2}u^{-1} & \text{if } \varepsilon = o, \\ (1 - \epsilon \overset{\omega}{eq}^{-(n-r)/2})^{-1}(1 - \epsilon \overset{\omega}{eq}^{(n+r)/2} - 2u^{2}) & \text{if } \varepsilon = e, \end{cases} \]

2. Let $r$ be even, and let $\varepsilon = ((-1)^{r+1/2}d, \varepsilon)$. Then

\[ Z_{n,\varepsilon}(u, t, d) = \frac{1}{|O^{*} : O^{x}|} (q^{-1})u(q^{-2})_{(n/2)}(q^{-2})_{(n-r/2)}^{-1}(q^{-2})_{(r/2)}^{-1} \times (1 - q^{n-2}u^{2})^{-1}(1 - u^{2})^{-1}(q^{n-2}u^{2}, q^{-2})_{(r/2)}(q^{-2}u^{2}, q^{-2})_{(r/2)}^{-1} \times \begin{cases} ((-1)^{r+1/2}, \varepsilon)q^{(n-r-1)/2}u & \text{if } \varepsilon = o, \\ (1 - \epsilon \overset{\omega}{eq}^{-(n-r)/2})^{-1}(1 - \epsilon \overset{\omega}{eq}^{(n-r)/2} - 1u^{2}) & \text{if } \varepsilon = e. \end{cases} \]
(2) Let \( r \) be even, and let \( \epsilon = ((-1)^{r/2}, \varnothing) \). Then

\[
Z_{n,a}(u, 1, d) = \frac{1}{|O^\times : O^{\times 2}|} (q^{-1})_{a}(q^{-2})_{[n/2]}(q^{-2})_{[r/2]}^{-1}(q^{-2})_{[a/2]}^{-1}(q^{-2})_{[r/2]}^{-1}
\times (1-q^{n-2}u^2)^{-1}(1-q^{n-r-1}u^2)^{-1}(q^{n-3}u^2, q^{-2})_{[r/2]}^{-1}(q^{r-1}u^2, q^{-2})_{[r/2]}^{-1}
\times \begin{cases} q^{n/2-1}u & \text{if } \epsilon = o, \\ (1-\epsilon q^{-r/2})^{-1}(1-\epsilon q^{-r/2-2} u^2) & \text{if } \epsilon = e. \end{cases}
\]

\[
Z_{n,a}(u, e, d) = \frac{1}{|O^\times : O^{\times 2}|} (q^{-1})_{a}(q^{-2})_{[n/2]}(q^{-2})_{[r/2]}^{-1}(q^{-2})_{[a/2]}^{-1}
\times (1-q^{r-1}u^2)^{-1}(1-u^2)^{-1}(q^{n-2}u^2, q^{-2})_{[r/2]}^{-1}(q^{r-2}u^2, q^{-2})_{[r/2]}^{-1}
\times \begin{cases} ((-1)^{r/2}, \varnothing)\epsilon q^{(n-2)/2}u & \text{if } \epsilon = o, \\ (1-\epsilon q^{-r/2})^{-1}(1-\epsilon q^{r/2-1} u^2) & \text{if } \epsilon = e. \end{cases}
\]

To describe the result in the case that \( n \) and \( r \) are even, we introduce a power series. Set

\[
D_1(u, d, \epsilon) = (-1-u^2)^{-1}(1-q^{r-1}u^2)^{-1}(1-\epsilon q^{r/2-1} u^2) = \sum_{m=0}^{\infty} a_1(m) u^m,
\]

\[
D_2(u, d, \epsilon) = (1-u^2)^{-1}(1-q^{n-r-1}u^2)^{-1}(1-\epsilon q^{(n-r)/2-1} u^2) = \sum_{m=0}^{\infty} a_2(m) u^m.
\]

Here \( \epsilon = \pm 1, \text{ or } 0 \). We define

\[
(D_1 \otimes D_2)(u, d, \epsilon) = (1-\epsilon q^{n/2-1} u^2)^{-1} \sum_{m=0}^{\infty} a_1(m) a_2(m) u^m.
\]

Then by some calculations, we find

**LEMMA 3.3.**

\[
(D_1 \otimes D_2)(u, d, \epsilon) = (1-u^2)^{-1}(1-q^{r-1}u^2)^{-1}(1-q^{n-r-1}u^2)^{-1}(1-q^{n-2}u^2)^{-1}
\times (1-\epsilon q^{r/2-1} + q^{n-r-2}u^2 - \epsilon q^{n/2-1} - q^{n/2+2}u^2 + \epsilon q^{n/2-1} + q^{n/2+2}u^2).
\]

Using this notation, we can state our result for even \( n \) as follows.

**THEOREM 3.4.** Let \( n \) be even, and let \( d \in O^\times \).

1. If \( r \) is odd, then

\[
Z(u, \omega, d) = \frac{u^{\epsilon}}{|O^\times : O^{\times 2}|} (q^{-1})_{a}(q^{-2})_{[n/2]}(q^{-2})_{[r/2]}^{-1}(q^{-2})_{[a/2]}^{-1}(1-\epsilon q^{-n/2})
\]

\[
\times \begin{cases} q^{n/2-1}u & \text{if } \epsilon = o, \\ (1-\epsilon q^{-r/2})^{-1}(1-\epsilon q^{-r/2-2} u^2) & \text{if } \epsilon = e. \end{cases}
\]

2. If \( r \) is even, then

\[
Z(u, e, d) = \frac{u^{\epsilon}}{|O^\times : O^{\times 2}|} (q^{-1})_{a}(q^{-2})_{[n/2]}(q^{-2})_{[r/2]}^{-1}(q^{-2})_{[a/2]}^{-1}(1-\epsilon q^{-n/2})
\]

\[
\times \begin{cases} ((-1)^{r/2}, \varnothing)\epsilon q^{(n-2)/2}u & \text{if } \epsilon = o, \\ (1-\epsilon q^{-r/2})^{-1}(1-\epsilon q^{-r/2-1} u^2) & \text{if } \epsilon = e. \end{cases}
\]
\[
\left\{ \begin{array}{ll}
(1-q^{n/2-1}u)^{-1}(1-\epsilon_q u)^{-1} \\
\times (q^{n-3}u^2, q^{-2})_{r/2}(q^{-1}u^2, q^{-2})_{r/2}^{-1} & \text{if } \omega = 1, \\
(1-q^{n-r-1/2}u)^{-1}(1-\epsilon_q q^{r-1/2}u)^{-1} \\
\times (q^{-2}u^2, q^{-2})_{r/2}(q^{-1}u^2, q^{-2})_{r/2}^{-1} & \text{if } \omega = e.
\end{array} \right.
\]

(2) If \( r \) is even, set
\[
\epsilon = \begin{cases} 
((-1)^{r/2}d, \omega) & \text{if } (-1)^{r/2}d \in U_{2e}O^{r/2}, \text{ and } * = e \\
0 & \text{otherwise}.
\end{cases}
\]

Then
\[
Z_{n,k}(u, 1, d) = \frac{q^n}{|O^* : O^{r/2}|} (q^{-1})_{n/2-1}(q^{-1})_{(n-r)/2-1}(q^{-1})_{r/2-1} \\
\times (1-\epsilon_q q^{n/2})(1-\epsilon_q q^{r/2})^{-1} \\
\times (q^{n-1}u^2, q^{-2})_{r/2-1}(q^{-1}u^2, q^{-2})_{r/2-1} \\
\times \left\{ \begin{array}{ll}
(q^{r/2-1}u)^{2e+1} & \text{if } * = 0, \\
(q^{r/2-1}u)^{2} & \text{if } * = e,
\end{array} \right.
\]

\[
Z_{d}(u, e, d) = \frac{1+\delta((-1)^{r/2}d))(-1)^{r/2}d, \omega)^{*}q^*}{2|O^* : O^{r/2}|} \\
\times (q^{-1})_{n/2-1}(q^{-1})_{(n-r)/2-1}(q^{-1})_{r/2-1} \\
\times (1-\epsilon_q q^{n/2})(1-\epsilon_q q^{r/2})^{-1} \\
\times (q^{-1}u^2, q^{-2})_{r/2-1}(q^{-1}u^2, q^{-2})_{r/2-1}.
\]

For the proof, we follow Igusa [12], [13]. Let \( i_X \) be the morphism of \( X \) to \( Aff^m \) for \( m = \binom{n}{r} \) such that the coordinates of \( i_X(x) \) are the \( r \)-minors of \( x \) arranged in a fixed way. As in [13], we set
\[
Y = i_X(X), \quad X' = X - i_X^{-1}(0), \quad Y' = i_X(X'), \quad U = Y(O).
\]

Then \( U \) is open subset of \( Y(O) \) and for \( y \in Y(O) \), \( y \in U \) if and only if one of its coordinates belongs to \( O^* \). Set \( f(x) = \det(xhx) \). Then \( f(x) \) depends only on \( y = i_X(x) \) and defines a quadratic form \( f_0(y) \) on \( Y \) such that \( f(x) = f_0(y) \) for \( y = i_X(x) \). On \( Y''(F) = Y'(F) - \{ f_0(y) = 0 \} \), we define a function \( \phi \) associated to \( \omega \) by
\[
\phi(y) = \omega(xhx)
\]
for \( i_X(x) = y \). Then it is easy to see \( \phi \) defines a continuous function on \( Y''(F) \). For \( a \in F^* \), take \( g \in GL_a(F) \) such that \( \det g = a \). Then \( i_X^{-1}(ai_X(x)) = xgSL_a(F) \), and for \( x' \in i_X^{-1}(ai_X(x)) \), \( \sigma(x'hx') = \sigma(xhx) \) and \( \sigma(x'hx') = \sigma(xhx) \). This implies \( \phi(ay) = \phi(y) \).

Applying Lemma 8 of [12] to \( \phi(y) / f_0(y)^p \), we obtain
\[ Z_n(q^{-\kappa} u, \omega, d) = (q^{-2}, q^{-1})_{r-1} \sum_{j_1, \ldots, j_r \geq 0} \prod_{1 \leq k \leq r} q^{-(n-k+1)j_k} \times \int_U \phi(m^{j_1+\cdots+j_r}) f_0(m^{j_1+\cdots+j_r}) \, dy \]

\[ = (q^{-2}, q^{-1})_{r-1} (q^{-n-r+1} u^2, q^{-1})_{r-1} \int_U \phi(y) f_0(y) \, dy. \]

Here \( dy \) is the measure of \( U \) invariant under the action of \( GL_n(O) \) through the exterior tensor representation of degree \( r \) normalized as

\[ vol(U) = (q^{-2}, q^{-1})_{r-1} (q^{-n-r+1}, q^{-1})_r. \]

To compute the integral in (3.1), we determine the orbits \( SO(Q) \backslash Y'(\bar{F}) \). To describe them, we introduce some notation. For matrices \( N_i \in M_n \), we denote the matrix

\[
\begin{pmatrix}
N_1 & 0 & \cdots & 0 \\
0 & N_2 & \cdots & 0 \\
\cdots & \cdots & \cdots \\
0 & 0 & \cdots & N_m
\end{pmatrix}
\]

by \( N_1 \oplus N_2 \oplus \cdots \oplus N_m \) and by \( mN_0 \) if \( N_1 = N_2 = \cdots = N_m = N_0 \). Let \( N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Let \( xu_1^2 + \beta v_1 v_2 + \gamma v_2^2 \) be a quadratic form determined by the norm on the maximal order of the unramified quadratic extension of \( F \), and set

\[ T = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}. \]

Let \( k \) be an integer such that \( 0 \leq k \leq r \), and let \( a \in O \). When \( k \) is odd, we define

\[ d_{k,a} = ((k-1)/2)N \oplus a \oplus O_{r-k}, \quad d_k = ((k-1)/2)N \oplus O_{r-k+1} \in M_\infty(F), \]

and

\[ \xi_{k,a} = \begin{pmatrix} 1_r \\ d_{k,a} \\ O_{n-2r,r} \end{pmatrix}, \quad \xi'_{k,a} = \begin{pmatrix} 1_{r-1} \oplus 0 \\ 0 \\ d_k \\ 0 \\ 0 \\ a \end{pmatrix} \in M_\infty(F). \]

When \( k \) is even, define

\[ d_0 = O_r, \quad d_k^+ = (k/2)N \oplus O_{r-k}, \quad d_k^- = (k/2-1)N \oplus T \oplus O_{r-k-2}, \]

\[ d_k^\times = (k/2+1)N \oplus T \oplus O_{r-k+2}. \]
\[ d_{r, a}^+ = (r/2 - 1)N \oplus aN, \]
\[ d_{r, a}^- = (r/2 - 1)N \oplus aT, \]
and define \( \xi_k^+, \xi_k^- \), etc. in the similar way as \( \xi_{k, a} \). The latter two elements are defined only when \( r \) is even. We set \( \eta_{k, a} = i_{\xi}(\xi_{k, a}) \), \( \eta_k^+ = i_{\xi}(\xi_k^+) \) etc. Let \( j_{\xi}(x) = \chi_{hx} \), and set
\[ \mathcal{D}(i_{\xi}(\xi)) = (-1)^{(k-1)/2} d(j_{\xi}(\xi)) \]
for \( \xi \) as above. Then \( \mathcal{D}(\eta_{k, a}) = 2a \), if \( k \) is odd, and
\[ \mathcal{D}(\eta_0) = \mathcal{D}(\eta_0^+) = \mathcal{D}(\eta_{r, a}^+) = 1, \quad \mathcal{D}(\eta_k^-) = \mathcal{D}(\eta_{r, a}^-) = \alpha_0 \]
with \( \alpha_0 \in A_{e+1} \), if \( k \) is even. Set \( \epsilon(\eta) = \epsilon(j_{\xi}(\xi)) \) for \( \eta = i_{\xi}(\xi) \). For \( \eta(\eta \neq \eta_{k, a}) \), we note \( \epsilon(\eta) = 1 \) if \( p \) is odd, and if \( p = 2 \), then \( \mathcal{D}(\eta) \in U_{2e} O^{*2} \) and
\[ \epsilon(\eta) = \begin{cases} 
( -1, -1)^{(k+1)/2} (2, \mathcal{D}(\eta)) & \text{if } k \text{ is even}, \\
( -1, -1)^{(k-1)/2} (-1)^{(k+1)/2} (2a) & \text{if } k \text{ is odd}.
\end{cases} \]

The following is Lemma 9 in [13] if \( p \) is odd.

**Lemma 3.5.** A complete set of representatives of \( O(Q)(F) \backslash Y(F) \) is given by
\[ \bar{n}_0 \cup \bigcup_{1 \leq k \leq r-1} \left( \bigcup_{a \in \bar{F}^+} \bar{n}_{k, a} \bigcup_{a \in \bar{F}^-} \bar{n}_{k, a}^{-1} \right) \cup \bigcup_{r \text{ even}} \left( \bigcup_{1 \leq k \leq r} \{ \bar{n}_k^+, \bar{n}_k^- \} \right) \]
\[ \cup \bigcup_{r \text{ odd}} \left( \bigcup_{1 \leq k \leq r} \{ \bar{n}_k^+, \bar{n}_k^- \} \right) \]
Let
\[ C(n, r, k) = q^{r(n-r) + 1 - (r-k)(r-k+1)/2} \left( q^{-1} \right)_{n/2} \left( q^{-1} \right)_{k/2} \]
\[ \times \left( q^{-1} \right)^{k-r} \left( q^{-2} \right)_{(n+2r+k)/2} \left( q^{-2} \right)_{k/2}. \]
and for \( \eta \) as above let
\[ A(n, r, k, \eta) = |O(Q)(F)\eta| / C(n, r, k). \]
Then for \( k, 0 \leq k < r \), \( A(n, r, k, \eta) \) is given by
\[ \begin{cases} 
\frac{1}{2} (1 + \epsilon q(D(\eta), \varpi) q^{- (n-2r+k)/2}) & \text{if } p \text{ is odd and } \eta = \eta_{k, a}, \\
1 - q^{- (n-2r+k)} & \text{if } p = 2 \text{ and } \eta = \eta_{k, a}, \\
q^{- (n-2r+k)} & \text{if } p = 2 \text{ and } \eta = \eta_{k, a},
\end{cases} \]
if \( n \) is odd and \( k \) is odd,
\[ \frac{1}{2} (1 + (D(\eta), \varpi) q^{-k/2}), \]
if \( n \) is odd, and \( k \) is even,
\[ \frac{1}{2} (1 - \epsilon q^{-n/2}) (1 - q^{-n})^{-1} \times \begin{cases} 1 & \text{if } p \text{ is odd}, \\
2 & \text{if } p = 2,
\end{cases} \]
if $n$ is even and $k$ is odd, and
\[
\frac{1}{2} (1 - \epsilon_0 q^{-n/2})(1 + (D(\eta), \varpi) q^{-k/2})(1 + \epsilon_0(D(\eta), \varpi) q^{-(n-2r+k)/2})
\]
if $n$ is even and $k$ is even. When $k = 0$, we understand $(D(\eta), \varpi) = 1$. For $k = r$, set $k = r$ in the above formula, and multiply
\[
\begin{cases}
2(q-1)^{-1} & \text{if } p \neq 2, \\
(q-1)^{-1} & \text{if } p = 2.
\end{cases}
\]

Proof. For $x \in X(\bar{F})$, let $Q_x$ be the quadratic form in $r$ variables defined by
\[Q_x(v) = \overline{Q(xv)},\]
for $v = (v_1, v_2, \ldots, v_r)$. If $x = (1, r, d, 0)$, and $d = (d_{ij}) \in M_r(F)$ is an upper triangular matrix, then
\[Q_x(v) = \sum_{i < j} d_{ij} v_i v_j.
\]
Using $Q_x$ instead of $(x^T x)/2$, we can proceed exactly in the same way as in [12] except in the case where $p = 2$, $n$ is odd, and $k$ is odd. In this case, let $v_0$ be a vector which is orthogonal to $\overline{F}^n$ with respect to the bilinear form $B$ defined by
\[B(v, w) = \overline{Q(v+w)} - \overline{Q(v)} - \overline{Q(w)}.
\]
Then $v_0$ is determined up to a non-zero constant. To apply the Witt’s theorem, we have to consider the cases $v_0 \in \langle x \rangle$ and $v_0 \notin \langle x \rangle$ separately, where $\langle x \rangle$ is the $\overline{F}$ vector space spanned by the column vectors of $x$. For a fixed $Q_0$, $x \in M_r(\overline{F})$ such that $Q_x = Q_0$ divides into two orbits with respect to $O(\overline{Q})$ according to whether $v_0 \in \langle x \rangle$ or $v_0 \notin \langle x \rangle$. The first case gives $\eta_{k,1}'$, and the second one gives $\eta_{k,1}$. The rest is the same as in the other cases.

The next lemma follows from the definition of the Hasse invariant.

**Lemma 3.6.** Let $w \in S_1(F, \overline{F}^r)$, $z \in S_m(F, \overline{F}^r)$. Then
\[\sigma(w \oplus wz) = \sigma(w)\sigma(z)(d(w), d(z))((\epsilon + 1)/2) d(w)^m d(z)^{m+1}, \varpi\]
\[\tilde{\sigma}(w \oplus wz) = \overline{\sigma(w)}\overline{\sigma(z)}(d(w), d(z))((1 + (m + 2)/2) d(w)^m d(z)^{m+1}, \varpi)
\times ((1 + (m + 2)/2) d(w)^m d(z)^{m+1}, \varpi).
\]

Let $\eta$ be one of the elements in Lemma 3.5. and let $U_\eta$ be the set of all $y \in U$ such that
\[y \equiv \eta \mod \varpi.
\]
In the following lemma, we denote by $Z_\eta(u, \omega, d)$, $Z_\eta(u, \omega, d)_k$ the zeta functions in §2. Then we have
LEMMA 3.7. Assume \( p \neq 2 \), when \( n \) is odd. Let \( \eta \) be an element in Lemma 3.5. Set \( k' = k - 1, \eta' = \eta_+^k \) if \( p = 2 \) and \( k \) is odd, and \( k' = k, \eta' = \eta \) otherwise. Then

\[
C(n, r, k) \int_{U_n} \phi(y)|f_0(y)|^qdy = (q^{-1})_1(q^{-2})_{\frac{q}{l_2}}q^{-(r-k)(r-k+1)/2}u^{r-1}(q^{-2})_{\frac{(r-2r+k+1)/2}{k_2}}(q^{-2})_{\frac{(r-2r+k+1)/2}{k_2}}
\]

\[
\times \epsilon(\omega, \eta)(q^{-1})_{r-k}^{-1}
\]

\[
\times \begin{cases}
Z_{r-k}^+u^{q^{-1}(r-k+1)/2}u, \omega', dd(\eta')) & \text{if } p \text{ is odd,} \\
q^{(r-k)(r-k+1)/2}u, \omega', dd(\eta')) & \text{if } p = 2, \text{ and } k \text{ is even,} \\
q^{(r-k)(r-k+1)/2}(1 - q^{-(r-k+1)/2})^{-1}u & \text{if } p = 2 \text{ and } k \text{ is odd.}
\end{cases}
\]

Here \( \epsilon(\omega, \eta) \) and \( \omega' \) are given by

\[
\epsilon(\omega, \eta) = \begin{cases}
1 & \text{if } \omega = 1, \\
\epsilon(\eta')((r-k)(r-k+1)/2) & \text{if } \omega = \epsilon, \\
\epsilon(\eta)((r-k)(r-k+1)/2, d(\eta')) & \text{if } \omega = \tilde{\epsilon},
\end{cases}
\]

\[
\omega'(z) = \begin{cases}
1 & \text{if } \omega = 1, \\
\epsilon(z)(d(\eta'), d(z)(\omega), d(z)) & \text{if } \omega = \epsilon, \\
\epsilon(z)((r-k)(r-k+1)/2) & \text{if } \omega = \tilde{\epsilon}.
\end{cases}
\]

For \( k = r \), we understand

\[
Z_{\theta}^+(u, 1, d) = Z_{\theta}^+(u, 1, d)_c = 1,
\]

and

\[
Z_{\theta}(u, \omega, d) = \frac{1}{2} (1 + (d, \omega)), \quad Z_{\theta}(u, \omega', d)_c = \frac{(1 + \delta(d))q^e}{2|O^\times : O^\times 2|} (1 + (d, \omega)),
\]

for \( \omega \neq 1 \). When \( p = 2, k = r \) and \( r \) is odd, the above integral should be replaced by

\[
q^{-1}(1 - q^{-1})^{-1} \sum_{a \in \mathbb{F}} \int_{U_{n, a}} \phi(y)|f_0(y)|^qdy.
\]

Proof: We can prove our assertion in the same way as in [13] using Prop.1.2. and Lemma 3.6 except in the case where \( p = 2 \) and \( n \) is even. We show this case.

As in [13], we take \( z \in M_{n-r, \ell}, g_1 \in GL_r \), and put

\[
x = \begin{pmatrix} 1 & \epsilon \\ \tilde{z} & 1 \end{pmatrix}, \quad i_x(x) = y, \quad \det g_1 = 1.
\]

Then \( t \) and \( z \) form local coordinates of \( Y \) around \( y \), and we write \( y = (t, z) \). For \( \zeta \in U_n \), let \( \zeta = \omega(t^*, z^*) \). Then the map \( \zeta \to (t^*, z^*) \) induces an \( K \)-analytic isomorphism of
$U_\eta$ to $O \times M_{n-r,r}(O)$. Under this isomorphism, the measure $dy$ corresponds to the measure $dt^* \times dz^*$ such that

$$\int_0^t dt^* = \int_{M_{n-r,r}(O)} dz^* = 1.$$ 

The Lie algebra of $SO(O)$ consists of

$$\begin{pmatrix} a_{11} & a_{12} & -'a_{32}h' \\ a_{21} & -a_{11} & -'a_{31}h' \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$ 

Here

$$'a_{12} = -a_{12}, \quad 'a_{21} = -a_{21}, \quad a_{11} \in M_r,$$

$$a_{31}, a_{32} \in M_{n-2r}, \quad 'a_{33}h' + h'a_{33} = 0, \quad a_{33} \in M_{n-2r}.$$ 

For $d \in M_r$, we see

$$(1_r + \varpi a) \begin{pmatrix} 1_r \\ d \\ O_{n-2r,r} \end{pmatrix} = \begin{pmatrix} 1_r \\ d + \varpi(a_{21} - da_{11} - 'a_{32}d - da_{12}d) \\ \varpi(a_{31} + a_{32}d) \end{pmatrix} (1_r + \varpi(a_{11} + a_{12}d) \text{mod } \varpi^2).$$

Let $T_m$ be the space of upper triangular matrices in $M_m$.

First let $k$ be even. Then we can take $0 \times (O_k \oplus T_{r-k}(O))$ as the complement $N$ in Prop.1.3. Then we can prove our assertion for $r > k$ in the same way as in [13]. The assertion for $r = k$ follows from Remark 2.6.

Let $k$ be odd. For $a = (a_1, a_2, \ldots, s_{r-1}) \in F^{r-1}$ other than 0, set

$$d_a = ((k-1)/2)N \oplus a_1 \oplus \cdots \oplus a_{r-1}, \quad \xi_a = \begin{pmatrix} 1_r \\ d_a \\ O_{n-2r,r} \end{pmatrix}, \quad \eta_a = i(x(\xi_a)).$$

Then $\eta_{k,1}$ and $\eta_a$ lie in the same orbit under $SO(F)$. Hence we have

$$\int_{U_{n_a}} \phi(y) | f_0(y) \|^a dy = \int_{U_{n_a}} \phi(y) | f_0(y) \|^a dy$$

$$= q^{-r-1} \sum_{a \in F^{r-1}} \int_{U_{n_a}} \phi(y) | f_0(y) \|^a dy.$$ 

As the complement $N$ in Prop.1.3, for each $a$, we can take $0 \times (O_{k-1} \oplus T_{r-1}(O))$. Since the union of
for all \( a \in F^{r-k+1} - \{0\} \) and \( m \in T_{r-k+1}(O) \) is equal to
\[
\mathfrak{w}(S_{r-k+1}(O)_{c-1} - S_{r-k+1}(O)_{c}).
\]
Our assertion for \( k < r \) follows from this. The same calculation gives the proof for \( k = r \).

We set
\[
I_{r-k} = C(n, r, k) \sum_{\eta} A(n, r, k, \eta) \int_{U_\eta} \phi(y) |f_\eta(y)|^s dy,
\]
where \( \eta \) runs through all representatives for \( k \) in Lemma 3.5. Now we begin the proof of our theorems.

**Proof of Th.3.2.** We give a proof only for (1). The case of (2) is similar. Let \( r-k = 2s \), or \( 2s+1 \) with \( 0 \leq s \leq \lceil r/2 \rceil \). First let \( \omega = \eta \). Since
\[
\sum_{D(\eta) \in \mathcal{O}^*/\mathcal{O}^{*2}} A(n, r, k, \eta)((1-q^{-1}u)^{-1}(1-q^{-2s-1}u) + \epsilon(D(\eta), \omega)q^{-s})
\]
\[
= \frac{1}{2} \sum_{D(\eta) \in \mathcal{O}^*/\mathcal{O}^{*2}} (1-\epsilon_\eta(D(\eta), \omega)q^{-(n-r-2s)/2})
\]
\[
\times ((1-q^{-1}u)^{-1}(1-q^{-2s-1}u) + \epsilon(D(\eta), \omega)q^{-s})
\]
\[
= (1-q^{-1}u)^{-1}(1-q^{-2s-1}u) + \epsilon_\eta q^{-s}q^{-(n-r)/2},
\]
for \( s \neq 0 \), by Lemma 3.5, 3.7 and Th.2.2, we see \( I_{2s} \) is equal to the product of (3.4) and (3.5)
\[
\frac{1}{2} (q^{-1}t_1(q^{-2})_{h/2}(q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2}),
\]
and we can check this holds also for \( s = 0 \). In the same way, \( I_{2s+1} \) is equal to the product of (3.5) and
\[
\frac{1}{2} (1-q^{-(n-r-2s)}) \sum_{\eta} (1+\epsilon(D(\eta), \omega)q^{-(h/2)})q^{-2s-1}u(1-q^{-1}u)^{-1}
\]
\[
= (1-q^{-(n-r-2s)})(1-q^{-1}u)^{-1}q^{-2s-1}u.
\]
Hence \( I_{2s} + I_{2s+1} \) is equal to the product of (3.5) and
\[
(1-q^{-1}u)^{-1}(1-q^{-(n-r+1})u + \epsilon_\eta q^{-(n-r)/2}
\]
\[
= (1-q^{-(n-r)}(1-q^{-2s-1}u)^{-1}
\]
\[
\times ((1-\epsilon_\eta q^{-(n-r)/2})^{-1}(1-\epsilon_\eta q^{-(n-r)/2}u) + q^{-1}u).
\]
Now (3.5) is equal to
\[
\frac{1}{2} (q^{-1}t_1(q^{-2})_{h/2}(q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2} - (q^{-2})_{h^{-1}/(n-r)/2}),
\]
\[
\times \left( \begin{array}{c}
(n-r)/2 \\
 s
\end{array} \right)_{q^{-2}} (q^{-2})^{r}(q^{-1}u^{2})^{r}(q^{-2s-3}u^{2}, q^{-2})_{(r/2)-s}(q^{-2})_{r/2}^{-1}-s.
\]

Here we used the relation \((q^{-3}u^{2}, q^{-2})_{s} = (q^{-3}u^{2}, q^{-2})_{r/2}(q^{-2s-3}u^{2}, q^{-2})_{r/2}^{-1}-s\). By (2) of Lemma 1.1, we have

\[
\sum_{s=0}^{l(r/2)} \left( \begin{array}{c}
(n-r)/2 \\
 s
\end{array} \right)_{q^{-2}} (q^{-2})^{r}(q^{-1}u^{2})^{r}(q^{-2s-3}u^{2}, q^{-2})_{(r/2)-s}
\]

\[
= \frac{(q^{-2})^{-(n-r+2)}u^{2}, q^{-2})_{r/2}}{(q^{-2})_{r/2}}.
\]

Note

\[
(q^{-2})^{-(n-r+1)}u^{2}, q^{-1})_{r}^{-1}(q^{-2})^{-(n-r+2)}u^{2}, q^{-2})_{r/2}
\]

\[
= (1 - q^{-2})^{-(n-r+1)}u^{2}, q^{-2})_{r/2}^{-1}.
\]

From this, by changing \(u\) by \(q^{-u}u = q^{u/2}u\) we obtain our assertion for \(i\).

Let \(\omega = \varepsilon\). For \(r-k=2s\), we have

\[
\varepsilon(\omega, \eta) = ((-1)^{s}, \omega), \quad \omega'(z) = \varepsilon(z)(\omega d(\eta), d(z)).
\]

Since \(Z_{d}(u, \omega, d)\) is an even function and

\[
(((-1)^{s}, \omega)(d(\eta))d, \omega) = \varepsilon(D(\eta), \omega),
\]

by Th.2.2, and Lemma 3.5, 3.7, \(I_{2s}\) is equal to the product of

\[
\frac{1}{2} \left( \begin{array}{c}
(n-r)/2 \\
 s
\end{array} \right)_{q^{-2}} (q^{-2})^{r}(q^{-2})_{(n-r)/2}(q^{-2})_{r/2}^{-1}
\]

\[
\times \left( \begin{array}{c}
(n-r)/2 \\
 s
\end{array} \right)_{q^{-2}} q^{-2s-3}u^{2}, q^{-2})_{(r/2)-s}(q^{-2})_{r/2}^{-1}-s
\]

and

\[
(\varepsilon\varepsilon^{-1}q^{-(n-r)/2-s} + q^{-s}).
\]

For \(r-k=2s+1\), we see

\[
\varepsilon(\omega, \eta) = ((-1)^{s+1}d(\eta), \omega),
\]

\[
\omega'(z) = \varepsilon(z)(d(\eta), d(z))
\]

\[
= \tilde{\varepsilon}(z)((-1)^{s+1}d(\eta), d(z)).
\]

Hence to obtain \(I_{2s+1}\), we need to replace \(u\) by \(((1)^{s+1}d(\eta), \omega)u\) in \(Z_{d}(u, \tilde{\varepsilon}, d(\eta))\) and multiply \(((1)^{s+1}d(\eta), \omega)\) to it. Noticing

\[
\frac{1}{2} \sum_{D(\eta) = O^{2}, 0} (1 + (D(\eta), \omega)q^{-k/2})((-1)^{s+1}d(\eta), \omega) + q^{-s-1}u)
\]

\[
= ((-1)^{r+1/2}, \omega)q^{-(r-2s-1)/2} + q^{-s-1}u,
\]
we find $I_{2s+1}$ is equal to the product of (3.6) and
\[
(1 - q^{-(2s+2)}u^2)^{-1}(1 - q^{-(n-r-2s)}q^{2s+1}u)
\times\left((-1)^{(r+1)/2}, \omega\right)q^{-(r-2s-1)/2} + q^{-s-1}u).
\]
Hence $I_{2s} + I_{2s+1}$ is equal to the product of (3.6) and
\[
(3.7)\quad ((-1)^{(r+1)/2}, \omega)q^{-(r+1)/2}u(q^{-s}(1 - q^{-2s-2}u^2) - (1 - q^{-(n-r+2)}u^2)q^{-(r-s)}
+ q^{-s}(1 - q^{-2s-2}u^2)^{-1}(1 - q^{-(n-r+2)}u^2) + \epsilon q^{-(r-s)/2})q^{-(r-s)}.
\]
By Lemma 1.1, we see
\[
\sum_s q^{-s} \left\lfloor \left(\frac{n-r}{2}\right) \right\rfloor q^{-s}(2s+1)u^{2s}(q^{-2s-2}u^2, q^{-2})_{[r/2]-s}(q^{-2})_{[r/2]-s}^{-1}
= (q^{-2})_{[r/2]}^{-1}q^{-2} \left\lfloor \left(\frac{n-r}{2}\right) \right\rfloor_{[r/2]}
\]
and
\[
\sum_s q^{-s}(1 - q^{-2s-2}u^2)^{-1} \left\lfloor \left(\frac{n-r}{2}\right) \right\rfloor q^{-s}(2s+1)u^{2s}
\times (q^{-2s-2}u^2, q^{-2})_{[r/2]-s}(q^{-2})_{[r/2]-s}^{-1}
= (1 - q^{-(r+1)}u^2)^{-1} \sum_s \left\lfloor \left(\frac{n-r}{2}\right) \right\rfloor q^{-s}(2s+1)u^{2s}
\times (q^{-2}u^2)(q^{-2s-2}u^2, q^{-2})_{[r/2]-s}(q^{-2})_{[r/2]-s}^{-1}
= (1 - q^{-(r+1)}u^2)^{-1}(q^{-2})_{[r/2]}^{-1}q^{-2} \left\lfloor \left(\frac{n-r}{2}\right) \right\rfloor_{[r/2]}
\]
Substituting these into (3.7), we obtain our assertion.

Proof of Th. 3.4. (1) Let $r$ be odd. First let $\omega = 1$. Set $r = 2s$, or $2s + 1$ with $0 \leq s \leq [r/2]$. Then by Lemma 3.5, 3.7, Th.2.2 and Cor.2.4, we see $I_{2s} + I_{2s+1}$ is equal to the product of
\[
\frac{u^e}{|O^s : O^{s+2}|} (1-q^{-1})(q^{-2})_{[r/2]-1}^{-1}(1-\epsilon q^{-n/2})(1-q^{-1}u)^{-1}(1+\epsilon q^{-n/2})u
\times (q^{-1})_{[r-n/2]}^{-1}(q^{-3}u^2, q^{-2})_{[r/2]}^{-1}
\]
and
\[
\sum_{s=0}^{[r/2]} \left\lfloor \left(\frac{n-r}{2}\right) \right\rfloor q^{-2s}(q^{-1}u^2)(q^{-2s-2}u^2, q^{-2})_{[r/2]-s}(q^{-2})_{[r/2]-s}^{-1}
= (q^{-2})_{[r/2]}^{-1}q^{-2} \left\lfloor \left(\frac{n-r}{2}\right) \right\rfloor_{[r/2]}
\]
In the same way as in the case of $n$ odd, we obtain our result.

Let $\omega = \hat{\epsilon}$. First let $p \neq 2$. For $r = 2s$, by Lemma 3.7, we find
\[
\epsilon(\omega, \eta) = ((-1)^{(r-k)/2}, \omega), \quad \omega'(z) = \hat{\epsilon}(z)(d(z), (1)^{(r+2)/2})d(\eta)\omega.
\]
Hence $I_{2s}$ is equal to the product of $q^{-s}$ and

$$
\frac{1}{2} \left( q^{-1} \right)_{[n/2]}^{-1} \left( q^{-2} \right)_{[n-r/2]}^{-1} \left( q^{-2} \right)_{[r/2]}^{-1} (1 - \epsilon qq^{-n/2})
$$

\[ (3.8) \]

$$
\times \left( \frac{[(n-r)/2]}{s} \right)_{q^{-2}} q^{-2s^2 - s} u^2 (q^{-2s^2 - 2u^2}, q^{-2})_{[r/2]}^{-1} (q^{-2})_{[r/2]}^{-1}.
$$

For $r-k=2s+1$, we find

$$
\epsilon(\omega, \eta) = (D(\eta), \varpi), \quad \delta(z) = \delta(z)(D(\eta), d(z)).
$$

Hence to obtain the integral in $I_{2s+1}$, in $Z_{2s+1}((q^{-s+1})u, \delta, d(\eta)d)$, we have to replace $u$ by $(D(\eta), \varpi)u$ and multiply $(D(\eta), \varpi)$ to it. We note

$$
\frac{1}{2} \sum_{D(\eta) \in O^* \times O^*} \left( 1 + (D(\eta), \varpi) \right) q^{-(r-2s-1)/2}
$$

\[ \times \left( 1 + \epsilon q(D(\eta), \varpi) \right) q^{-(n-r-2s-1)/2} \right) (D(\eta), \varpi) + q^{-s-1} u
$$

$$
= q^{-s-1} u \left( 1 + \epsilon qq^{-n-2s+2u^2} \right) + q^{-(r-2s-1)/2} + \epsilon qq^{-(n-r-2s-1/2)}.
$$

Hence $I_{2s+1}$ is equal to the product of (3.8) and

$$
\left( 1 - q^{-2s^2 - 2u^2} \right)^{-1} \left( q^{-2} \right)^{1} (1 + \epsilon qq^{-n-2s+2u^2}) + q^{-2s^2 - 2u^2} (1 + \epsilon qq^{-n-2s+2u^2})
$$

$$
+ q^{-2s^2 - 2u^2} (1 + \epsilon qq^{-n-2s+2u^2}) = q^{-s} \left( 1 - q^{-2s^2 - 2u^2} \right)^{-1} (1 + \epsilon qq^{-n-2s+2u^2}).
$$

Now we see

$$
(1 - q^{-2s^2 - 2u^2})^{-1} \sum_{s} \left( \frac{[(n-r)/2]}{s} \right)_{q^{-2}} q^{-2s^2 - 2s} u^2 q^{-2s^2 - 2u^2} q^{-2})_{[r/2]}^{-1} (q^{-2})_{[r/2]}^{-1}
$$

$$
= (1 - q^{-(r+1)u^2})^{-1} \sum_{s} \left( \frac{[(n-r)/2]}{s} \right)_{q^{-2}} (q^{-2})^{s} (q^{-2}u^2)^{s}
$$

$$
\times (q^{-2s^2 - 2u^2}, q^{-2})_{[r/2]}^{-1} (q^{-2})_{[r/2]}^{-1}.
$$

Our assertion follows from this.

Let $p=2$. For $r-k=2s$, $k$ is odd. Set $k'=k-1$. Then we have

$$
\epsilon(\omega, \eta) = \delta(k'/2)H((-1)^{(r+1)/2}, (-1)^{(k'/2)}), \quad \omega'(z) = \delta(z).
$$

From this we see $I_{2s}$ is equal to that in the case of $p$ odd multiplied by

$$
(3.9) \quad (-1, -1)^{(r^2-1)/8} \frac{u^e}{|O^* : O^2|}.
$$

If $r-k=2s+1$, then $k$ is even. In this case, we have

$$
\epsilon(\omega, \eta) = \delta(k/2)H(\varpi^{-1}, D(\eta))((-1)^{(r+1)/2}, (-1)^{(k/2)}), \quad \omega'(z) = \delta(z)(D(\eta), d(z)).
$$
If we replace \( u \) by \((D(\eta), \varpi)u\), then the term \((1+u)u^e\) in \(Z_{2s+1}(q^{-(2s+1)/2}u, \xi, d)\) changes to \((D(\eta), \varpi)u^{e+1}(D(\eta), \varpi) + u)u^e\). Taking summation of \(D(\eta)\) over \(U_{2e}/U_{2e+1}\), \(I_{2s+1}\) is equal to that in the case of \( p \) odd up to the factor \((3.9)\). From this we obtain our result in the case \( p = 2 \).

(2) Let \( r \) be even. First let \( p \not= 2 \). Then \( r-k = 2s, 2s+1 \) for \( s \), \( 0 \leq s \leq r/2 - 1 \) or \( r-k = 2s \) for \( s=r/2 \). Let \( \omega = i \). Then for \( s \), \( 0 \leq s \leq r/2 - 1 \), we find \( I_{2s} + I_{2s+1} \) is equal to the product

\[
\frac{1}{2} (q^{-1})_{(n/2) - 1}(q^{-2})_{(n-r)/2 - 1}(1 - \epsilon q^{-n/2})
\times (1 - q^{-2}u^2)^{-1}(q^{-3}u^2, q^{-2})_{r/2 - 1}
\]

(3.10)

and

\[
\left[ (n-r)/2 \right]_s (q^{-2})_{(n-r)/2}(q^{-2}u^2)'(q^{-2}u^2, q^{-2})_{r/2 - 1}(q^{-2})_{r/2 - s}
\times (1 - (\epsilon q^{-n/2} - q^{-r} - q^{-r})u^2)
+ \epsilon (1 - q^{-2}u^2)(q^{-r/2} + \epsilon q^{-r+n/2} + q^{-n/2}u^2)
+ q^{-1}u((1 - \epsilon q^{-n/2} - q^{-r} - q^{-r})u^2) + q^{-2}u)^2(q^{-n/2} + \epsilon q^{-r+n/2}).
\]

We can check this for \( s=r/2 \) is equal to \( I_{2s} \). Let us consider the odd part. By (1) and (2) of Lemma 1.1, we see the sum of the odd terms of \( I_{2s} + I_{2s+1} \) over \( s, 0 \leq s < r/2 \) and \( I_r \) is equal to the product of (3.10) and

\[
q^{-1}u((1 - \epsilon q^{-n/2} - q^{-r} - q^{-r+n/2})(q^{-n/2} - 3u^2, q^{-2})_{r/2}(q^{-2})_{r/2 - 1}
+ (q^n + \epsilon q^{-n/2})(q^{-n/2} - 3u^2, q^{-2})_{r/2}(q^{-2})_{r/2 - 1}
+ q^{-1}u^2(q^{-n/2} - 3u^2, q^{-2})_{r/2 - 1}(q^{-2})_{r/2 - 1})\]

\[
=q^{-1}u((q^{-2})_{r/2}(q^{-n/2} - 3u^2, q^{-2})_{r/2 - 1}
\times ((1 - \epsilon q^{-n/2} - q^{-r} - q^{-n/2})(1 - q^{-n/2}u^2)
+ (q^n + \epsilon q^{-n/2})(1 - q^{-n/2}u^2) + (1 - q^{-r})q^{-1}u^2))
\]

\[
=q^{-1}u((q^{-2})_{r/2}(q^{-n/2} - 3u^2, q^{-2})_{r/2 - 1}(1 - q^{-r})(1 - q^{-n/2}(1 + \epsilon q^{-n/2} - 1)u^2).
\]

We find that when \( u \) is replaced by \( q^n/2u \) the last factor of the last formula is equal to

(3.11) \((D_1 \otimes D_2)(u, d, 0)(1-u^2)(1-q^{-1}u^2)(1-q^{n-r}u^2)(1-q^{n-2}u^2)\).

From

\[
(1 - q^{-n/2}u^2)^{-1}(1 - u^2)(1 - q^{-1}u^2)(1 - q^{n-r}u^2)(1 - q^{n-2}u^2)
\times (q^{n-3}u^2, q^{-2})_{r/2 - 1}(q^{r-3}u^2, q^{-2})_{r/2 - 1}(q^{-1}u^2, q^{-1})_{r/2 - 1}
\]

\[
=(q^{n-1}u^2, q^{-2})_{r/2 - 1}(q^{-r}u^2, q^{-2})_{r/2 - 1}
\]

by some calculations, we obtain our assertion for \(* = 0\). In the same way, we can
treat the case of \( * = e \). We omit the details.

Let \( p = 2 \). For \( m \) even, we note the following. The zeta function \( q^mZ_{m,d}(u, t, d) \) is equal to \((q^{m-1}/2)u^{r/2}Z_{m,d}(u, t, d)\) in the case of \( p \) odd as rational functions in \( u, q \) by Cor.2.3. If \((-1)^{m/2}d \not\in U_{2e}O^{r/2} \), then also by Cor.2.3 \( q^mZ_{m,d}(u, t, d) \) is equal to \( Z_{m,d}(u, t, d) \) in the case of \( p \) odd as rational functions in \( u, q \). If \((-1)^{m/2}d \not\in U_{2e}O^{r/2} \), then it is equal to \((q^{1/2}u)^{1-1/r}(q^{1/2}u^{1/2})^{-1}Z_{m,d}(u, t, d)\). By Cor.2.4, we see similar results holds for \( Z_{m,d}(u, t, d) \). The calculation proceeds in the same way as in the case of \( p \) odd. This completes the proof for \( \omega = i \).

Let \( \omega = e \). First let \( p \neq 2 \). For \( r - k = 2s \) with \( s \neq r/2 \), we have

\[
\epsilon(\omega, \eta) = ((-1)^{r-k}/2, \omega), \quad \omega'(z) = \overline{\epsilon}(d(z), d(\eta)\omega).
\]

We note

\[
((-1)^{r-k}/2, \omega)(d(\eta)d, d(\eta)\omega) = (D(\eta), \omega)((-1)^{r/2}d, \omega).
\]

From this, we see \( I_{2s} \) is equal to the product of

\[
\frac{1}{2} (q^{-1})(q^{-1})(n/2)! (q^{-1})(n-r/2)! (1 - \epsilon q^{1/2})(q^{-1/2}u^{2}, q^{-1/2}u^{2})
\]

\[
\times \left( \begin{array}{c}
\frac{n-r}{2} \\
s
\end{array} \right) q^{2s^{2}-s^{2}u^{2}}(q^{-1/2}u^{2}, q^{-1/2}u^{2})
\]

\[
(1 - q^{-1/2+s} + \epsilon q^{-(n-r)/2+s} + q^{-s} + \epsilon q^{-n/2+s}).
\]

For \( r - k = 2s + 1 \), since

\[
\epsilon(\omega, \eta) = ((-1)^{r-k+1}/2, d(\eta), \omega),
\]

\[
\epsilon'(z) = \overline{\epsilon}(d(z)((-1)^{r-k+1}/2, d(\eta), d(z)),
\]

\( I_{2s+1} \) is equal to the product of (3.12) and

(3.13) \((1 - q^{-2s^{2}-2u^{2}})(1 - q^{-3s^{2}-2u^{2}})(1 - q^{-r^{2}}u^{2})(1 - q^{-(n-r)+2s}).\)

Hence \( I_{2s} + I_{2s+1} \) for \( s, 0 \leq s < r/2 \), is equal to the product of (3.12) and

\[
q^4(\epsilon q^{-r/2} + \epsilon q^{-(n-r)/2} + \epsilon q^{-n/2} + q^{-n-2u^{2}})
\]

\[
+ q^{-r}(1 - q^{-2s^{2}-2u^{2}})^{-1}(1 - q^{-(n-r)-2u^{2}})(1 - q^{r^{2}-u^{2}}).
\]

We see for \( s = r/2 \) this is equal to \( I_{2s} \). In the same way as above, we obtain our result.

Let \( p = 2 \). For \( r - k = 2s \), we have

\[
\epsilon(\omega, \eta) = \epsilon(k/2H)(2, D(\eta))((-1)^{r-k}/2, \omega), \quad \omega'(z) = \overline{\epsilon}(d(z), d(\eta)\omega, d(z)).
\]

We may assume \( D(\eta), (-1)^{r/2}d \in U_{2e} \). We note

\[
(d(\eta)\omega, d(\eta)d)((-1)^{r-k}/2, \omega) = (D(\eta), \omega)((-1)^{r/2}d, \omega)((-1)^{k/2}, (-1)^{r-k}/2),
\]

\[
\frac{1}{2} q^{2s^{2}-s^{2}u^{2}}(q^{-1/2}u^{2}, q^{-1/2}u^{2})
\]

\[
(1 - q^{-1/2+s} + \epsilon q^{-(n-r)/2+s} + q^{-s} + \epsilon q^{-n/2+s}).
\]

For \( r - k = 2s + 1 \), since

\[
\epsilon(\omega, \eta) = ((-1)^{r-k+1}/2, d(\eta), \omega),
\]

\[
\epsilon'(z) = \overline{\epsilon}(d(z)((-1)^{r-k+1}/2, d(\eta), d(z)),
\]

\( I_{2s+1} \) is equal to the product of (3.12) and

(3.13) \((1 - q^{-2s^{2}-2u^{2}})(1 - q^{-3s^{2}-2u^{2}})(1 - q^{-r^{2}}u^{2})(1 - q^{-(n-r)+2s}).\)

Hence \( I_{2s} + I_{2s+1} \) for \( s, 0 \leq s < r/2 \), is equal to the product of (3.12) and

\[
q^4(\epsilon q^{-r/2} + \epsilon q^{-(n-r)/2} + \epsilon q^{-n/2} + q^{-n-2u^{2}})
\]

\[
+ q^{-r}(1 - q^{-2s^{2}-2u^{2}})^{-1}(1 - q^{-(n-r)-2u^{2}})(1 - q^{r^{2}-u^{2}}).
\]

We see for \( s = r/2 \) this is equal to \( I_{2s} \). In the same way as above, we obtain our result.

Let \( p = 2 \). For \( r - k = 2s \), we have

\[
\epsilon(\omega, \eta) = \epsilon(k/2H)(2, D(\eta))((-1)^{r-k}/2, \omega), \quad \omega'(z) = \overline{\epsilon}(d(z), d(\eta)\omega, d(z)).
\]

We may assume \( D(\eta), (-1)^{r/2}d \in U_{2e} \). We note

\[
(d(\eta)\omega, d(\eta)d)((-1)^{r-k}/2, \omega) = (D(\eta), \omega)((-1)^{r/2}d, \omega)((-1)^{k/2}, (-1)^{r-k}/2),
\]

\[
\frac{1}{2} q^{2s^{2}-s^{2}u^{2}}(q^{-1/2}u^{2}, q^{-1/2}u^{2})
\]
and
\[((-1)^{(r-k)/2}d(\eta)d, \varpi) = (D(\eta)(-1)^{r/2}d, \varpi))\,.
\]

Taking summation of $D(\eta)$ over $U_{2e}/U_{2e+1}$, we see $I_{2e}$ can be obtained by multiplying
\[(3.14)\]
\[(-1, -1)^{n+2}/8((-1)^{r/2}d, \varpi)e^{-q^2(1+\delta((-1)^{r/2}d))}2|O^\times: O^{\times 2}|\]
to that in the case of $p$ odd. For $r-k=2s+1$, let $k'=k-1$. Then
\[\epsilon(\omega, \eta) = \epsilon((k'/2)H)((-1)^{r/2}d, \varpi), \quad \omega'(z) = \epsilon(z)((-1)^{k'/2}\varpi, d(z)).\]

We note for $d$ with $(-1)^{r/2}d \in U_{2e}$
\[((-1)^{r/2}, \varpi)((-1)^{k'/2}\varpi, (-1)^{k'/2}d) = ((-1)^{r/2}d, \varpi)((-1)^{k'/2}, (-1)^{(r-k)/2}).\]

Hence the difference between the case of $p$ odd and that of $p=2$ is given by (3.14). This proves our assertion. This completes the whole proof of Th.3.4.

§4. The case (III) (the case of the space of hermitian matrices)

Let $K$ be a quadratic extension of $F$ or $F\oplus F$. When $K$ is a quadratic extension of $F$, let $O_K$, $\sigma_K$ be the ring of integers, a prime element of $K$, and let $f_K$ the exponent of the conductor of $K/F$. For $a \in K$, we denote by $\bar{a}$ the conjugate of $a$ over $F$, and by $Na$ the norm of $a$. We assume $\sigma_K = \sigma$ if $K/F$ is unramified, and $N\sigma_K = \sigma$ if $K/F$ is ramified. Let $\chi_K$ the quadratic character of $F^\times$ corresponding to the quadratic extension $K/F$. When $K = F \oplus F$, we set
\[O_K = O \oplus O, \quad \sigma_K = (\sigma, \sigma), \quad f_K = 0,\]
and for $a = (s_1, a_2) \in K$, set $\bar{a} = (a_2, a_1)$, $Na = a_1a_2$. For a subset $R$ of $K$, we set
\[H_n(R) = \{x \in M_n(R) | x = x\}\]
and for a subset $S$ of $F$, we define
\[H_n(R, S) = \{x \in H_n(R) | \det x \in S\} .\]

When $f_K \neq 0$, for $k$, $0 \leq k \leq f_K$, we define
\[H_n(R, S)_k = \{x = (x_{ij}) \in H_n(R, S) | x_{ii} \in \sigma_K^kO\} .\]

The following is an easy consequence of the theory of hermitian forms.

**Proposition 4.1.** Let $(G, \rho, X)$ be of type (III). Then the $G(F)$ orbits of $X^{\sigma}(F)$ are given by
\[H_n(K, F^\times)\]
if $K = F \oplus F$ or $n$ is odd, and by
\[\bigcup_{d \in F^+/NK^\times} H_n(K, dNK^\times)\]
otherwise.

In this section, we calculate

\[ \int_{D} |\det x|^{s-k}dx , \]

for

\[ D = \bigcup_{t=0}^{\infty} H_{n}(O_{K}, d\sigma^{t}NO_{K}^{*}) . \]

Here \( \kappa = n \) and the measure \( dx \) on \( H_{n}(K) \) is defined so that the volume of \( H_{n}(O_{K}) \) is 1. This integral converges for a sufficiently large \( \text{Re}(s) \), and was shown to be a rational function in \( u = q^{-s} \). We denote it by \( Z_{n}(u, d) \). To include the dual lattice of \( H_{n}(O_{K}) \) into consideration, we also calculate the integral for

\[ D = \bigcup_{t=0}^{\infty} H_{n}(O_{K}, d\sigma^{t}NO_{K}^{*})_{k} , \quad 0 \leq k \leq f_{K}/2 \]

when \( f_{K} \) is even, and

\[ D = \bigcup_{t=0}^{\infty} H_{n}(\sigma O_{K}, d\sigma^{t}NO_{K}^{*})_{k} , \quad 1 \leq k \leq (f_{K}+1)/2 , \]

when \( f_{K} \) is odd. They will be denoted by \( Z_{n}(u, d)_{k} \), \( Z_{n}^{*}(u, d)_{k} \) respectively. For \( k = 0 \), \( Z_{n}(u, d) = Z_{n}(u, d)_{0} \). Our result is

**Theorem 4.2.** Let \( d \in O^{*} \).

1. Let \( K = F \oplus F \). Then

\[ Z_{n}(u, d) = (q^{-1})_{d}(u, q)_{n}^{-1} \]

2. Let \( K \) be an unramified quadratic extension of \( F \). Then

\[ Z_{n}(u, d) = (q^{-1}, q^{-2})_{n+1/2}(-q^{-2}, q^{-2})_{n/2}((-1)^{n-1}u, -q)^{-1} . \]

3. Let \( K \) be a ramified quadratic extension of \( F \). Assume \( f_{K} \) is even, or \( f_{K} \) is odd and \( k = 0 \). Then

\[ Z_{n}(u, d)_{k} = \frac{1}{2} (q^{-1}, q^{-2})_{n+1/2} \]

\[ \times \begin{cases} (u, q^{2})_{(n+1)/2}u^{k} & \text{if } n \text{ is odd}, \\ ((qu, q^{2})_{n/2}q^{-kn} + Z_{K}((-1)^{n/2}d)q^{-f_{K}n/2}(u, q^{2})_{n/2}^{-1}) & \text{if } n \text{ is even}. \end{cases} \]

If \( f_{K} \) is odd and \( k \geq 1 \), then

\[ Z_{n}(u, d)_{k} = \]
\[ Z_n^*(u, d)_k = \frac{1}{2} (q^{-1}, q^{-2})_{(n+1)/2} \]
\[ \times \left\{ \begin{array}{ll}
(u, q^{-2})_{(n+1)/2} u^{n-1/2 + k} & \text{if } n \text{ is odd}, \\
u^{n/2} (q^{-2(k-1)n/2} (qu, q^{-2})_{n/2}^{-1} + \chi_k ((-1)^{n/2} d) q^{-\zeta n/2} (u, q^{-2})_{n/2}^{-1}) & \text{if } n \text{ is even.}
\end{array} \right. \]

The assertions (1) and (2) are well-known (cf. [11] for (2)), and they are included in the theorem for the comparison of the three cases and for the convenience for calculating global zeta functions. Hence hereafter we assume \( f_k \geq 1 \). In all the cases considered in Th.4.1, we have the Jordan decomposition (cf. [S], [T]), that is, for \( x \in H_n(O_K) \), there exists \( g \in GL_n(O_K) \) such that

\[ t \hat{g} x g = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\
0 & x_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_m \end{pmatrix}, \]

where \( x_i \in \varpi_K^i GL_n(O_K) \) for a partition \( \{n_i\} \) of \( n \) and a sequence \( \{t_i\} \) associated with it. Here \( \{n_i\} \) and \( \{t_i\} \) are independent of the choice of \( g \). We call \( x \) has the Jordan decomposition of type \( \{n_i\}, \{t_i\} \). We denote by \( H_n(O_K, S, \{n_i\}, \{t_i\})_k \) the set of elements of type \( \{n_i\}, \{t_i\} \) in \( H_n(O_K, S)_k \), and we denote by \( H_n(O_K, S, \{n_i\}, \{t_i\})_\mathfrak{k}^0 \) the set of all elements in \( H_n(O_K, S)_k \) of the form (4.1). For \( d \in \mathcal{O}^* \), define

\[ \lambda_n(d, \{n_i\}, \{t_i\})_k = \int_{H_n(O_K, d \varpi^t NO_K^t, \{n_i\}, \{t_i\})_k} |\det x|^{-K} dx \]

and define \( \lambda_n^*(d, \{n_i\}, \{t_i\})_k \) for \( H_n(\varpi_K^t O_K, d \varpi^t NO_K^t, \{n_i\}, \{t_i\})_k \) similarly. Here \( 2t = \sum_{i=1}^m n_i t_i \). Then

\[ \lambda_n^*(d, \{n_i\}, \{t_i\})_k = \begin{cases} \lambda_n(d, \{n_i\}, \{t_i\})_k & \text{if } t_i \geq 1, \\
0 & \text{otherwise}, \end{cases} \]

and we have

\[ Z_n(u, d)_k = \sum_{\{n_i\}, \{t_i\}} \lambda_n(d, \{n_i\}, \{t_i\})_k \]

and a similar expression of \( Z_n^*(u, d) \) by \( \lambda_n^*(d, \{n_i\}, \{t_i\})_k \).

For non-negative integers \( t, t' \), we denote by

\[ H_n(O_K, d \varpi^t NO_K^t)^{\mathfrak{S}}_k \]

the set of all elements \( x \) in \( H_n(O_K, d \varpi^t NO_K^t) \) of the form \( \varpi_K^t g \) for \( g \in GL_n(O_K) \), and define \( H_n(\varpi_K^t O_K, d \varpi^t NO_K^t)^{\mathfrak{S}}_k \) similarly. These sets are empty unless \( nt' = 2t \).

For a positive integer \( v \), we set \( O_{K,v} = O_K/\varpi^v O_K \). We denote by \( H_n(O_K, v) \) the subset of \( H_n(O_K) \mod \varpi^v \) of \( M_n(O_K,v) \), and for a subset of \( S \) of \( O_v \), we denote by \( H_n(O_K, v, S) \) its subset consisting of all elements \( x \) satisfying \( \det x \in S \). For a sufficiently
large \nu, we can define also the sets $H_n(O_{K,\nu}, d\sigma^t N(O_{K,\nu}^t))$, $H_n(O_{K,\nu}, d\sigma^t N(O_{K,\nu}^t), \{n_i\}, \{t_i\})^0$, etc. Let

$$\lambda_n(d, t_k) = \lim_{\nu \to \infty} \frac{q^{n^2 \nu + nt_i/2}}{|GL_n(O_{K,\nu})|} \left| H_n(O_{K,\nu}, d\sigma^{nt_i/2} N(O_{K,\nu}^t))^0 \right|$$

and note

$$\frac{n}{2} \sum_{i=1}^m n_i t_i - \frac{1}{2} \sum_{i=1}^m n_i^2 t_i - \sum_{j<i} t_j n_i n_j = \tilde{Q}(\{n_i\}, \{t_i\}).$$

Then as in §2, using $\lim_{\nu \to \infty} |GL_n(O_{K,\nu})|/q^{2n^2} = (q^{-1})_n$, we can prove

$$\lambda_n(d, \{n_i\}, \{t_i\}) = (q^{-1})_n \epsilon_{\nu=1} \sum_{\nu=1}^m \epsilon_{\nu/2} q^{\tilde{Q}(\{n_i\}, \{t_i\})} \prod_{i=1}^m \lambda_n(d, t_i),$$

where $d_1, d_2, \cdots, d_m$ run through $O^* / NO_{K}^*$ satisfying $d_1 d_2 \cdots d_m \in dN O_{K}^*$.

Before proceeding to the calculation of $\lambda_n(d, \{n_i\}, \{t_i\})$, we prove a simple lemma. For $d \in O^*$, and $k, 1 \leq k \leq (f K + 1)/2$, we define

$$A(d)_{k, \nu} = q^{-\nu} \sum_{u \in O_{K,\nu}, a, b, c \in \mathfrak{m}^{k}O_{\nu}} \psi((ab - ax - d)\sigma^{-\nu}),$$

when $f K$ is even, and

$$A^*(d)_{k, \nu} = q^{-\nu} \sum_{u \in O_{K,\nu}, a, b, c \in \mathfrak{m}^{k}O_{\nu}} \psi((ab - ax - \sigma d)\sigma^{-\nu}),$$

where $f K$ is odd. Then we have

**Lemma 4.3.** The notation being as above, one has

$$A(d)_{k, \nu} = q^{3\nu - 2k(1 + \chi_k)(-d)} q^{-s f (f - 2k)},$$

$$A^*(d)_{k, \nu} = q^{3\nu - 2k(1 + \chi_k)(-d)} q^{-s f (f - 2k + 1)}.$$

**Proof.** We give a proof only for $A(d)_{k, \nu}$. The other case is similar. Considering the summation over $a$ and setting $b = \sigma b'$, we find

$$A(d)_{k, \nu} = q^{-k} \sum_{a \in O_{K,\nu}, a \in \mathfrak{m}^{k}O_{\nu}, b' \in \mathfrak{m}^{k}O_{\nu}} \psi((-ax - d)\sigma^{-\nu})$$

$$= q^{-k} \left( (q - 1) \sum_{i=0}^{v - 2k - 1} q^{v - k - i} \psi((ax + i)\sigma^{-\nu}) \right. + q^k \sum_{u \in \mathfrak{m}^{v - 1}O_{\nu}} \psi((ax + d)\sigma^{-\nu}) \right).$$

Now the sum

$$\sum_{a \in O_{K,\nu}, a \in \mathfrak{m}^{v - 1}O_{\nu}} \psi((ax + d)\sigma^{-\nu})$$
is equal to
\[ q^{|\{ x \in O^*_K \mid x \bar{x} \equiv -d \mod (\mathfrak{m}^i) \}|} = q^{2^r} \begin{cases} 1 & \text{if } l \leq f_K, \\ (1 + \chi_K(-d))q^{2^r} & \text{if } f_K \leq l. \end{cases} \]

Substituting these into (4.2), we obtain our lemma.

For \( d \in O^* \), we denote by \( H_n(O_{K,v}, dN(O^*_K))_o \) the set of all odd elements, that is, which can be diagonalized by an element in \( GL_n(O_{K,v}) \), in \( H_n(O_{K,v}, dN(O^*_K)) \). Then it is easy to see
\[ H_n(O_{K,v}, dN(O^*_K)) = H_n(O_{K,v}, dN(O^*_K))_1 \cup H_n(O_{K,v}, dN(O^*_K))_o \]
is a disjoint union, and \( H_n(O_{K,v}, dN(O^*_K))_1 \) is empty if \( n \) is odd. An element in
\[ H_n(O_{K,v}, O^*_K)_1 \]
is equivalent to a diagonal matrix with \( n/2 \) diagonal elements in \( H_2(O_{K,v}, O^*_K)_1 \). As in the case of symmetric matrices, we have
\[ |H_n(O_{K,v}, O^*_K)| = q^{-vn^2(q^n - 2)^{-1}} |GL_n(O_{K,v})| \]
and for \( n \) even and \( k \geq 1 \)
\[ |H_n(O_{K,v}, O^*_K)| |H_n(O_{K,v}, O^*_K)| = (1 - q^{-n}), 
|H_n(O_{K,v}, O^*_K)| |H_n(O_{K,v}, O^*_K)| = q^{-kn}. \]

On \( \lambda_n(d, t)_k \), we have

**Lemma 4.4.** Let \( t \) be even. Then
\[ \lambda_n(d, t)_k = 2^{-1}(q^{n^2})^{-1}_{[n/2]} \times \begin{cases} 0 & \text{if } t \leq 2k - 2, \text{ n odd}, \\ q^{-(k-t/2)n} + \chi_k((-1)^{n/2}d)q^{-f_Kn/2} & \text{if } t \leq 2k - 2, \text{ n even}, \\ 1 & \text{if } 2k \leq t, \text{ n odd}, \\ 1 + \chi_k((-1)^{n/2}d)q^{-f_Kn/2} & \text{if } 2k \leq t, \text{ n even}. \end{cases} \]

Let \( t \) be odd. Then
\[ \lambda_n(d, t)_k = 2^{-1}(q^{n^2})^{-1}_{[n/2]} \times \begin{cases} 0 & \text{if } n \text{ is odd}, \\ q^{-(k-t/2)n} + \chi_k((-1)^{n/2}d)q^{-f_Kn/2} & \text{if } t \leq 2k - 1, \text{ n even}, \\ q^{-n/2} + \chi_k((-1)^{n/2}d)^{-f_Kn/2} & \text{if } 2k + 1 \leq t, \text{ n even}. \end{cases} \]

**Proof.** Let \( t \) be even. Assume \( k = 0 \). Then we have
\[ |H_n(O_{K,v}, dN(O^*_K))^{(t)}| = |H_n(O_{K,v-t/2}, dN(O^*_K-v-t/2))|. \]
When \( f_K = 1 \), that is, \( K/F \) is tamely ramified, using
\[ |H_n(O_K, dN(O_K^x)) \mod \sigma_K| = |S_n(O_1, dO_1^{x,2})|, \]

we can easily verify our assertion. Assume \( f_K \geq 2 \), hence \( N((O_K/\sigma_K O_K)^x) = O_1^x \). Then we see

\[
\lambda_n(d, t) = \lim_{v \to \infty} (q^{-2})_{[n/2]} q^{(1/2)n^2 t} \left( \frac{|H_n(O_{K,v}, dN(O_{K,v}^{x,t/2}))_0|}{|H_n(O_{K,v}, O_v)|} + \frac{|H_n(O_{K,v}, dN(O_{K,v}^{x,t/2}))_1|}{|H_n(O_{K,v}, O_v)|} \right)
\]

\[
= \lim_{v \to \infty} (q^{-2})_{[n/2]} \left( \frac{|H_n(O_{K,v}, dN(O_{K,v}^{x,t/2}))_0|}{|H_n(O_{K,v}, O_v)|} \right)
\]

\[
+ \frac{|H_n(O_{K,v}, dN(O_{K,v}^{x,t/2}))_1|}{|H_n(O_{K,v}, O_v)|} .
\]

The second term in the last bracket vanishes if \( n \) is odd. By Lemma 1.2, we see the first term is the bracket is equal to 1/2 if \( n \) is odd. When \( n \) is even, by (4.3), this is equal to

\[
\lim (q^{-2})_{[n/2]} (1-q^{-n}) \left( \frac{|H_n(O_{K,v}, dN(O_{K,v}^{x,t/2}))_0|}{|H_n(O_{K,v}, O_v)|} \right)
\]

\[
\qquad + q^{-n} \left( \frac{|H_n(O_{K,v}, dN(O_{K,v}^{x,t/2}))_1|}{|H_n(O_{K,v}, O_v)|} \right) .
\]

The first term in the bracket is equal to \((1-q^{-n})/2\). Since \( |H_3(O_{K,v}, O_v^x)| = q^{4v-2(1-q^{-1})} \), by Lemma 1.2 and 4.3, the limit of the second term of (4.4) is equal to

\[
\lim q^{-n} \sum_{d_1 d_2 \cdots d_n = n} \prod_{i=1}^{n/2} |A(d_i)|_v ||NO_{K,v}^x|| H_2(O_{K,v}, O_v^x)|
\]

\[
= 2^{-n} q^{-n} \sum_{d_1 d_2 \cdots d_n = n} \prod_{i=1}^{n/2} (1+\chi_K(-d_i)q^{-f_K-2})
\]

\[
= 2^{-1} q^{-n} (1+\chi_K(-1)^{n/2} d) q^{-f_K-2} n/2 .
\]

Our assertion for \( k=0 \) follows form this.

Assume \( k \geq 1 \). If \( t \geq 2k \), then

\[ |H_n(O_{K,v}, d\sigma^{n/2} O_{K,v}^{x})_0| = |H_n(O_{K,v}, d\sigma^{n/2} N(O_{K,v}))_0| , \]

and if \( t \leq 2k \), then

\[ |H_n(O_{K,v}, d\sigma^{n/2} N(O_{K,v})_k)| = |H_n(O_{K,v}, dN(O_{K,v}^{x,t/2}))_k| . \]

Our assertion easily follows from this in the same way as above by Lemma 1.2 and 4.3.

Let \( t \) be odd. Then \( n \) is even. First assume \( k=0 \). We have
\[ |H_n(O_{K,v}, d\sigma^{\nu/2} N(O^n_{K,v}))^{(1)}| = |H_n(O_{K,v-(t-1)/2}, d\sigma^{\nu/2} N(O^n_{K,v-(t-1)/2}))^{(1)}| \]

Using \[|H_n(O_{K,v}, d\sigma^{\nu/2} O^v_{K,v})^{(1)}| = |S_n(O_{K,v}, O^v_{K,v})|, \] by induction on \(v\), we find
\[ |H_n(O_{K,v}, d\sigma^{\nu/2} O^v_{K,v})^{(1)}| = |GL_n(O_{K,v})|q^{-\nu(v^2-n(n+1)/2)(v^2-n^2)^{-1}}/(|n/2|), \]

hence
\[
\frac{q^{\nu(n^2+n)/2}}{|GL_n(O_{K,v})|} = (q^{-2})^{-1}_{[n/2]} \frac{q^{(t-1)n^2/2-n/2}}{|H_n(O_{K,v}, d\sigma^{\nu/2} O^v_{K,v})^{(1)}|} = (q^{-2})^{-1}_{[n/2]} \frac{q^{-n/2}}{|H_n(O_{K,v-(t-1)/2}, d\sigma^{\nu/2} O^v_{K,v-(t-1)/2})^{(1)}|} .
\]

By Lemma 1.2 and 4.3, we can prove
\[ |H_n(O_{K,v}, d\sigma^{\nu/2} NO^v_{K,v})^{(1)}| = |H_n(O_{K,v}, d\sigma^{\nu/2} O^v_{K,v})^{(1)}| = 1 + \chi((-1)^n d)q^{-(f_K-1)n/2} .
\]

By this, we obtain our result. For \(k > 0\), we have
\[ |H_n(O_{K,v}, d\sigma^{\nu/2} NO^v_{K,v})^{(k)}| = \begin{cases} |H_n(O_{K,v}, d\sigma^{\nu/2} NO^v_{K,v})^{(1)}| & \text{if } k \leq t/2 , \\ |H_n(O_{K,v-(t-1)/2}, d\sigma^{\nu/2} NO^v_{K,v-(t-1)/2})^{(1)}| & \text{if } t/2 < k . \end{cases} \]

From this, in the same way as above, follows our assertion for \(k > 0\), and this completes the proof.

By this lemma, in the same way as in §2, we can prove

**Proposition 4.5.** Let \(K\) be a ramified quadratic extension of \(F\) and let \(d \in O^v\). When \(\{n_1\}\) is odd, \(\lambda_n(d, \{n_1\}, \{t_1\}) = 0\) unless \(n_1 t_1 \equiv 0 \mod 2\) and \(2k \leq t_1\) for \(n_1\) odd. Assume this condition. Then
\[
\lambda_n(d, \{n_1\}, \{t_1\})_k = 2^{-1}(q^{-1})_h P(q^{-2}, \{n_1\}) u^{t/2} q^{	ilde{d}(\{n_1\}, \{t_1\})} \\
\times \begin{cases} \prod_{t < 2k} q^{-(k-t/2)n_1} \prod_{n_1 \text{ even}, 2k \leq t_1, 2k \text{ odd}} q^{-n_1/2} & \text{if } \{n_1\} \text{ is odd} , \\
 + \chi_k((-1)^{n/2} d)q^{-f_K n/2} & \text{if } \{n_1\} \text{ is even} . \end{cases}
\]

**Proof of Th. 4.2.** We give a proof only for a ramified \(K\) with an even \(f_K\). By Prop.4.5, we see if \(n\) is odd,
\[ Z_n(u, d)_k = \frac{1}{2} (q^{-1})_n X_n^0(u^{1/2}, t)_{2k} \]
and if \(n\) is even
\[ Z_n(u, d)_k = \frac{1}{2} (q^{-1})_n (X_n^0(u^{1/2}, t)_{2k} + \chi_k((-1)^{n/2} d)q^{-f_K n/2} Y_n^0(u^{1/2}, t)) .\]
Our assertion follows from Lemma 2.8 and 2.9. This completes the proof.

References


[O] O. T. O'Meara; Introduction to quadratic forms, Springer.
